Electronic Journal of Differential Equations, Vol. 2023 (2023), No. 75, pp. 1–20.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2023.75

CONCENTRATION OF NODAL SOLUTIONS FOR SEMICLASSICAL QUADRATIC CHOQUARD EQUATIONS

LU YANG, XIANGQING LIU, JIANWEN ZHOU

ABSTRACT. In this article concerns the semiclassical Choquard equation $-\varepsilon^2\Delta u + V(x)u = \varepsilon^{-2}(\frac{1}{|\cdot|} * u^2)u$ for $x \in \mathbb{R}^3$ and small ε. We establish the existence of a sequence of localized nodal solutions concentrating near a given local minimum point of the potential function V, by means of the perturbation method and the method of invariant sets of descending flow.

1. Introduction

In the past two decades, attention has devoted to the study of the existence, multiplicity, and properties of the solutions for the nonlinear Choquard equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{\alpha-N}\Big(\frac{1}{|\cdot|^\alpha} * u^p\Big)u^{p-1},\ x \in \mathbb{R}^N, \tag{1.1}$$

where $0 < \alpha < N$, $\frac{2N-\alpha}{N} < p < \frac{2N-\alpha}{N-2}$, and $\varepsilon > 0$ is a small positive parameter. When $N = 3$, $\alpha = 1$ and $\varepsilon = 1$, as an important model, the problem

$$\Delta u + V(x)u = \big(\frac{1}{|\cdot|} * u^2\big)u, \quad x \in \mathbb{R}^3 \tag{1.2}$$

was introduced by Pekar [30] to describe the quantum theory of a polaron at rest, and then used by Choquard [18] to study steady states of the one-component plasma approximation to the Hartree-Fock theory. Later, the same equation re-emerged as a model of self-gravitating matter [29], and in that context it is referred as to the Schrödinger-Newton system.

For the existence and qualitative properties of solutions for the nonlinear Choquard equation (1.1), we refer the reader to [2, 3, 5, 9, 10, 14, 17, 26, 27, 31, 32, 34, 35] and references therein. In particular, for $p > 2$, the existence of nodal solutions for the Choquard equation is an appealing aspect which is investigated in [5, 8, 11, 13, 15, 16, 23] by the variational method. In the physical case, for $p = 2$, the existence of nodal solutions for (1.1) only has few results. For $p \geq 2$ and V is a radial symmetry function, Gui [13] show that, for any positive integer k, the equation(1.1) has a sign-changing solution u_k which changes signs exactly k times. When $V \equiv 1$ and $p = 2$, Ghimenti[12] proved the existence of the least action

2020 *Mathematics Subject Classification.* 35B05, 35B45.
Key words and phrases. Choquard equation; sign-changing solution; perturbation method; Fredholm operator; nodal solution.

Submitted May 18, 2023. Published October 30, 2023.

nodal solutions. However, without symmetry or periodicity assumptions on the potential function V, there is no result of the existence of infinitely many sign-changing solutions for the equation (1.1) with $p = 2$. Motivated by the works mentioned above, we consider the existence of infinitely sign-changing solutions for the following equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{-2}\Big(\frac{1}{|\cdot|} * u^2\Big)u,\ x \in \mathbb{R}^3, \tag{1.3}$$

where the potential function V satisfies the assumptions:

(A1) $V \in C^1(\mathbb{R}^3, \mathbb{R})$ and there exist constants $b > a > 0$ such that
$$a \le V(x) \le b, \quad \forall\, x \in \mathbb{R}^3\,.$$

(A2) There exists a bounded domain $\mathcal{M} \subset \mathbb{R}^3$ with smooth boundary $\partial\mathcal{M}$ such that
$$\langle \overrightarrow{n}(x),\, \nabla V(x)\rangle > 0\,, \quad \forall\, x \in \partial\mathcal{M}\,,$$
where $\overrightarrow{n}(x)$ is the outer normal of $\partial\mathcal{M}$ at x.

Under the assumption (A2), in view of the critical set

$$\mathcal{A} = \{x \in \mathcal{M} : \nabla V(x) = 0\} \neq \emptyset, \tag{1.4}$$

without loss of generality we assume $0 \in \mathcal{A}$. For each set $B \subset \mathbb{R}^N$ and any $\delta > 0$, we set

$$B_\delta = \{x \in \mathbb{R}^3 : \delta x \in B\},$$
$$B^\delta = \{x \in \mathbb{R}^3 : \operatorname{dist}(x, B) := \inf_{y\in B} |x - y| < \delta\}.$$

The main result of this paper reads as follows.

Theorem 1.1. *Assume that* (A1), (A2) *hold. For each positive integer k, there exists $\varepsilon'_k > 0$ such that if $0 < \varepsilon < \varepsilon'_k$, then* (1.3) *has at least k pairs of sign-changing solutions $\pm v_{j,\varepsilon}$, $j = 1, \dots, k$. Moreover, for any $\delta > 0$ there exist $\mu > 0$, $C = C_k > 0$, and $\varepsilon'_k(\delta) > 0$ such that if $0 < \varepsilon < \varepsilon'_k(\delta)$, then*

$$|v_{j,\varepsilon}(x)| \le Ce^{-\frac{\mu}{\varepsilon}\operatorname{dist}(x,\mathcal{A}^\delta)} \quad \text{for } x \in \mathbb{R}^3,\ j = 1, \dots, k. \tag{1.5}$$

In this article, we can also obtain the existence and concentration phenomenon of the sign-changing solution of the equation(1.1). Here, we only consider the case with $\alpha = 1$ and $N = 3$.

By making the change of variable $\varepsilon y = x$, equation (1.3) is equivalent to

$$-\Delta u + V(\varepsilon x)u = \Big(\frac{1}{|\cdot|} * u^2\Big)u, \quad x \in \mathbb{R}^3 \tag{1.6}$$

and the corresponding functional is

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(\varepsilon x)u^2)\,dx - \frac{1}{4}\int_{\mathbb{R}^3}\Big(\frac{1}{|\cdot|} * u^2\Big)u^2\,dx\,. \tag{1.7}$$

We will use the method of invariant sets of descending flow to prove the existence of sign-changing solutions for (1.3), but the setting of invariant sets of descending flow can not fit well for the Choquard equation. In [15], we used the perturbation method [24] to overcome this difficulty for Choquard equation (1.1) with $2 < p < \frac{2N-\alpha}{N-2}$. However, the method described in [15] becomes invalid for the case $p = 2$.

To obtain compactness for the functional I_ε, we use the penalization method in [4, 36]. Let $G \in C^\infty(\mathbb{R}, \mathbb{R})$, satisfy $G'(s) \in [0, 1]$, $G''(s) \in [0, 2]$, $G(s) = 0$ for

$s \leq 1/2$ and $G(s) = s - 1$ for $s \geq 3/2$. We also require that $|G(s) - G'(s)s| \leq 3/2$. We define

$$\chi_\varepsilon(x) = \begin{cases} 0, & \text{if } x \in \mathcal{M}_\varepsilon \\ \varepsilon^{-6}\zeta(\operatorname{dist}(x, \mathcal{M}_\varepsilon)), & \text{if } x \notin \mathcal{M}_\varepsilon, \end{cases}$$

where $\zeta \in C^\infty$ is a cut-off function such that $\zeta(t) = 0$ if $t \leq 0$; $\zeta(t) = 1$ if $t \geq 1$ and $0 \leq \zeta'(t) \leq 2$, $0 \leq \zeta(t) \leq 1$. For each $\varepsilon > 0$, $p \in (2, p_0)$, $p_0 \in (2, 5)$ is a fixed constant, $u \in H^1(\mathbb{R}^3)$, we consider functionals:

$$\begin{aligned} \Gamma_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2)\,dx + \frac{1}{2}G\Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x)u^2\,dx\Big) \\ &\quad - \frac{1}{4}\int_{\mathbb{R}^3} \big(\frac{1}{|\cdot|} * u^2\big)u^2\,dx, \quad u \in H^1(\mathbb{R}^3), \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \Gamma_{\varepsilon,p}(u) &= \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2)\,dx + \frac{1}{2}G\Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x)u^2\,dx\Big) \\ &\quad - \frac{1}{2p}\int_{\mathbb{R}^3} \big(\frac{1}{|\cdot|} * |u|^p\big)|u|^p\,dx, \ \ u \in H^1(\mathbb{R}^3). \end{aligned} \tag{1.9}$$

Note that

$$\begin{aligned} &\langle D\Gamma_\varepsilon(u), \varphi\rangle \\ &= \int_{\mathbb{R}^3} (\nabla u \nabla \varphi + V(\varepsilon x)u\varphi)\,dx + G'\Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x)u^2\,dx\Big)\int_{\mathbb{R}^3} \chi_\varepsilon(x)u\varphi\,dx \\ &\quad - \int_{\mathbb{R}^3} \big(\frac{1}{|\cdot|} * u^2\big)u\varphi\,dx, \quad \forall \varphi \in H^1(\mathbb{R}^3), \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} &\langle D\Gamma_{\varepsilon,p}(u), \varphi\rangle \\ &= \int_{\mathbb{R}^3} (\nabla u \nabla \varphi + V(\varepsilon x)u\varphi)\,dx + G'\Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x)u^2\,dx\Big)\int_{\mathbb{R}^3} \chi_\varepsilon(x)u\varphi\,dx \\ &\quad - \int_{\mathbb{R}^3} \big(\frac{1}{|\cdot|} * |u|^p\big|u|^{p-2}u\varphi\,dx, \quad \forall \varphi \in H^1(\mathbb{R}^3). \end{aligned} \tag{1.11}$$

We also note that the critical points of Γ_ε and $\Gamma_{\varepsilon,p}$ are, respectively, solutions of

$$-\Delta u + V(\varepsilon x)u + G'\Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x)u^2\,dx\Big)\chi_\varepsilon(x)u = \big(\frac{1}{|\cdot|} * u^2\big)u, \tag{1.12}$$

$$-\Delta u + V(\varepsilon x)u + G'(\int_{\mathbb{R}^3} \chi_\varepsilon(x)u^2\,dx)\chi_\varepsilon(x)u = \big(\frac{1}{|\cdot|} * |u|^p\big)|u|^{p-2}u, \tag{1.13}$$

for all $u \in H^1(\mathbb{R}^3)$. If u is a critical point of Γ_ε and $\int_{\mathbb{R}^3} \chi_\varepsilon(x)u^2\,dx < \frac{1}{2}$, then u is a solution of (1.6).

