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EXISTENCE AND DECAY OF SOLUTIONS TO COUPLED SYSTEMS OF NONLINEAR WAVE EQUATIONS WITH VARIABLE EXPONENTS

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Abstract. In this article, we consider a coupled system of two hyperbolic equations with variable exponents in the damping and source terms, where the dampings are modilated with time-dependent coefficients $\alpha(t), \beta(t)$. First, we state and prove an existence result of a global weak solution, using Galerkin's method with compactness arguments. Then, by a Lemma due to Martinez, we establish the decay rates of the solution energy, under suitable assumptions on the variable exponents m and r and the coefficients α and β. To illustrate our theoretical results, we give some numerical examples.

1. Introduction

In this work, we study the initial-boundary-value problem

$$\begin{gathered} u_{tt} - \Delta u + \alpha(t)|u_t|^{m(x)-2}u_t + |u|^{p(x)-2}u|v|^{p(x)} = 0 \quad \text{in } \Omega \times (0,T), \\ v_{tt} - \Delta v + \beta(t)|v_t|^{r(x)-2}v_t + |v|^{p(x)-2}v|u|^{p(x)} = 0 \quad \text{in } \Omega \times (0,T), \\ u = v = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \quad u_t(0) = u_1 \quad \text{in } \Omega, \\ v(0) = v_0, \quad v_t(0) = v_1 \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $T > 0$ and Ω is a bounded domain of $\mathbb{R}^n (n \in \mathbb{N}^*)$ with a smooth boundary $\partial\Omega$; $\alpha, \beta : [0,\infty) \to (0,\infty)$ are two non-increasing C^1-functions and m, r and p are given continuous functions on $\overline{\Omega}$ satisfying some conditions to be specified later.

The wave equations with variable exponents of the nonlinearity occur in mathematical models of various physical phenomena such as flows of electro-rheological fluids or fluids with temperature dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media and image processing, thermorheological fluids, or robotics, etc. For more details on the subject, the reader can see [1, 10]. In fact, several works concerning hyperbolic problems with nonlinearities of variable-exponent type have appeared, of which we mention some recent ones.

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For a class of one wave equation, Antontsev [4] studied the equation

$$u_{tt} - \operatorname{div}(a|\nabla u|^{p(x,t)-2}\nabla u) - \alpha\Delta u_t - bu|u|^{\sigma(x,t)-2} = f, \quad \text{in } \Omega \times (0,T),$$

where $\alpha > 0$ is a constant and a, b, p, σ are given functions. Under specific conditions on the exponents, he proved the existence of local and global weak solutions and a blow-up result. Guo and Gao [12] took $\sigma(x,t) = r > 2$ and established a finite-time blow-up result for certain solutions with positive initial energy. After that, Guo [13] applied an interpolation inequality and some energy inequalities to obtain an estimate of the lower bound for the blow-up time when the source is super-linear. Sun et al. [27] study the equation

$$u_{tt} - \operatorname{div}(a(x,t)\nabla u) + c(x,t)u_t|u_t|^{q(x,t)-1} = b(x,t)u|u|^{p(x,t)-2}, \quad \text{in } \Omega \times (0,T),$$

established a blow-up result and gave lower and upper bounds for the blow-up time, under some conditions on the initial data. In addition, they provided numerical illustrations for their results. Messaoudi and Talahmeh [19] studied the equation

$$u_{tt} - \operatorname{div}(|\nabla u|^{r(x)-2}\nabla u) + au_t|u_t|^{m(x)-2} = bu|u|^{p(x)-2}, \quad \text{in } \Omega \times (0,T),$$

where $a, b > 0$ are two constants and m, r, p are given functions. They proved a finite-time blow-up result. In the absence of source term ($b = 0$), the same authors in [21] obtained decay estimates of solutions and presented two numerical applications as illustration for their theoretical results. After that, they gave in [22] an overview of results concerning decay and blow up for nonlinear wave equations involving variable and constant exponents. Recently, Xiaolei et al. [28] used some energy estimates and a Komornik's inequality to establish an asymptotic stability of solutions to quasilinear hyperbolic equations with variable source and damping terms.

Concerning coupled systems of hyperbolic equations with variable exponents, we mention the work of Bouhoufani and Hamchi [9], where they proved the existence of a global weak solution and established decay estimates of the energy depending on the variable exponents. Messaoudi and Talahmeh [24] considered a system of wave equations, with damping and source terms of variable-exponent nonlinearities, and proved a blow-up result for solutions with negative initial energy. Recently, Messaoudi et al. [25] studied a coupled hyperbolic system with variable exponents. They obtained an existence and uniqueness result of a weak solution, showed that certain solutions, with positive initial energy, blow up in finite time and gave some numerical applications.

For the case of equations and systems with constant exponents, we can cite the works of Mustafa and Messaoudi [26], Benaissa and Mimouni [5], Benaissa and Mokaddem [6], Zennir [29], Bociu [7], Bociu and Lasiecka [8], Agre and Rammaha [2], and Jianghao Hao and Li Cai [15]. In particular, Bociu [7] considered, in a three-dimensional bounded domain, the wave equation with interior and boundary nonlinear sources and dampings. She classified the "polynomial-type" sources in four categories and established some local and global existence results. In her case, the super-supercase, the exponent of the nonlinearity could approach 6, however, in our case, the nonlinearity exponent p cannot exceed 2 when $n = 3$, due to the nature of the source we have in our problem. See (H3) below.

In this work, we intend to prove the local and global existence for our problem (1.1) and establish explicit decay rates of the solution energy depending on the range of the variable exponents m, r and the time-dependent coefficients α and β. This article consists of six sections. After the introduction, we recall the definitions of the variable-exponent Lebesgue and Sobolev spaces as well as some useful Lemmas. In section 3, we state and prove the local and global existence of a weak solution of (1.1). Section 4 is devoted to the statement and the proof of our aim result. In section 5, we give numerical examples to illustrate our theoretical findings. We conclude with some remarks in section 6.

2. Preliminaries

In this section, we present some essential facts from [3, 11, 17] related to the Lebesgue and Sobolev spaces with variable exponents. Let $q : \Omega \to [1, \infty)$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$. We define the Lebesgue space with a variable exponent by

$$L^{q(\cdot)}(\Omega) = \{f : \Omega \to \mathbb{R} \text{ measurable in } \Omega : \varrho_{q(\cdot)}(\lambda f) < +\infty, \text{ for some } \lambda > 0\},$$

where

$$\varrho_{q(\cdot)}(f) = \int_\Omega |f(x)|^{q(x)} dx.$$

Endowed with the Luxembourg-type norm

$$\|f\|_{q(\cdot)} := \inf\{\lambda > 0 : \int_\Omega |\frac{f(x)}{\lambda}|^{q(x)} dx \le 1\}.$$

$L^{q(\cdot)}(\Omega)$ is a Banach space (see [3, 11, 17]). We, also, define the variable exponent Sobolev space

$$W^{1,q(\cdot)}(\Omega) = \{f \in L^{q(\cdot)}(\Omega) : \nabla f \text{ exists and } |\nabla f| \in L^{q(\cdot)}(\Omega)\}.$$

This is a Banach space with respect to the norm

$$\|f\|_{W^{1,q(\cdot)}(\Omega)} = \|f\|_{q(\cdot)} + \|\nabla f\|_{q(\cdot)}.$$

Definition 2.1. We say that a function $q : \Omega \to \mathbb{R}$ is log-Hölder continuous on Ω, if there exists a constant $\theta > 0$ such that for all $0 < \delta < 1$, we have

$$|q(x) - q(y)| \le -\frac{\theta}{\log|x-y|}, \quad \text{for a.e. } x, y \in \Omega, \text{ with } |x-y| < \delta.$$

Furthermore, for q satisfying the log-Hölder continuity, we denote by $W_0^{1,q(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,q(\cdot)}(\Omega)$ and by $W^{-1,q'(\cdot)}(\Omega)$ the dual space of $W_0^{1,q(\cdot)}(\Omega)$, in the same way as the usual Sobolev spaces, where $\frac{1}{q(\cdot)} + \frac{1}{q'(\cdot)} = 1$.

