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STABILITY OF GROUND STATES OF NONLINEAR SCHRÖDINGER SYSTEMS

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Abstract. In this article, we study existence and stability of ground states for a system of two coupled nonlinear Schrödinger equations with logarithmic nonlinearity. Moreover, global well-posedness is verified for the Cauchy problem in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ and in an appropriate Orlicz space.

1. Introduction

We consider the coupled system of logarithmic nonlinear Schrödinger equations

$$\begin{gathered} i\partial_t u + \partial_x^2 u + u \log |u|^2 + u|u|^{p-1}|v|^{p+1} = 0 \\ i\partial_t v + \partial_x^2 v + v \log |v|^2 + v|v|^{p-1}|u|^{p+1} = 0 \\ u(x,0) = u_0(x) \quad v(x,0) = v_0(x), \end{gathered} \tag{1.1}$$

where $u = u(x,t)$ and $v = v(x,t)$ are complex-valued functions of $(x,t) \in \mathbb{R} \times \mathbb{R}$, and $1 \leq p < 2$.

The motivation for studying coupled NLS equations mainly comes from their applications in various physical fields; for example, in quantum optics, quantum mechanics, nuclear physics, fluid dynamics, plasma physics and Bose-Einstein condensation (for the principal references see [26, 25]). Such systems (1.1) appear in the study of interactions between short and long dispersive waves (see [1, 14, 14]). Moreover, numerical studies describing the dynamics of gaussons collisions of (1.1) were reported in [23]. For local well-posedness in the energy space, existence and stability of standing waves for coupled nonlinear Schrödinger system with power-type nonlinearities, the reader is referred to [2, 11, 12, 15, 20, 21, 5, 6, 7, 22, 3, 4] and references therein.

System (1.1) has the conserved quantities

$$\begin{aligned} &E(u,v) \\ &= \frac{1}{2}\int_{\mathbb{R}} \left\{ |\partial_x u|^2 + |\partial_x v|^2 - |u|^2 \log |u|^2 - |v|^2 \log |v|^2 - \frac{2}{p+1}|u|^{p+1}|v|^{p+1} \right\} dx \end{aligned} \tag{1.2}$$

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and

$$Q(u,v)=\frac{1}{2}\int_{\mathbb{R}}(|u|^2+|v|^2)\,dx. \tag{1.3}$$

That is, when applied to sufficiently regular solutions $(u(x,t),v(x,t))$ of 1.1, the energy functional E and the mass Q are independent of t.

Notice that the function $z\mapsto |z|^2\mathrm{Log}|z|^2$ has a singularity at the origin, and hence the energy functional $E(u,v)$ fails to be differentiable on $H^1(\mathbb{R})\times H^1(\mathbb{R})$. Thus, to overcome the singularity of the logarithm at the origin, we need to define an energy space where the functional E is well-defined. Indeed, we define the energy space $\mathcal{B}:=W(\mathbb{R})\times W(\mathbb{R})$,

$$W(\mathbb{R})=\{u\in H^1(\mathbb{R}):|u|^2\log|u|^2\in L^1(\mathbb{R})\},$$

endowed with a Luxemburg type norm. In Section 2 we show that the energy functional E is of class C^1 on $\mathcal{B}$. Now we state our first result concerning the global well-posedness of the Cauchy problem (1.1). The proof is contained in Section 2.

Theorem 1.1. *Assume that $1\le p$. For every $(u_0,v_0)\in\mathcal{B}$ the Cauchy problem* (1.1) *is locally well posed in $\mathcal{B}$, i.e. there exist $T>0$ and a unique solution $(u,v)\in C([0,T],\mathcal{B})$ such that $(u(x,0),v(x,0))=(u_0,v_0)$. For each $T_0\in(0,T)$ the mapping $(u_0,v_0)\in\mathcal{B}\mapsto(u,v)\in C([0,T_0],\mathcal{B})$ is continuous. In addition, the mass associated with* (1.1)*, as well as energy are conserved in time, namely, for all* $[0,T]$

$$\|u(x,t)\|_{L^2}^2+\|v(x,t)\|_{L^2}^2=\|u_0\|_{L^2}^2+\|v_0\|_{L^2}^2,\quad E(u(x,t),v(x,t))=E(u_0,v_0).$$

From a mathematical and physical point of view, an important type of solutions for the system (1.1) are the so-called standing waves. In this article we are interested in the existence and stability of standing waves of (1.1); namely solutions to (1.1) of the form

$$(f(x,t),g(x,t))=(e^{i\theta t}\phi(x),e^{i\omega t}\varphi(x)),\quad \theta,\omega\in\mathbb{R}\text{ and }(\phi,\varphi)\in\mathcal{B},$$

where (ϕ,φ) has to satisfy the system of ordinary differential equations

$$\begin{aligned}-\partial_x^2\phi+\theta\phi&=\phi\log|\phi|^2+\phi|\phi|^{p-1}|\varphi|^{p+1}\\ -\partial_x^2\varphi+\omega\varphi&=\varphi\log|\varphi|^2+\varphi|\varphi|^{p-1}|\phi|^{p+1}.\end{aligned} \tag{1.4}$$

The most common approach to construct orbitally stable standing waves to (1.1) is to consider the variational problem

$$J(\eta,\zeta):=\inf\Big\{E(u,v):(u,v)\in\mathcal{B},\int_{\mathbb{R}}|u|^2=\eta\text{ and }\int_{\mathbb{R}}|v|^2=\zeta\Big\}. \tag{1.5}$$

A minimizer of problem (1.5) is called a ground state solution of (1.4). The corresponding set of (non-trivial) minimizers for $J(\eta,\zeta)$ is defined by

$$\mathcal{G}(\eta,\zeta)=\inf\{(\phi,\varphi)\in\mathcal{B}:E(\phi,\varphi)=J(\eta,\zeta),\|\phi\|_{L^2}^2=\eta\text{ and }\|\varphi\|_{L^2}^2=\zeta\}.$$

From the logarithmic Sobolev inequality, it is not difficult to show that problem (1.5) is well-defined; that is $J(\eta,\zeta)>-\infty$ (see Lemma 3.1 below). Moreover, by using the concentrated compactness principle [18], the existence of ground states will be obtained as a consequence of the stronger statement that any minimizing sequence for the problem $J(\eta,\zeta)$ is, up to translation, precompact in $\mathcal{B}$. More precisely, we have the following result.