Let $b \in C^\infty(\mathbb{R}^+, [0,1])$ such that $b(t) = 1$ if $t \leq 1$; $b(t) = 0$ if $t \geq 2$ and $0 \leq b(t) \leq 1$, $b'(t) \leq 0$. Let $0 < \lambda < 1$, $b_\lambda(t) = b(\lambda t), m_\lambda(t) = \int_0^t b_\lambda(\tau)\,d\tau$, $g_\lambda(t) = \frac{m_\lambda(t)}{t}$. We define

$$\psi(u) = \int_{\mathbb{R}^3} \big(\frac{1}{|\cdot|} * |u|^p\big)|u|^p\,dx$$

and

$$\begin{aligned}\Gamma^{(\lambda)}_{\varepsilon,p}(u) &= \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+V(\varepsilon x)u^2)\,dx+\frac{1}{2}G\Big(\int_{\mathbb{R}^3}\chi_\varepsilon(x)u^2\,dx\Big)\\ &\quad -\frac{1}{2p}g_\lambda(\psi^{1/2}(u))\psi(u).\end{aligned}\tag{1.14}$$

For each $\varphi\in H^1(\mathbb{R}^3)$, since $g_\lambda'(t)t+g_\lambda(t)=b_\lambda(t)$, we have

$$\begin{aligned}&\langle D\Gamma^{(\lambda)}_{\varepsilon,p}(u),\varphi\rangle\\ &=\int_{\mathbb{R}^3}\nabla u\nabla\varphi+V(\varepsilon x)u\varphi\,dx+G'\Big(\int_{\mathbb{R}^3}\chi_\varepsilon(x)u^2\,dx\Big)\int_{\mathbb{R}^3}\chi_\varepsilon(x)u\varphi\,dx\\ &\quad-\frac{1}{2}(b_\lambda(\psi^{1/2}(u))+g_\lambda(\psi^{1/2}(u)))\int_{\mathbb{R}^3}\big(\frac{1}{|\cdot|}*|u|^p\big)|u|^{p-2}u\varphi\,dx.\end{aligned}\tag{1.15}$$

Define $\|u\|^2=\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)\,dx$ for $u\in H^1(\mathbb{R}^3)$. By Hardy-Littlewood-Sobolev inequality we know that there exists $C_p>0$ such that $\psi^{1/2}(u)\le C_p\|u\|^p$ and C_p independent of u. It is easy to know that when $\|u\|\le(\frac{1}{C_p\lambda})^{\frac{1}{p}}$, we have $\Gamma_{\varepsilon,p}(u)=\Gamma^{(\lambda)}_{\varepsilon,p}(u)$ and $D\Gamma_{\varepsilon,p}(u)=D\Gamma^{(\lambda)}_{\varepsilon,p}(u)$.

This article is organized as follows. In Section 2, we prove $(PS)_c$ condition for $\Gamma_{\varepsilon,p}$ and give some uniform estimates (independent of p) on the critical points of $\Gamma_{\varepsilon,p}$. In Section 3, we prove the existence of sign-changing solutions for Γ_ε. Section 4 is devoted to the proof of Theorem 1.1.

For a Banach space E, we denote its dual space by E'. Throughout the paper, $c,c_0,c_1,\dots$ denote different constants and c_λ,C_λ denote constants depending on λ.

2. $(PS)_c$ condition for $\Gamma_{\varepsilon,p}$

In this section, we first collect elementary properties of the Choquard term, and then we prove that $\Gamma_{\varepsilon,p}$ satisfies the $(PS)_c$ condition.

Lemma 2.1 (Hardy-Littlewood-Sobolev inequality [19]). *Suppose $\alpha\in(0,N)$, and $s,r>1$ with $\frac{1}{s}+\frac{1}{r}+\frac{\alpha}{N}=2$. Let $g\in L^s(\mathbb{R}^N),h\in L^r(\mathbb{R}^N)$, there exists a sharp constant $C(s,\alpha,r,N)$, independently of g,h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{g(x)h(y)}{|x-y|^\alpha}\,dx\,dy\le C\|g\|_{L^r(\mathbb{R}^N)}\|h\|_{L^s(\mathbb{R}^N)}.$$

Lemma 2.2. *Assume $u_n\rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then*

$$\int_{\mathbb{R}^3}\big(\frac{1}{|\cdot|}*|u_n|^p\big)|u_n|^p\,dx-\int_{\mathbb{R}^3}\big(\frac{1}{|\cdot|}*|u_n-u|^p\big)|u_n-u|^p\,dx\to\int_{\mathbb{R}^3}\big(\frac{1}{|\cdot|}*|u|^p\big)|u|^p\,dx$$

and

$$\big(\frac{1}{|\cdot|}*|u_n|^p\big)|u_n|^{p-2}u_n-\big(\frac{1}{|\cdot|}*|u_n-u|^p\big)|u_n-u|^{p-2}(u_n-u)\to\big(\frac{1}{|\cdot|}*|u|^p\big)|u|^{p-2}u$$

in $(H^1(\mathbb{R}^3))'$.

The above lemma can be proved as in [1, Lemma 3.4].

Lemma 2.3. *It holds that for $t>0$ and $0<\lambda<1$:*

1. $g_\lambda(t)=1$, $g_\lambda'(t)=0$ *if* $0<t<1/\lambda$;
2. $-g_\lambda'(t)t\le g_\lambda(t)\le c_\lambda/t$, *where* $c_\lambda=\int_0^\infty b(\tau)\,d\tau/\lambda$;
3. $b_\lambda(t)t\le g_\lambda(t)t\le c_\lambda$.

The above lemma can be obtained by direct calculation.

Lemma 2.4. *Assume* $\|D\Gamma^{(\lambda)}_{\varepsilon,p}(u)\| \le 1, \Gamma^{(\lambda)}_{\varepsilon,p}(u) \le L$, *then there exists* $\lambda_{L,p} > 0$ *such that if* $0 < \lambda < \lambda_{L,p}$, *then* $D\Gamma^{(\lambda)}_{\varepsilon,p}(u) = D\Gamma_{\varepsilon,p}(u), \Gamma^{(\lambda)}_{\varepsilon,p}(u) = \Gamma_{\varepsilon,p}(u)$.

Proof. By (1.14),(1.15) and Lemma 2.3, we have

$$
\begin{aligned}
L + \|u\| &\ge \Gamma^{(\lambda)}_{\varepsilon,p}(u) - \frac{1}{p}\langle D\Gamma^{(\lambda)}_{\varepsilon,p}(u), u\rangle \\
&= (\frac{1}{2} - \frac{1}{p}) \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2)\, dx + \frac{1}{2} G\Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x) u^2\, dx\Big) \\
&\quad - \frac{1}{p} G'\Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x) u^2\, dx\Big) \int_{\mathbb{R}^3} \chi_\varepsilon(x) u^2\, dx \\
&\quad + \frac{1}{2p} b_\lambda(\psi^{1/2}(u)) \int_{\mathbb{R}^3} (\frac{1}{|\cdot|} * |u|^p)|u|^p\, dx \\
&\ge c_p \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2)\, dx + c_p G\Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x) u^2\, dx\Big) - c\,.
\end{aligned} \tag{2.1}
$$

As a result, there exists a constant $c_{L,p}$ such that $\psi^{1/2}(u) \le c\|u\|^p \le c_{L,p}$ and $c_{L,p} \to \infty$ as $p \to 2$. Choose $\lambda_{L,p} \le \frac{1}{2c_{L,p}}$, by Lemma 2.3, we have $D\Gamma^{\lambda}_{\varepsilon,p}(u) = D\Gamma_{\varepsilon,p}(u), \Gamma^{\lambda}_{\varepsilon,p}(u) = \Gamma_{\varepsilon,p}(u)$. ◻

Lemma 2.5. *For each* $L > 0$, *there exists* $\varepsilon_L > 0$ *such that, for* $0 < \varepsilon < \varepsilon_L$, *if* $c < L$, *the following statements hold*

(1) *The* $\Gamma_{\varepsilon,p}$ *satisfies the* $(PS)_c$ *condition.*
(2) *Let* $p_n \subset (2, 2^*)$ *be a sequence such that* $p_n \to 2$ *as* $n \to \infty$. *For* $\{u_n\} \subset H^1(\mathbb{R}^3)$ *such that*

$$\Gamma_{\varepsilon,p_n}(u_n) \to c, D\Gamma_{\varepsilon,p_n}(u_n) = o_n(1),$$

there exists a critical point $u \in H^1(\mathbb{R}^3)$ *of* Γ_ε, *such that* $u_n \to u \in H^1(\mathbb{R}^3)$ *up to a subsequence.*

Proof. (1) The proof of this part is similar to that in [15].
(2) By (1.9) and (1.11), we have

$$
\begin{aligned}
L &\ge \Gamma_{\varepsilon,p_n}(u_n) - \frac{1}{2p_n}\langle D\Gamma_{\varepsilon,p_n}(u_n), u_n\rangle \\
&= (\frac{1}{2} - \frac{1}{2p_n}) \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(\varepsilon x)u_n^2)\, dx + \frac{1}{2} G\Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x) u_n^2\, dx\Big) \\
&\quad - \frac{1}{2p_n} G'\Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x) u_n^2\, dx\Big) \int_{\mathbb{R}^3} \chi_\varepsilon(x) u_n^2\, dx \\
&\ge c \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(\varepsilon x)u_n^2)\, dx + c G\Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x) u_n^2\, dx\Big) - c\,.
\end{aligned} \tag{2.2}
$$

By (2.2) we know that there exists $\hat\eta_L > 0$ independent of ε, p such that $\|u_n\| \le \hat\eta_L$ and $G(\int_{\mathbb{R}^3} \chi_\varepsilon(x) u_n^2\, dx) \le \hat\eta_L$. Up to a subsequence, we assume that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$,

$$\zeta_n := G'\Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x) u_n^2\, dx\Big) \to \zeta.$$

It is easy to show that u is a solution of the equation

$$- \triangle u + V(\varepsilon x)u + \zeta\chi_\varepsilon(x)u = (\frac{1}{|\cdot|} * u^2)u. \tag{2.3}$$

