

QUALITATIVE PROPERTIES OF SOLUTIONS TO A REACTION-DIFFUSION EQUATION WITH WEIGHTED STRONG REACTION

RAZVAN GABRIEL IAGAR, ANA I. MUÑOZ, ARIEL SÁNCHEZ

ABSTRACT. We study the existence and qualitative properties of solutions to the Cauchy problem associated to the quasilinear reaction-diffusion equation

$$\partial_t u = \Delta u^m + (1 + |x|)^\sigma u^p,$$

posed for $(x, t) \in \mathbb{R}^N \times (0, \infty)$, where $m > 1$, $p \in (0, 1)$ and $\sigma > 0$. Initial data are taken to be bounded, non-negative and compactly supported. In the range when $m + p \geq 2$, we prove existence of local solutions with a finite speed of propagation of their supports for compactly supported initial conditions. We also show in this case that, for a given compactly supported initial condition, there exist infinitely many solutions to the Cauchy problem, by prescribing the evolution of their interface. In the complementary range $m + p < 2$, we obtain new Aronson-Bénilan estimates satisfied by solutions to the Cauchy problem, which are of independent interest as a priori bounds for the solutions. We apply these estimates to establish infinite speed of propagation of the supports of solutions if $m + p < 2$, that is, $u(x, t) > 0$ for any $x \in \mathbb{R}^N$, $t > 0$, even in the case when the initial condition u_0 is compactly supported.

1. INTRODUCTION

This article concerns the qualitative theory of the weak solutions to the Cauchy problem for the reaction-diffusion equation

$$\partial_t u = \Delta u^m + (1 + |x|)^\sigma u^p, \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \quad N \geq 1, \quad (1.1)$$

supplemented with the initial condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N. \quad (1.2)$$

The exponents in (1.1) are considered throughout the paper to belong to the range

$$m > 1, \quad 0 < p < 1, \quad 0 < \sigma < \infty, \quad (1.3)$$

although we also give alternative proofs or even slight improvements of known results with $\sigma = 0$. We consider bounded, compactly supported, non-negative and non-trivial initial conditions, more precisely

$$u_0 \in L^\infty(\mathbb{R}^N), \quad \text{supp } u_0 \subseteq B(0, R), \quad u_0(x) \geq 0, \quad \forall x \in \mathbb{R}^N, \quad u_0 \not\equiv 0, \quad (1.4)$$

2020 *Mathematics Subject Classification.* 35B44, 35B45, 35K57, 35K59.

Key words and phrases. Reaction-diffusion equations; weighted reaction; strong reaction; Aronson-Bénilan estimates.

©2023. This work is licensed under a CC BY 4.0 license.

Submitted June 13, 2023. Published October 23, 2023.

and further regularity assumptions (such as continuity) will be specified at the points where they are needed. In nonlinear diffusion problems it is standard to consider data belonging to the space $L^1_{\text{loc}}(\mathbb{R}^N)$ and we are sure that some of our results can be extended to this weaker space than $L^\infty(\mathbb{R}^N)$. However, for simplicity and also as in the range of exponents (1.3) finite time blow-up of bounded solutions is expected (as established, for example, for self-similar solutions in recent works such as [18, 19, 16]), we decided to avoid possible pointwise singularities at finite points and thus require (1.4).

A deep study of (1.1) in the range $p > 1$ has been performed by Andreucci and DiBenedetto in [1] for weights with any $\sigma \in \mathbb{R}$, that is, both positive and negative. Properties such as existence of local weak solutions under optimal growth conditions on the initial data, estimates, regularity of them and Harnack-type inequalities are obtained. The authors also specify in [1] that some of their results can be extended to the range $0 < p < 1$ but only when $\sigma < 0$ and leave open the case $\sigma > 0$ with $0 < p < 1$. We find this latter case very interesting to study due to the merging of the non-Lipschitz reaction term (which does not produce finite time blow-up by itself) with the unbounded weight.

The non-weighted equation

$$u_t = \Delta u^m + u^p, \quad (1.5)$$

corresponding to $\sigma = 0$ in (1.1) is nowadays quite well understood (at least in dimension $N = 1$) after a series of works by de Pablo and Vázquez [26, 27, 28, 25] and the outcome of them is very interesting, despite the fact that we are dealing with an ill-posed problem. It is shown that solutions exist always and they are global in time if u_0 satisfies a growth condition similar to the one required for existence in the porous medium equation, namely $u_0(x) = o(|x|^{2/(m-1)})$ [27] and that there is always a minimal and a maximal solution, both constructed via approximations. The most interesting and surprising features of this problem with $\sigma = 0$ are related with the uniqueness. This property depends strongly on two aspects: the sign of the critical exponent $m + p - 2$ and the positivity or not of the initial condition u_0 . More precisely we have the following:

- If $m + p - 2 \geq 0$, uniqueness of solutions to the Cauchy problem (1.5)-(1.2) holds if and only if $\text{supp } u_0(x) = \mathbb{R}^N$ [26]. If u_0 is compactly supported, then its support has the property of finite propagation and interfaces appear. Moreover, the extent of the non-uniqueness property in this case is addressed in [28] where it is shown that giving a rather general compactly supported initial condition u_0 and a function of time $\xi(t)$ advancing faster than the interface of the (unique) minimal solution constructed in [27], there is always a solution to the Cauchy problem with data u_0 and interface at time $t > 0$ given by $\xi(t)$. This is a very strong and sharp non-uniqueness property, giving rise in fact to an infinity of solutions.
- If $m + p - 2 < 0$, things change radically because of the infinite speed of propagation: even if u_0 is compactly supported it is shown in [26] that $u(x, t) > 0$ for any $x \in \mathbb{R}^N$ and $t > 0$, thus we have a property known as *quasi-uniqueness*: there exists a unique solution to the Cauchy problem for any initial condition u_0 except for the trivial one $u_0 \equiv 0$, where two different solutions are constructed.

Considering weighted reaction terms came as a natural extension of the already well developed knowledge on the “classical” reaction-diffusion equations with reaction of the form u^p or more general functions resembling it (see [32, 34] as important monographs on this subject). Many results were achieved for the semilinear case

$m = 1$ and unbounded weights of the form $|x|^\sigma u^p$ (and sometimes even more general weights $V(x)$ instead of pure powers), always with $p > 1$. Fujita-type exponents and conditions on the data for finite time blow-up to occur were studied in celebrated papers by Baras and Kersner, Bandle and Levine, Pinsky et al., see for example [6, 5, 29, 30]. More recently, still with $m = 1$, an interesting question was addressed: considering the equation

$$u_t = \Delta u + |x|^\sigma u^p, \quad (1.6)$$

it is natural to ask ourselves whether $x = 0$ (or more generally, any zero of a power-like weight $V(x)$) can be a blow-up point. Examples of both possible situations (when $x = 0$ is a blow-up point and when it is not) were constructed (mostly for Cauchy-Dirichlet problems posed in bounded domains) in the series of recent papers by Guo and collaborators [11, 12, 13, 14]. Finer analysis on how blow-up occurs, with rates and local asymptotic behavior in self-similar forms near the blow-up time and points were performed by Filippas and Tertikas [10] and in the very recent work by Mukai and Seki [24] for different ranges of the exponent $p > 1$.

Because of their further complexity and the fact that even for the non-weighted case $\sigma = 0$ there are some difficult open problems (see for example [34, Chapter 4]), equations such as (1.1) or its close relative

$$u_t = \Delta u^m + |x|^\sigma u^p, \quad (1.7)$$

with $m > 1$ have been less considered in literature. Apart from the quoted paper [1], the Fujita-type exponent and rather sharp conditions on the initial data for the finite time blow-up to hold have been obtained by Qi [31] and then Suzuki [35] for the case $p > m > 1$ (including also a part of the fast diffusion range $m < 1$ in [31]). We recommend Suzuki's paper to the reader as a well-written basic work on the qualitative theory for these equations, while blow-up rates as $t \rightarrow T$ also for $p > m$ are proved in [2].

In recent years, the authors of the present work started a larger project of understanding the patterns (in self-similar form) that solutions may take either in the case of global solutions, or close to the blow-up time if this occurs, for (1.7). This is an important part of the study, since it is well-known that such patterns are a prototype of the general behavior of the equation, in the form of asymptotic profiles as $t \rightarrow \infty$ or $t \rightarrow T$ and also bring a deeper understanding on the blow-up sets and rates. A series of papers [17, 20, 22, 15] addressed the question of the blow-up profiles to (1.7) for $m > 1$ and $1 \leq p \leq m$, where interesting and rather unexpected behaviors were established. In all these cases, solutions blow up in finite time when $\sigma > 0$, but their specific blow-up behavior it is shown to depend strongly on the magnitude of σ . Going back to the case of interest for us, $p \in (0, 1)$, we have proved that blow-up profiles exist if the following condition is fulfilled:

$$L := \sigma(m - 1) + 2(p - 1) > 0. \quad (1.8)$$

We classified such blow-up profiles in [16] in general dimensions $N \geq 2$, following previous results restricted to dimension $N = 1$ in [19, 21], obtaining again that the sign of $m + p - 2$ is fundamental for their existence and behavior.

- When $m + p - 2 > 0$, all the blow-up self-similar profiles present an interface (that is, they are compactly supported) and there are two different types of possible interface behaviors. This is both a manifestation of non-uniqueness (expected for

compactly supported data) and a suggestion of possible non-existence of solutions when $u_0(x) > 0$, as there is no pattern they can approach when $t \rightarrow T$.