Theorem 1.2. *Let $\eta>0$, $\zeta>0$ and $1\le p<2$. Then the following assertions hold.*

(i) *If* $\{(f_n, g_n)\}$ *is a minimizing sequence of* $J(\eta, \zeta)$*, then there exists a sequence* $\{y_n\}$ *of real numbers such that* $\{(f_n(\cdot + y_n), g_n(\cdot + y_n))\}$ *contains a convergent subsequence in* $\mathcal{B}$*. Which means that* $\{(f_n, g_n)\}$ *is relativity compact in* $\mathcal{B}$ *up to translations. Hence, the set* $\mathcal{G}(\eta, \zeta)$ *is non-empty, since there exists a minimizer for problem* (1.5)*.*

(ii) *If* $(f, g) \in \mathcal{G}(\eta, \zeta)$*, then there exist* θ *and* $\omega \in \mathbb{R}$ *such that* $f(x) = e^{i\theta}\phi(x)$ *and* $g(x) = e^{i\omega}\varphi(x)$*, where* $\phi, \varphi \in C^2(\mathbb{R})$ *and* $\phi(x), \varphi(x) > 0$ *for all* $x \in \mathbb{R}$*.*

(iii) *The set of minimizers* $\mathcal{G}(\eta, \zeta)$ *forms a true two-parameter family; that is, the two sets* $\mathcal{G}(\eta_1, \zeta_1)$ *and* $\mathcal{G}(\eta_2, \zeta_2)$ *are disjoint if* $(\eta_1, \zeta_1) \neq (\eta_2, \zeta_2)$*.*

To prove Theorem 1.2, we will use variational methods and the concentration compactness method of P.L. Lions [18]. Similar techniques have been used previously by Albert and Bhattarai [2] (see also[21, 7]) to prove the existence and orbital stability of standing wave solutions to NLS-KdV systems.

It is standard that the minimizers of the variational problem (1.5) are solutions to the stationary problem (1.4). The following is our orbital stability result, which is a direct consequence of the precompactness of the minimizing sequences of (1.5) (see Theorem 1.2). This result shows that if the initial data of a solution of (1.1) is near $\mathcal{G}(\eta, \zeta)$, then the solution will remain near $\mathcal{G}(\eta, \zeta)$ for every time $t \geq 0$.

Corollary 1.3. *Let* $\eta, \zeta > 0$*. Then the set* $\mathcal{G}(\eta, \zeta)$ *of minimizers for* $J(\eta, \zeta)$ *is* $\mathcal{B}$*-stable in the following sense. Given* $\epsilon > 0$ *there exist* $\delta > 0$ *such that, if* $(f_0, g_0) \in \mathcal{B}$ *satisfies*

$$\inf_{(\varphi,\phi)\in\mathcal{G}(\eta,\zeta)} \|(f_0, g_0) - (\varphi, \phi)\|_{\mathcal{B}} < \delta$$

then the solution $(f(x, t), g(x, t))$ *of the Cauchy problem* (1.1) *with initial data* $(f(x, 0), g(x, 0)) = (f_0(x), g_0(x))$ *satisfies*

$$\inf_{(\varphi,\phi)\in\mathcal{G}(\eta,\zeta)} \|(f(\cdot, t), g(\cdot, t)) - (\varphi, \phi)\|_{\mathcal{B}} < \epsilon$$

for all $t \in [0, +\infty)$*.*

To the best of our knowledge, this is the first work concerning the existence and stability of ground states for the system (1.1) in the L^2-subcritical case ($1 \leq p < 2$). We mention here that the well-posedness of the Cauchy problem (1.1), existence and stability/ instability of standing waves is open problem in the L^2-critical case ($p = 2$) and L^2-supercritical case ($p > 2$).

This article is organized as follows. In Section 2 we address the well-posedness of Cauchy problem for system (1.1) and we give the proof of Theorem 1.1. Section 3 is devoted to study existence and stability of standing waves (Theorem 1.2 and Corollary 1.3). Throughout this paper, the letter C will denote positive constants.

2. Existence results

This section is devoted to establish the local well-posedness of system (1.1). First we need to introduce some notation. For every $x \in [0, +\infty)$, we define the functions $\Phi(x)$, $\Psi(x) \in C^1([0, +\infty)) \cap C^2((0, +\infty))$ as follows,

$$\begin{gathered} \Phi(x) = \begin{cases} -x^2 \log(x^2), & \text{if } 0 \leq x \leq e^{-3}; \\ 3x^2 + 4e^{-3}x - e^{-6}, & \text{if } x \geq e^{-3}; \end{cases} \\ \Psi(x) = F(x) + \Phi(x), \end{gathered} \tag{2.1}$$

where $F(x) = x^2 \mathrm{Log}\, x^2$. Notice that Φ is a nonnegative convex and increasing function on $[0, +\infty)$. Next we define the Orlicz space $L^\Phi(\mathbb{R})$ associated to Φ as the completion of the $C_0^\infty(\mathbb{R})$-functions under the Luxemburg norm

$$\|f\|_{L^\Phi(\mathbb{R})} = \inf\big\{k > 0 : \int_{\mathbb{R}} \Phi(k^{-1}|f(x)|)dx \le 1\big\}.$$

It is well-known that (see [8, Section 2]),

$$L^\Phi(\mathbb{R}) = \{f \in L^1_{\rm loc}(\mathbb{R}) : \Phi(|u|) \in L^1(\mathbb{R})\}.$$

From [8, Lemma 2.1] we see that $(L^\Phi(\mathbb{R}), \|\cdot\|_{L^\Phi(\mathbb{R})})$ is a reflexive Banach space. In addition, we can prove the following properties.