By Lemma 2.2, for any $v \in H^1(\mathbb{R}^3)$,

$$\begin{aligned} o(\|v\|) &= \langle D\Gamma_{\varepsilon,p_n}(u_n), v\rangle \\ &= \int_{\mathbb{R}^3} (\nabla(u_n - u)\nabla v + V(\varepsilon x)(u_n - u)v)\,dx \\ &\quad + \zeta \int_{\mathbb{R}^3} \chi_\varepsilon(x)(u_n - u)v\,dx + (\zeta_n - \zeta)\int_{\mathbb{R}^3} \chi_\varepsilon(x)u_n v\,dx \\ &\quad - \int_{\mathbb{R}^3} (\frac{1}{|\cdot|} * |u_n - u|^{p_n})|u_n - u|^{p_n-2}(u_n - u)v\,dx \\ &\quad - \int_{\mathbb{R}^3} \Big((\frac{1}{|\cdot|} * |u|^{p_n})|u|^{p_n-2}u - (\frac{1}{|\cdot|} * u^2)u\Big)v\,dx + o(\|v\|), \end{aligned} \tag{2.4}$$

as $n \to \infty$. Since $\|u_n\| \le \hat\eta_L$, we have $\|u\| \le \hat\eta_L$. Hence, there exists a constant such that

$$\frac{1}{|\cdot|} * |u|^{p_n} \le c, \frac{1}{|\cdot|} * u^2 \le c,$$
$$\big|(\frac{1}{|\cdot|} * |u|^{p_n})|u|^{p_n-2}u - (\frac{1}{|\cdot|} * u^2)u\big|^2 \le c(|u|^{p_n-1} + u)^2 \le c(|u|^{2(p_0-1)} + u^2).$$

Using the dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^3} \big((\frac{1}{|\cdot|} * |u|^{p_n})|u|^{p_n-2}u - (\frac{1}{|\cdot|} * u^2)u\big)v\,dx = o(\|v\|).$$

Choose $R_0 > 0$ such that $\mathcal{M} \subset B(0, R_0)$. Let ϕ_ε be a C^∞ function such that $\phi_\varepsilon(x) = 0$ for $|x| \le \varepsilon^{-1}(R_0+1)+1$; $\phi_\varepsilon(x) = 1$ for $|x| \ge \varepsilon^{-1}(R_0+1)+2$; $0 \le \phi_\varepsilon \le 1$ and $|\nabla\phi_\varepsilon| \le 4$. Take $v = \phi_\varepsilon^2(u_n - u)$ in (2.4), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3} (|\nabla(\phi_\varepsilon(u_n - u))|^2 + V(\varepsilon x)\phi_\varepsilon^2(u_n - u)^2)\,dx - \int_{\mathbb{R}^3} (u_n - u)^2|\nabla\phi_\varepsilon|^2\,dx \\ &+ \zeta\int_{\mathbb{R}^3} \chi_\varepsilon(x)\phi_\varepsilon^2(u_n - u)^2\,dx + (\zeta_n - \zeta)\int_{\mathbb{R}^3} \chi_\varepsilon(x)\phi_\varepsilon^2 u_n(u_n - u)\,dx \\ &- \int_{\mathbb{R}^3} \big(\frac{1}{|\cdot|} * |u_n - u|^{p_n}\big)|u_n - u|^{p_n-2}\phi_\varepsilon^2(u_n - u)^2\,dx = o(1), \quad n \to \infty. \end{aligned} \tag{2.5}$$

Since $\zeta_n \to \zeta, n \to \infty$ and $|\nabla\phi_\varepsilon|^2$ has a compact support, by $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} (u_n - u)^2|\nabla\phi_\varepsilon|^2\,dx = o(1), \quad (\zeta_n - \zeta)\int_{\mathbb{R}^3} \chi_\varepsilon(x)\phi_\varepsilon^2 u_n(u_n - u)\,dx = o(1),$$

as $n \to \infty$. Then by (A1), $\|u\| \le \hat{\eta}_L$ and (2.5), we obtain

$$
\begin{aligned}
&\min\{1,a\}\|\phi_\varepsilon(u_n-u)\|^2\\
&\le \int_{\mathbb{R}^3} (\frac{1}{|\cdot|} * |u_n-u|^{p_n})|u_n-u|^{p_n-2}\phi_\varepsilon^2(u_n-u)^2\,dx + o(1)\\
&\le \int_{\mathbb{R}^3}\int_{|y|\le \varepsilon^{-1}R_0+1} \frac{|u_n(y)-u(y)|^{p_n}|u_n(x)-u(x)|^{p_n}\phi_\varepsilon^2(x)}{|x-y|}\,dx\,dy\\
&\quad + \int_{\mathbb{R}^3}\int_{|y|\ge \varepsilon^{-1}R_0+1} \frac{|u_n(y)-u(y)|^{p_n}|u_n(x)-u(x)|^{p_n}\phi_\varepsilon^2(x)}{|x-y|}\,dx\,dy + o(1)\\
&\le c\|u_n-u\|_{L^{\frac{6p_n}{5}}(B_{\varepsilon^{-1}R_0+1}(0))}\|u_n-u\|^{p_n-2}_{L^{\frac{6p_n}{5}}(\mathbb{R}^3)}\|\phi_\varepsilon(u_n-u)\|^2\\
&\quad + c\|\phi_\varepsilon(u_n-u)\|^2\Big(\int_{|x|\ge \varepsilon^{-1}R_0+1}|u_n|^{\frac{6p_n}{5}}\,dx\\
&\quad + \int_{|x|\ge \varepsilon^{-1}R_0+1}|u|^{\frac{6p_n}{5}}\,dx\Big)^{\frac{5(2p_n-2)}{6p_n}} + o(1)\\
&\le o(1)\|\phi_\varepsilon(u_n-u)\|^2 + c_1\|\phi_\varepsilon(u_n-u)\|^2\Big(\int_{|x|\ge \varepsilon^{-1}R_0+1}|u_n|^{\frac{6p_n}{5}}\,dx\\
&\quad + \int_{|x|\ge \varepsilon^{-1}R_0+1}|u|^{\frac{6p_n}{5}}\,dx\Big)^{\frac{5(2p_n-2)}{6p_n}} + o(1), \quad n\to\infty.
\end{aligned} \tag{2.6}
$$

Since $(\mathcal{M}_\varepsilon)^1 \subset B(0,\varepsilon^{-1}R_0+1)$ and $G(\int_{\mathbb{R}^3}\chi_\varepsilon(x)u_n^2\,dx) \le \hat{\eta}_L$, we have

$$
\int_{|x|\ge \varepsilon^{-1}R_0+1} u_n^2\,dx \le c(\frac{1}{2}+\hat{\eta}_L)\varepsilon^6. \tag{2.7}
$$

By Fatou's Lemma, we have

$$
\int_{|x|\ge \varepsilon^{-1}R_0+1} u^2\,dx \le c(\frac{1}{2}+\hat{\eta}_L)\varepsilon^6. \tag{2.8}
$$

Using the interpolation inequality, for $2<\frac{6p_n}{5}<q<2^*$, we have

$$
\|u_n\|_{L^{\frac{6p_n}{5}}(\mathbb{R}^3)} \le \|u_n\|^{t_n}_{L^2(\mathbb{R}^3)}\|u_n\|^{1-t_n}_{L^q(\mathbb{R}^3)} \le c\|u_n\|^{t_n}_{L^2(\mathbb{R}^3)}\|u_n\|^{1-t_n}, \tag{2.9}
$$

where $\frac{5}{6p_n}=\frac{t_n}{2}+\frac{(1-t_n)}{q}, 0<t_0<t_n<1$. Combining with $\|u_n\|\le\hat{\eta}_L$, (2.7) and (2.8) we obtain

$$
\int_{|x|\ge \varepsilon^{-1}R_0+1}|u_n|^{\frac{6p_n}{5}}\,dx \le C_L\varepsilon^{\frac{18p_nt_0}{5}}, \quad \int_{|x|\ge \varepsilon^{-1}R_0+1}|u|^{\frac{6p_n}{5}}\,dx \le C_L\varepsilon^{\frac{18p_nt_0}{5}}, \tag{2.10}
$$

where C_L is independent of ε. Choose $\varepsilon_L>0$ such that $c_1\cdot(C_L^{\frac{5(2p_n-2)}{6p_n}}\varepsilon_k^{3(2p_n-2)t_0}) \le \min\{1,a\}/2$. Then by (2.6), for $0<\varepsilon<\varepsilon_L$, we have

$$
\lim_{n\to\infty}\|\phi_\varepsilon(u_n-u)\| = 0. \tag{2.11}
$$

Set $v=(1-\phi_\varepsilon)^2(u_n-u)$ in (2.4), it is easy to obtain

$$
\lim_{n\to\infty}\|(1-\phi_\varepsilon)(u_n-u)\| = 0. \tag{2.12}
$$

The result of the lemma follows from (2.11) and (2.12). $\square$

3. Existence of sign-changing critical points for Γ_ε

To obtain multiple sign-changing critical points of $\Gamma^\lambda_{\varepsilon,p}$, we introduce the abstract critical point theorem [22, Theorem 2.5], see also [6, Theorem 3.2].

Let X be a Hilbert space, f be an even C^2-functional on X. Let P, Q be open convex sets of X, $Q=-P$. Set

$$W = P\cup Q, \quad \Sigma = \partial P\cap \partial Q.$$

For a critical point $x\in X$ of f, the augmented Morse index

$$m^*(x) = \max\{\dim X_0 : X_0\subset X \text{is a subspace such that } D^2 f(x)(h,h)\le 0, \forall h\in X\}.$$

Assume

- (A3) there exists $L>0$ such that f satisfies the $(PS)_c$ condition, for $c<L$;
- (A4) $c^* = \inf_{x\in\Sigma} f(x) > 0$;
- (A5) For every critical point x of f, $D^2 f$ is a Fredholm operator.

Also assume there exists an odd continuous map $A: X\to X$ satisfying

- (A6) given c_0, $b_0>0$, there exists $b=b(c_0,b_0)>0$ such that if $\|Df(x)\|\ge b_0$, $|f(x)|\le c_0$, then
$$\langle Df(x),\, x-Ax\rangle \ge b\|x-Ax\| > 0\,;$$
- (A7) $A(\partial P_j)\subset P_j$, $A(\partial Q_j)\subset Q_j$, $j=1,\dots,k$.