- When $m+p = 2$, self-similar blow-up patterns present a rather similar panorama, but with a single type of interface behavior.

- When $m+p < 2$, a rather striking non-existence of any kind of blow-up profiles (either with interfaces or not) occurs [19]. Such an outcome gives the intuition of an infinite speed of propagation and a complete non-existence of non-trivial solutions.

In this paper we thus begin the qualitative study of the Cauchy problem for (1.1) posed in \mathbb{R}^N . We thus address the quite interesting (in view of precedents such as [26, 28]) question of speed of propagation and non-uniqueness for compactly supported data, deriving in the process some new Aronson-Bénilan estimates for the solutions to (1.1) when $m+p < 2$. We stress here that uniform lower bounds on the weight are essential in the forthcoming proofs, thus technical complications introduced by the weight in a neighborhood of $x = 0$ (where it is no longer uniformly positive) appear when trying to adapt the same proofs to the close relative (1.7). But let us present below in more detail our main results.

Main results. To state our results concerning the qualitative theory of solutions to (1.1), we first have to introduce the notion of *weak solution* that will be used throughout the paper. Let us denote by $u(t)$ the mapping $x \mapsto u(x, t)$ for $t > 0$ fixed. We will slightly modify the functional framework from Andreucci and DiBenedetto [1] by passing in our weak formulation the full Laplacian to the test function.

Definition 1.1. A non-negative function $u : \mathbb{R}^N \times (0, T) \mapsto [0, \infty)$ is said to be a *weak solution* to the Cauchy problem (1.1)-(1.2) with initial condition u_0 as in (1.4) if the following assumptions are satisfied by u :

(a) *Regularity assumption:* we have

$$u \in C(0, T; L^1_{\text{loc}}(\mathbb{R}^N)) \cap L^\infty_{\text{loc}}(\mathbb{R}^N \times (0, T)). \quad (1.9)$$

(b) *Weak formulation of (1.1):* For any test function $\eta \in C^\infty_0(\mathbb{R}^N \times (0, T))$ and for any $t \in (0, T)$ we have

$$\begin{aligned} & \int_{\mathbb{R}^N} u(x, t) \eta(x, t) dx + \int_0^t \int_{\mathbb{R}^N} (-u(x, \tau) \eta_t(x, \tau) - u^m(x, \tau) \Delta \eta(x, \tau)) dx d\tau \\ &= \int_0^t \int_{\mathbb{R}^N} (1 + |x|)^\sigma u^p(x, \tau) \eta(x, \tau) dx d\tau. \end{aligned} \quad (1.10)$$

(c) *Taking the initial condition:* this is done in L^1 sense, more precisely

$$\lim_{t \rightarrow 0} u(t) = u_0, \quad \text{with convergence in } L^1_{\text{loc}}(\mathbb{R}^N) \quad (1.11)$$

This change relaxes the functional assumptions of regularity for a solution, making it easier to obtain weak solutions by limiting processes. We will also need throughout the paper the notions of (weak) sub- and supersolution to (1.1). We say that u is a weak subsolution (respectively weak supersolution) to (1.1) if condition (a) is fulfilled and condition (b) is modified in the sense that the equal sign is replaced by \leq (respectively \geq) for any test functions $\eta \in C^\infty_0(\mathbb{R}^N \times (0, T))$ such that $\eta \geq 0$ and for any $t \in (0, T)$. Moreover, the notions of weak solution, subsolution, supersolution to (1.1) (without the initial condition) can be defined in an obvious way on time intervals $[t_1, t_2] \subset (0, T)$ instead of $(0, T)$ by just removing assumption (c).

Once defined the notion of weak solution, we are in a position to give below the main theorems of this work. As explained above, it is expected from the results in papers such as [26, 27, 28] for equations with non-weighted reaction that, in our range of exponents, ill-posedness of the Cauchy problem will still hold and both existence and uniqueness of solutions are an issue. As we shall see below, if we consider initial conditions u_0 as in (1.4), these basic properties of existence and uniqueness of weak solutions are strongly related to the finite or infinite speed of propagation of their edge of the support. Throughout the paper, we will denote by $C_0(\mathbb{R}^N)$ the space of continuous, compactly supported functions on \mathbb{R}^N .

Range $m + p \geq 2$. Let us first focus on the range of exponents for which $m + p \geq 2$ and $\sigma > 0$. In this case, we shall prove that, at least for some interval of time $t \in (0, T)$, there exist infinitely many weak solutions to (1.1). We begin with the existence of at least one local (in time) weak solution, as stated in the following result.

Theorem 1.2 (Existence of compactly supported solutions). *In our framework and notation, assume that $m + p \geq 2$ and let u_0 be an initial condition as in (1.4) satisfying moreover that $u_0 \in C_0(\mathbb{R}^N)$. Then there exists $T > 0$ and there exists at least a weak solution u to the Cauchy problem (1.1)-(1.2) for $t \in (0, T)$ which remains continuous and compactly supported: $u(t) \in C_0(\mathbb{R}^N)$ for any $t \in (0, T)$.*

The proof relies on the construction of so-called *minimal solutions* to the Cauchy problem (1.1)-(1.2), which are obtained through a limit process from a family of approximating (regularized) Cauchy problems. It will be shown that such a minimal solution exists for any compactly supported condition u_0 and stays compactly supported for $t \in (0, T)$, a property known as finite speed of propagation. The (local in time) finite speed of propagation will be proved with the aid of comparison with solutions and supersolutions in self-similar form introduced in the recent works [19, 21, 16].

The statement of Theorem 1.2 and precedents in the non-weighted case [28] give the idea that uniqueness does not hold. In fact, we infer from Theorem 1.2 that at intuitive level the weight $V(x) = (1 + |x|)^\sigma$ is equivalent to a constant for times $t \in (0, T)$ (that is, while the support of $u(t)$ remains finite), thus a similar property to the non-weighted case concerning non-uniqueness of solutions is expected. The next result characterizes the extent of this non-uniqueness, by showing that we can prescribe in infinitely many ways the evolution of the interface of a solution stemming from the same initial condition.

Theorem 1.3 (Non-uniqueness of compactly supported solutions). *In our framework and notation, assume that $m + p \geq 2$ and let u_0 be an initial condition as in (1.4) satisfying moreover that $u_0 \in C_0(\mathbb{R})$. Then there exists $T > 0$ and infinitely many weak solutions to the Cauchy problem (1.1)-(1.2) for $t \in (0, T)$. The same existence of infinitely many weak solutions holds true for radially symmetric initial conditions $u_0 \in C_0(\mathbb{R}^N)$ satisfying (1.4).*

This result is rather similar to the non-weighted case $\sigma = 0$, although the proof will be technically more involved: it is not clear whether there exists when $\sigma > 0$ a maximal solution, in order to get almost for free a different, second solution (as it holds true for $\sigma = 0$, see [26, 27]), thus we have to use a different approach. The statement of Theorem 1.3 will be thus enforced and made more precise in Section 3, where we show that the existence of infinitely many solutions is linked to a

prescribed evolution of their interface in time, adapting but also slightly improving techniques from [28].

Range $m + p < 2$. In this complementary range, our main goal is to prove that solutions to the Cauchy problem for (1.1) with initial conditions as in (1.4) have infinite speed of propagation, that is, they become immediately positive at any point $x \in \mathbb{R}^N$. This fact extends to the non-homogeneous range $\sigma > 0$ a similar result holding true for $\sigma = 0$, established in [26], but we use a different approach which gives, along the way, a result of independent interest in the study of the homogeneous equation (1.5). We begin by establishing the following Aronson-Bénilan estimates (whose name stems from the celebrated short note [3] on solutions to the porous medium equation) for solutions to the homogeneous case $\sigma = 0$. We thus introduce the pressure function

$$v = \frac{m}{m-1} u^{m-1}.$$

Theorem 1.4 (Aronson-Bénilan estimates when $m + p < 2$ and $\sigma = 0$). *Assume that $m + p < 2$. Let u be a weak solution to (1.5) in $\mathbb{R}^N \times (0, T)$ with a continuous initial condition $u_0(x) = u(x, 0)$ satisfying (1.4) and let v be the pressure variable introduced above. Then the inequality*

$$\Delta v \geq -\frac{K}{t}, \quad K = \frac{N}{N(m-1)+2}, \quad (1.12)$$

holds in the sense of distributions in \mathbb{R}^N , that means,

$$\int_0^T \int_{\mathbb{R}^N} \left(v(x, t) \Delta \varphi(x, t) + \frac{K}{t} \varphi(x, t) \right) dx dt \geq 0, \quad (1.13)$$

for any test function $\varphi \in C_0^\infty(\mathbb{R}^N \times (0, T))$ such that $\varphi \geq 0$ in $\mathbb{R}^N \times (0, T)$.

Remark. Theorem 1.4 is expected to hold also for any $\sigma > 0$, provided $N \geq 2$. We give a formal proof of this fact at the end of Section 4. However, transforming it into a rigorous proof is impossible by now for $\sigma > 0$, as we are lacking a well-posedness result for the Cauchy problem (1.1)-(1.2) in the range $m + p < 2$. This is why, we introduce this formal proof as a remark.