Lemma 2.1. *Let $\{f_m\}$ be a sequence in $L^\Phi(\mathbb{R}^+)$, we have the following.*

(i) *If $\|f_m - f\|_{L^\Phi(\mathbb{R})} \to 0$ as $n \to +\infty$, then $\|\Phi(|f_n|) - \Phi(|f|)\|_{L^1(\mathbb{R})}$ as $n \to +\infty$.*

(ii) *Let $f \in L^\Phi(\mathbb{R})$. If $f_n(x) \to f(x)$ a.e. $x \in \mathbb{R}$ and if*

$$\lim_{n\to\infty} \int_{\mathbb{R}} \Phi(|f_n(x)|)dx = \int_{\mathbb{R}} \Phi(|f(x)|)dx,$$

then $\|f_m - f\|_{L^\Phi(\mathbb{R})} \to 0$ as $n \to +\infty$.

(iii) *For each $f \in L^\Phi(\mathbb{R})$, we have*

$$\min\{\|f\|_{L^\Phi(\mathbb{R})}, \|f\|^2_{L^\Phi(\mathbb{R})}\} \le \int_{\mathbb{R}} \Phi(|f(x)|)dx \le \max\{\|f\|_{L^\Phi(\mathbb{R})}, \|f\|^2_{L^\Phi(\mathbb{R})}\}. \tag{2.2}$$

Also, notice that there exists a constant $C > 0$ such that (see [8, Eq. (2.6), p. 1131])

$$\int_{\mathbb{R}} |\Psi(|f|) - \Psi(|g|)|dx \le C\big(1 + \|f\|^2_{H^1(\mathbb{R})} + \|g\|^2_{H^1(\mathbb{R})}\big)\|f - g\|_{L^2(\mathbb{R})}, \tag{2.3}$$

for all $f, g \in H^1(\mathbb{R})$. Finally, we define the reflexive Banach space

$$W(\mathbb{R}) = H^1(\mathbb{R}) \cap L^\Phi(\mathbb{R})$$

equipped with the usual norm $\|f\|_{W(\mathbb{R})} = \|f\|_{H^1(\mathbb{R})} + \|f\|_{L^\Phi(\mathbb{R})}$, for any $f \in W(\mathbb{R})$. Combining (2.2) and (2.3), it is not hard to show that (see [8, Proposition 2.2])

$$W(\mathbb{R}) = \{u \in H^1(\mathbb{R}) : |u|^2 \log |u|^2 \in L^1(\mathbb{R})\}.$$

It is important to note that the energy functional (1.2) is Fréchet differentiable on $\mathcal{B} = W(\mathbb{R}) \times W(\mathbb{R})$. Moreover, the Fréchet derivative is

$$E'(f, g) = \begin{bmatrix} -\partial_x^2 f - f \log |f|^2 - f|f|^{p-1}|g|^{p+1} \\ -\partial_x^2 g - g \log |g|^2 - g|g|^{p-1}|f|^{p+1} \end{bmatrix} \tag{2.4}$$

for $(f, g) \in \mathcal{B}$. The proof of (2.4) is similar to that of [8, Proposition 2.7] and we omit here.

Now we sketch the basic points of the standard theory of Log NLS (see [9, Chapter 9] and [8]) ensuring the local well-posedness of the Cauchy Problem for (1.1) in the energy space $\mathcal{B}$. First, we regularize the logarithmic nonlinearity near the origin. Let $z \in \mathbb{C}$ and $n \in \mathbb{N}$. Bearing in mind (2.1), we consider the functions

$$\zeta_n(z) = \begin{cases} a(z), & \text{if } |z| \ge \frac{1}{n}; \\ n\, z\, a(\frac{1}{n}), & \text{if } |z| \le \frac{1}{n}; \end{cases} \quad \text{and} \quad \beta_n(z) = \begin{cases} b(z), & \text{if } |z| \le n; \\ \frac{z}{n} b(n), & \text{if } |z| \ge n, \end{cases}$$

where

$$a(z) = \frac{z}{|z|^2}\, \Phi(|z|), \quad b(z) = \frac{z}{|z|^2}\, \Psi(|z|) \quad \text{for } z \in \mathbb{C},\ z \ne 0.$$

We introduce the family of regularized nonlinearities in the form $g_n(z) = \beta_n(z) - \zeta_n(z)$ for every $z \in \mathbb{C}$. Notice that we have the point-wise limit $g_n(z) \to z \log |z|^2$ as n goes to $+\infty$. Let $(u_0, v_0) \in \mathcal{B}$ (in particular u_0, $v_0 \in H^1(\mathbb{R})$). For every $n \in \mathbb{N}$, we consider the initial value problem

$$\begin{gathered}
i\partial_t u^n + \partial_x^2 u^n + g_n(u^n) + u^n|u^n|^{p-1}|v^n|^{p+1} = 0\\
i\partial_t v^n + \partial_x^2 v^n + g_n(v^n) + v^n|v^n|^{p-1}|u^n|^{p+1} = 0\\
u^n(x,0) = u_0(x) \quad v^n(x,0) = v_0(x).
\end{gathered} \tag{2.5}$$

Lemma 2.2. *Assume that $1 \leq p < 2$. Then the initial value problem (2.5) is well posed in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, i.e., there exists a unique global solution $(u, v) \in C(\mathbb{R}, H^1(\mathbb{R}) \times H^1(\mathbb{R}))$ such that $(u(0), v(0)) = (u_0, v_0)$. In addition, conservation of energy and of L^2-norm holds: for all $t \in \mathbb{R}$,*

$$\|u^n(t)\|_{L^2}^2 + \|v^n(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2, \quad E_n(u^n(t), v^n(t)) = E_n(u_0, v_0)$$

where

$$E_n(u,v) = \frac{1}{2}\int_{\mathbb{R}} \big\{|\partial_x u|^2 + |\partial_x v|^2 - G_n(u) - G_n(v) - \frac{2}{p+1}|u|^{p+1}|v|^{p+1}\big\}\,dx,$$

and

$$G_n(z) = \int_0^{|z|} g_n(s)\,ds.$$

Proof. First, since g_n is globally Lipschitz continuous $\mathbb{C} \to \mathbb{C}$, the proof of the local wellposedness in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ for $1 \leq p < 2$ is a contraction argument based on Strichartz estimates; see Remark [9, 3.3.12] and [13, Theorem 1] for more details.