Lemma 2.2 (Young's Inequality [3, 11, 17]). *Let $r, q, s \ge 1$ be measurable functions defined on Ω, such that*

$$\frac{1}{s(y)} = \frac{1}{r(y)} + \frac{1}{q(y)}, \quad \textit{for a.e. } y \in \Omega.$$

Then for all $a, b \ge 0$ we have

$$\frac{(ab)^{s(\cdot)}}{s(\cdot)} \le \frac{(a)^{r(\cdot)}}{r(\cdot)} + \frac{(b)^{q(\cdot)}}{q(\cdot)}.$$

Lemma 2.3 ([3, 11, 17]). *If $1 < q^- \le q(y) \le q^+ < +\infty$ holds, then for each $f \in L^{q(\cdot)}(\Omega)$ we have (i)*

$$\min\{\|f\|_{q(\cdot)}^{q^-}, \|f\|_{q(\cdot)}^{q^+}\} \le \varrho_{q(\cdot)}(f) \le \max\{\|f\|_{q(\cdot)}^{q^-}, \|f\|_{q(\cdot)}^{q^+}\}$$

(ii)
$$\varrho_{q(\cdot)}(f) \leq \|f\|_{q^-}^{q^-} + \|f\|_{q^+}^{q^+},$$
where
$$q^- = \operatorname{ess\,inf}_{x\in\Omega} q(x), \quad q^+ = \operatorname{ess\,sup}_{x\in\Omega}\ q(x).$$

Lemma 2.4 (Poincaré's inequality [3, 17])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If q satisfies the log-Hölder continuity condition, then there exists a positive constant C depending on Ω and q only, such that*
$$\|f\|_{q(\cdot)} \leq C\|\nabla f\|_{q(\cdot)}, \quad \text{for all } f \in W_0^{1,q(\cdot)}(\Omega).$$
In particular, the space $W_0^{1,q(\cdot)}(\Omega)$ has an equivalent norm,
$$\|f\|_{W_0^{1,q(\cdot)}(\Omega)} = \|\nabla f\|_{q(\cdot)}.$$

Corollary 2.5 ([3, 17])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$. Assume that $q : \overline{\Omega} \to (1,\infty)$ is a continuous function such that*
$$2 \leq q^- \leq q(x) \leq q^+ < \frac{2n}{n-2},\ n \geq 3,$$
then the embedding $H_0^1(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

To prove our decay result, we need the following Lemma.

Lemma 2.6 ([18])**.** *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing C^1-function, with $\sigma(0) = 0$ and $\sigma(t) \to +\infty$ as $t \to \infty$. Assume that there exists $q \geq 0$ and $C > 0$ such that*
$$\int_S^\infty \sigma'(t)E(t)^{q+1}dt \leq CE(S), \quad 0 \leq S < \infty.$$
Then, there exist two positive constants c and w such that for all $t \geq 0$,
$$E(t) \leq \begin{cases} ce^{-\omega\sigma(t)}, & \text{if } q = 0,\\ \frac{c}{[1+\sigma(t)]^{1/q}}, & \text{if } q > 0.\end{cases}$$

Now, we specify the assumptions on the variable-exponent functions. We assume that for all $x \in \overline{\Omega}$:
$$\begin{gathered} 2 \leq m(x), \quad \text{if } n = 1,2,\\ 2 \leq m^- \leq m(x) \leq m^+ \leq \frac{2n}{n-2}, \quad \text{if } n \geq 3, \end{gathered} \tag{2.1}$$
$$\begin{gathered} 2 \leq r(x), \quad \text{if } n = 1,2,\\ 2 \leq r^- \leq r(x) \leq r^+ \leq \frac{2n}{n-2}, \quad \text{if } n \geq 3 \end{gathered} \tag{2.2}$$
$$\begin{gathered} 1 \leq p(x), \quad \text{if } n = 1,2,\\ 1 \leq p^- \leq p(x) \leq p^+ \leq \frac{n-1}{n-2}, \quad \text{if } n \geq 3, \end{gathered} \tag{2.3}$$
with
$$\begin{gathered} m^- = \inf_{x\in\overline{\Omega}} m(x), \quad m^+ = \sup_{x\in\overline{\Omega}} m(x),\\ r^- = \inf_{x\in\overline{\Omega}} r(x), \quad r^+ = \sup_{x\in\overline{\Omega}} r(x), \end{gathered}$$

$$p^- = \inf_{x\in\overline{\Omega}} p(x), \quad p^+ = \sup_{x\in\overline{\Omega}} p(x).$$

Since m, r, p are $C^1(\overline{\Omega})$, they satisfy the log-Hölder continuity condition.

3. Existence of global solutions

Definition 3.1. Consider $u_0, v_0 \in H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$. A pair of functions (u, v) is said to be a weak solution of (1.1) on $[0, T)$, if

$$\begin{gathered} u, v \in L^\infty((0,T), H_0^1(\Omega)), \quad u_t, v_t \in L^\infty((0,T), L^2(\Omega)), \\ u_t \in L_\alpha^{m(\cdot)}(\Omega \times (0,T)), \quad v_t \in L_\beta^{r(\cdot)}(\Omega \times (0,T)) \end{gathered}$$

and (u, v) satisfies

$$\begin{aligned} &\int_\Omega u_t \phi\,dx - \int_\Omega u_1 \phi\,dx + \int_0^t \int_\Omega \alpha(\tau)|u_t|^{m(x)-2} u_t \phi\,dx\,d\tau \\ &+ \int_0^t \int_\Omega \nabla u.\nabla \phi\,dx\,d\tau + \int_0^t \int_\Omega |u|^{p(x)-2} u |v|^{p(x)} \phi\,dx\,d\tau = 0 \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega v_t \psi\,dx - \int_\Omega v_1 \psi\,dx + \int_0^t \int_\Omega \beta(\tau)|v_t|^{r(x)-2} v_t \psi\,dx\,d\tau \\ &+ \int_0^t \int_\Omega \nabla v.\nabla \psi\,dx\,d\tau + \int_0^t \int_\Omega |v|^{p(x)-2} v |u|^{p(x)} \psi\,dx\,d\tau = 0, \end{aligned}$$

for all $\phi, \psi \in H_0^1(\Omega)$ and all $t \in (0, T)$, with

$$(u(\cdot, 0), v(\cdot, 0)) = (u_0, v_0), \quad (u_t(\cdot, 0), v_t(\cdot, 0)) = (u_1, v_1).$$

Here,

$$\begin{aligned} L_\alpha^{m(\cdot)}(\Omega \times (0,T)) &= \{w : \Omega \times (0,T) \to \mathbb{R} : \int_0^T \int_\Omega \alpha(\tau)|w(x,\tau)|^{m(x)}\,dx\,d\tau < +\infty\}, \\ L_\beta^{r(\cdot)}(\Omega \times (0,T)) &= \{w : \Omega \times (0,T) \to \mathbb{R} : \int_0^T \int_\Omega \beta(\tau)|w(x,\tau)|^{r(x)}\,dx\,d\tau < +\infty\}. \end{aligned}$$

Theorem 3.2. *Assume that* (2.1)–(2.3) *are satisfied. Then, for any initial data* $u_0, v_0 \in H_0^1(\Omega)$ *and* $u_1, v_1 \in L^2(\Omega)$*, there exists a weak solution* (u, v) *of* (1.1) *(in the sense of Definition* 3.1*) defined in* $[0, T)$*, for all* $T > 0$.

Proof. We use the Faedo-Galerkin approximations combined with arguments by Aubin-Lions.