We then employ the Aronson-Bénilan estimates established in Theorem 1.4 and some consequences of them in order to prove the infinite speed of propagation of the supports of solutions to (1.1) when $m + p - 2 < 0$, which is strongly contrasting to the results established in the range $m + p \geq 2$. We include the range $\sigma = 0$ in the statement, as our approach gives an alternative proof to the one in [26].

Theorem 1.5 (Infinite speed of propagation when $m + p < 2$). *If the exponents m and p satisfy $m + p < 2$, Equation (1.1) has the property of infinite speed of propagation for any $\sigma \geq 0$, that is, every weak solution (if it exists) to the Cauchy problem associated to (1.1) with continuous initial condition u_0 as in (1.4) satisfies $u(x, t) > 0$ for any $x \in \mathbb{R}^N$ and $t > 0$.*

The rest of this article is devoted to the proofs of the main theorems, following the outlines explained in the comments near their statements. We then give at the end of the paper a list of open problems and possible extensions of our study that we believe to be interesting for future developments.

2. EXISTENCE AND FINITE SPEED OF PROPAGATION WHEN $m + p \geq 2$

The goal of this section is to prove Theorem 1.2 in the range of parameters $m + p \geq 2$. Let u_0 be a continuous initial condition as in (1.4). The scheme of the proof is based on the construction of a minimal solution via an approximation process and showing that this minimal solution is a weak solution to the Cauchy problem (1.1)-(1.2) with compact support at any time $t \in (0, T)$ for some $T > 0$.

Proposition 2.1. *There exists some $T > 0$ and a continuous weak solution \underline{u} defined and compactly supported for $t \in (0, T)$ to the Cauchy problem (1.1)-(1.2) with u_0 as above such that for any other weak solution u to the Cauchy problem (1.1)-(1.2) (if it exists), we have*

$$\underline{u}(x, t) \leq u(x, t), \quad \text{for all } (x, t) \in \mathbb{R}^N \times (0, T).$$

Proof. We divide the proof into three steps for the reader's convenience. Let us stress at this point that, while Step 1 of the proof is an adaptation of an analogous construction in [27], the idea in Steps 2 and 3 strongly departs from the one used in the previously mentioned work and employs results on self-similar solutions to (1.7) published recently by the authors.

Step 1. Construction of the minimal solution. We approximate the Cauchy problem (1.1)-(1.2) by the following sequence of Cauchy problems for any positive integer $k \geq 1$

$$\begin{aligned} w_t &= \Delta w^m + \min\{(1 + |x|)^\sigma, k\} f_k(w), & (x, t) \in \mathbb{R}^N \times (0, \infty), \\ w(x, 0) &= u_0(x), & x \in \mathbb{R}^N, \end{aligned} \quad (2.1)$$

where

$$f_k(w) = \begin{cases} (\frac{1}{k})^{p-1} w, & \text{if } 0 \leq w \leq \frac{1}{k}, \\ w^p, & \text{if } w \geq \frac{1}{k}. \end{cases}$$

Since the nonlinearity in (2.1) is of the form $g(x)h(w)$ with $g \in L^\infty(\mathbb{R}^N)$, $g(x) \geq 1$ for any $x \in \mathbb{R}^N$ and h is a Lipschitz function, we infer by standard results for quasi-linear parabolic equations (see for example [9, 33]) that the Cauchy problem (2.1) admits a unique solution w_k defined for $(x, t) \in \mathbb{R}^N \times (0, \infty)$, which is compactly supported and continuous. The comparison principle is in force for the Cauchy problem (2.1) and w_{k+1} is a supersolution to the problem (2.1), thus $w_{k+1} \geq w_k$ for any $k \geq 1$. This allows us to introduce the pointwise (and monotone increasing) limit (which might become infinite starting from some finite time)

$$\underline{u}(x, t) = \lim_{k \rightarrow \infty} w_k(x, t) < \infty, \quad (x, t) \in \mathbb{R}^N \times (0, T_\infty),$$

which is well defined provided that $T_\infty > 0$. This fact will follow from the construction of a ‘‘universal’’ family of supersolutions in self-similar form which is postponed to Step 2 (in dimension $N = 1$) and Step 3 (in dimension $N \geq 2$) below. Moreover, the solutions w_k are thus uniformly bounded on $\mathbb{R}^N \times (0, T)$ for some $T = T(u_0) > 0$ depending on u_0 , and this uniform boundedness, together with classical results in [9, 33], imply that the family $(w_k)_{k \geq 1}$ is uniformly equicontinuous in $\mathbb{R}^N \times [0, T]$, hence there exists a subsequence (re-labeled also w_k for simplicity) which converges locally uniformly to the same function $\underline{u}(x, t)$.

Then the fact that \underline{u} is a continuous weak solution to (1.1)-(1.2) for $(x, t) \in \mathbb{R}^N \times (0, T_\infty)$ follows now readily from the previous convergences (assuming for now the outcome of Steps 2 and 3 below): indeed, Lebesgue's monotone convergence

theorem ensures the assumptions (b) and (c) in Definition 1.1, while the uniform bound by the supersolutions constructed in Steps 2 and 3 below, together with the equicontinuity, give that \underline{u} satisfies the regularity assumption (a) in Definition 1.1. Moreover, if u is another weak solution to the Cauchy problem (1.1)-(1.2), then it is a supersolution to the problem (2.1) for any $k \geq 1$, whence $u(x, t) \geq w_k(x, t)$ for any $k \geq 1$ and $(x, t) \in \mathbb{R}^N \times (0, \infty)$ and by passing to the limit $u \geq \underline{u}$ in $\mathbb{R}^N \times (0, T_\infty)$, proving the minimality of \underline{u} .

Step 2. Supersolutions in dimensions $N = 1$. We are left with the task of obtaining a uniform bound from above for all solutions w_k to the Cauchy problems (2.1), $k \geq 1$, at least up to some (short) finite time. This follows by comparison with suitable supersolutions in self-similar form constructed in recent works by the authors such as [16] for $N \geq 2$ and $m + p \geq 2$, respectively [19, 21] in dimension $N = 1$ and either $m + p > 2$ or $m + p = 2$. If we restrict ourselves only to dimension $N = 1$ for this step, it is shown in the previously quoted works that, in our range of exponents together with the extra condition $\sigma > 2(1-p)/(m-1)$, there are radially symmetric blow-up self-similar supersolutions to (1.7) in the form

$$u(x, t) = (T - t)^{-\alpha} f(|x|(T - t)^\beta), \quad \alpha = \frac{\sigma + 2}{L}, \quad \beta = \frac{m - p}{L}, \quad (2.2)$$

where $L > 0$ has been defined in (1.8), such that their self-similar profiles f solve the differential equation

$$\begin{aligned} (f^m)''(\xi) - \alpha f(\xi) + \beta \xi f'(\xi) + \xi^\sigma f(\xi)^p &= 0, \\ \xi &= |x|^\sigma (T - t)^\beta. \end{aligned} \quad (2.3)$$

Moreover, it is established in [19, Proposition 4.1] if $m + p > 2$ and in [21, Proposition 3.1] if $m + p = 2$ that the self-similar profiles f of the supersolutions in the form (2.2) fulfill the following two additional properties:

- f is strictly decreasing until reaching the zero level: $f(0) = A > 0$, $f'(\xi) < 0$ at points $\xi \geq 0$ where $f(\xi) > 0$.
- f presents an interface at some finite point $\xi_0 \in (0, \infty)$, that is, $f(\xi_0) = 0$, $f(\xi) > 0$ for any $\xi \in (0, \xi_0)$ and $(f^m)'(\xi_0) = 0$.

Here the blow-up time $T > 0$ is a free parameter and the functions defined in (2.2) are actually weak solutions to (1.7) except at the point $x = 0$ where the condition $(f^m)'(0) = 0$ is not fulfilled in order to be a weak solution. We adapt these supersolutions to our equation (1.1) by defining

$$z(x, t) = (T - t)^{-\alpha} f((1 + |x|)(T - t)^\beta), \quad (2.4)$$

with α , β and f as in (2.2). Since f is a supersolution to the differential equation (2.3), it is straightforward to check that z is a supersolution to (1.1). The amplitude $s(t)$ of the support of $z(t)$ at some $t \in (0, T)$ is given by $\xi_0 = (1 + s(t))(T - t)^\beta$ or equivalently

$$s(t) = (T - t)^{-\beta} \xi_0 - 1 \rightarrow \infty, \quad \text{as } t \rightarrow T.$$

Moreover, the above supersolutions have been established in [19, 21] only under the condition $\sigma > 2(1-p)/(m-1)$. However, if $0 < \sigma \leq 2(1-p)/(m-1)$, it is obvious from the fact that $1 + |x| \geq 1$, that $(1 + |x|)^\sigma \leq (1 + |x|)^{\sigma_1}$ for any $\sigma_1 > 2(1-p)/(m-1)$ and for any $x \in \mathbb{R}$. It thus follows that the supersolutions constructed for (1.1) with such an exponent $\sigma_1 > 2(1-p)/(m-1)$ as above, will serve also as supersolutions to (1.1) with exponents σ smaller. Since T is a free

parameter and u_0 is bounded and compactly supported, we can choose some $T_0 > 0$ sufficiently small such that

$$z(x, 0) = T_0^{-\alpha} f((1 + |x|)T_0^\beta) \geq \|u_0\|_\infty \geq u_0(x) \tag{2.5}$$

for any $x \in \text{supp } u_0$. This, together with the fact that any function z as in (2.4) is a supersolution to (1.1), gives that z is also a supersolution to the Cauchy problem (2.1) for any $k \geq 1$ for which the comparison principle applies to give that

$$w_k(x, t) \leq z(x, t), \quad \text{for all } (x, t) \in \mathbb{R} \times (0, T_0).$$

In particular we infer on the one hand that $T_\infty \geq T_0 > 0$ as claimed, and on the other hand by passing to the limit as $k \rightarrow \infty$ we obtain

$$\underline{u}(x, t) \leq z(x, t), \quad \text{for all } (x, t) \in \mathbb{R} \times (0, T_0),$$

which proves the finite speed of propagation of the support of \underline{u} at least for some short interval of time.