We need only show that the maximal solution of (2.5) is global. The proof relies on the following Gagliardo-Nirenberg inequality [13, Section 3]: there exists a constant $C > 0$ such that

$$\|u\|_{L^{2p+2}}^{2p+2} + \|v\|_{L^{2p+2}}^{2p+2} + \|uv\|_{L^{p+1}}^{p+1} \leq C(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2)^{p/2}(\|u\|_{L^2}^2 + \|v\|_{L^2}^2)^{p+1-\frac{p}{2}} \tag{2.6}$$

Indeed, by the local theory, we just need to control the $H^1(\mathbb{R}) \times H^1(\mathbb{R})$-norm for every $t \in \mathbb{R}$. It is clear that $\int_{\mathbb{R}} G_n(u)\,dx \leq C\|u\|_{L^2}^2$. From (2.6), in view of the conservation of charge and energy, we see

$$\begin{aligned}
E_n(u^n, v^n) \geq &\frac{1}{2}(\|\nabla u^n\|_{L^2}^2 + \|\nabla v^n\|_{L^2}^2) - C(\|u^n\|_{L^2}^2 + \|v^n\|_{L^2}^2)\\
&- C(\|\nabla u^n\|_{L^2}^2 + \|\nabla v^n\|_{L^2}^2)^{p/2}(\|u^n\|_{L^2}^2 + \|v^n\|_{L^2}^2)^{p+1-\frac{p}{2}}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&(\|\nabla u^n\|_{L^2}^2 + \|\nabla v^n\|_{L^2}^2)\big[\frac{1}{2} - C(\|\nabla u^n\|_{L^2}^2 + \|\nabla v^n\|_{L^2}^2)^{\frac{p}{2}-1}(\|u^n\|_{L^2}^2 + \|v^n\|_{L^2}^2)^{p+1-\frac{p}{2}}\big]\\
&\leq E_n(u^n, v^n) + C(\|u^n\|_{L^2}^2 + \|v^n\|_{L^2}^2)
\end{aligned}$$

Since $p < 2$, by mass and energy conservation, we see easily that the $H^1(\mathbb{R}) \times H^1(\mathbb{R})$-norm of the solution $(u^n(t), v^n(t))$ is bounded. The continuity argument implies that all solutions of (2.5) are global, which completes the proof. □

To show the uniqueness of the Cauchy problem (1.1), we will use Lemma 2.3 below.

Lemma 2.3. (i) *Let u, v, z, $w \in \mathbb{C}$ be such that $|u|$, $|v|$, $|z|$, $|w| \le K$, where K is a positive constant. Then there exists $C > 0$ such that*
$$|u|u|^{p-1}|v|^{p+1} - z|z|^{p-1}|w|^{p+1}| \le C(|z-u| + |w-v|).$$
(ii) *For every z_1, $z_2 \in \mathbb{C}$ we have*
$$|\operatorname{Im}((z_1 \log|z_1|^2 - z_2 \log|z_2|^2)(\overline{z_1} - \overline{z_2}))| \le 4|z_1 - z_2|^2.$$

Proof. Statement (ii) follows immediately from [9, Lemma 9.3.5]. Next we prove (i). A simple calculation shows that
$$\begin{aligned} &|u||z|(u|u|^{p-1}|v|^{p+1} - z|z|^{p-1}|w|^{p+1}) \\ &= u|z|(|u|^p|v|^{p+1} - |z|^p|w|^{p+1}) + [u(|z| - |u|) + |u|(u-z)]|z|^p|w|^{p+1}. \end{aligned} \tag{2.7}$$
On the other hand, since $p \ge 1$ and $|u|$, $|v|$, $|z|$, $|w| \le K$, it is not hard to show that there exists a constant $C > 0$ such that
$$\big||u|^p|v|^{p+1} - |z|^p|w|^{p+1}big| \le C(|u-z| + |v-w|). \tag{2.8}$$
Therefore, combining (2.7) and (2.8) we infer that
$$\begin{aligned} |u|u|^{p-1}|v|^{p+1} - z|z|^{p-1}|w|^{p+1}| &\le ||u|^p|v|^{p+1} - |z|^p|w|^{p+1}| + 2|u-z||z|^{p-1}|w|^{p+1} \\ &\le C(|u-z| + |v-w|), \end{aligned}$$
and this completes the proof. □

Proof of Theorem 1.1. Our approach is inspired by Cazenave's approach for the single logarithmic Schrödinger equation in [9, Theorems 9.3.4, 3.3.5, 3.3.9].

Step 1. First, we regularize the logarithmic nonlinearity near the origin. Let $z \in \mathbb{C}$ and $n \in \mathbb{N}$. Bearing in mind (2.1), we consider the functions
$$\zeta_n(z) = \begin{cases} a(z), & \text{if } |z| \ge \frac{1}{n}; \\ nza(\frac{1}{n}), & \text{if } |z| \le \frac{1}{n}; \end{cases} \quad \text{and} \quad \beta_n(z) = \begin{cases} b(z), & \text{if } |z| \le n; \\ \frac{z}{n} b(n), & \text{if } |z| \ge n, \end{cases}$$
where
$$a(z) = \frac{z}{|z|^2}\Phi(|z|), \quad b(z) = \frac{z}{|z|^2}\Psi(|z|) \quad \text{for } z \in \mathbb{C},\ z \ne 0.$$
We introduce the family of regularized nonlinearities in the form $g_n(z) = \beta_n(z) - \zeta_n(z)$ for every $z \in \mathbb{C}$. Notice that we have the point-wise limit $g_n(z) \to z \log|z|^2$ as $n \to +\infty$.