Step 1. Consider $T > 0$ fixed but arbitrary. Let $\{\omega_j\}_{j=1}^\infty$ be an orthonormal basis of $H_0^1(\Omega)$ and $V_k = \operatorname{span}\{\omega_1, \omega_2, \dots, \omega_k\}$ be the subspace generated by the k first vectors $\omega_1, \omega_2, \dots, \omega_k$. Consider

$$u^k(t) = \Sigma_{j=1}^k a_j(t)\omega_j \quad \text{and} \quad v^k(t) = \Sigma_{j=1}^k b_j(t)\omega_j, \quad t \in (0,T),$$

such that (u^k, v^k) is an approximate solution of problem (1.1), satisfying

$$\begin{aligned}
&\int_\Omega u^k_{tt}(t)\omega_j\,dx + \int_\Omega \nabla u^k(t)\nabla\omega_j\,dx + \int_\Omega \alpha(t)|u^k_t(t)|^{m(x)-2}u^k_t(t)\omega_j\,dx\\
&= -\int_\Omega |u^k(t)|^{p(x)-2}u^k(t)|v^k(t)|^{p(x)}\omega_j\,dx,\\
&\int_\Omega v^k_{tt}(t)\omega_j\,dx + \int_\Omega \nabla v^k(t)\nabla\omega_j\,dx + \int_\Omega \beta(t)|v^k_t(t)|^{r(x)-2}v^k_t(t)\omega_j\,dx\\
&= -\int_\Omega |v^k(t)|^{p(x)-2}v^k(t)|u^k(t)|^{p(x)}\omega_j\,dx,
\end{aligned} \tag{3.1}$$

for all $j = 1, 2, \dots, k$, and

$$\begin{gathered}
u^k(0) = u^k_0 \to u_0, \quad v^k(0) = v^k_0 \to v_0 \quad \text{in } H^1_0(\Omega),\\
u^k_t(0) = u^k_1 \to u_1 \quad v^k_t(0) = v^k_1 \to v_1 \quad \text{in } L^2(\Omega).
\end{gathered} \tag{3.2}$$

By ODE standard existence theory, problem (3.1) ,(3.2) has a unique local solution (u^k, v^k) defined on $[0, t_k)$, $0 < t_k \le T$, for all $k \ge 1$. In the following step and by a priory estimates, we extend these solutions to the interval $[0, T)$ for all $k \ge 1$.

Step 2. Multiplying both sides of $(3.1)_1$ and $(3.1)_2$ by $a'_j(t)$ and $b'_j(t)$, respectively, using Green's formula and the boundary conditions, and then summing each result over j, from 1 to k, we obtain, for all $0 < t \le t_k$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|u^k_t\|_2^2 + \|\nabla u^k\|_2^2] + \int_\Omega \alpha(t)|u^k_t(x,t)|^{m(x)}dx\\
&= -\int_\Omega |u^k|^{p(x)-2}u^k|v^k|^{p(x)}u^k_t dx
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[\|v^k_t\|_2^2 + \|\nabla v^k\|_2^2] + \int_\Omega \beta(t)|v^k_t(x,t)|^{r(x)}dx\\
&= -\int_\Omega |v^k|^{p(x)-2}v^k|u^k|^{p(x)}v^k_t dx.
\end{aligned} \tag{3.4}$$

Adding (3.3) and (3.4), and then integrating the result, over $(0, t)$, with $t \le t_k$, it yields

$$\begin{aligned}
&\frac{1}{2}[\|u^k_t\|_2^2 + \|v^k_t\|_2^2 + \|\nabla u^k\|_2^2 + \|\nabla v^k\|_2^2]\\
&+ \int_0^t \int_\Omega (\alpha(\tau)|u^k_t(x,\tau)|^{m(x)} + \beta(\tau)|v^k_t(x,\tau)|^{r(x)})\,dx\,d\tau\\
&\le \frac{1}{2}[\|u^k_1\|_2^2 + \|v^k_1\|_2^2 + \|\nabla u^k_0\|_2^2 + \|\nabla v^k_0\|_2^2]\\
&\quad + \int_0^t \int_\Omega [|u^k|^{p(x)-1}|v^k|^{p(x)}|u^k_t| + |v^k|^{p(x)-1}|u^k|^{p(x)}|v^k_t|]\,dx\,d\tau.
\end{aligned}$$

Recalling (3.2), for some $C > 0$ we have

$$\begin{aligned} &\frac{1}{2}[\|u_t^k\|_2^2 + \|v_t^k\|_2^2 + \|\nabla u^k\|_2^2 + \|\nabla v^k\|_2^2] \\ &+ \int_0^t \int_\Omega (\alpha(\tau)|u_t^k(x,\tau)|^{m(x)} + \beta(\tau)|v_t^k(x,\tau)|^{r(x)})\,dx\,d\tau \\ &\le C + \int_0^t \int_\Omega [|u^k|^{p(x)-1}|v^k|^{p(x)}|u_t^k| + |v^k|^{p(x)-1}|u^k|^{p(x)}|v_t^k|]\,dx\,d\tau. \end{aligned} \tag{3.5}$$

Now, we handle the last term in the right-hand side of (3.5). Applying Young's inequality, with

$$q(x) = \frac{2p(x)-1}{p(x)-1} \text{ and } q'(x) = \frac{2p(x)-1}{p(x)},$$

we obtain, that for a.e. $x \in \Omega$,

$$|u^k|^{p(x)-1}|v^k|^{p(x)} \le \frac{1}{2}|u^k|^{2p(x)-1} + C(x)|v^k|^{2p(x)-1},$$

where

$$C(x) = \frac{p(x)}{2p(x)-1}\Big(\frac{2p(x)}{(2p(x)-1)}\Big)^{\frac{p(x)-1}{p(x)}}.$$

From assumption (2.3), p is bounded on Ω. Therefore, $C(x)$ is bounded too. Hence, for some $C_1 > 0$ and for a.e. $x \in \Omega$ it follows that

$$|u^k|^{p(x)-1}|v^k|^{p(x)} \le C_1[|u^k|^{2p(x)-1} + |v^k|^{2p(x)-1}] \tag{3.6}$$

and, similarly,

$$|v^k|^{p(x)-1}|u^k|^{p(x)} \le C_1[|v^k|^{2p(x)-1} + |u^k|^{2p(x)-1}]. \tag{3.7}$$

Under condition (2.3), we recall (3.6), (3.7), Lemma 2.3 and the embeddings result (Corollary 2.5), to find that for all $t \le t_k$,

$$\begin{aligned} &\int_\Omega |u^k|^{p(x)-1}|v^k|^{p(x)}|u_t^k|\,dx \\ &\le \frac{1}{2}\|u_t^k\|_2^2 + \frac{C_1^2}{2}\int_\Omega [|u^k|^{2p(x)-1} + |v^k|^{2p(x)-1}]^2 dx \\ &\le \frac{1}{2}\|u_t^k\|_2^2 + C\int_\Omega \Big(|u^k|^{2(2p^+-1)} + |u^k|^{2(2p^--1)} + |v^k|^{2(2p^+-1)} \\ &\quad + |v^k|^{2(2p^-)}\Big)\,dx \\ &\le \frac{1}{2}\|u_t^k\|_2^2 + C\Big(\|\nabla u^k\|_2^{2(2p^+-1)} + \|\nabla u^k\|_2^{2(2p^--1)} + \|\nabla v^k\|_2^{2(2p^+-1)} \\ &\quad + \|\nabla v^k\|_2^{2(2p^--1)}\Big). \end{aligned} \tag{3.8}$$

where $C > 0$ is a generic positive constant. Again, by (3.2), for some $C > 0$, we have

$$E^k(t) \le E^k(0) \le C, \ \forall t \le t_k, \tag{3.9}$$

since

$$\frac{d}{dt}E^k(t) = -\int_\Omega |u_t^k(t)|^{m(x)+1}\,dx - \int_\Omega |v_t^k(t)|^{r(x)+1}\,dx \le 0,$$

where

$$E^k(t) = \frac{1}{2}[\|u_t^k\|_2^2 + \|v_t^k\|_2^2 + \|\nabla u^k\|_2^2 + \|\nabla v^k\|_2^2] + \int_\Omega \frac{|u^k v^k|^{p(x)}}{p(x)} dx.$$