Step 3. Supersolutions in dimension $N \geq 2$. In this case, there are no longer bounded and decreasing supersolutions in the self-similar form used in Step 2. We thus construct suitable supersolutions to (1.7) by joining two different self-similar profiles. We begin again by fixing $\sigma > 2(1 - p)/(m - 1)$ as a first case. We look for self-similar solutions to (1.7) in the same form (2.2) as above, with the same exponents α and β , but whose profiles solve the differential equation

$$(f^m)''(\xi) + \frac{N - 1}{\xi} (f^m)'(\xi) - \alpha f(\xi) + \beta \xi f'(\xi) + \xi^\sigma f(\xi)^p = 0, \tag{2.6}$$

$$\xi = |x|^\sigma (T - t)^\beta.$$

On the one hand, the analysis in [16, Proposition 4.1] if $m + p > 2$, respectively [16, Proposition 4.2 and Lemma 4.3] if $m + p = 2$, ensure that, given $\xi_0 \in (0, \xi_*)$ sufficiently small, there exists a decreasing self-similar profile $f_2(\xi)$ solution to the differential equation (2.6) having an interface exactly at $\xi = \xi_0$ and a vertical asymptote as $\xi \rightarrow 0$ with local behavior

$$f_2(\xi) \sim \begin{cases} C \xi^{-(N-2)/m}, & \text{if } N \geq 3, \\ C (-\ln \xi)^{1/m}, & \text{if } N = 2, \end{cases}$$

as it follows from [16, Lemma 3.2 and Lemma 3.5], where $C > 0$ designs a positive constant that might change from one case to another. On the other hand, the analysis performed in [16, Lemma 3.1] implies that, for any $A > 0$, there exists a profile $f_1(\xi; A)$ local solution to (2.6) such that

$$f_1(0; A) = A, \quad f_1(\xi; A) \sim \left[A^{m-1} + \frac{\alpha(m-1)}{2mN} \xi^2 \right]^{1/(m-1)}, \quad \text{as } \xi \rightarrow 0,$$

which is increasing in a right-neighborhood of the origin up to some maximum point $\xi_1(A) > 0$. Thus, given an initial condition u_0 as in (1.4), one can choose for example $A = \|u_0\|_\infty$ and fix some $\xi_0 \in (0, \xi_1(A)) \cap (0, \xi_*)$ such that there exists a decreasing profile $f_2(\xi)$ as above with vertical asymptote as $\xi \rightarrow 0$ and edge of the support at $\xi = \xi_0$. The profiles $f_1(\cdot; A)$ and f_2 have to cross at some point $\bar{\xi} \in (0, \xi_1(A))$. We finally define a self-similar supersolution to (1.1) as follows:

$$z(x, t) = (T - t)^{-\alpha} f((1 + |x|)(T - t)^\beta), \quad f(\xi) = \begin{cases} f_1(\xi; A), & \xi \in [0, \bar{\xi}], \\ f_2(\xi), & \xi \geq \bar{\xi}, \end{cases} \tag{2.7}$$

and notice that this is indeed a supersolution for any $T > 0$, as it can be described alternatively as

$$z(x, t) = \min\{z_1(x, t), z_2(x, t)\}, \quad z_i(x, t) = (T - t)^{-\alpha} f_i((1 + |x|)(T - t)^\beta),$$

for $i = 1, 2$, and z_1, z_2 are in fact solutions. The same considerations about the magnitude of σ as in the end of Step 2 ensure that the supersolution defined in (2.7) works also for values of σ smaller than $2(1-p)/(m-1)$. We thus complete the proof by choosing a sufficiently small $T_0 > 0$ such that (2.5) holds true on the support of u_0 , and notice that z is a supersolution to the approximating problems (2.1) leading to the minimal solution. This again implies that $T_\infty \geq T_0 > 0$, completing the proof. \square

The solution \underline{u} constructed in the proof of Proposition 2.1 will be referred as *the minimal solution* to the Cauchy problem (1.1)-(1.2) and denoted by $M(u_0)$ in the sequel. Notice that the above proof does not imply that necessarily the minimal solution blows up in finite time. In fact, it might blow up or not according to whether σ is larger or smaller than $2(1-p)/(m-1)$, but this is not easy to prove once we miss a comparison principle, and we refer the reader to the section of open problems at the end.

3. NON-UNIQUENESS FOR $m + p \geq 2$

A natural question raised by the previous section is whether in the case $m + p \geq 2$ and for compactly supported and continuous data u_0 the minimal solution \underline{u} constructed in Proposition 2.1 is the only solution to the Cauchy problem (1.1)-(1.2). For $\sigma = 0$ non-uniqueness follows easily from the construction of a different solution called maximal, which is shown to be strictly positive for any $t > 0$ and thus different from \underline{u} , see [27]. We cannot construct such a maximal solution to (1.1), since strictly positive solutions might not even exist at all in some ranges of σ (see a comment with a formal intuition for such non-existence in the final Section of this work). In order thus to prove the existence of multiple solutions (and in fact an infinite number of them) we adapt to our case the deeper results in [28], where infinitely many solutions to the Cauchy problem (1.5)-(1.2) are constructed, based on a prescribed evolution of the interface of them. As a preliminary fact, let us recall that (1.5) admits in dimension $N = 1$ an absolute minimal solution in self-similar form

$$E(x, t) = t^{1/(1-p)} \varphi(|x|t^{-\gamma}), \quad \gamma = \frac{m-p}{2(1-p)} \quad (3.1)$$

with zero initial condition $E(x, 0) = 0$ for any $x \in \mathbb{R}$, according to [26]. It is also shown in [26] that such solution lies below any solution (and supersolution) to (1.5) and consequently also below any solution to (1.1) (which is a strict supersolution to (1.5)). Moreover, the profile φ of E is non-increasing and compactly supported, thus $\text{supp } E \subseteq [-\varrho_0 t^\gamma, \varrho_0 t^\gamma]$ where $\varrho_0 > 0$ is the right-interface point of the profile φ .

With these elements and notation in mind, we can prove Theorem 1.3 as an immediate consequence of a stronger result adapting a construction from [28]. More precisely, by prescribing the behavior of the interface (under some limitations) we can obtain a solution to the Cauchy problem (1.1)-(1.2) with exactly that given interface at every (small) time $t \in (0, T)$. We formalize this below in dimension

$N = 1$, but the result for radially symmetric solutions in dimension $N \geq 2$ will be then completely analogous.

Proposition 3.1. *Let $u_0 \in C_0(\mathbb{R})$ be a compactly supported initial condition such that $\text{supp } u_0 = [r_0, R_0]$ for some $r_0, R_0 \in \mathbb{R}$. Let $M(u_0)$ be the minimal solution to the Cauchy problem (1.1)-(1.2) with initial condition u_0 defined for $t \in (0, T_0)$ and denote by $s_l(t), s_r(t)$ the left and right interfaces of $M(u_0)(t)$ for $t \in (0, T_0)$, that is, $\text{supp } M(u_0)(t) = [s_l(t), s_r(t)]$. Let $\xi_l(t), \xi_r(t)$ be two continuous functions of time such that $\xi_l(0) \leq r_0, \xi_r(0) \geq R_0$ and*

$$\xi_l(t_1) - \xi_l(t_2) \geq s_l(t_1) - s_l(t_2), \quad \xi_r(t_2) - \xi_r(t_1) \geq s_r(t_2) - s_r(t_1),$$

for any $t_1, t_2 \in (0, T_0)$ such that $t_1 < t_2$. Then there exists at least a shorter time interval $(0, T) \subset (0, T_0)$ and a solution u to the Cauchy problem (1.1)-(1.2) with initial condition u_0 defined for $t \in (0, T)$ such that its left and right interfaces are given exactly by $\xi_l(t)$ and $\xi_r(t)$ for any $t \in (0, T)$.

Proof. We divide the proof into two steps.