Further, we consider $J_n = (I - \frac{1}{n}\Delta)^{-1}$. Observe that J_n satisfies the following properties
$$\|J_n\|_{\mathcal{L}(H^{-1}, H^1)} \le n, \quad \|J_n\|_{\mathcal{L}(X,X)} \le 1, \tag{2.9}$$
where X coincides with one of the spaces $H^1(\mathbb{R})$, $H^{-1}(\mathbb{R})$ and $L^2(\mathbb{R})$. Moreover,
$$\begin{gathered} J_n u \underset{n\to\infty}{\longrightarrow} u \text{ in } X \text{ for all } u \in X; \\ \text{if } \sup_n \|u^n\|_X < \infty, \text{ then } J_n u^n - u^n \rightharpoonup 0 \text{ in } X \text{ as } n \to \infty. \end{gathered} \tag{2.10}$$
We define
$$g_{1,n}(u,v) = J_n g_1(J_n u, J_n v), \quad g_{2,n}(u,v) = J_n g_2(J_n u, J_n v),$$
where
$$g_1(u,v) = u|u|^{p-1}|v|^{p+1}, \quad g_2(u,v) = v|v|^{p-1}|u|^{p+1}.$$

Let $(u_0, v_0) \in \mathcal{B}$ (in particular $u_0,\ v_0 \in H^1(\mathbb{R})$). For every $n \in \mathbb{N}$, we consider the initial value problem

$$
\begin{gathered}
i\partial_t u^n + \partial_x^2 u^n + g_n(u^n) + g_{1,n}(u^n, v^n) = 0 \\
i\partial_t v^n + \partial_x^2 v^n + g_n(v^n) + g_{2,n}(u^n, v^n) = 0 \\
u^n(x,0) = u_0(x), \quad v^n(x,0) = v_0(x).
\end{gathered}
\tag{2.11}
$$

Since $g_n, g_{1,n}, g_{2,n}$ are locally L^2-Lipschitz continuous, by [9, Theorem 3.3.1], we infer that there exists a unique global solution $(u^n, v^n) \in C(\mathbb{R}, H^1(\mathbb{R}) \times H^1(\mathbb{R}))$ of (2.11), for every $n \in \mathbb{N}$, such that

$$
\|u^n(t)\|_{L^2}^2 + \|v^n(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2, \quad E_n(u^n(t), v^n(t)) = E_n(u_0, v_0) \tag{2.12}
$$

where

$$
\begin{aligned}
E_n(u,v) &= \frac{1}{2}\int_{\mathbb{R}} \big\{|\partial_x u^n|^2 + |\partial_x v^n|^2\big\} dx + G_n(u^n, v^n), \\
G_n(u^n, v^n) &= \frac{1}{2}\int_{\mathbb{R}} \big\{\Pi_n(|u^n|) + \Pi_n(|v^n|) - \Xi_n(|u^n|) - \Xi_n(|v^n|) \\
&\quad - \frac{2}{p+1}|J_n u^n|^{p+1}|J_n v^n|^{p+1}\big\}\, dx,
\end{aligned}
\tag{2.13}
$$

and

$$
\Pi_n(z) = \frac{1}{2}\int_0^{|z|} \zeta_n(s) ds, \quad \Xi_n(z) = \frac{1}{2}\int_0^{|z|} \beta_n(s) ds. \tag{2.14}
$$

Step 2. We set

$$
\theta_n = \sup\{\tau > 0 : \|(u^n(t), v^n(t))\|_{H^1 \times H^1} \le 2M \text{ on } (-\tau, \tau)\}
$$

It is easily seen that

$$
\|g_j(u,v) - g_j(w,z)\|_{L^2 \times L^2} \le C(M)\|(u,v) - (w,z)\|_{L^2 \times L^2}, \quad j = 1, 2, \tag{2.15}
$$

for all $(u,v), (w,z) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ such that $\|(u,v)\|_{H^1 \times H^1} + \|(w,z)\|_{H^1 \times H^1} \le M$.

By (2.9), $g_{1,n}, g_{2,n}$ satisfy (2.15) with $C(M)$ independent on n. Hence, from (2.11), we obtain

$$
\sup_{n \in \mathbb{N}} \|(\partial_t u^n, \partial_t v^n)\|_{L^\infty((-\theta_n, \theta_n), H^{-1}(\mathbb{R}) \times H^{-1}(\mathbb{R}))} \le C(M). \tag{2.16}
$$

Using [9, Lemma 3.6] and (2.16), we conclude that

$$
\|(u^n(t), v^n(t)) - (u_0, v_0)\|_{L^2 \times L^2} \le C(M)|t|^{1/2} \quad \text{for all } t \in (-\theta_n, \theta_n) \tag{2.17}
$$

Applying (2.17), the conservation of energy and charge, and that

$$
G_n' = (g_n + g_{1,n}, g_n + g_{2,n}),
$$

we obtain

$$
\begin{aligned}
\|(u^n(t), v^n(t)\|_{H^1 \times H^1}^2 &\le \|(u_0, v_0)\|_{L^2 \times L^2}^2 + \|(u_0', v_0')\|_{L^2}^2 \\
&\quad + 2|G_n(u^n(t), v^n(t)) - G_n(u_0, v_0)| \\
&\le \|(u_0, v_0)\|_{H^1 \times H^1}^2 + C(M)|t|^{1/2}, \quad t \in (-\theta_n, \theta_n).
\end{aligned}
$$