We define

$$\begin{aligned}
&\Gamma_j = \{E|\, E\subset X,\, E \text{ compact},\, -E=E,\, \gamma(E\cap \eta^{-1}(\Sigma))\ge j \text{ for } \eta\in\Lambda\},\\
&\Lambda = \big\{\eta : \eta\in C(X,X) : \eta \text{ is odd},\, \eta(P)\subset P,\, \eta(Q)\subset Q,\, j=1,\dots,k,\\
&\qquad \eta(x) = x \text{ if } f(x)<0\big\}
\end{aligned}$$

where γ is the genus of symmetric sets,

$$\gamma(E) = \inf\big\{n : \text{there exists an odd map } \eta : E\to \mathbb{R}^n\backslash\{0\}\big\}\,.$$

Assume that

- (A8) Γ_j is nonempty.

Define

$$\begin{gathered}
c_j = \inf_{E\in\Gamma_j}\sup_{x\in E\backslash W} f(x),\ j=1,2,\dots\,,\\
K_c = \{x : Df(x)=0,\, f(x)=c\},\, K_c^* = K_c\setminus W\,.
\end{gathered}$$

Theorem 3.1. *Assume* (A3)–(A8) *hold. If $c_j<L, j=1,\dots,k$, then*

1. *$c_j\ge c^*$, $K^*_{c_j}\ne\emptyset$;*
2. *There exists $x\in K_{c_j}\backslash W$ with $m^*(x)\ge j$.*

For $u\in H^1(\mathbb{R}^3)$, we define $v=Au$ by the unique solution to

$$\begin{aligned}
&-\Delta v + V(\varepsilon x)v + G'\Big(\int_{\mathbb{R}^3}\chi_\varepsilon(x)u^2\,dx\Big)\chi_\varepsilon(x)v\\
&= \frac12 (b_\lambda(\psi^{1/2}(u)) + g_\lambda(\psi^{1/2}(u)))(\frac{1}{|\cdot|} * |u|^p)|u|^{p-2}u\,.
\end{aligned}\tag{3.1}$$

Note that A is odd, well defined, and continuous on $H^1(\mathbb{R}^3)$; see [15, Lemma 3.1].

Lemma 3.2. *Let $u\in H^1(\mathbb{R}^3)$. If $v=Au$, then*

(1) $\langle D\Gamma_{\varepsilon,p}^{(\lambda)}(u), u-v\rangle \geq c\|u-v\|^2$;
(2) $\|D\Gamma_{\varepsilon,p}^{(\lambda)}(u)\| \leq c(1+|\Gamma_{\varepsilon,p}^{(\lambda)}(u)| + \|u-v\|)^\gamma \|u-v\|$.

This lemma can be proved as in [15, Lemma 3.2]. Using Lemma 3.2, it is easy to prove assumption (A6).

For $\delta > 0$, let

$$P := \{u \in H^1(\mathbb{R}^3) : \|u^+\|_{L^{6p/5}(\mathbb{R}^3)} < \delta\},$$
$$Q := \{u \in H^1(\mathbb{R}^3) : \|u^-\|_{L^{6p/5}(\mathbb{R}^3)} < \delta\}.$$

Lemma 3.3. *For $0 < \lambda < 1$, there exists $\delta_\lambda > 0$ such that for $0 < \delta < \delta_\lambda$,*

$$A(\partial P) \subset P, \quad A(\partial Q) \subset Q, \quad \delta_\lambda \to 0 \quad \text{as } \lambda \to 0.$$

Moreover, there exists $\delta_0 \in (0, \delta_\lambda)$ and $c^* = c^*(\delta_0) > 0$ such that

$$\Gamma_{\varepsilon,p}^{(\lambda)}(u) \geq c^*, \quad \text{for } u \in \partial P \cap \partial Q.$$

The proof of the above lemma is similar to that of [15, Lemmas 3.4 and 3.5].

Let

$$J_0(u) = \frac{1}{2}\int_{B(0,1)} (|\nabla u|^2 + bu^2)\,dx - c\Big(\int_{B(0,1)} u^2\,dx\Big)^2,$$

where c independent of p, λ. Let $\{e_n\} \subset H_0^1(B(0,1))$ be an orthogonal basis and $H_n := \operatorname{span}\{e_1, \dots, e_n\}$. Then there exists an increasing sequence $\{R_n\}$ such that

$$J_0(u) < 0, \quad \forall u \in H_n,\ \|u\| \geq R_n.$$

Choose an appropriate ε such that

$$B(0,1) \subset \mathcal{M}_\varepsilon. \tag{3.2}$$

Define $\varphi_n \in C(B_n, H_0^1(B(0,1)))$ as

$$\varphi_n(t) = R_n \sum_{i=1}^n t_i e_i, \quad t = (t_1, \dots, t_n) \in B_n.$$

Let

$$\begin{gathered}\Gamma_j = \{E \subset H^1(\mathbb{R}^3) : E \text{ is compact}, -E = E,\ \gamma(E \cap \eta^{-1}(\Sigma)) \geq j \text{ for } \eta \in \Lambda\},\\ \Lambda = \big\{\eta \in C(H^1(\mathbb{R}^3) : H^1(\mathbb{R}^3)),\ \eta \text{ is odd},\ \eta(P) \subset P,\ \eta(Q) \subset Q,\\ \eta(u) = u \text{ if } \Gamma_{\varepsilon,p}^{(\lambda)}(u) < 0\big\}.\end{gathered}$$

Lemma 3.4. *There exists $\tilde\lambda_k > 0$, such that, for $0 < \lambda < \tilde\lambda_k$ and sufficiently small ε, $E_j = \varphi_{j+1}(B_{j+1}) \subset \Gamma_j$, $j = 1, \dots, k$.*

Proof. For $x \in \mathcal{M}_\varepsilon$, we have $\chi_\varepsilon(x) = 0$ and $V(\varepsilon x) \leq b$. Then for $u \in E_j$, we have $G(\int_{\mathbb{R}^3} \chi_\varepsilon u^2\,dx) = 0$. For $u \in E_j$, choose $\tilde\lambda_k = c(R_k)$ such that, for $0 < \lambda < \tilde\lambda_k$, $g_\lambda(\psi^{1/2}(u)) = 1$. Then we have

$$\begin{aligned}\Gamma_{\varepsilon,p}^{(\lambda)}(u) &= \frac{1}{2}\int_{B(0,1)} (|\nabla u|^2 + V(\varepsilon x)u^2)\,dx - \frac{1}{2p}\psi(u)\\ &\leq \frac{1}{2}\int_{B(0,1)} (|\nabla u|^2 + bu^2)\,dx - \frac{1}{4}\Big(\int_{B(0,1)} |u|^p\,dx\Big)^2\\ &\leq \frac{1}{2}\int_{B(0,1)} (|\nabla u|^2 + bu^2)\,dx - c\Big(\int_{B(0,1)} |u|^2\,dx\Big)^2\end{aligned}$$

$$\le J_0(u)\,.$$

As for [21, Lemma 5.6], we can complete the proof. □

Lemma 3.5. *For every critical point u of $\Gamma_{\varepsilon,p}$, $D^2\Gamma_{\varepsilon,p}(u)$ is a Fredholm operator.*

Proof. Note that every critical point of $\Gamma_{\varepsilon,p}$ is a weak solution of

$$\begin{gathered}-\Delta u+V(\varepsilon x)u+G'\Big(\int_{\mathbb{R}^3}\chi_\varepsilon(x)u^2\,dx\Big)\chi_\varepsilon(x)u=(\frac{1}{|\cdot|}*|u|^p)|u|^{p-2}u,\\ u\in H^1(\mathbb{R}^3).\end{gathered}\tag{3.3}$$

Assume u solves (3.3), by the sub-solution estimates, we have $u\in L^\infty(\mathbb{R}^3)$ and $u(x)\to 0$ as $x\to\infty$. For $\psi,\varphi\in H^1(\mathbb{R}^3)$, we have

$$\begin{aligned}
&\langle D^2\Gamma_{\varepsilon,p}(u)\psi,\varphi\rangle\\
&=\int_{\mathbb{R}^3}(\nabla\psi\nabla\varphi+V(\varepsilon x)\psi\varphi)\,dx+G'\Big(\int_{\mathbb{R}^3}\chi_\varphi(x)u^2\,dx\Big)\int_{\mathbb{R}^3}\chi_\varepsilon(x)\psi\varphi\,dx\\
&\quad+G''\Big(\int_{\mathbb{R}^3}\chi_\varepsilon(x)u^2\,dx\Big)\int_{\mathbb{R}^3}\chi_\varepsilon(x)u\psi\,dx\int_{\mathbb{R}^3}\chi_\varepsilon(x)u\varphi\,dx\\
&\quad+(p-1)\int_{\mathbb{R}^3}(\frac{1}{|\cdot|}*|u|^p)|u|^{p-2}\psi\varphi\,dx+p\int_{\mathbb{R}^3}(\frac{1}{|\cdot|}*(|u|^{p-2}u\psi))|u|^{p-2}u\varphi\,dx
\end{aligned}$$

Note $-\Delta+V(\varepsilon x)+G'(\int_{\mathbb{R}^3}\chi_\varepsilon(x)u^2\,dx)\chi_\varepsilon(x):H^1(\mathbb{R}^3)\to H^{-1}(\mathbb{R}^3)$ is the Fredholm operator. On the other hand, the linear operator $Q:H^1(\mathbb{R}^3)\to H^{-1}(\mathbb{R}^3)$ defined by

$$\begin{aligned}
\langle Q\psi,\varphi\rangle&=G''\Big(\int_{\mathbb{R}^3}\chi_\varepsilon(x)u^2\,dx\Big)\int_{\mathbb{R}^3}\chi_\varepsilon(x)u\psi\,dx\int_{\mathbb{R}^3}\chi_\varepsilon(x)u\varphi\,dx\\
&\quad+(p-1)\int_{\mathbb{R}^3}(\frac{1}{|\cdot|}*|u|^p)|u|^{p-2}\psi\varphi\,dx\\
&\quad+p\int_{\mathbb{R}^3}\big(\frac{1}{|\cdot|}*(|u|^{p-2}u\psi)\big)|u|^{p-2}u\varphi\,dx
\end{aligned}\tag{3.4}$$

is compact. Hence $D^2\Gamma_{\varepsilon,p}(u)$ is the Fredholm operator. □

We define

$$c_j(\varepsilon,p,\lambda)=\inf_{E\in\Gamma_j}\sup_{u\in E\setminus W}\Gamma^{(\lambda)}_{\varepsilon,p}(u),\quad \tilde c_j=\sup_{u\in E_{j+1}}J_0(u),\ j=1,\dots,k.$$

Theorem 3.6. *Let $L>0$, $0<\lambda<\lambda^{(k)}_{L,p}$, $0<\varepsilon<\varepsilon_L$, k be such that $\tilde c_k<L$. $\Gamma_{\varepsilon,p}$ has at least k pairs of sign-changing critical points $\{\pm u_{j,\varepsilon,p},\,1\le j\le k\}$ such that*

$$\Gamma_{\varepsilon,p}(u_{j,\varepsilon,p})=c_j(\varepsilon,p)\le\tilde c_k,\quad 1\le j\le k.$$

Moreover, there exists $u_{j,\varepsilon,p}\in K^*_{c_j(\varepsilon,p)}$ such that $m^*(u_{j,\varepsilon,p})\ge j$.