So, thanks to (3.9), estimate (3.8) becomes

$$\begin{aligned} &\int_\Omega |u^k|^{p(x)-1}|v^k|^{p(x)}|u_t^k|\,dx \\ &\le \frac{1}{2}\|u_t^k\|_2^2 + C\max\left\{(E^k(0))^{4(p_2-1)}, (E^k(0))^{4(p_1-1)}\right\}[\|\nabla u^k\|_2^2 + \|\nabla v^k\|_2^2] \\ &\le \frac{1}{2}\|u_t^k\|_2^2 + C[\|\nabla u^k\|_2^2 + \|\nabla v^k\|_2^2] \\ &\le C[\|u_t^k\|_2^2 + \|\nabla u^k\|_2^2 + \|\nabla v^k\|_2^2]. \end{aligned} \tag{3.10}$$

In a similar way, for all $t \le t_k$, we have

$$\int_\Omega |v^k|^{p(x)-1}|u^k|^{p(x)}|v_t^k|\,dx \le C[\|v_t^k\|_2^2 + \|\nabla v^k\|_2^2 + \|\nabla u^k\|_2^2]. \tag{3.11}$$

By substituting (3.10) and (3.11) into (3.5), we arrive at

$$\begin{aligned} &\frac{1}{2}[\|u_t^k\|_2^2 + \|v_t^k\|_2^2 + \|\nabla u^k\|_2^2 + \|\nabla v^k\|_2^2] \\ &+ \int_0^t \int_\Omega (\alpha(\tau)|u_t^k(x,\tau)|^{m(x)} + \beta(\tau)|v_t^k(x,\tau)|^{r(x)})\,dx\,d\tau \\ &\le C + C\int_0^t [\|u_t^k\|_2^2 + \|v_t^k\|_2^2 + \|\nabla u^k\|_2^2 + \|\nabla v^k\|_2^2]\,d\tau, \end{aligned} \tag{3.12}$$

for all $t \le t_k$ ($t_k \le T$). Invoking Gronwall's Lemma, inequality (3.12) yields

$$\begin{aligned} &\|u_t^k\|_2^2 + \|v_t^k\|_2^2 + \|\nabla u^k\|_2^2 + \|\nabla v^k\|_2^2 \\ &+ \int_0^t \int_\Omega (\alpha(\tau)|u_t^k(x,\tau)|^{m(x)} + \beta(\tau)|v_t^k(x,\tau)|^{r(x)})\,dx\,d\tau \le C_T, \end{aligned}$$

for all $0 \le t \le t_k$, where C_T is a constant independent of t and k. Therefore, we can extend $(u_k)_k$ and $(v_k)_k$ on $[0,T)$. Moreover, we have

$$\begin{gathered} (u^k)_k \text{ and } (v^k)_k \text{ are bounded in } L^\infty((0,T), H_0^1(\Omega)), \\ (u_t^k)_k \text{ is bounded in } L^\infty((0,T), L^2(\Omega)) \cap L_\alpha^{m(\cdot)}(\Omega \times (0,T)), \\ (v_t^k)_k \text{ is bounded in } L^\infty((0,T), L^2(\Omega)) \cap L_\beta^{r(\cdot)}(\Omega \times (0,T)). \end{gathered} \tag{3.13}$$

Step 3. From (3.13), there exist subsequences of $(u^k)_k$ and $(v^k)_k$, still denoted by $(u^k)_k$ and $(v^k)_k$, (for simplicity), and two functions $u, v : \Omega \times [0,T) \to \mathbb{R}$, such that

$$\begin{gathered} u^k \rightharpoonup^* u \quad \text{and} \quad v^k \rightharpoonup^* v \quad \text{in } L^\infty((0,T), H_0^1(\Omega)), \\ u_t^k \rightharpoonup^* u_t \quad \text{and} \quad v_t^k \rightharpoonup^* v_t \quad \text{in } L^\infty((0,T), L^2(\Omega)). \end{gathered}$$

Next, we show that

$$|u^k|^{p(\cdot)-2}u^k|v^k|^{p(\cdot)} \to |u|^{p(\cdot)-2}u|v|^{p(\cdot)} \quad \text{a.e. in } \Omega \times (0,T), \tag{3.14}$$

$$|v^k|^{p(\cdot)-2}v^k|u^k|^{p(\cdot)} \to |v|^{p(\cdot)-2}v|u|^{p(\cdot)} \quad \text{a.e. in } \Omega \times (0,T). \tag{3.15}$$

Since $H_0^1(\Omega) \hookrightarrow^{compact} L^2(\Omega)$ and by the Aubin-Lions theorem, there are subsequences of $(u^k)_k$ and $(v^k)_k$, still denoted by $(u^k)_k$ and $(v^k)_k$, respectively, such that

$$\begin{gathered} u^k \to u \quad \text{and} \quad v^k \to v \quad \text{strongly in } L^2((0,T), L^2(\Omega)), \\ u^k \to u \quad \text{and} \quad v^k \to v \quad \text{a.e. in } \Omega \times (0,T). \end{gathered} \tag{3.16}$$

The continuity of the function

$$(u,v) \mapsto \big(|u|^{p(\cdot)-2}u|v|^{p(\cdot)}, |v|^{p(\cdot)-2}v|u|^{p(\cdot)}\big)$$

and the convergence (3.16) allow us to establish (3.14) and (3.15). Also, from (3.10) and (3.13), it follows that

$$\int_0^T \||u^k|^{p(\cdot)-2}u^k|v^k|^{p(\cdot)}\|_2^2 \, d\tau \le C \int_0^T [\|\nabla u^k\|_2^2 + \|\nabla v^k\|_2^2] \, d\tau \le C_T,$$

which means that $|u^k|^{p(\cdot)-2}u^k|v^k|^{p(\cdot)}$ is bounded in $L^2(\Omega \times (0,T))$. Combining this result with (3.14), and invoking Lions' Lemma, we deduce that

$$|u^k|^{p(\cdot)-2}u^k|v^k|^{p(\cdot)} \to |u|^{p(\cdot)-2}u|v|^{p(\cdot)} \quad \text{in } L^2(\Omega \times (0,T)).$$

Similarly,

$$|v^k|^{p(\cdot)-2}v^k|u^k|^{p(\cdot)} \to |v|^{p(\cdot)-2}v|u|^{p(\cdot)} \quad \text{in } L^2(\Omega \times (0,T)).$$

By repeating the same steps of [25] for the sequences $(S_k)_k, (\tilde{S}_k)_k$ defined, for all $k \ge 1$, by

$$S_k = \int_0^T \alpha(t) \int_\Omega (h(u_t^k) - h(z))(u_t^k - z) \, dx \, dt,$$

for $z \in L_\alpha^{m(\cdot)}((0,T), H_0^1(\Omega))$ and $h(z) = |z|^{m(\cdot)-2}z$, and

$$\tilde{S}_k = \int_0^T \beta(t) \int_\Omega (h(v_t^k) - h(z))(v_t^k - z) \, dx \, dt,$$

for $z \in L_\beta^{r(\cdot)}((0,T), H_0^1(\Omega))$ and $h(z) = |z|^{r(\cdot)-2}z$, we easily show that

$$\begin{gathered} \alpha(\cdot)|u_t^k|^{m(\cdot)-2}u_t^k \rightharpoonup \alpha(\cdot)|u_t|^{m(\cdot)-2}u_t \quad \text{in } L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega \times (0,T)), \\ \beta(\cdot)|v_t^k|^{r(\cdot)-2}v_t^k \rightharpoonup \beta(\cdot)|v_t|^{r(\cdot)-2}v_t \quad \text{in } L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega \times (0,T)) \end{gathered}$$

and establish that (u,v) satisfies the two identities of Definition 3.1, for all test functions $\phi, \psi \in H_0^1(\Omega)$, all $t \in (0,T)$ and all $T > 0$.