We define $T(M)$ by

$$
C(M)T(M)^{1/2} = 2M^2.
$$

Then

$$\|(u^n, v^n)\|_{L^\infty((-T,T),H^1)} < 2M \quad \text{for } T = \min\{T(M), \theta_n\}.$$

Hence $T(M) \leq \theta_n$. Therefore

$$\|(u^n, v^n)\|_{L^\infty((-T(M),T(M)),H^1\times H^1)} \leq 2M, \tag{2.18}$$

$$\|(\partial_t u^n, \partial_t v^n)\|_{L^\infty((-T(M),T(M)),H^{-1}\times H^{-1})} \leq C(M) \tag{2.19}$$

Step 3. From (2.18), (2.19) and [9, Proposition 1.3.14], we conclude that there exist $(u(t), v(t))$ in

$$L^\infty((-T(M), T(M)), H^1 \times H^1) \cap W^{1,\infty}((-T(M), T(M)), H^{-1} \times H^{-1}),$$

and a subsequence, which we still denote by (u^n, v^n), such that

$$(u^n(t), v^n(t)) \underset{n\to\infty}{\rightharpoonup} u(t) \quad \text{in } H^1(\mathbb{R}),$$

for all $t \in [-T(M), T(M)]$,

Now since $E_n(u_0, v_0) \to E(u_0, v_0)$ as n goes to $+\infty$ and $\Pi_n \geq 0$, it follows that

$$\begin{aligned}\|u^n(t)\|_{H^1}^2 + \|v^n(t)\|_{H^1}^2 &\leq C + \|\Xi_n(u^n(t))\|_{L^1} + \|\Xi_n(v^n(t))\|_{L^1} \\ &\quad + C\|u^n(t)v^n(t)\|_{L^{p+1}}^{p+1}\end{aligned} \tag{2.20}$$

In addition, a simple calculation shows that there exists a constant C such that (see [9, p. 296])

$$\begin{aligned}\|\Xi_n(u^n(t))\|_{L^1} + \|\Xi_n(v^n(t))\|_{L^1} &\leq \frac{1}{4}(\|u^n(t)\|_{H^1}^2 + \|v^n(t)\|_{H^1}^2) \\ &\quad + C(\|u^n(t)\|_{L^2}^2 + \|v^n(t)\|_{L^2}^2).\end{aligned} \tag{2.21}$$

Thus, combining (2.6), (2.20), and (2.21) we infer that (u^n, v^n) is bounded in $L^\infty(\mathbb{R}, H^1(\mathbb{R}) \times H^1(\mathbb{R}))$. In particular, we easily verify that (see [9, Step 2 in Theorem 9.3.4])

$$u^n|u^n|^{p-1}|v^n|^{p+1} \text{ and } v^n|v^n|^{p-1}|u^n|^{p+1} \text{ are bounded in } L^\infty(\mathbb{R}, H^{-1}(\mathbb{R})), \tag{2.22}$$

$$g_n(u^n) \text{ and } g_n(v^n) \text{ are bounded in } L^\infty(\mathbb{R}, H^{-1}(\Omega_k)) \text{ for every } k \in \mathbb{R}, \tag{2.23}$$

where $\Omega_k = \{x \in \mathbb{R} : |x| \leq k\}$. It follows from (2.11), (2.22) and (2.23) that $\partial_t u^n\big|_{\Omega_k}$ and $\partial_t v^n\big|_{\Omega_k}$ are bounded in $L^\infty(\mathbb{R}, H^{-1}(\Omega_k))$. By the Sobolev's embedding and Arzela-Ascoli compactness criterion there exist $(u, v) \in L^\infty(\mathbb{R}, H^1(\mathbb{R}) \times H^1(\mathbb{R}))$ and a subsequence, which we still denote by (u^n, v^n), such that (see [9, Lemma 9.3.6])

(i) $(u^n|_{\Omega_k}, v^n|_{\Omega_k}) \in W^{1,\infty}(\mathbb{R}, H^{-1}(\Omega_k)) \times W^{1,\infty}(\mathbb{R}, H^{-1}(\Omega_k))$ for all $k \in \mathbb{R}$.
(ii) $(u^n(t), v^n(t)) \rightharpoonup (u, v)$ in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ as $n \to +\infty$ for every $t \in \mathbb{R}$.
(iii) $(u^n(x,t), v^n(x,t)) \rightharpoonup (u(x,t), v(x,t))$ as $n \to +\infty$, for almost every $x \in \mathbb{R}$ and for every $t \in \mathbb{R}$.

Now, since $g_n(z) \to z\log|z|^2$ as n goes to $+\infty$ and by properties (i)-(iii) above, it is not hard to see that the limiting function $(u, v) \in L^\infty(\mathbb{R}, W(\mathbb{R}) \times W(\mathbb{R}))$ and it is a weak solution to the equation (1.1) such that $(u(0), v(0)) = (u_0, v_0)$. Moreover, by weak lower semicontinuity, Fatou's lemma and arguing in the same manner as in [9, Step 3 in Theorem 9.3.4]) it is easy to see that $\|u^n(t)\|_{L^2}^2 + \|v^n(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2$ and

$$E(u(t), v(t)) \leq E(u_0, v_0) \quad \text{for every } t \in \mathbb{R}.$$

Next we show uniqueness of the weak solution $(u, v) \in L^\infty(\mathbb{R}, W(\mathbb{R}) \times W(\mathbb{R}))$ to the equation (1.1). Assume the existence of two solutions (u, v) and (z, w) of (1.1) with the same initial data (u_0, v_0) in the class $L^\infty(\mathbb{R}, W(\mathbb{R}) \times W(\mathbb{R}))$. We will prove that $(u, v) \equiv (z, w)$. It suffices to show that $u - z$ and $v - w$ satisfy $\|u - z\|_{L^2}^2 = \|v - w\|_{L^2}^2 = 0$ for every $t \in [0, T]$, where T is a fixed positive number. Indeed, on taking the difference of the two equations