Step 1. Fundamental extension. This step adapts to our problem Step 1 in [28, Proof of Corollary 1.2] and at its end in fact we already have the proof of Theorem 1.3. The goal here is to construct a solution to our Cauchy problem whose support has an instantaneous jump to the right from $R_0 = s_r(0)$ to $R_0 + r$ at time $t = 0$ for a given (fixed) $r > 0$ (and of course, a perfectly similar construction can be done for the left interface). Let then $r > 0$ be given. For any $\epsilon > 0$ sufficiently small (it is enough to start for example from $\epsilon < \|u_0\|_\infty/2$) there exists a last, closest point (that we denote by R_ϵ) to the right interface R_0 of u_0 such that $u_0(R_\epsilon) = \epsilon$. Let us introduce the following compactly supported and continuous initial condition

$$u_{\epsilon,0}(x) = \begin{cases} u_0(x), & \text{for } x \in (-\infty, R_\epsilon), \\ \frac{\epsilon(R_0+r-x)}{R_0+r-R_\epsilon}, & \text{for } x \in [R_\epsilon, R_0+r], \\ 0, & \text{for } x \in (R_0+r, \infty), \end{cases} \tag{3.2}$$

that is, adding a linear extension to u_0 from R_ϵ to the new edge of the support $R_0 + r$. Let $u_\epsilon = M(u_{\epsilon,0})$ be the minimal solution associated to this new condition. By comparison between minimal solutions (which holds since it is obvious that any solution to (1.1) with a larger initial condition than the given u_0 becomes a supersolution to any of the problems (2.1) approximating the minimal solution $M(u_0)$) it readily follows that $u_{\epsilon_2}(x, t) \geq u_{\epsilon_1}(x, t)$ for any $\epsilon_2 > \epsilon_1 > 0$ and at any $(x, t) \in \mathbb{R} \times (0, T)$ where $T > 0$ is for example the lifetime of the solution with the biggest ϵ chosen. Recalling the construction of the absolute minimal solution $E(x, t)$ to (1.5) (see the details in [26, Theorem 4]) and the fact that any non-trivial solution to our Eq. (1.1) is a strict supersolution to (1.5) we easily conclude that for any $(x_0, t_0) \in \mathbb{R} \times (0, T)$ with $x_0 \in [R_\epsilon, R_0 + r]$ we have

$$u_\epsilon(x, t) \geq E(x - x_0, t - t_0), \quad t > t_0. \tag{3.3}$$

We can then define

$$\bar{u}(x, t) = \lim_{\epsilon \rightarrow 0} u_\epsilon(x, t), \quad (x, t) \in \mathbb{R} \times (0, T),$$

which is a monotone limit and it is easy to see that \bar{u} is a weak solution to (1.1) using the Monotone Convergence Theorem (in fact we even have uniform convergence by Dini's Theorem since the limit is continuous). Let us stress here that we do not

need in this construction a bound from above to guarantee that the limit is finite, since we are dealing with a decreasing limit. It is then obvious that

$$\bar{u}(x, 0) = \lim_{\epsilon \rightarrow 0} u_{\epsilon, 0}(x) = u_0(x)$$

and the comparison from above with E in (3.3) transfers to the limit \bar{u} , proving that for any $t > 0$ we have $\bar{u}(x, t) > 0$ for $x \in (R_0, R_0 + r)$.

Step 2. Iterative construction. We perform now an iterative construction based on a discretization in time and the application of Step 1 in each interval of this discretization to perform an instantaneous jump in the supports. Let then $\xi_l(t)$ and $\xi_r(t)$ be two continuous, increasing functions of time as in the statement of Proposition 3.1. Fix a time $t \in (0, T)$ for a $T > 0$ sufficiently small (that will be chosen later). For any positive integer n we construct an approximate solution u_n as in [28, Proof of Corollary 1.2]. We briefly and sketchy describe the construction here for the sake of completeness. Consider a partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t, \quad t_j = \frac{jt}{n}$$

and construct the function u_n by induction. More precisely, assume first that u_n is already constructed for $t \in [0, t_j]$ and we want to pass to t_{j+1} . To this end, we begin from the edges of the support of $u_n(t_j)$ and we perform a jump of them by applying Step 1 both to the right (with $r = \xi_r(t_{j+1}) - \xi_r(t_j)$) and to the left (with $r = \xi_l(t_j) - \xi_l(t_{j+1})$), using here the fact that the speed of advance of the prescribed interfaces to the left and to the right is higher than the ones of the minimal solution with initial condition u_0 . The precise details are easy and given in the above mentioned proof of [28, Corollary 1.2]. To pass to the limit as $n \rightarrow \infty$ in the iteration we need an uniform bound from above to show that the limit solution does not escape to infinity. We cannot use a translation of a minimal solution or a construction based on it (as it was done in [28, Lemma 2.4]) since our equation is not invariant to translations, but instead we can bound uniformly from above the iterated solutions by a sufficiently big non-increasing supersolution in self-similar form

$$U(x, t) = (T - t)^{-\alpha} f((1 + |x|)(T - t)^\beta), \quad \alpha = \frac{\sigma + 2}{L}, \quad \beta = \frac{m - p}{L}$$

similar to the ones introduced in (2.2) with a profile $f(\xi)$ solving (2.3) and having a right interface at $\xi_0 \in (0, \infty)$. Indeed, comparison from above with such a supersolution U can be performed as the iterative construction of u_n is based on adding up at each iteration step only minimal solutions, provided that at our fixed time $t > 0$ for which we have built u_n we have ordered supports between u_n and U , that is

$$\xi_r(t) < \xi_0(T - t)^{-\beta}, \quad \xi_l(t) > -\xi_0(T - t)^{-\beta}, \quad (3.4)$$

which also gives a limitation for the lifetime $T > 0$ depending on the two prescribed functions ξ_r , ξ_l . We notice that for “faster” advancing interface functions $\xi_l(t)$, $\xi_r(t)$, smaller lifetime T is expected. Once satisfied this condition, we can pass to the limit in the discretization as $n \rightarrow \infty$ and obtain the desired weak solution

$$v(x, t) = \lim_{n \rightarrow \infty} u_n(x, t), \quad t \in (0, T),$$

with $T > 0$ chosen sufficiently small according to (3.4). The bound from below by E at every point of positivity (as done in Step 1) together with the construction and the continuity of the prescribed functions ξ_l , ξ_r ensure that the left and right

interfaces of v at every time $t \in (0, T)$ are given exactly by the continuous functions $\xi_l(t)$ and $\xi_r(t)$. In the meantime, the universal bound from above by the supersolution U in self-similar form ensures that at least in the time interval $(0, T)$ we have $v(x, t) < \infty$ for any $x \in \mathbb{R}$. \square

Remarks. (1) Step 1 above already completes the proof of Theorem 1.3 in dimension $N = 1$. Indeed, for every $r > 0$ we can construct a different solution to the same Cauchy problem (1.1)-(1.2). A totally analogous extension can be constructed for radially symmetric initial conditions $u_0 \in C_0(\mathbb{R}^N)$ by replacing x by $r = |x|$, which allows us to work with the radial variable exactly as in dimension $N = 1$ but using for comparison from above the supersolutions introduced in Step 3 of the proof of Proposition 2.1, thus completing the proof of Theorem 1.3 also in dimension $N \geq 2$.

(2) Our Proposition 3.1 also holds for $\sigma = 0$ and improves slightly the result for (1.5) in [28, Theorem 1.1 and Theorem 5.1] since we do not need to consider the non-increasing majorant \tilde{u}_0 of the initial condition as considered in the above mentioned work.

4. ARONSON-BÉNILAN ESTIMATES WHEN $m + p < 2$

This section is devoted to the deduction of the Aronson-Bénilan estimates (1.12) in the homogeneous case $\sigma = 0$. Let us recall here the pressure function

$$v = \frac{m}{m-1} u^{m-1},$$

since most of the forthcoming work will be performed on this function.

Proof of Theorem 1.4. The proof is inspired from the one for [36, Proposition 9.4] but technically more involved. We first derive after rather straightforward calculations the pressure equation which will be used further in the present work, that is, the parabolic PDE satisfied by the function v introduced above:

$$\begin{aligned} v_t &= (m-1)v\Delta v + |\nabla v|^2 + K(m, p)v^{(m+p-2)/(m-1)}, \\ K(m, p) &= m \left(\frac{m-1}{m} \right)^{(m+p-2)/(m-1)}. \end{aligned} \quad (4.1)$$

Let us notice here the strong influence of the sign of $m + p - 2$ on this equation, because of its last term. To go further, we set $w = \Delta v$ and we next derive the partial differential equation solved by w . To this end, we calculate the terms separately. On the one hand, for the reaction term we obtain

$$\frac{\partial}{\partial x_i} K(m, p)v^{(m+p-2)/(m-1)} = K(m, p) \left[\frac{m+p-2}{m-1} v^{(p-1)/(m-1)} \frac{\partial v}{\partial x_i} \right]$$

hence

$$\begin{aligned} & \frac{\partial^2}{\partial x_i^2} K(m, p)v^{\frac{m+p-2}{m-1}} \\ &= K(m, p) \left[\frac{m+p-2}{m-1} v^{\frac{p-1}{m-1}} \frac{\partial^2 v}{\partial x_i^2} + \frac{(2-m-p)(1-p)}{(m-1)^2} v^{\frac{p-m}{m-1}} \left(\frac{\partial v}{\partial x_i} \right)^2 \right]. \end{aligned}$$

The contribution of the reaction term thus gives

$$\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} K(m, p)v^{\frac{m+p-2}{m-1}}$$

$$\begin{aligned}
&= K(m, p) \frac{m+p-2}{m-1} v^{\frac{p-1}{m-1}} w + K(m, p) \frac{(2-m-p)(1-p)}{(m-1)^2} v^{\frac{p-m}{m-1}} |\nabla v|^2 \\
&= \mathcal{R}_1(x, v)w + \mathcal{R}_2(x, v),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}_1(x, v) &= K(m, p) \frac{m+p-2}{m-1} v^{\frac{p-1}{m-1}} < 0, \\
\mathcal{R}_2(x, v) &= K(m, p) \frac{(2-m-p)(1-p)}{(m-1)^2} v^{\frac{p-m}{m-1}} |\nabla v|^2 \geq 0.
\end{aligned}$$