Proof. By the definition of $c_j(\varepsilon,\lambda),\tilde c_j$ and Lemma 3.4, for $0<\lambda<\tilde\lambda_k$, we obtain

$$c_1(\varepsilon,p,\lambda)\le\dots\le c_k(\varepsilon,p,\lambda)\le\tilde c_k,\ j=1,\dots,k.$$

By Lemma 2.4, there exists $\lambda_{L,p}>0$ such that if $0<\lambda<\lambda^{(k)}_{L,p}:=\min\{\lambda_{L,p},\tilde\lambda_k\}$, $D\Gamma^{(\lambda)}_{\varepsilon,p}(u)=o(1)$ and $\Gamma^{(\lambda)}_{\varepsilon,p}(u)\le L$, then we have $D\Gamma^{(\lambda)}_{\varepsilon,p}(u)=D\Gamma_{\varepsilon,p}(u),\Gamma^{(\lambda)}_{\varepsilon,p}(u)=\Gamma_{\varepsilon,p}(u)$. By Lemma 2.5 and Lemma 3.5, for $p\in(2,p_0)$, $0<\lambda<\lambda^{(k)}_{L,p}$, $\Gamma^{(\lambda)}_{\varepsilon,p}$ satisfies $(PS)_c$ condition with $c<L$ and $D^2\Gamma^{(\lambda)}_{\varepsilon,p}(u)$ is the Fredholm operator for some u

such that $D\Gamma^{(\lambda)}_{\varepsilon,p}(u)=0, \Gamma^{(\lambda)}_{\varepsilon,p}(u)<L$. Hence, we have verified that $\Gamma^{(\lambda)}_{\varepsilon,p}$ satisfies all assumptions of Theorem 3.1. By Theorem 3.1, for $0<\varepsilon<\varepsilon_L$, we obtain $\Gamma^{(\lambda)}_{\varepsilon,p}, 0<\lambda<\lambda^{(k)}_{L,p}$ has at least k pairs of sign-changing critical points

$$\{\pm u^{(\lambda)}_{j,\varepsilon,p} | 1\le j\le k\}$$

such that

$$\Gamma^{(\lambda)}_{\varepsilon,p}(u^{(\lambda)}_{j,\varepsilon,p})=c_j(\varepsilon,p,\lambda)\le \tilde{c}_k, 1\le j\le k.$$

Moreover, there exists $x\in K_{c_j(\varepsilon,p,\lambda)}\backslash W$ with $m^*(u^{(\lambda)}_{j,\varepsilon,p})\ge j$, then we also have $\Gamma^{\lambda}_{\varepsilon,p}(u^{(\lambda)}_{j,\varepsilon,p})=\Gamma_{\varepsilon,p}(u^{(\lambda)}_{j,\varepsilon,p}), D\Gamma^{\lambda}_{\varepsilon,p}(u^{(\lambda)}_{j,\varepsilon,p})=D\Gamma_{\varepsilon,p}(u^{(\lambda)}_{j,\varepsilon,p})=0, j=1,\dots,k$. In this case, we can write $u^{(\lambda)}_{j,\varepsilon,p}$ as $u_{j,\varepsilon,p}$ and the theorem is proved. □

Assume $\Gamma_{\varepsilon,p_n}(u_n)\le L, D\Gamma_{\varepsilon,p_n}(u_n)=0$ and $p_n\to 2$ as $n\to\infty$. By Lemma 2.5 (2), there exists a critical point $u\in H^1(\mathbb{R}^3)$ of Γ_ε, such that $u_n\to u\in H^1(\mathbb{R}^3)$ up to subsequence.

Lemma 3.7. *Assume u_n is sign-changing critical points of Γ_{ε,p_n} and $u_n\to u$ in $H^1(\mathbb{R}^3)$, then u is a sign-changing critical point of Γ_ε.*

Proof. Since $\langle D\Gamma_{\varepsilon,p_n}(u_n),u_n\rangle=0$, we have

$$\begin{aligned} \|u_n\|^2_{H^1(\mathbb{R}^3)} &= \int_{\mathbb{R}^3}(|\nabla u_n|^2+V(\varepsilon x)u_n^2)\,dx \\ &\le \int_{\mathbb{R}^3}(\frac{1}{|\cdot|}*u_n^{p_n})u_n^{p_n}\,dx \\ &\le c\|u_n\|^{2p_n}_{L^{\frac{12}{5}}(\mathbb{R}^3)}\le c\|u_n\|^{2p_n}_{H^1(\mathbb{R}^3)}. \end{aligned} \tag{3.5}$$

So there exists $m>0$ such that $\|u_n\|_{H^1(\mathbb{R}^3)}>m$ and $0\ne\|u\|_{H^1(\mathbb{R}^3)}\ge m$. Without loss of generality, we assume that

$$u^+=0 \quad\text{and}\quad u^-\ne 0. \tag{3.6}$$

We define the normalized part as

$$v_n=\frac{u_n^+}{\|u_n^+\|}\,. \tag{3.7}$$

Then, up to a subsequence, there exists $v\in H^1(\mathbb{R}^3)$ such that $v_n\rightharpoonup v$ weakly in $H^1(\mathbb{R}^3)$. Since

$$\begin{aligned} &\int_{\mathbb{R}^3}(\frac{1}{|\cdot|}*|u_n|^{p_n})|v_n|^{p_n}\,dx \\ &\ge \int_{\mathbb{R}^3}(\frac{1}{|\cdot|}*|u_n^+|^{p_n})|v_n|^{p_n}\,dx \\ &\ge \int_{\mathbb{R}^3}(|\nabla u_n^+|^2+V(\varepsilon x)(u_n^+)^2)\,dx\|u_n^+\|^{-p_n} \\ &= \|u_n^+\|^{2-p_n}\ge\|u_n\|^{2-p_n}\to 1 \quad\text{as } p_n\to 2^+, \end{aligned} \tag{3.8}$$

we have

$$\int_{\mathbb{R}^3}(\frac{1}{|\cdot|}*|u|^2)|v|^2\,dx=\lim_{p_n\to 2}\int_{\mathbb{R}^3}(\frac{1}{|\cdot|}*|u_n|^{p_n})|v_n|^{p_n}\,dx\ge 1. \tag{3.9}$$

Then, by Hardy-Littlewood Sobolev inequality

$$\int_{\mathbb{R}^3}(\frac{1}{|\cdot|}*|u|^2)|v|^2\,dx \le c\|u\|^2_{L^{\frac{12}{5}}(\mathbb{R}^3)}\|v\|^2_{L^{\frac{12}{5}}(\mathbb{R}^3)}, \tag{3.10}$$

we have $\|v\|_{L^{\frac{12}{5}}(\mathbb{R}^3)} > 0$. Therefore $v \neq 0$ and

$$S = \{x \in \mathbb{R}^3 : v(x) > 0\} \neq \emptyset.$$

Since u_n satisfies $D\Gamma_{\varepsilon,p_n}(u_n) = 0$, in the weak sense, we have

$$\begin{aligned} &-\Delta u_n + V(\varepsilon x)u_n + G'\Big(\int_{\mathbb{R}^3}\chi_\varepsilon(x)u_n^2\,dx\Big)\chi_\varepsilon(x)u_n \\ &= (\frac{1}{|\cdot|}*|u_n|^{p_n})|u_n|^{p_n-2}u_n. \end{aligned} \tag{3.11}$$

By elliptic regularity theory, we have $u_n \in C^2_{loc}(\mathbb{R}^3)$. Therefore, let $x_0 \in S$, we have $u_n(x_0) > 0$ for n large enough and $u(x_0) = \lim_{n\to\infty} u_n(x_0) \ge 0$.

On the other hand, since $D\Gamma_\varepsilon(u) = 0$, we have

$$-\Delta u + V(\varepsilon x)u + G'\Big(\int_{\mathbb{R}^3}\chi_\varepsilon(x)u^2\,dx\Big)\chi_\varepsilon(x)u = (\frac{1}{|\cdot|}*|u|^2)u, \text{ in } \mathbb{R}^3. \tag{3.12}$$

Note that assumption (3.6) implies $u \le 0$ in $\mathbb{R}^3$. Hence, by the classical regularity argument and the strong maximum principle on (3.12), we have $u < 0$ or $u \equiv 0$ in $\mathbb{R}^3$. Since $u \neq 0$, we obtain $u < 0$ in $\mathbb{R}^3$. This leads to $u(x_0) < 0$, which contradicts $u(x_0) \ge 0$. Thus, the lemma is proved. □

4. Proof of Theorem 1.1

Assume $u_{n,p} \to u_n$ in $H^1(\mathbb{R}^3)$ as $p \to 2$. Let $\varepsilon_n \to 0$, assume $D\Gamma_{\varepsilon_n,p}(u_{n,p}) = D\Gamma_{\varepsilon_n}(u_n) = 0, \Gamma_{\varepsilon_n,p}(u_{n,p}), \Gamma_{\varepsilon_n}(u_n) \le L$. The following two lemmas can be proved in a similar way as in [15].