$$i\partial_t(u - z) + \partial_x^2(u - z) + u \log|u|^2 - z \log|z|^2 + u|u|^{p-1}|v|^{p+1} - z|z|^{p-1}|w|^{p+1} = 0$$
$$i\partial_t(v - w) + \partial_x^2(v - w) + v \log|v|^2 - w \log|w|^2 + v|v|^{p-1}|u|^{p+1} - w|w|^{p-1}|z|^{p+1} = 0$$

and multiplying this equation by $(i(u - z), i(v - w))$ we infer that

$$\begin{aligned}
&-\frac{1}{2}\partial_t(\|u - z\|_{L^2}^2 + \|v - w\|_{L^2}^2) \\
&= \operatorname{Im}\int_{\mathbb{R}} (u \log|u|^2 - z \log|z|^2)(\overline{u} - \overline{z})dx + \operatorname{Im}\int_{\mathbb{R}} (v \log|v|^2 - w \log|w|^2)(\overline{v} - \overline{w}) \\
&\quad + \operatorname{Im}\int_{\mathbb{R}} (u|u|^{p-1}|v|^{p+1} - z|z|^{p-1}|w|^{p+1})(\overline{u} - \overline{z})dx \\
&\quad + \operatorname{Im}\int_{\mathbb{R}} (v|v|^{p-1}|u|^{p+1} - w|w|^{p-1}|z|^{p+1})(\overline{v} - \overline{w})dx.
\end{aligned}$$

Then from Lemma 2.3 we see that there exists a constant $C > 0$ such that

$$\|u(t) - z(t)\|_{L^2}^2 + \|v(t) - w(t)\|_{L^2}^2 \le C\int_0^t \{\|u(t) - z(t)\|_{L^2}^2 + \|v(t) - w(t)\|_{L^2}^2\}\, dt.$$

Therefore, uniqueness of solution follows by Gronwall's inequality. Finally, the conservation of energy, and the continuity of the solution $(u, v) \in C(\mathbb{R}, W(\mathbb{R}) \times W(\mathbb{R}))$ follow from the same kind of arguments as in [9, Step 4 of Theorem 9.3.3]. This completes of proof. □

3. Existence and stability of standing waves

In this section, we study the existence of minimizers of problem (1.5) and the orbital stability of $\mathcal{G}(\eta, \zeta)$. Thus, the aim is to prove Theorem 1.2 and Corollary 1.3. First, we need some preliminary lemmas.

Lemma 3.1. *Let $\eta, \zeta > 0$. Every minimizing sequence for $J(\eta, \zeta)$ is bounded in $\mathcal{B}$. Moreover, $J(\eta, \zeta) > -\infty$.*

Proof. Let $\{(u_n, v_n)\}$ be a minimizing sequence for $J(\eta, \zeta)$, then we find that $\|u_n\|_{L^2}^2$ and $\|v_n\|_{L^2}^2$ are bounded. Now, from the Gagliardo-Nirenberg inequality (see (2.6)) we obtain that

$$\|u_n v_n\|_{L^{p+1}}^{p+1} \le C(\|u_n\|_{L^2}^2 + \|v_n\|_{L^2}^2)^{\frac{p+2}{2}}(\|\partial_x u_n\|_{L^2}^2 + \|\partial_x v_n\|_{L^2}^2)^{p/2}. \tag{3.1}$$

Moreover, by logarithmic Sobolev inequality with $\alpha^2 = \pi/2$ we have

$$\begin{aligned}
\int_{\mathbb{R}} |u_n|^2 \log|u_n|^2 dx &\le \frac{1}{2}\|\partial_x u_n\|_{L^2}^2 + (\log\|u_n\|_{L^2}^2 - (1 + \log\sqrt{\pi/2}))\|u_n\|_{L^2}^2 \\
&\le \frac{1}{2}\|\partial_x u_n\|_{L^2}^2 + \eta\log\eta + C\eta,
\end{aligned} \tag{3.2}$$

and

$$\int_{\mathbb{R}} |v_n|^2 \log|v_n|^2 dx \le \frac{1}{2}\|\partial_x v_n\|_{L^2}^2 + \zeta\log\zeta + C\zeta. \tag{3.3}$$

Combining (3.1), (3.2), and (3.3) we have

$$\begin{aligned}&2E(u_n,v_n)+\eta\log\eta+\zeta\log\zeta+C(\eta+\zeta)+C(\|\partial_x u_n\|_{L^2}^2+\|\partial_x v_n\|_{L^2}^2)^{p/2}\\&\geq \frac{1}{2}(\|\partial_x u_n\|_{L^2}^2+\|\partial_x v_n\|_{L^2}^2)\end{aligned}$$

Since $E(u_n,v_n)$ is bounded and $1 \leq p < 2$, it follows that (u_n,v_n) is bounded in $H^1 \times H^1$. Moreover, by (2.1), (2.3), and (3.1) we have

$$\begin{aligned}&\int_{\mathbb{R}}\{\Phi(|u_n(x)|)+\Phi(|v_n(x)|)\}dx\\&\leq 2\sup_n E(u_n,v_n)+\int_{\mathbb{R}}\{\Psi(|u_n(x)|)+\Psi(|v_n(x)|)\}dx+\frac{2}{p+1}|u_n|^{p+1}|v_n|^{p+1}\leq C.\end{aligned}$$

Thus, by Lemma 2.1, we have that $\|u_n\|_{W(\mathbb{R})}^2$ and $\|v_n\|_{W(\mathbb{R})}^2$ are bounded, hence $\|(u_n,v_n)\|_{\mathcal{B}}$ is bounded. Finally, if $(u,v)\in\mathcal{B}$, $\|u\|_{L^2}^2=\eta$, and $\|v\|_{L^2}^2=\zeta$, by (3.1), (3.2), and (3.3), it follows that $E(u,v)$ is bounded below, hence $J(\eta,\zeta) > -\infty$. This completes the proof. $\square$