On the other hand, the diffusion term can be worked out as in [36, Proposition 9.4] to obtain

$$\begin{aligned}
w_t &= (m-1)v\Delta w + 2m\nabla v \cdot \nabla w + (m-1)w^2 + 2 \sum_{i,j=1}^N \left(\frac{\partial^2 v}{\partial x_i \partial x_j} \right)^2 \\
&\quad + \mathcal{R}_1(x, v)w + \mathcal{R}_2(x, v) \\
&\geq (m-1)v\Delta w + 2m\nabla v \cdot \nabla w + \left(m-1 + \frac{2}{N} \right) w^2 + \mathcal{R}_1(x, v)w + \mathcal{R}_2(x, v),
\end{aligned}$$

after a straightforward application of the Cauchy-Schwartz inequality. This can be written equivalently as $\mathcal{L}w \geq \mathcal{R}_2(x, v)$, where

$$\mathcal{L}w := w_t - (m-1)v\Delta w - 2m\nabla v \cdot \nabla w - \left(m-1 + \frac{2}{N} \right) w^2 - \mathcal{R}_1(x, v)w$$

is a uniformly parabolic operator. Since $\mathcal{R}_2(x, v) \geq 0$, we deduce that $\mathcal{L}w \geq 0$ and we aim to find a subsolution for \mathcal{L} depending only on time. We thus take for $t > 0$

$$\underline{W}(x, t) = -\frac{C}{t}, \quad C = \frac{N}{N(m-1) + 2}$$

and calculate

$$\mathcal{L}\underline{W} = \frac{C}{t^2} - \frac{C}{t^2} + \frac{C\mathcal{R}_1(x, v)}{t} < 0,$$

since $m+p-2 < 0$ in our range of exponents. Applying the comparison principle to the parabolic operator \mathcal{L} we infer that

$$\Delta v(x, t) = w(x, t) \geq \underline{W}(t) = -\frac{N}{(N(m-1) + 2)t},$$

for any $(x, t) \in \mathbb{R}^N \times (0, \infty)$, as stated.

All the previous calculations and the application of the maximum principle to the operator \mathcal{L} are fully justified for solutions u such that (in the pressure variable) \mathcal{L} is uniformly parabolic, that is, when $v, \nabla v$ are bounded and $v > 0$ uniformly. In order to extend the Aronson-Bénilan estimates to general weak solutions, we proceed by approximation. Let us first consider u_0 to be an initial condition as in (1.4) and continuous. Then, there exists a unique solution u to the Cauchy problem (1.5)-(1.2) in a time interval $(0, T)$, according to [27]. Let u_k be the solution to the Cauchy problem with initial condition

$$u_{0,k}(x) = u_0(x) + \frac{1}{k},$$

for each positive integer k . We infer from [27, Theorem 2.1] and its proof that there exists a unique solution u_k to (1.5) with initial condition $u_{0,k}$, and it satisfies $u_k(x, t) \geq 1/k$ for all $x \in \mathbb{R}^N, t > 0$. We further find from [23, Theorem 8.1,

Chapter V] (which applies for our approximating solutions u_k since they are now bounded from below by a positive constant) that u_k has the regularity required for (1.5) to hold in a classical sense and all the space derivatives of u_k up to the second order and time derivatives of u_k up to order one are uniformly bounded. In this case, the previous calculation applies rigorously for u_k and we obtain that

$$\Delta v_k(x, t) \geq -\frac{K}{t}, \quad v_k = \frac{m}{m-1} u_k^{m-1}. \quad (4.2)$$

Moreover, the comparison principle for (1.5) (see [27, Theorem 2.1]) entails that solutions u_k for $k \geq 1$ form a non-increasing sequence of functions, thus there exists a limit

$$\bar{u}(x, t) = \lim_{k \rightarrow \infty} u_k(x, t), \quad (x, t) \in \mathbb{R}^N \times (0, T),$$

and the uniform bound of u_k and their derivatives up to second order together with the Arzelá-Ascoli Theorem imply that $u_k \rightarrow \bar{u}$ locally uniformly and the same holds true for their first order derivatives with respect to the space variables. Then, the uniform boundedness of u_k and $\partial_t u_k$ gives the continuity with respect to the time variable over $(0, T)$ of the limit function \bar{u} , while the fact that \bar{u} belongs to L_{loc}^∞ is obvious, as it is bounded from above by any u_k . We thus fulfill the regularity assumption (a) in Definition 1.1. The monotone convergence theorem then easily gives that \bar{u} satisfies assumptions (b) and (c) in Definition 1.1, hence, since $u_k(x, 0) = u_{0,k}(x) \rightarrow u_0(x)$ as $k \rightarrow \infty$ uniformly on \mathbb{R}^N , we readily infer that \bar{u} is a weak solution to the Cauchy problem (1.5)-(1.2). Uniqueness of solutions to the latter Cauchy problem, established in [26] for continuous and bounded initial conditions, then proves that $\bar{u} = u$. We then come back to (4.2), which, after multiplication by a non-negative test function and integration by parts, reads

$$\int_0^T \int_{\mathbb{R}^N} \left(v_k(x, t) \Delta \varphi(x, t) + \frac{K}{t} \varphi(x, t) \right) dx dt \geq 0, \quad (4.3)$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N \times (0, T))$, $\varphi \geq 0$. We pass to the limit in (4.3) as $k \rightarrow \infty$, taking into account that $v_k \rightarrow v$ locally uniformly as $k \rightarrow \infty$, and obtain the claimed distributional form (1.13). \square

We end this section with a corollary which will be used in the sequel.

Corollary 4.1. *In the same conditions as in Theorem 1.4, we have*

$$u_t \geq -\frac{Ku}{t}, \quad K = \frac{N}{N(m-1) + 2},$$

in the sense of distributions in \mathbb{R}^N .

Proof. At a formal level, we infer from (4.1) that $v_t \geq (m-1)v\Delta v$, hence

$$(m-1) \frac{u_t}{u} = \frac{v_t}{v} \geq (m-1)\Delta v \geq -\frac{N(m-1)}{(N(m-1) + 2)t}$$

and we reach the conclusion with the same constant K as in (1.12). For general weak solutions the estimate is proved by using the same approximation as in the proof of Theorem 1.4. \square

Remark. Formal proof of Aronson-Bénilan estimates for $\sigma > 0$. At a formal level, the Aronson-Bénilan estimates (1.12) or (1.13) hold also for $\sigma > 0$ and $N \geq 2$. Indeed, a slightly longer but straightforward calculation along the first few

lines of the proof of Theorem 1.4 by computing the derivatives up to second order of the reaction term, but applied to (1.1) with $\sigma > 0$, gives

$$\begin{aligned} & \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} K(m, p)(1 + |x|)^\sigma v^{\frac{m+p-2}{m-1}} \\ &= K(m, p) \frac{m+p-2}{m-1} (1 + |x|)^\sigma v^{\frac{p-1}{m-1}} w \\ & \quad + K(m, p)(1 + |x|)^{\sigma-2} v^{\frac{p-m}{m-1}} \left[\frac{(2-m-p)(1-p)}{(m-1)^2} (1 + |x|)^2 |\nabla v|^2 \right. \\ & \quad \left. - 2\sigma \frac{2-m-p}{m-1} (1 + |x|) v \frac{x}{|x|} \cdot \nabla v + \sigma \left(\sigma - 1 + (N-1) \frac{1+|x|}{|x|} \right) v^2 \right] \\ &= \mathcal{R}_1(x, v)w + \mathcal{R}_2(x, v), \end{aligned}$$

where

$$\mathcal{R}_1(x, v) = K(m, p) \frac{m+p-2}{m-1} (1 + |x|)^\sigma v^{\frac{p-1}{m-1}} < 0$$

and $\mathcal{R}_2(x, v)$ gathers the rest of the terms. In order to proceed with the comparison principle as we did in the body of the proof of Theorem 1.4, we still need to have $\mathcal{R}_2(x, v) \geq 0$. To this end, we write \mathcal{R}_2 as a square and we examine the remainders. More precisely, using once more a standard Cauchy-Schwarz inequality for the scalar product $\nabla v \cdot x/|x|$ we find that

$$\begin{aligned} \frac{\mathcal{R}_2(x, v)}{K(m, p)(1 + |x|)^{\sigma-2} v^{\frac{p-m}{m-1}}} &\geq \left[\frac{2-m-p}{m-1} (1 + |x|) |\nabla v| - \sigma v \right]^2 \\ &\quad + \frac{2-m-p}{m-1} (1 + |x|)^2 |\nabla v|^2 \\ &\quad + \sigma \left[(N-1) \frac{1+|x|}{|x|} - 1 \right] v^2 \end{aligned}$$

and taking into account that we are in the range $m+p-2 < 0$, it follows that $\mathcal{R}_2(x, v) \geq 0$ provided that

$$\sigma \left[(N-1) \frac{1+|x|}{|x|} - 1 \right] \geq 0,$$

which holds for $\sigma > 0$ if $N \geq 2$. We thus conclude, at a formal level, that (1.13) should hold in this case. However, we left this part out of the statement of Theorem 1.4 since the final approximation argument leading to the rigorous proof for $\sigma = 0$ cannot be performed, as we are missing an existence and uniqueness result for solutions to the Cauchy problem (1.1)-(1.2) when $\sigma > 0$.