Lemma 4.1. *Up to a subsequence, there exists an integer $m > 0$, $y_{n,i} \subset (\mathcal{M}_\varepsilon)^1$, $y_i \in \overline{\mathcal{M}}$, $U_i \in H^1(\mathbb{R}^3)\backslash\{0\}$, $i = 1,\dots,m$ such that*

(1) $\lim_{n\to\infty}|y_{n,i} - y_{n,l}| \to \infty$ *if* $i \neq l$, $\lim_{n\to\infty}|y_{n,i}| \to \infty$.
(2) $y_i = \lim_{n\to\infty}\varepsilon_n y_{n,i} \in \overline{\mathcal{M}}$.
(3) *For* $1 \le i \le m$, $\lim_{n\to\infty}\operatorname{dist}(y_{n,i}, \partial\mathcal{M}_{\varepsilon_n}) \to \infty$, U_i *is the weak limit of* $u_n(\cdot + y_{n,i})$ *in* $H^1(\mathbb{R}^3)$ *and satisfies*

$$-\Delta U + V(y_i)U = (\frac{1}{|\cdot|}*U^2)U,\ U \in H^1(\mathbb{R}^3). \tag{4.1}$$

(4) $\lim_{n\to\infty}\|u_n - \sum_{i=1}^m U_i(\cdot - y_{n,i})\|_{L^s(\mathbb{R}^3)} = 0$ *for* $2 < s < 2^*$.

Assume that the sequence $\{u_n\}$ satisfies the condition of Lemma 4.1, and define

$$\Omega_R^{(n)} = \mathbb{R}^3\backslash\{\cup_{i=1}^m B(y_{n,k}, R)\}.$$

Lemma 4.2. *There exist c, μ independent of n, p, such that*

$$\int_{\Omega_R^{(n)}}(|\nabla u_n|^2 + u_n^2)\,dx \le ce^{-\mu R}, \quad \zeta_n\int_{\Omega_R^{(n)}}\chi_\varepsilon(x)u_n^2\,dx \le ce^{-\mu R},$$

where $\zeta_n := G'\big(\int_{\mathbb{R}^3}\chi_\varepsilon(x)u_n^2\,dx\big)$. Moreover,

$$|u_n(x)| \le ce^{-\mu R} \quad \textit{for } x \in \Omega_R^{(n)}.$$

Corollary 4.3. *There exist c, μ, p_1, independent of n, such that, for $2 < p < p_1$, we have*

$$|u_{n,p}(x)| \le c \sum_{i=1}^{m} e^{-\mu|x-y_{n,i}|} \quad \text{for } x \in \mathbb{R}^3.$$

Proof. The Proof of Lemma 4.2 can be done by the same method as [15, Lemma 4.3]. We only need to prove that there exists $p_1 > 2$ such that, for $2 < p < p_1$,

$$(\frac{1}{|\cdot|} * |u_{n,p}|^p) u_{n,p}^{p-2} \le \frac{a}{2} \quad \text{for } x \in \Omega_R^{(n)} .$$

By Moser's iteration, there exists a $c > 0$ such that $\|u_{n,p}\|_{L^\infty(\mathbb{R}^3)} \le c$ for $2 < p < p_0$. By Lemma 4.2 and $u_{n,p} \to u_n$ in $H^1(\mathbb{R}^3)$ as $p \to 2$, for $x \in \Omega_R^{(n)}$, we have

$$\begin{aligned}
&(\frac{1}{|\cdot|} * |u_{n,p}|^p)|u_{n,p}|^{p-2} u_{n,p} \\
&\le c \int_{\mathbb{R}^3} \frac{u_{n,p}^2}{|x-y|}\, dy u_{n,p} \\
&\le c\Big(\int_{\mathbb{R}^3} \frac{u_n^2}{|x-y|}\, dy + o_p(1)\Big) u_{n,p} \\
&\le c\big(\frac{2}{R} \int_{|x-y|>R/2} u_n^2\, dy + e^{-\mu R} \int_{|x-y|\le R/2} \frac{1}{|x-y|}\, dy + o_p(1)\big) u_{n,p} \\
&\le c(o_R(1) + o_p(1)) u_{n,p} .
\end{aligned} \tag{4.2}$$

Hence, there exists $p_1 > 2$ such that for $2 < p < p_1$, we have $co_p(1) \le \frac{a}{4}$ and

$$(\frac{1}{|\cdot|} * |u_{n,p}|^p) u_{n,p}^{p-2} \le \frac{a}{2} \quad \text{for } x \in \Omega_R^{(n)} .$$

$\square$

Lemma 4.4. *If $1 \le i \le m$, then $y_i^* = \lim_{n\to\infty} \varepsilon_n y_{n,i} \in \bar{\mathcal{A}}$.*

Proof. If not, we assume that there exists i such that $1 \le i \le m$ and $\varepsilon_n > 0$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $\operatorname{dist}(y_i, \bar{\mathcal{A}}) > 0$. Let $t_k = \nabla V(y_i) \ne 0$, by (A2) we deduce that there exists $\delta_1 > 0$ such that

$$(t_i, \nabla V(x)) \ge \frac{1}{2}|t_i|^2 > 0, \quad (t_i, \nabla \operatorname{dist}(x, \mathcal{M})) \ge 0 \quad \text{for } x \in B_{\delta_1}(y_i). \tag{4.3}$$

Set

$$\delta_2 = \min\{|y_i - y_l|,\, y_i \ne y_l,\ i, l = 1, \dots, m\}. \tag{4.4}$$

Let

$$0 < \delta < \min\{\frac{1}{2}\delta_1, \frac{1}{100}\delta_2\}.$$

Denote

$$B_n = \{x||x - y_{n,i}| \le 2\delta\varepsilon_n^{-1}\},$$
$$T_n = \{x|\delta\varepsilon_n^{-1} \le |x - y_{n,i}| \le 2\delta\varepsilon_n^{-1}\}.$$

Choose $\eta \in C_0^\infty(\mathbb{R}^3)$ such that $\eta(x)=0$ if $|x-y_{n,i}| \ge 2\delta\varepsilon_n^{-1}$; $\eta(x)=1$ if $|x-y_{n,i}| \le \delta\varepsilon_n^{-1}$ and $|\nabla\eta| \le \frac{2}{\delta}\varepsilon_n (\le 1)$. By $\langle D\Gamma_{\varepsilon_n}(u_n), \varphi\rangle = 0$ for $\varphi \in H^1(\mathbb{R}^3)$, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} (\nabla u_n \nabla\varphi + V(\varepsilon x) u_n\varphi)\,dx + \zeta_n \int_{\mathbb{R}^3} \chi_\varepsilon(x) u_n\varphi\,dx \\ &= \int_{\mathbb{R}^3} (\frac{1}{|\cdot|} * |u_n|^2) u_n\varphi\,dx, \end{aligned} \tag{4.5}$$

where $\zeta_n = G'(\int_{\mathbb{R}^3} \chi_\varepsilon(x) u_n^2\,dx)$. Choosing $\varphi = t_k \cdot \nabla u_n \cdot \eta$ as test function in (4.5), we obtain the local Poho$\hat{z}$aev identity

$$\begin{aligned} &\frac{1}{2}\varepsilon_n \int_{\mathbb{R}^3} (t_k, \nabla V(\varepsilon_n x)) u_n^2\eta\,dx + \frac{1}{2}\zeta_n \int_{\mathbb{R}^3} (\nabla\chi_{\varepsilon_n}(x), t_k) u_n^2\eta\,dx \\ &= \int_{\mathbb{R}^3} (\nabla u_n, \nabla\eta)(t_k, \nabla u_n)\,dx - \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2)(t_k, \nabla\eta)\,dx \\ &\quad - \frac{1}{2}\zeta_n \int_{\mathbb{R}^3} \chi_{\varepsilon_n}(x) u_n^2 (t_k, \nabla\eta)\,dx + \frac{1}{2}\int_{\mathbb{R}^3} (\frac{1}{|\cdot|} * u_n^2) u_n^2 (t_k, \nabla\eta)\,dx \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3} (\nabla_x(\frac{1}{|\cdot|} * u_n^2), t_k) u_n^2\eta\,dx. \end{aligned} \tag{4.6}$$

Next, we estimate all terms of (4.6). By (4.3), we have

$$\begin{gathered} \varepsilon_n \int_{\mathbb{R}^3} (t_k, \nabla V(\varepsilon_n x)) u_n^2\eta\,dx \ge c\varepsilon_n, \\ \frac{1}{2}\zeta_n \int_{\mathbb{R}^3} (\nabla\chi_{\varepsilon_n}(x), t_k) u_n^2\eta\,dx \ge 0. \end{gathered}$$

Hence the left-hand side of (4.6) is greater than or equal to $c\varepsilon_n$.

Since

$$\frac{1}{|\cdot|} * u_n^2 = \int_{\mathbb{R}^3} \frac{u_n^2(y)}{|x-y|}\,dy$$

and

$$\nabla(\frac{1}{|\cdot|} * u_n^2) = -\int_{\mathbb{R}^3} \frac{u_n(y)^2}{|x-y|^3}(x-y)\,dy,$$

we have

$$\int_{\mathbb{R}^3} (\nabla(\frac{1}{|\cdot|} * u_n^2), t_k) u_n^2\eta\,dx = -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|^3}(t_k, x-y)\eta(x)\,dx\,dy.$$

Since

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|^3}(t_k, x-y)\eta(x)\eta(y)\,dx\,dy = 0\,,$$

we have

$$\begin{aligned} &\int_{\mathbb{R}^3} (\nabla(\frac{1}{|\cdot|} * u_n^2), t_k) u_n^2\eta\,dx \\ &= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_n^2(x)u_n^2(y)}{|x-y|^3}(t_k, x-y)\eta(x)(1-\eta(y))\,dx\,dy. \end{aligned}$$