Step 4. As in [25], we easily establish that (u,v) satisfies the initial conditions. Finally, we conclude that (u,v) is a global weak solution of (P) in the sense of Definition 3.1. □

Note that the uniqueness of the solution remains an open question. However, if $\alpha(\cdot) = \beta(\cdot)$, we can obtain uniqueness by repeating the same steps of [23].

4. Stability result

We first introduce the energy functional associated with system (1.1),

$$E(t) =: \frac{1}{2}[\|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2] + \int_\Omega \frac{|uv|^{p(x)}}{p(x)} dx,$$

for all $t \in [0, T)$.

Lemma 4.1. *Along the solution of* (1.1), *the energy functional satisfies*

$$E'(t) = -\alpha(t) \int_\Omega |u_t|^{m(x)} dx - \beta(t) \int_\Omega |v_t|^{r(x)} dx \leq 0,$$

for all $t \in [0, T)$.

Theorem 4.2. *Suppose that* (2.1)-(2.3) *hold. Assume, further, that* $\int_0^\infty \alpha(s)ds = \infty$ *and* $\int_0^\infty \beta(s)ds = \infty$. *Then, there exist two constants* $c, \omega > 0$ *such that for all* $t \geq 0$, *the solution of* (1.1) *satisfies*

$$E(t) \leq \begin{cases} ce^{-\omega \int_0^t \gamma(s)ds}, & \text{if } \lambda^+ = 2, \\ \frac{c}{(1+\int_0^t \gamma(s)ds)^{2/(\lambda^+-2)}}, & \text{if } \lambda^+ > 2, \end{cases}$$

where $\lambda^+ = \max\{m^+, r^+\}$ *and* $\gamma = \min\{\alpha, \beta\}$.

Proof. Let $T > S > 0$ and $q \geq 0$ be specified later. Multiplying the first differential equation of (1.1) by $\gamma E^q u$, the second one by $\gamma E^q v$, integrating each result over $\Omega \times (S, T)$ and using Green's formula, we obtain

$$\begin{aligned} &\int_S^T \gamma(t)E^q(t) \int_\Omega [(uu_t)_t - u_t^2 + |\nabla u|^2 + \alpha(t)|u_t|^{m(x)-2}u_t u]\, dx\, dt \\ &= -\int_S^T \gamma(t)E^q(t) \int_\Omega |uv|^{p(x)}\, dx\, dt \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\int_S^T \gamma(t)E^q(t) \int_\Omega [(vv_t)_t - v_t^2 + |\nabla v|^2 + \beta(t)|v_t|^{r(x)-2}v_t v]\, dx\, dt \\ &= -\int_S^T \gamma(t)E^q(t) \int_\Omega |uv|^{p(x)}\, dx\, dt. \end{aligned} \tag{4.2}$$

We add and subtract the following two terms

$$-\int_S^T \gamma(t)E^q(t) \int_\Omega u_t^2\, dx\, dt \quad \text{and} \quad -\int_S^T \gamma(t)E^q(t) \int_\Omega v_t^2\, dx\, dt$$

to (4.1) and (4.2), respectively. The addition of the two results implies

$$\begin{aligned} &\int_S^T \gamma E^q \int_\Omega (u_t^2 + v_t^2 + |\nabla u|^2 + |\nabla v|^2)\, dx\, dt \\ &= -\int_S^T \gamma E^q \int_\Omega (uu_t + vv_t)_t\, dx\, dt + 2\int_S^T \gamma E^q \int_\Omega (u_t^2 + v_t^2)\, dx\, dt \\ &\quad - \int_S^T \gamma E^q \int_\Omega \Big(\alpha|u_t|^{m(x)-2}u_t u + \beta|v_t|^{r(x)-2}v_t v\Big)\, dx\, dt \\ &\quad - 2\int_S^T \gamma E^q \int_\Omega |uv|^{p(x)}\, dx\, dt. \end{aligned} \tag{4.3}$$

Recalling the expression of E, (4.3) leads to

$$\begin{aligned} 2\int_S^T \gamma E^{q+1}dt &= -\int_S^T \gamma E^q \int_\Omega (uu_t+vv_t)_t\,dx\,dt + 2\int_S^T \gamma E^q \int_\Omega (u_t^2+v_t^2)\,dx\,dt \\ &\quad - \int_S^T \gamma E^q \int_\Omega \alpha\Big(|u_t|^{m(x)-2}u_t u + \beta|v_t|^{r(x)-2}v_t v\Big)\,dx\,dt \\ &\quad + \int_S^T \gamma E^q \int_\Omega \big(\frac{2}{p(x)}-2\big)|uv|^{p(x)}\,dx\,dt. \end{aligned}$$

Since $p(x)>1$, for all $x\in\Omega$, it follows that

$$\begin{aligned} &2\int_S^T \gamma E^{q+1}dt \\ &\le -\int_S^T \gamma E^q \int_\Omega (uu_t+vv_t)_t\,dx\,dt + 2\int_S^T \gamma E^q \int_\Omega (u_t^2+v_t^2)\,dx\,dt \\ &\quad - \int_S^T \gamma E^q \int_\Omega \Big(\alpha|u_t|^{m(x)-2}u_t u + \beta|v_t|^{r(x)-2}v_t v\Big)\,dx\,dt\,. \end{aligned} \tag{4.4}$$

On the other hand, for a.e. $t\in[S,T]$, we have

$$\frac{d}{dt}\Big(\gamma E^q \int_\Omega (uu_t+vv_t)\,dx\Big) = (\gamma E^q)' \int_\Omega (uu_t+vv_t)\,dx + \gamma E^q \int_\Omega (uu_t+vv_t)_t dx$$

which gives

$$\begin{aligned} &\gamma E^q \int_\Omega (uu_t+vv_t)_t\,dx \\ &= \frac{d}{dt}\Big(\gamma E^q \int_\Omega (uu_t+vv_t)\,dx\Big) - (\gamma E^q)' \int_\Omega (uu_t+vv_t)\,dx. \end{aligned} \tag{4.5}$$

Substituting (4.5) into (4.4), we arrive at

$$2\int_S^T \gamma E^{q+1}dt \le I_1+I_2+I_3+I_4, \tag{4.6}$$

where

$$\begin{gathered} I_1 = -[\gamma E^q \int_\Omega (uu_t+vv_t)\,dx]_S^T, \\ I_2 = \int_S^T (\gamma' E^q + q\gamma E^{q-1}E') \int_\Omega (uu_t+vv_t)\,dx\,dt, \\ I_3 = 2\int_S^T \gamma E^q \int_\Omega (u_t^2+v_t^2)\,dx\,dt, \\ I_4 = -\int_S^T \gamma E^q \int_\Omega \Big(\alpha|u_t|^{m(x)-2}u_t u + \beta|v_t|^{r(x)-2}v_t v\Big)\,dx\,dt. \end{gathered}$$

In what follows, we estimate I_i, for $i=1,\dots,4$. First, using Young's and Poincaré's inequalities and the definition of E, we obtain

$$|\int_\Omega (uu_t+vv_t)\,dx| \le \frac{c_e}{2}[\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u_t|_2^2 + \|v_t\|_2^2] \le CE(t), \tag{4.7}$$

where c_e is the Poincaré constant. Therefore, recalling Lemma 4.1,

$$\begin{aligned} I_1 &= \gamma(S)E^q(S)\int_\Omega (u(x,S)u_t(x,S)+v(x,S)v_t(x,S))\,dx \\ &\quad -\gamma(T)E^q(T)\int_\Omega (u(x,T)u_t(x,T)+v(x,T)v_t(x,T))\,dx \\ &\le C[\gamma(S)E^{q+1}(S)+\gamma(T)E^{q+1}(T)] \le C\gamma(S)E^{q+1}(S) \\ &\le CE(S), \end{aligned} \tag{4.8}$$

where C is a generic positive constant. Next, using $E'(t)\le 0$, we obtain

$$\begin{aligned} I_2 &\le C\int_S^T (\gamma' E^q + q\gamma E^{q-1}E')E(t)\,dt \\ &\le C\Big|\int_S^T \gamma' E^{q+1}\,dt\Big| + C\Big|\int_S^T q\gamma E^q E'\,dt\Big| \\ &\le CE^{q+1}(S)\Big|\int_S^T \gamma' dt\Big| + Cq\gamma(S)\Big|\int_S^T E^q E'\,dt\Big| \\ &\le CE^{q+1}(S)[\gamma(S)-\gamma(T)] + CE(S) \le CE(S). \end{aligned} \tag{4.9}$$