5. INFINITE SPEED OF PROPAGATION WHEN $m+p < 2$

In this part we use the Aronson-Bénilan estimates in Theorem 1.4 to establish the infinite speed of propagation of the supports of solutions to (1.1) when $m+p-2 < 0$ and thus complete the proof of Theorem 1.5. Let us stress again here that Theorem 1.5 has been proved in [26, Lemma 2.4] for $\sigma = 0$. We give here an independent proof, based on a completely different argument, and extend it to exponents $\sigma > 0$.

Proof of Theorem 1.5. In a first step, let $\sigma = 0$ and assume for contradiction that for some compactly supported initial condition u_0 as in (1.4), $u(t)$ remains compactly supported for $t \in (0, T_0)$. Recalling the pressure equation (4.1), at a formal

level, since $m + p - 2 < 0$, one reaches easily a contradiction. Indeed, picking an arbitrary $t \in (0, T_0)$, since $\Delta v(x, t) \geq -K/t$ and $|\nabla v(x, t)|^2 \geq 0$ at any point $x \in \mathbb{R}^N$, it follows that at the interface point $x = s(t)$ we obtain $v_t(s(t), t) = +\infty$ in order to compensate the negative power $(m + p - 2)/(m - 1)$ in the last term of the right-hand side. This is obviously equivalent to the infinite speed of propagation of the supports. More rigorously, since $m + p < 2$ we multiply by $v^{(2-m-p)/(m-1)}$ in (4.1) and also by a test function $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$, then we integrate on \mathbb{R}^N and on any time interval $(\tau_0, \tau_1) \subset (0, T_0)$ and we drop the second term in the right hand side (which is always positive) to obtain

$$\begin{aligned} & \frac{m-1}{1-p} \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}^N} (v^{(1-p)/(m-1)})_t \varphi \, dx \, dt \\ & \geq (m-1) \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}^N} v^{(1-p)/(m-1)} \Delta v \varphi \, dx \, dt + K(m,p) \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}^N} \varphi \, dx \, dt \\ & \geq -\frac{N(m-1)}{N(m-1)+2} \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}^N} \frac{1}{t} v^{(1-p)/(m-1)} \varphi \, dx \, dt \\ & \quad + K(m,p) \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}^N} \varphi \, dx \, dt. \end{aligned} \tag{5.1}$$

We now consider a sequence of test functions $(\varphi_n)_{n \geq 1}$ defined as follows

$$\varphi_n(x) = 1, \text{ for } |x| \leq n, \quad 0 \leq \varphi_n(x) \leq 1 \quad \forall x \in \mathbb{R}^N, \quad \text{supp } \varphi_n \subseteq B(0, 2n),$$

where $B(0, 2n) = \{x \in \mathbb{R}^N : |x| \leq 2n\}$, and let $\varphi = \varphi_n$ in (5.1) for any positive integer $n \geq 1$. Since the support of v is uniformly localized for $t \in [\tau_0, \tau_1]$ (as $\tau_1 < T_0$), it follows that the right-hand side of (5.1) tends to $+\infty$ as $n \rightarrow \infty$ due to the last integral only of φ_n . We thus infer that

$$\lim_{n \rightarrow \infty} \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}^N} (v^{(1-p)/(m-1)})_t \varphi_n \, dx \, dt = +\infty$$

or equivalently

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} [v^{(1-p)/(m-1)}(\tau_1) - v^{(1-p)/(m-1)}(\tau_0)] \varphi_n \, dx = +\infty,$$

which is a contradiction with the localization of the supports of $v(t)$ for $t \in [\tau_0, \tau_1]$. Let us notice here that the chosen range of exponents $p \in (0, 1)$ and $m + p < 2$ were decisive, as after multiplication by a positive power $v^{(2-m-p)/(m-1)}$, we got in the left-hand side also a positive power $v^{(1-p)/(m-1)}$ of v , thus we do not create new singularities at the edges of the supports.

We pass now to $\sigma > 0$. Assume that there is a weak solution u to the Cauchy problem Eq. (1.1)-(1.2) defined for $t \in (0, T)$ with some $T > 0$. Since $(1 + |x|)^\sigma \geq 1$ for any $x \in \mathbb{R}^N$ we deduce that u is a supersolution to the Cauchy problem (1.5)-(1.2). We infer from the comparison principle (which holds true for (1.5) and non-trivial initial data in the range $m + p < 2$, [27]) that $u(x, t) > 0$ for any $(x, t) \in \mathbb{R}^N \times (0, T)$. □

EXTENSIONS AND OPEN PROBLEMS

We gather here some extensions related to the previous results, that we consider interesting.

1. Finite time blow-up. A natural question is whether any solution to (1.1) blows up in finite time or there are some initial conditions u_0 producing (minimal) solutions that are global in time. Our conjecture is that, if $L > 0$, each non-trivial solution is expected to blow up in finite time, while if $L \leq 0$, there are initial conditions producing solutions that are global in time (we recall that L is defined in (1.8)). A formal argument about general finite time blow-up if $L > 0$ is based on comparison with subsolutions in self-similar form obtained in our recent papers [19, 21, 16]. More precisely, it goes by contradiction as follows: assume that there exists $u_0 \in C_0(\mathbb{R}^N)$ as in (1.4) such that the minimal solution $M(u_0)$ is defined for $t \in (0, \infty)$. We infer by comparison with the absolute minimal solution E defined in [26] and (3.1) that for large t , solution $M(u_0)(t)$ is as large as we want both in amplitude and support. We can thus find a blow-up self-similar solution to (1.7) as in [19, 21, 16] (which is a subsolution to (1.1)) below it. If comparison would be allowed, then we would get an easy contradiction proving that any minimal solution (and then any other solution) blows up in finite time.

But as we see by considering the subsolution \underline{U} with initial condition $\underline{U}(x, 0) = 0$ for any $x \in \mathbb{R}^N$, which is defined explicitly by

$$\underline{U}(x, t) = \left(\frac{1}{1-p} \right)^{1/(p-1)} t^{1/(1-p)} (1 + |x|)^{\sigma/(1-p)}, \quad (5.2)$$

comparison does not hold in general, as otherwise any solution (even the ones with compact support) could have been compared to \underline{U} in order to force it to become positive everywhere, contradicting the finite speed of propagation at least in the range $m + p \geq 2$. This opens the question on whether the self-similar solutions to (1.1) are minimal solutions in the sense of the construction performed in Proposition 2.1.

2. Non-existence of positive solutions if $\sigma > 2(1-p)/(m-1)$. Strongly connected to the first comment, and expecting (at a formal level) that we might compare a strictly positive solution to (1.1) with the subsolution \underline{U} introduced in (5.2), we conjecture that, if $L > 0$ (that is, $\sigma > 2(1-p)/(m-1)$) there are no solutions to (1.1) such that $u(x, t) > 0$ for any $x \in \mathbb{R}^N$. Indeed, assuming that the comparison can be performed rigorously if $u(x, t) > 0$ for any $x \in \mathbb{R}^N$, we would get a solution with local behavior

$$u(x, t) \geq C(1 + |x|)^{\sigma/(1-p)}, \quad \text{as } |x| \rightarrow \infty, \text{ for all } t > 0.$$

But any solution to (1.1) is a supersolution to the standard porous medium equation and classical results on the porous medium equation (see for example [4, 7, 8]) state that there are no solutions (and it seems to us that the proofs can be extended to supersolutions) to it increasing at infinity faster than $|x|^{2/(m-1)}$. Since $\sigma/(1-p) > 2/(m-1)$ if $L > 0$, we would be in this case. In particular, since infinite speed of propagation is in force for $m + p < 2$ by Theorem 1.5, we expect complete non-existence of non-trivial solutions if $L > 0$ and $m + p < 2$.

3. Establishing which self-similar solutions are minimal. An immediate adaptation of the result in [19] proves that (1.1) with $m + p > 2$ presents two types of blow-up self-similar solutions

$$U(x, t) = (T - t)^{-\alpha} f((1 + |x|)(T - t)^\beta), \quad \alpha = \frac{\sigma + 2}{L}, \quad \beta = \frac{m - p}{L},$$

which differ with respect to the local behavior of the profile $f(\xi)$ near the interface point $\xi_0 \in (0, \infty)$, namely

$$f(\xi) \sim (\xi_0 - \xi)^{1/(m-1)} \text{ (Type I)} \quad \text{or} \quad f(\xi) \sim (\xi_0 - \xi)^{1/(1-p)} \text{ (Type II)},$$

both taken in the limit $\xi \rightarrow \xi_0$, $\xi < \xi_0$. It is also shown at least at a formal level that these solutions satisfy two different interface equations. Thus, a natural question would be whether any of these self-similar solutions is minimal in the sense of the construction in Proposition 2.1, which would allow us to compare and conclude on the finite time blow-up. This is not an easy question and our intuition suggests that minimality has to do with the interface equation: we might expect that the solutions with interface of Type I are minimal, while the other ones are not. This conjecture is supported by the analogy with the minimality of the traveling wave solutions to (1.5), see for example [27, Theorem 4.1].