Then

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^3}(\nabla(\frac{1}{|\cdot|}*u_n^2),t_k)u_n^2\eta\,dx\Big| &\le c\iint_{\substack{\{|y-y_{n,i}|\ge\delta\varepsilon_n^{-1}\}\\ \{|x-y_{n,i}|\le 2\delta\varepsilon_n^{-1}\}}}\frac{u_n^2(x)u_n^2(y)}{|x-y|^2}\,dx\,dy\\
&\le c\iint_{\substack{\{\delta\varepsilon_n^{-1}\le|y-y_{n,i}|\le 3\delta\varepsilon_n^{-1}\}\\ \{|x-y_{n,i}|\le 2\delta\varepsilon_n^{-1}\}}}\frac{u_n^2(x)u_n^2(y)}{|x-y|^2}\,dx\,dy\\
&\quad+c\iint_{\substack{\{|y-y_{n,i}|\ge 3\delta\varepsilon_n^{-1}\}\\ \{|x-y_{n,i}|\le 2\delta\varepsilon_n^{-1}\}}}\frac{u_n^2(x)u_n^2(y)}{|x-y|^2}\,dx\,dy\\
&=: I+II\,,
\end{aligned}
\tag{4.7}
$$

where

$$
II\le c\iint_{\substack{\{|y-y_{n,i}|\ge 3\delta\varepsilon_n^{-1}\}\\ \{|x-y_{n,i}|\le 2\delta\varepsilon_n^{-1}\}}}u_n^2(y)u_n^2(x)\cdot\frac{1}{\delta^2}\varepsilon_n^2\,dx\,dy\le c\varepsilon_n^2\,.
$$

The region $\tilde{T_n}=\{y|\delta\varepsilon_n^{-1}\le|y-y_{n,k}|\le 3\delta\varepsilon_n^{-1}\}$ is contained in $\Omega^{(n)}_{\delta\varepsilon_n^{-1}}$, and we have

$$
|u_n(y)|\le ce^{-\mu\delta\varepsilon_n^{-1}},\quad y\in\tilde{T_n}.
$$

Then

$$
\begin{aligned}
I&\le ce^{-\mu\delta\varepsilon_n^{-1}}\iint_{\substack{\{\delta\varepsilon_n^{-1}\le|y-y_{n,k}|\le 3\delta\varepsilon_n^{-1}\}\\ \{|x-y_{n,i}|\le 2\delta\varepsilon_n^{-1}\}}}\frac{u_n^2(x)}{|x-y|^2}\,dx\,dy\\
&\le ce^{-\mu\delta\varepsilon_n^{-1}}\iint_{\substack{\{|x-y|\le 5\delta\varepsilon_n^{-1}\}\\ \{|x-y_{n,i}|\le 2\delta\varepsilon_n^{-1}\}}}\frac{u_n^2(x)}{|x-y|^2}\,dy\,dx\\
&\le ce^{-\mu\delta\varepsilon_n^{-1}}\varepsilon_n^{-1}\le c\varepsilon_n^2\,.
\end{aligned}
$$

From the above estimates, by Lemma 4.2,

$$
\begin{aligned}
\text{RHS of (4.6)}&\le c\int_{\mathbb{R}^3}(|\nabla u_n|^2+u_n^2)|\nabla\eta|\,dx+\zeta_n\int_{\mathbb{R}^3}\chi_\varepsilon(x)u_n^2|\nabla\eta|\,dx+c\varepsilon_n^2\\
&\le\int_{T_n}(|\nabla u_n|^2+u_n^2)|\nabla\eta|\,dx+\zeta_n\int_{T_n}\chi_\varepsilon(x)u_n^2|\nabla\eta|\,dx+c\varepsilon_n^2\\
&\le ce^{\mu\delta\varepsilon_n^{-1}}+c\varepsilon_n^2\le c\varepsilon_n^2.
\end{aligned}
\tag{4.8}
$$

Therefore, $c\varepsilon_n\le c\varepsilon_n^2$ as $n\to\infty$. We arrived at a contradiction and completes the proof. $\square$

Lemma 4.5. *For each $\delta>0$, there exists $c=c(L)>0$ such that*

$$
|u_n(x)|\le ce^{-\mu\,\mathrm{dist}(x,(\mathcal{A}^\delta)_{\varepsilon_n})}.
$$

Proof. By Lemma 4.2, $|u_n(x)|\le ce^{-\mu R}$ for $x\in\Omega_R^{(n)}$. Let $R_n(x)=\min\{|x-y_{n,i}|: i=1,\dots,m\}$, then $|u_n(x)|\le ce^{-\mu R_n(x)}$. Since $\varepsilon_n y_{n,i}\to y_i\in\mathcal{A}$, there exists $\varepsilon(\delta)$ such that for $\varepsilon_n\le\varepsilon(\delta)$, $\varepsilon_n y_{n,i}\in\mathcal{A}^\delta$, hence $R_n(x)\ge\mathrm{dist}(x,(\mathcal{A}^\delta)_{\varepsilon_n})$ and

$$
|u_n(x)|\le ce^{-\mu\,\mathrm{dist}(x,(\mathcal{A}^\delta)_{\varepsilon_n})},\quad x\in\mathbb{R}^3\,.
\tag{4.9}
$$

$\square$

Proposition 4.6. *Assume $D\Gamma_\varepsilon(u)=0$, $\Gamma_\varepsilon(u)\le L$. Then there exists $\bar\varepsilon=\bar\varepsilon(L)$ such that $\Gamma_\varepsilon(u)=I_\varepsilon(u)$ and $DI_\varepsilon(u)=0$ if $0<\varepsilon<\bar\varepsilon$.*

Proof. By Lemma 4.5, there exist $c = c(L)$ and $\mu = \mu(L)$ such that

$$|u(x)| \le c\, e^{-\mu \operatorname{dist}(x,(\mathcal{A}^\delta)_\varepsilon)} \le c\, e^{-\mu \operatorname{dist}(x,\mathcal{M}_\varepsilon)}\,. \tag{4.10}$$

Denote $d = \operatorname{dist}(\mathcal{A}^\delta, \partial\mathcal{M})$, then for $x \notin \mathcal{M}_\varepsilon$, we have

$$\operatorname{dist}(x,\, (\mathcal{A}^\delta)_\varepsilon) \ge \operatorname{dist}(x, \mathcal{M}_\varepsilon) + d\varepsilon^{-1}\,. \tag{4.11}$$

Hence

$$\begin{aligned}
\int_{\mathbb{R}^3} \chi_\varepsilon(x) u^2\, dx &\le \varepsilon^{-6} \int_{\mathbb{R}^3\setminus\mathcal{M}_\varepsilon} u^2\, dx \\
&\le c\varepsilon^{-6} \int_{\mathbb{R}^3\setminus\mathcal{M}_\varepsilon} e^{-2\mu \operatorname{dist}(x,(\mathcal{A}^\delta)_\varepsilon)}\, dx \\
&\le c\varepsilon^{-6} e^{-\mu d\varepsilon^{-1}} \int_{\mathbb{R}^3\setminus\mathcal{M}_\varepsilon} e^{-\mu \operatorname{dist}(x,(\mathcal{A}^\delta)_\varepsilon)}\, dx \\
&\le c\varepsilon^{-6} e^{-\mu d\varepsilon^{-1}} \int_{\mathbb{R}^3\setminus\mathcal{M}} e^{-\mu \operatorname{dist}(x,\mathcal{A}^\delta)}\, dx \\
&\le c\varepsilon^{-6} e^{-\mu d\varepsilon^{-1}} \to 0 \quad \text{as } \varepsilon \to 0\,.
\end{aligned}$$

In particular there exists $\bar\varepsilon$ such that for $0 < \varepsilon \le \bar\varepsilon$ we have

$$G\Big(\int_{\mathbb{R}^3} \chi_\varepsilon(x) u^2\, dx\Big) = 0\,. \tag{4.12}$$

Hence, $I_\varepsilon(u) = \Gamma_\varepsilon(u)$ and $DI_\varepsilon(u) = D\Gamma_\varepsilon(u) = 0$. $\square$

Lemma 4.7. *There is a direct sum $H^1(\mathbb{R}^3) = X_0 \oplus X_0^\perp$ such that $\dim X_0 < \infty$ and for all $\varphi \in X_0^\perp$,*

$$\begin{aligned}
&\int_{\mathbb{R}^3} (|\nabla\varphi|^2 + a\varphi^2) - c_1 \int_{\mathbb{R}^3} (\frac{1}{|\cdot|} * e^{-c_2|x|})\varphi^2\, dx - c_1 \int_{\mathbb{R}^3} (\frac{1}{|\cdot|} * (e^{-c_2|x|}\varphi))e^{-c_2|x|}\varphi\, dx \\
&\ge \frac{a}{2} \int_{\mathbb{R}^3} \varphi^2\, dx\,,
\end{aligned}$$

where $c_1, c_2 > 0$.

Proof. It suffices to prove that if X_0 is a subspace of $H^1(\mathbb{R}^3)$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^3} (|\nabla\varphi|^2 + \frac{a}{2}\varphi^2) - c_1 \int_{\mathbb{R}^3} (\frac{1}{|\cdot|} * e^{-c_2|x|})\varphi^2\, dx \\
&- c_1 \int_{\mathbb{R}^3} (\frac{1}{|\cdot|} * (e^{-c_2|x|}\varphi))e^{-c_2|x|}\varphi\, dx \le 0, \quad \varphi \in X_0,
\end{aligned} \tag{4.13}$$

then X_0 is finite-dimensional.