For the third integral, we have

$$I_3 = 2\int_S^T \gamma E^q \int_\Omega |u_t|^2\,dx\,dt + 2\int_S^T \gamma E^q \int_\Omega |v_t|^2\,dx\,dt = J_1 + J_2.$$

To estimate J_1, we consider the following partition of Ω,

$$\Omega_+ = \{x\in\Omega : |u_t(x,t)|\ge 1\}, \quad \Omega_- = \{x\in\Omega : |u_t(x,t)|<1\}.$$

Therefore, by Hölder's inequality and the definition of λ^+, we obtain

$$\begin{aligned} J_1 &= 2\int_S^T \gamma E^q \Big[\int_{\Omega_-}|u_t|^2 dx + \int_{\Omega_+}|u_t|^2 dx\Big]\,dt \\ &\le C\int_S^T \gamma E^q \Big(\int_{\Omega_-}|u_t|^{\lambda^+}dx\Big)^{2/\lambda^+} dt + C\int_S^T \gamma E^q \int_{\Omega_+}|u_t|^{m(x)}\,dx\,dt \\ &\le C\int_S^T \gamma E^q \Big(\int_{\Omega_-}|u_t|^{m(x)}\,dx\Big)^{2/\lambda^+} dt + C\int_S^T E^q\Big(\gamma\int_{\Omega_+}|u_t|^{m(x)}dx\Big)\,dt. \end{aligned}$$

This yields

$$\begin{aligned} J_1 &\le C\int_S^T \gamma^{(1-\frac{2}{\lambda^+})}E^q\Big(\gamma\int_\Omega |u_t|^{m(x)}dx\Big)^{2/\lambda^+} + C\int_S^T E^q\Big(\gamma\int_\Omega |u_t|^{m(x)}dx\Big)\,dt \\ &\le C\int_S^T \gamma^{(1-\frac{2}{\lambda^+})}E^q\Big(\alpha\int_\Omega |u_t|^{m(x)}dx\Big)^{2/\lambda^+} + C\int_S^T E^q\Big(\alpha\int_\Omega |u_t|^{m(x)}dx\Big)\,dt \\ &\le C\int_S^T \gamma^{(1-\frac{2}{\lambda^+})}E^q(-E')^{2/\lambda^+}\,dt + C\int_S^T E^q(-E')dt \\ &\le C\int_S^T \gamma^{(1-\frac{2}{\lambda^+})}E^q(-E')^{2/\lambda^+}\,dt + CE(S), \end{aligned}$$

using Lemma 4.1, and the definition of γ. Similarly, we find that

$$J_2 \le C\int_S^T \gamma^{(1-\frac{2}{\lambda^+})}E^q(-E')^{2/\lambda^+}\,dt + CE(S).$$

By addition,

$$I_3 \le C \int_S^T \gamma^{(1-\frac{2}{\lambda^+})} E^q (-E')^{2/\lambda^+}\, dt + CE(S).$$

Two cases are possible:

Case 1. If $\lambda^+ = 2$, then

$$\begin{aligned} I_3 &\le C \int_S^T E^q(-E^{'})dt + CE(S) \\ &\le C[E^{q+1}(S) - E^{q+1}(T)] + CE(S) \le CE(S). \end{aligned}$$

Case 2. if $\lambda^+ > 2$, by Young's inequality with $\delta = q+1$ and $\delta' = (q+1)/q$, we have that for all $\varepsilon > 0$,

$$I_3 \le \varepsilon C \int_S^T \gamma^{(1-\frac{2}{\lambda^+})(\frac{q+1}{q})} E^{q+1} dt + C_\varepsilon \int_S^T (-E')^{\frac{2(q+1)}{\lambda^+}}\, dt + CE(S).$$

If we take $\varepsilon = \frac{1}{2C}$ and $q = \frac{\lambda^+}{2} - 1$, then

$$\begin{aligned} I_3 &\le \frac{1}{2} \int_S^T \gamma E^{q+1} dt + C_\varepsilon \int_S^T (-E')dt + CE(S) \\ &\le \frac{1}{2} \int_S^T \gamma E^{q+1} dt + CE(S). \end{aligned}$$

Therefore, for $\lambda^+ \ge 2$,

$$I_3 \le \frac{1}{2} \int_S^T \gamma E^{q+1} dt + CE(S). \tag{4.10}$$

Finally, we handle I_4 as follows. Since α and β are bounded functions on $\mathbb{R}_+$, then

$$\begin{aligned} I_4 &\le C \int_S^T \gamma E^q \int_\Omega |u||u_t|^{m(x)-1}\, dx\, dt + C \int_S^T \gamma E^q \int_\Omega |v||v_t|^{r(x)-1}\, dx\, dt \\ &= J_3 + J_4. \end{aligned}$$

Now, as in [21], applying Young's inequality with

$$\delta(x) = \frac{m(x)}{m(x)-1} \quad \text{and} \quad \delta'(x) = m(x),$$

we obtain that for all $\varepsilon > 0$,

$$J_3 \le \int_S^T \gamma E^q \Big[\varepsilon \int_\Omega |u|^{m(x)} dx + \int_\Omega C_\varepsilon(x)|u_t|^{m(x)} dx\Big]\, dt,$$

where

$$C_\varepsilon(x) = \frac{[m(x)-1]^{m(x)-1}}{[m(x)]^{m(x)} \varepsilon^{m(x)-1}}.$$

Similarly,

$$J_4 \le \int_S^T \gamma E^q \Big[\varepsilon \int_\Omega |v|^{r(x)} dx + \int_\Omega C'_\varepsilon(x)|v_t|^{r(x)} dx\Big]\, dt,$$

where

$$C'_\varepsilon(x) = \frac{[r(x)-1]^{r(x)-1}}{[r(x)]^{r(x)} \varepsilon^{r(x)-1}}.$$

By addition, we find

$$I_4 \le \int_S^T \gamma E^q \int_\Omega \Big(\varepsilon|u|^{m(x)} + \varepsilon|v|^{r(x)} + C_\varepsilon(x)|u_t|^{m(x)} + C'_\varepsilon(x)|v_t|^{r(x)}\Big)\,dx\,dt. \quad (4.11)$$

Using (2.1) and recalling that $m^-, r^- \ge 2$, we have the estimate

$$\begin{aligned}
J_5 &= \varepsilon \int_S^T \gamma E^q \int_\Omega \big(|u|^{m(x)} + |v|^{r(x)}\big)\,dx\,dt \\
&\le \varepsilon C \int_S^T \gamma E^q \int_\Omega \big(|u|^{m_-} + |u|^{m_+} + |v|^{r_-} + |v|^{r_+}\big)\,dx\,dt \\
&\le \varepsilon C \int_S^T \gamma E^q \big(\|\nabla u\|_2^{m_-} + \|\nabla u\|_2^{m_+} + \|\nabla v\|_2^{r_-} + \|\nabla v\|_2^{r_+}\big)\,dt \\
&\le \varepsilon C \int_S^T \gamma E^{q+1} \big(E^{\frac{m_-}{2}-1} + E^{\frac{m_+}{2}-1} + E^{\frac{r_-}{2}-1} + E^{\frac{r_+}{2}-1}\big)\,dt \\
&\le \varepsilon C \big(E(0)^{\frac{m_-}{2}-1} + E(0)^{\frac{m_+}{2}-1} + E(0)^{\frac{r_-}{2}-1} + E(0)^{\frac{r_+}{2}-1}\big) \int_S^T \gamma E^{q+1}\,dt.
\end{aligned}$$