4. Connection between non-uniqueness and blow-up time. A much deeper open question is related to whether, in the range $\sigma > 2(1-p)/(m-1)$, prescribing a blow-up time T and a function u_0 , there exists a unique solution to the Cauchy problem (1.1)-(1.2) with condition u_0 blowing up in finite time exactly at the given time T . More precisely, taking $u_0 \in C_0(\mathbb{R})$ satisfying (1.4), there exists a minimal solution $M(u_0)$ which (assuming that point 1 in this enumeration of open problems holds true, as we strongly expect) comes with a finite blow-up time $T_0 \in (0, \infty)$. We have also proved in Section 3 that the Cauchy problem (1.1)-(1.2) has an infinite number of compactly supported solutions with interfaces advancing faster than the minimal one, and estimate (3.4) shows that faster advancing speed of the interface implies shorter lifetime before blow-up. Thus one can wonder naturally whether, given $T \in (0, T_0)$, there exists one solution to the Cauchy problem (1.1)-(1.2) blowing up exactly at this time T . We do not have by now a suggestion of how to approach this problem, but it is in our opinion an interesting and deep open question.

Acknowledgments. R. G. I. and A. S. were partially supported by the Project PID2020-115273GB-I00 and by the Grant RED2022-134301-T (Spain).

REFERENCES

- [1] D. Andreucci, E. DiBenedetto; On the Cauchy problem and initial traces for a class of evolution equations with strongly nonlinear sources, *Ann. Scuola Norm. Sup. Pisa*, **18** (1991), 363-441.
- [2] D. Andreucci, A. F. Tedeev; Universal bounds at the blow-up time for nonlinear parabolic equations, *Adv. Differential Equations*, **10** (2005), no. 1, 89-120.
- [3] D. G. Aronson, Ph. Bénilan; Régularité des solutions de l'équation des milieux poreux dans \mathbb{R}^N (French), *CR Acad. Sci. Paris Sér A*, **288** (1979), 103-105.
- [4] D. G. Aronson, L. A. Caffarelli; The initial trace of a solution of the porous medium equation, *Trans. Amer. Math. Soc.*, **280** (1983), no. 1, 351-366.
- [5] C. Bandle, H. Levine; On the existence and nonexistence of global solutions of reaction-diffusion equations in sectorial domains, *Trans. Amer. Math. Soc.*, **316** (1989), 595-622.
- [6] P. Baras, R. Kersner; Local and global solvability of a class of semilinear parabolic equations, *J. Differential Equations*, **68** (1987), 238-252.
- [7] P. Bénilan, M. G. Crandall, M. Pierre; Solutions of the porous medium in \mathbb{R}^N under optimal conditions on the initial values, *Indiana Univ. Math. Jour.*, **33** (1984), no. 1, 51-87.
- [8] L. A. Caffarelli, J. L. Vázquez, N. I. Wolanski; Lipschitz continuity of solutions and interfaces of the N -dimensional porous medium equation, *Indiana Univ. Math. J.*, **36** (1987), no. 2, 373-401.
- [9] E. DiBenedetto; Continuity of weak solutions to a general porous medium equation, *Indiana Univ. Math. J.*, **32** (1983), no. 1, 83-118.

- [10] S. Filippas, A. Tertikas; On similarity solutions of a heat equation with a nonhomogeneous nonlinearity, *J. Differential Equations*, **165** (2000), no. 2, 468-492.
- [11] J.-S. Guo, C.-S. Lin, M. Shimojo; Blow-up behavior for a parabolic equation with spatially dependent coefficient, *Dynam. Systems Appl.*, **19** (2010), no. 3-4, 415-433.
- [12] J.-S. Guo, C.-S. Lin, M. Shimojo; Blow-up for a reaction-diffusion equation with variable coefficient, *Appl. Math. Lett.*, **26** (2013), no. 1, 150-153.
- [13] J.-S. Guo, M. Shimojo; Blowing up at zero points of potential for an initial boundary value problem, *Commun. Pure Appl. Anal.*, **10** (2011), no. 1, 161-177.
- [14] J.-S. Guo, P. Souplet; Excluding blowup at zero points of the potential by means of Liouville-type theorems, *J. Differential Equations*, **265** (2018), no. 10, 4942-4964.
- [15] R. G. Iagar, M. Latorre, A. Sánchez; Blow-up patterns for a reaction-diffusion equation with weighted reaction in general dimension, *Adv. Differential Equations* (accepted November 2022), Preprint ArXiv no. 2205.09407.
- [16] R. G. Iagar, A. I. Muñoz, A. Sánchez; Self-similar blow-up patterns for a reaction-diffusion equation with weighted reaction in general dimension, *Commun. Pure Appl. Analysis*, **21** (2022), no. 3, 891-925.
- [17] R. G. Iagar, A. Sánchez; Blow up profiles for a quasilinear reaction-diffusion equation with weighted reaction with linear growth, *J. Dynam. Differential Equations*, **31** (2019), no. 4, 2061-2094.
- [18] R. G. Iagar, A. Sánchez; Blow up profiles for a reaction-diffusion equation with critical weighted reaction, *Nonlinear Anal.*, **191** (2020), paper no. 111628, 24 pages.
- [19] R. G. Iagar, A. Sánchez; Self-similar blow-up profiles for a reaction-diffusion equation with strong weighted reaction, *Adv. Nonl. Studies*, **20** (2020), no. 4, 867-894.
- [20] R. G. Iagar, A. Sánchez; Blow up profiles for a quasilinear reaction-diffusion equation with weighted reaction, *J. Differential Equations*, **272** (2021), 560-605.
- [21] R. G. Iagar, A. Sánchez; Self-similar blow-up profiles for a reaction-diffusion equation with critically strong weighted reaction, *J. Dynam. Differential Equations*, **34** (2022), no. 2, 1139-1172.
- [22] R. Iagar, A. Sánchez; Separate variable blow-up patterns for a reaction-diffusion equation with critical weighted reaction, *Nonlinear Anal.*, **217** (2022), Article ID 112740, 33 pages.
- [23] O. A. Ladyzhenskaya, M. A. Solonnikov, N. N. Uraltseva; Linear and quasilinear equations of parabolic type, *Transl. Math. Monographs*, **23**, Amer. Math. Society, Providence, 1968.
- [24] A. Mukai, Y. Seki; Refined construction of Type II blow-up solutions for semilinear heat equations with Joseph-Lundgren supercritical nonlinearity, *Discrete Cont. Dynamical Systems*, **41** (2021), no. 10, 4847-4885.
- [25] A. de Pablo; Large-time behaviour of solutions of a reaction-diffusion equation, *Proc. Roy. Soc. Edinburgh Sect. A*, **124** (1994), no. 2, 389-398.
- [26] A. de Pablo, J. L. Vázquez; The balance between strong reaction and slow diffusion, *Comm. Partial Differential Equations*, **15** (1990), no. 2, 159-183.
- [27] A. de Pablo, J. L. Vázquez; Travelling waves and finite propagation in a reaction-diffusion equation, *J. Differential Equations*, **93** (1991), no. 1, 19-61.
- [28] A. de Pablo, J. L. Vázquez; An overdetermined initial and boundary-value problem for a reaction-diffusion equation, *Nonlinear Anal.*, **19** (1992), no. 3, 259-269.
- [29] R. G. Pinsky; Existence and nonexistence of global solutions for $u_t = \Delta u + a(x)u^p$ in \mathbb{R}^d , *J. Differential Equations*, **133** (1997), no. 1, 152-177.
- [30] R. G. Pinsky; The behavior of the life span for solutions to $u_t = \Delta u + a(x)u^p$ in \mathbb{R}^d , *J. Differential Equations*, **147** (1998), no. 1, 30-57.
- [31] Y.-W. Qi; The critical exponents of parabolic equations and blow-up in \mathbb{R}^N , *Proc. Roy. Soc. Edinburgh Section A*, **128** (1998), no. 1, 123-136.
- [32] P. Quittner, Ph. Souplet; Superlinear parabolic problems. Blow-up, global existence and steady states, *Birkhauser Advanced Texts*, Birkhauser Verlag, Basel, 2007.
- [33] P. E. Sacks; The initial and boundary problem for a class of degenerate parabolic equations, *Comm. Partial Differential Equations*, **8** (1983), no. 7, 693-733.
- [34] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, A. P. Mikhailov; Blow-up in quasilinear parabolic problems, *de Gruyter Expositions in Mathematics*, **19**, W. de Gruyter, Berlin, 1995.
- [35] R. Suzuki; Existence and nonexistence of global solutions of quasilinear parabolic equations, *J. Math. Soc. Japan*, **54** (2002), no. 4, 747-792.

- [36] J. L. Vázquez; The porous medium equation. Mathematical theory, *Oxford Monographs in Mathematics*, Oxford University Press, 2007.

RAZVAN GABRIEL IAGAR

DEPARTAMENTO DE MATEMÁTICA APLICADA, CIENCIA E INGENIERIA DE MATERIALES Y TECNOLOGIA ELECTRÓNICA, UNIVERSIDAD REY JUAN CARLOS, MÓSTOLES, 28933, MADRID, SPAIN

Email address: `razvan.iagar@urjc.es`

ANA I. MUÑOZ

DEPARTAMENTO DE MATEMÁTICA APLICADA, CIENCIA E INGENIERIA DE MATERIALES Y TECNOLOGIA ELECTRÓNICA, UNIVERSIDAD REY JUAN CARLOS, MÓSTOLES, 28933, MADRID, SPAIN

Email address: `anaisabel.munoz@urjc.es`

ARIEL SÁNCHEZ

DEPARTAMENTO DE MATEMÁTICA APLICADA, CIENCIA E INGENIERIA DE MATERIALES Y TECNOLOGIA ELECTRÓNICA, UNIVERSIDAD REY JUAN CARLOS, MÓSTOLES, 28933, MADRID, SPAIN

Email address: `ariel.sanchez@urjc.es`