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\frac{1}{|\cdot|} * (e^{-c_2|x|}\varphi))e^{-c_2|x|}\varphi\, dx \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{e^{-c_2|x|}\varphi(x)e^{-c_2|y|}\varphi(y)}{|x-y|}\, dx\, dy \\
&= \Big(\int_{\mathbb{R}^3\setminus B_R(0)}\int_{\mathbb{R}^3\setminus B_R(0)} + 2\int_{\mathbb{R}^3\setminus B_R(0)}\int_{B_R(0)} \\
&\quad + \int_{B_R(0)}\int_{B_R(0)} \Big) \frac{e^{-c_2|x|}\varphi(x)e^{-c_2|y|}\varphi(y)}{|x-y|}\, dx\, dy \\
&=: I_1 + 2I_2 + I_3,
\end{aligned} \tag{4.14}$$

where

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^3\setminus B_R(0)} \int_{\mathbb{R}^3\setminus B_R(0)} \frac{e^{-c_2|x|}\varphi(x)e^{-c_2|y|}\varphi(y)}{|x-y|}\,dx\,dy \\
&\le \int_{\mathbb{R}^3\setminus B_R(0)} e^{-c_2|x|}\varphi(x)\Big(\int_{\mathbb{R}^3\setminus B_R(0)} \frac{e^{-2c_2|y|}}{|x-y|^2}\,dy\Big)^{1/2}dx\Big(\int_{\mathbb{R}^3\setminus B_R(0)} \varphi^2\,dx\Big)^{1/2} \\
&\le c\int_{\mathbb{R}^3\setminus B_R(0)} e^{-c_2x}\varphi(x)\,dx\Big(\int_{\mathbb{R}^3\setminus B_R(0)} \varphi^2\,dx\Big)^{1/2} \\
&\le o(\frac{1}{R})\int_{\mathbb{R}^3\setminus B_R(0)} \varphi^2\,dx\,.
\end{aligned}
$$

Similarly, we have

$$
\begin{gathered}
I_2 \le o\big(\frac{1}{R}\big)\int_{\mathbb{R}^3\setminus B_R(0)} \varphi^2\,dx + c\int_{B_R(0)} \varphi^2\,dx, \\
I_3 \le c\int_{B_R(0)} \varphi^2\,dx.
\end{gathered}
$$

Hence, we have

$$
\int_{\mathbb{R}^3} (\frac{1}{|\cdot|} * (e^{-c_2|x|}\varphi))e^{-c_2|x|}\varphi\,dx \le o(\frac{1}{R})\int_{\mathbb{R}^3\setminus B_R(0)} \varphi^2\,dx + c\int_{B_R(0)} \varphi^2\,dx.
$$

Since $\lim_{|x|\to\infty} \frac{1}{|x|} * e^{-c|x|} = 0$, we choose $R > 0$ such that

$$
\begin{aligned}
&c\int_{\mathbb{R}^3} (\frac{1}{|\cdot|} * e^{-c_2|x|})\varphi^2\,dx + c\int_{\mathbb{R}^3} (\frac{1}{|\cdot|} * (e^{-c_2|x|}\varphi))e^{-c_2|x|}\varphi\,dx \\
&\le \frac{a}{4}\int_{\mathbb{R}^3\setminus B_R(0)} \varphi^2\,dx + c\int_{B_R(0)} \varphi^2\,dx.
\end{aligned} \tag{4.15}
$$

By (4.13) and (4.15), we have, for $\varphi \in X_0$,

$$
\int_{\mathbb{R}^3} (|\nabla\varphi|^2 + \frac{a}{4}\varphi^2)\,dx - c\int_{B_R(0)} \varphi^2\,dx \le 0.
$$

Hence, we obtain

$$
\int_{B_R(0)} |\nabla\varphi|^2\,dx + \int_{\mathbb{R}^3\setminus B_R(0)} (|\nabla\varphi|^2 + \frac{a}{4}\varphi^2)\,dx \le c\int_{B_R(0)} \varphi^2\,dx. \tag{4.16}
$$

Now define the restriction operator P from $L^2(\mathbb{R}^3)$ to $L^2(B_R(0))$ by $P\varphi = \varphi|_{B_R(0)}$. Since (4.16) holds, it is easy to see that P is injective. Let $\tilde{X}_0 = PX_0$, it suffice to prove $\tilde{X}_0$ is finite-dimensional. It also follows from (4.16) that

$$
\|\varphi\|_{H^1(B_R(0))} \le c\|\varphi\|_{L^2(B_R(0))}, \quad \varphi \in \tilde{X}_0. \tag{4.17}
$$

Let $S := \{\varphi \in \tilde{X}_0 | \|\varphi\|_{L^2(B_R(0))} = 1\}$, then the set S is compact by (4.17). Hence, we obtain that $\tilde{X}_0$ is finite-dimensional subspace. $\square$

Proposition 4.8. *For each positive integer k, there exists $\varepsilon_k' > 0$ such that for $0 < \varepsilon < \varepsilon_k'$, Γ_ε has at least k pairs of sign-changing critical points $\pm u_{n_j}$, $j = 1, \dots, k$.*

Proof. Choose $\tilde{\varepsilon}_n$ small enough to satisfy Theorem 3.6 and Lemma 2.5. We denote $\varepsilon_n := \{\tilde{\varepsilon}_n, \frac{1}{n}\}$. By Lemma 2.5, without loss of generality, we may assume $c_n(\varepsilon, p) \to c_n(\varepsilon)$ and $u_{n,\varepsilon,p} \to u_{n,\varepsilon}$ in $H^1(\mathbb{R}^3)$ as $p \to 2$. By Lemma 3.7, $c_n(\varepsilon)$ is a critical value of Γ_ε with sign-changing critical point $u_{n,\varepsilon} \in H^1(\mathbb{R}^3)$.

We claim that for any $M>0$, there is n such that $c_{n,\varepsilon}>M$ for any $\varepsilon\in(0,\varepsilon_n)$. Then for every positive integer k, we can choose $(n_j)_{j=1}^k$ and $\varepsilon'_k=\min\{\varepsilon_{n_j}\}_{j=1}^k$ such that for $\varepsilon\in(0,\varepsilon'_k)$, $c_{n_j,\varepsilon}>\tilde c_j\geq c_{n_{j-1},\varepsilon}$, $2\leq j\leq k$. As a result, we can find k different critical values $(c_{n_j,\varepsilon})_{j=1}^k$ of Γ_ε as $\varepsilon\in(0,\varepsilon'_k)$.

By contradiction, we assume there exists $M>0$, $\bar\varepsilon_n\to\infty$ as $n\to\infty$ such that $c_n:=c_{n,\bar\varepsilon_n}\leq M$. Hence, there exists $p_n\in(2,p_0)$, such that $c_{n,p}:=c_n(\bar\varepsilon_n,p)\leq M+1$ for all $p\in(2,p_n)$. By Theorem 3.6, let $u_{n,p}\in H^1(\mathbb{R}^3)$ be such that

$$\Gamma_{\bar\varepsilon_n,p}(u_{n,p})=c_{n,p}\leq M+1,\quad D\Gamma_{\bar\varepsilon_n,p}(u_{n,p})=0,\quad m^*(u_{n,p})\geq n\,.$$

By Corollary 4.3, we have

$$\frac{p-1}{2}\int_{\mathbb{R}^3}(\frac{1}{|\cdot|}*(|u_{n,p}|^p))|u_{n,p}|^{p-2}\varphi^2\,dx\leq\sum_{i=1}^m c\int_{\mathbb{R}^3}(\frac{1}{|\cdot|}*e^{-c|x-y_{n,i}|})\varphi^2\,dx$$

and

$$\begin{aligned}
&\frac{p}{2}\int_{\mathbb{R}^3}(\frac{1}{|\cdot|}*(|u_{n,p}|^{p-2}u_{n,p}\varphi))|u_{n,p}|^{p-2}u_{n,p}\varphi\,dx\\
&\leq\sum_{i=1}^m c\int_{\mathbb{R}^3}\big(\frac{1}{|\cdot|}*(e^{-c|x-y_{n,i}|}\varphi)\big)e^{-c|x-y_{n,i}|}\varphi\,dx\\
&\quad+\sum_{i\neq j}^m c\int_{\mathbb{R}^3}\big(\frac{1}{|\cdot|}*(e^{-c|x-y_{n,j}|}\varphi)\big)e^{-c|x-y_{n,i}|}\varphi\,dx\\
&\leq\sum_{i=1}^m c\int_{\mathbb{R}^3}\big(\frac{1}{|\cdot|}*(e^{-c|x-y_{n,i}|}\varphi)\big)e^{-c|x-y_{n,i}|}\varphi\,dx+o(1)\int_{\mathbb{R}^3}\varphi^2\,dx\,.
\end{aligned}$$

Let

$$X_{i,n}=\{\varphi(x-y_{n,i})|\varphi\in X_0\},\quad X_n=\{\sum_{i=1}^m\varphi_i|\varphi_i\in X_{i,n}\}.$$

Then $\dim X_{i,n}=\dim X_0$, $\dim X_n\leq m\dim X_0<\infty$, $H^1(\mathbb{R}^3)=X_n\oplus X_n^\perp$, where $X_n^\perp=\cap_{i=1}^m X_{i,n}^\perp$. By Lemma 4.7, for $\varphi\in X_n^\perp$, we have

$$\begin{aligned}
&\langle D^2\Gamma_{\bar\varepsilon_n,p}(u_{n,p})\varphi,\varphi\rangle\\
&\geq\int_{\mathbb{R}^3}|\nabla\varphi|^2+a\varphi^2\,dx-\int_{\mathbb{R}^3}\big(\frac{1}{|\cdot|}*(|u_{n,p}|^p)\big)|u_{n,p}|^{p-2}\varphi^2\,dx\\
&\quad-\int_{\mathbb{R}^3}\big(\frac{1}{|\cdot|}*(|u_{n,p}|^{p-2}u_{n,p}\varphi)\big)|u_{n,p}|^{p-2}u_{n,p}\varphi\,dx\\
&\geq\int_{\mathbb{R}^3}|\nabla\varphi|^2+a\varphi^2\,dx-\sum_{i=1}^m c\int_{\mathbb{R}^3}(\frac{1}{|\cdot|}*e^{-c|x-y_{n,i}|})\varphi^2\,dx\\
&\quad-\sum_{i=1}^m c\int_{\mathbb{R}^3}(\frac{1}{|\cdot|}*(e^{-c|x-y_{n,i}|}\varphi))e^{-c|x-y_{n,i}|}\varphi\,dx-o(1)\int_{\mathbb{R}^3}\varphi^2\,dx\\
&\geq(\frac{a}{2}-o(1))\int_{\mathbb{R}^3}\varphi^2\,dx\\
&\geq\frac{a}{4}\int_{\mathbb{R}^3}\varphi^2\,dx\,.
\end{aligned}$$

As a result, we can get $m^*(u_{n,p})\leq m\dim X_0\leq C$, for some $C>0$ independent of n, which contradicts to $m^*(u_{n,p})\geq n\to\infty$. $\square$

The proof Theorem 1.1 follows from Propositions 4.6 and 4.8.

Acknowledgments. The authors are grateful to the referees for their helpful comments. This work is supported by NSFC 12161093 and 11961078, and by the Yunnan Key Laboratory of Modern Analytical Mathematics and Applications, Yunnan Province, China.

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Lu Yang
Department of Mathematics, Yunnan University, Kunming, 650091, China
Email address: 2973434415@qq.com

Xiangqing Liu
Department of Mathematics, Yunnan Normal University, Kunming, 650500, China
Email address: lxq8u8@163.com

Jianwen Zhou (corresponding author)
Department of Mathematics, Yunnan University, Kunming, 650091, China
Email address: jwzhou@ynu.edu.cn