by taking

$$\varepsilon = \frac{1}{2C}\Big(E(0)^{\frac{m_-}{2}-1} + E(0)^{\frac{m_+}{2}-1} + E(0)^{\frac{r_-}{2}-1} + E(0)^{\frac{r_+}{2}-1}\Big)^{-1},$$

it yields

$$J_5 \le \frac{1}{2}\int_S^T \gamma E^{q+1} dt.$$

Moreover, $C_\varepsilon(\cdot)$ and $C'_\varepsilon(\cdot)$ are bounded since $m(\cdot)$ and $r(\cdot)$ are bounded. Consequently, (4.11) becomes

$$\begin{aligned}
I_4 &\le \frac{1}{2}\int_S^T \gamma E^{q+1} dt + C\int_S^T \gamma E^q \big(|u_t|^{m(x)} + |v_t|^{r(x)}\big)\,dx\,dt \\
&\le \frac{1}{2}\int_S^T \gamma E^{q+1} dt + C\int_S^T E^q \big(\alpha|u_t|^{m(x)} + \beta|v_t|^{r(x)}\big)\,dx\,dt \\
&\le \frac{1}{2}\int_S^T \gamma E^{q+1} dt + C\int_S^T E^q(-E'(t))dt \\
&\le \frac{1}{2}\int_S^T \gamma E^{q+1} dt + CE(S).
\end{aligned} \quad (4.12)$$

Finally, by inserting (4.8), (4.9), (4.10) and (4.12) into (4.6), we have

$$\int_S^T \gamma E^{q+1}(t) dt \le CE(S).$$

Taking $T \to \infty$, it follows that

$$\int_S^\infty \gamma E^{q+1}(t) dt \le CE(S).$$

Invoking Lemma 2.6 with $\sigma(t) = \int_0^t \gamma(s)ds$, we obtain the desired result. □

As a special case, when α and β are constants, we obtain the result of [23]. More precisely, we have the following result.

Corollary 4.3. *Assume that* (2.1)-(2.3) *hold. Then there exist two constants* $c, \omega > 0$ *such that the solution of* (1.1) *satisfies*

$$E(t) \leq \begin{cases} ce^{-\omega t}, & \text{if } \lambda^+ = 2, \\ \frac{c}{(1+t)^{2/(\lambda^+-2)}}, & \text{if } \lambda^+ > 2. \end{cases}$$

for all $t \geq 0$.

We end this section with examples illustrating our stability result.

Example 4.4. if $\alpha(t) = \beta(t) = \frac{1}{1+t}$, then the estimate in Theorem 4.2 gives

$$E(t) \leq \begin{cases} \frac{c}{(1+t)^{\omega}}, & \text{if } \lambda^+ = 2, \\ \frac{c}{(1+\ln(1+t))^{2/(\lambda^+-2)}}, & \text{if } \lambda^+ > 2. \end{cases}$$

Example 4.5. If $\alpha(t) = \frac{1}{(1+t)^a}, \beta(t) = \frac{1}{(1+t)^b}$, for $0 \leq b < a < 1$ then the estimate in Theorem 4.2 gives

$$E(t) \leq \begin{cases} ce^{\frac{-\omega}{1-a}(1+t)^{(1-a)}}, & \text{if } \lambda^+ = 2, \\ c/(1 + \frac{1}{1-a}[(1+t)^{(1-a)} - 1])^{2/(\lambda^+-2)}, & \text{if } \lambda^+ > 2. \end{cases}$$

Example 4.6. If $\alpha(t) = 1/(2+t)\ln(2+t), \beta(t) = 1/(2+t)^2(\ln(2+t))^2$, then the estimate in Theorem 4.2 gives

$$E(t) \leq \begin{cases} c(\frac{\ln 2}{\ln(2+t)})^{\omega}, & \text{if } \lambda^+ = 2, \\ c/[1 + \ln(\frac{\ln(2+t)}{\ln 2})]^{2/(\lambda^+-2)}, & \text{if } \lambda^+ > 2. \end{cases}$$

5. Numerical tests

In this section, we illustrate numerically the theoretical results of the present work. We solve the system (1.1) under the corresponding initial and boundary conditions. The nonlinear system (1.1) is discritized using the classical second order finite difference method in time and space. In addition, we implement the stable and conservative scheme of Lax-Wendroff. For more details and similar techniques, we refer to [16]. Here we give five performed tests, for $\Omega = (0, 1)$ and $[0, T] = [0, 20]$:

Test 1. Based on Theorem 4.2 and the result explained in Example 4.4, we obtain the polynomial decay of the energy

$$E_1(t) \leq E_f^1(t) = \frac{c}{(1+t)^w},$$

for two positive constants c and w. For this test, we use the functions

$$m(x) = r(x) = 2, \quad p(x) = 2 - \frac{1}{1+x}, \quad \forall x \in \Omega,$$
$$\alpha(t) = \beta(t) = \frac{1}{1+t}, \quad \forall t > 0.$$

Test 2. Examining the second result explained in Example 4.4, we obtain a logarithmic-polynomial decay of the energy

$$E_2(t) \leq E_f^2(t) = \frac{c}{(1 + \ln(1+t))^2},$$

for a positive constant c. For this test, we use the functions

$$m(x) = 2, \quad r(x) = 2 + \frac{1}{1+x}, \quad p(x) = 2 - \frac{1}{1+x}, \quad \forall x \in \Omega,$$

$$\alpha(t) = \beta(t) = \frac{1}{1+t}, \quad \forall t > 0.$$

Test 3. Examining the result explained in Example 4.5, we obtain an exponential-type decay of the energy

$$E_3(t) \le E_f^3(t) = c_1 e^{-c_2\sqrt{t}},$$

for two positive constants c_1 and c_2. For this test, we use the functions

$$m(x) = r(x) = 2, \quad p(x) = 1 + \frac{1}{1+x}, \quad \forall x \in \Omega,$$
$$\alpha(t) = \beta(t) = \frac{1}{\sqrt{1+t}}, \quad \forall t > 0.$$

Test 4. Examining the second result explained in Example 4.5, we obtain a polynomial type decay of the energy

$$E_4(t) \le E_f^4(t) = \frac{c}{(1+t)^w},$$

for two positive constants c and w. For this test, we use the functions

$$m(x) = r(x) = 2 + \frac{1}{1+x}, \quad p(x) = 1 + \frac{1}{1+x}, \quad \forall x \in \Omega,$$
$$\alpha(t) = \beta(t) = \frac{1}{\sqrt{1+t}}, \quad \forall t > 0.$$

Test 5: Examining the results obtained in Example 4.6, we obtain a logarithmic-polynomial decay of the energy

$$E_5(t) \le E_f^5(t) = \frac{c}{(\ln(2+t))^w},$$

for two positive constants c and w. For this test, we use the functions

$$m(x) = r(x) = p(x) = 2, \quad \forall x \in \Omega,$$
$$\alpha(t) = \frac{1}{(2+t)\ln(1+t)}, \beta(t) = \frac{1}{((2+t)\ln(1+t))^2}, \quad \forall t > 0.$$

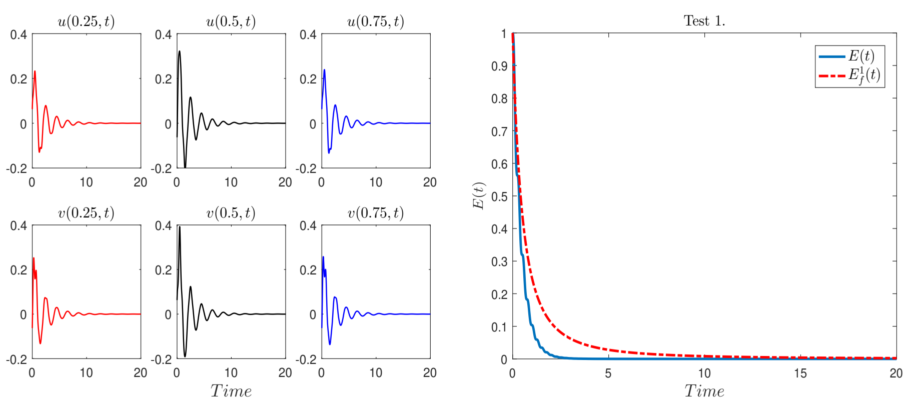

FIGURE 1. Test 1: Damping cross section waves, energy decay and upper bound function $E_f^1(t) = \frac{1}{(1+t)^2}$.

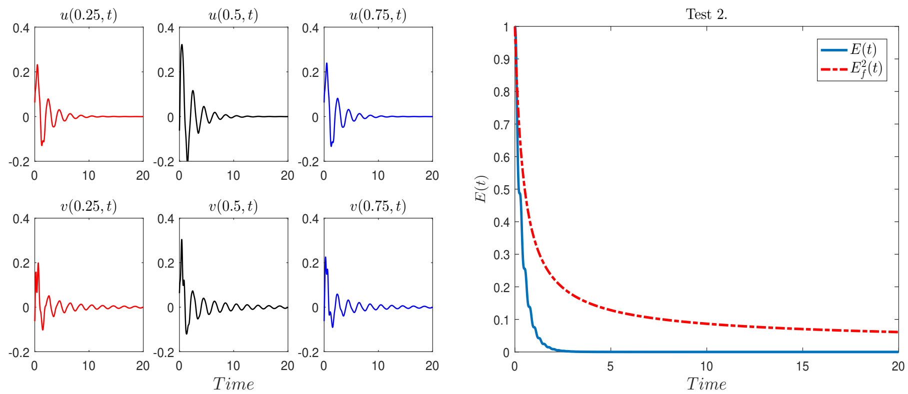

FIGURE 2. Test 2: Damping cross section waves, energy decay and upper bound function $E_f^2(t) = \frac{1}{(1+\ln(1+t))^2}$.

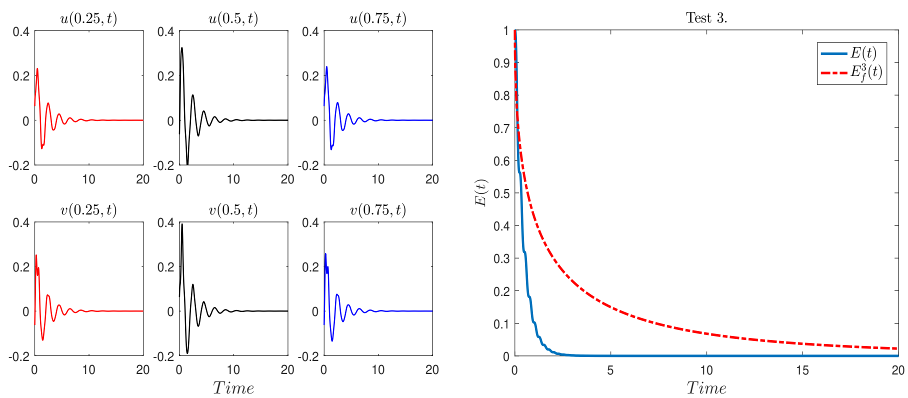

FIGURE 3. Test 3: Damping cross section waves, energy decay and upper bound function $E_f^3(t) = e^{-0.85\sqrt{t}}$.

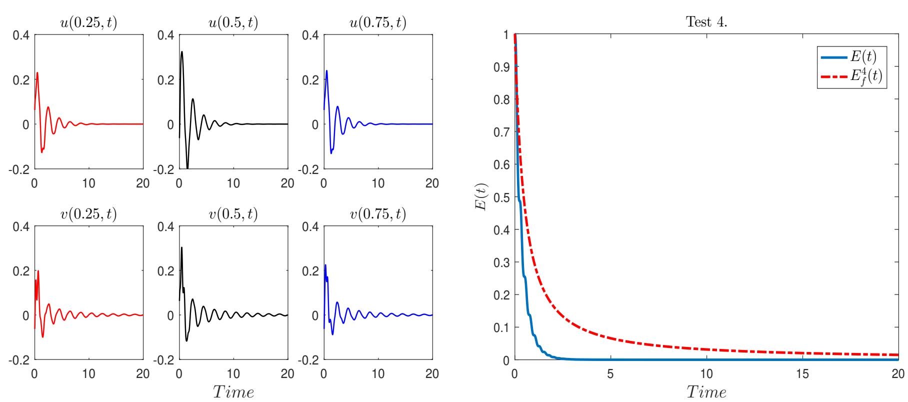

FIGURE 4. Test 4: Damping cross section waves, energy decay and upper bound function $E_f^4(t) = \frac{1}{(-1+2\sqrt{1+t})^2}$.

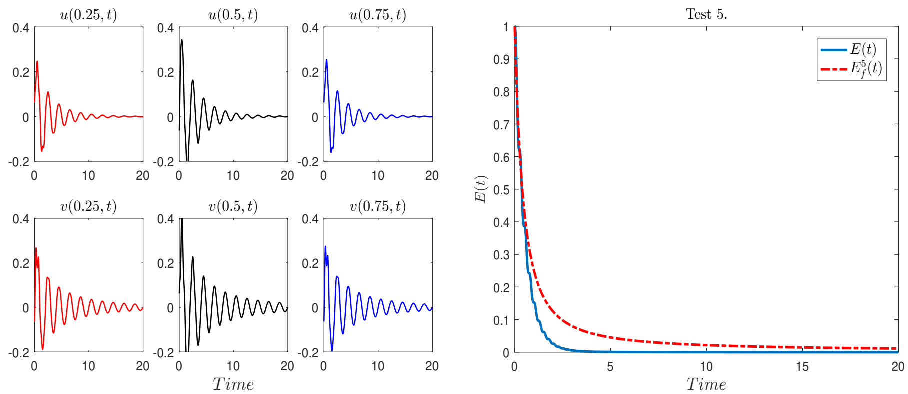

FIGURE 5. Test 5: Damping cross section waves, energy decay and the upper bound function $E_f^5(t) = \frac{(\ln(2))^3}{(\ln(2+t))^3}$.

It should be stressed that the numerical stability of the method implemented is ensured by taking in consideration the Courant-Friedrichs-Lewy (CFL) inequality $\Delta t \ll 0.5\Delta x$, where Δt represents the numerical time step and Δx the numerical spatial step. The spatial interval $\Omega = (0, 1)$ is subdivided into 200 subintervals and the temporal interval $[0, T] = [0, 20]$ is deduced from the stability condition above. We run our code for 10000 time steps $\Delta t = 2 \cdot 10^{-3}$, using the initial conditions

$$\begin{gathered} u(x,0) = \sin(\pi x), \quad v(x,0) = -\sin(\pi x) \quad \text{in } \Omega, \\ u_t(x,0) = 1, \quad v_t(x,0) = 1 \quad \text{in } \Omega. \end{gathered}$$

Our computational simulations show in Figures 1–5(left) all decay types. We restricted our plotings to three cross-section cuts for the numerical solution (u, v) at $x = 0.25$, $x = 0.5$ and at $x = 0.75$. For all components of the solutions, the decay behavior is clearly demonstrated in all tests. Moreover, the dotted curves in Figures 1–5 (right) represent the corresponding upper bound of the energy function $E_f^i(t)$ for $i = 1, \dots, 5$.

6. Concluding remarks

In this work, we studied a coupled system of two weakly damped wave equations, where the coupling terms and the dampings are of non-standard forms. We first proved the existence of a weak global solution then established some decay results in terms of the damping exponents and the damping coefficients. We also give some examples and presented various numerical tests to illustrate our theoretical findings. All numerical tests came in agreement with the theoretical results. This work generalizes many other works in the literature. In particular, we obtain the results of [9, 23], if $\alpha(\cdot) = \beta(\cdot)$.

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