For each minimizing sequence $\{(f_n,g_n)\}$ of $J(\eta,\zeta)$, we define

$$\gamma=\lim_{r\to\infty}\lim_{n\to\infty}\sup_{y\in\mathbb{R}}\int_{y-r}^{y+r}(|f_n|^2+|g_n|^2)\,dx.$$

Notice that γ satisfies $0\leq\gamma\leq\eta+\zeta$. It follows from the concentration compactness principle due to Lions [18] that there are three mutually exclusive cases:

(i) (Vanishing) $\gamma=0$. This means that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}}\int_{y-r}^{y+r}(|f_n|^2+|g_n|^2)\,dx=0,\quad\text{for every } r\geq 0.$$

(ii) (Dichotomy) $0<\gamma<\eta+\zeta$, or

(iii) (Compactness) $\gamma=\eta+\zeta$. That is, given $\epsilon>0$ there exist $r_\epsilon>0$ and a sequence $\{y_n\}$ in $\mathbb{R}$ such that, for all n, we have

$$\int_{y_n-r_\epsilon}^{y_n+r_\epsilon}(|f_n|^2+|g_n|^2)\,dx\geq\eta+\zeta-\epsilon.$$

Let us study each case separately. First, we rule out the vanishing case (case (i)). The following lemma can be proved by almost the same way as for [7, Lemma 3.2]. We omit the proof.

Lemma 3.2. *Let $\{(f_n,g_n)\}$ be a minimizing sequence for $J(\eta,\zeta)$. If $p>2$, then there are positive constants $M=M(p,\eta)$ and $C=C(p,\zeta)$ such that $\|f_n\|_{L^p}^p\geq M$ and $\|g_n\|_{L^p}^p\geq C$ for all n.*

The following classical lemma is needed to rule out the case of vanishing. For a proof, see [16, Lemma 3.9].

Lemma 3.3. *Let $\{f_n\}$ be a bounded sequence in $H^1(\mathbb{R})$. If $p>2$ and*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}}\int_{y-r}^{y+r}|f_n(x)|^2\,dx=0\tag{3.4}$$

for some $r>0$, then $\lim_{n\to\infty}\|f_n\|_{L^p}=0$.

The following lemma states that $\gamma=0$ does not occur.

Lemma 3.4. *If $\{(f_n, g_n)\}$ is a minimizing sequence for $J(\eta, \zeta)$, then $\gamma > 0$.*

Proof. If $\gamma = 0$ then there are exist $r > 0$ and a subsequence $\{(f_{n_k}, g_{n_k})\}$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} (|f_{n_k}|^2 + |g_{n_k}|^2)\,dx = 0.$$

Since the sequences $\{f_n\}$ and $\{g_n\}$ are bounded in $H^1(\mathbb{R})$ by Lemma 3.1, then, by Lemma 3.3, we infer that $\lim_{n\to\infty} \|f_{n_k}\|_{L^p} = 0$ and $\lim_{n\to\infty} \|g_{n_k}\|_{L^p} = 0$ for $p > 2$, which contradicts Lemma 3.2. □

We now establish the following sequence of lemmas to rule out the dichotomy case.

Lemma 3.5. *If $\{(f_n, g_n)\}$ is a minimizing sequence for $J(\eta, \zeta)$, then there exist positive constants κ and δ such that, for n large enough,*

(1) $\|\partial_x f_n\|_{L^2}^2 \ge \kappa$, *if $\eta > 0$ and $\zeta \ge 0$.*
(2) $\|\partial_x g_n\|_{L^2}^2 \ge \delta$, *if $\eta \ge 0$ and $\zeta > 0$.*

Proof. Suppose (1) is false. Then there exists a subsequence $\{f_{n_k}\}$ such that $\lim_{n\to\infty} \|\partial_x f_{n_k}\|_{L^2}^2 = 0$. Since (f_n) is bounded in $H^1(\mathbb{R})$ by Lemma 3.1, then from Gagliardo-Nirenberg inequality we have that

$$\|f_{n_k}\|_{L^6}^6 \le C\|\partial_x f_{n_k}\|_{L^2}^2 \|f_{n_k}\|_{L^2}^4 \to 0 \text{ if } n \to \infty,$$

and this contradicts Lemma 3.2. Arguing as in the proof of (1), we obtain (2). This completes the proof. □

The proof of the following lemma can be found in [7, Lemma 3.6].

Lemma 3.6. *Let $\eta > 0$. Denote $E_1(f) = E(f, 0)$ and consider $J_1(\eta)$ given by*

$$J_1(\eta) = \inf\big\{E_1(f) : f \in W(\mathbb{R}) textand \|f\|_{L^2}^2 = \eta\big\}. \tag{3.5}$$

Let $\{f_n\}$ be a sequence in $W(\mathbb{R})$ such that $\lim_{n\to\infty} \|f_n\|_{L^2}^2 = \eta$ and $\lim_{n\to\infty} E_1(f) = J_1(\eta)$, then there exists a real number θ, a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a sequence $\{y_k\}$ in $\mathbb{R}$ such that $\{e^{i\theta} f_{n_k}(\cdot + y_k)\}$ converges strongly to $\phi_\eta(x)$ in $W(\mathbb{R})$, where

$$\phi_\eta(x) := \big(\frac{1}{\pi}\big)^{1/4} \eta^{1/2} e^{-x^2/2}, \quad x \in \mathbb{R}. \tag{3.6}$$

In particular, $\|\phi_\eta\|_{L^2}^2 = \eta$ and $J_1(\eta) = E_1(\phi_\eta)$.

Recall that given a measurable function $h : [0,\infty) \to \mathbb{R}$, h^* denotes the symmetric-decreasing rearrangement of h,

$$h^*(x) = \int_0^\infty \chi^*_{\{|h|>t\}}(x)\,dt,$$

where $\chi^*_{\{|h|>t\}}$ denotes the characteristic function of a ball of volume $m(\{x : |h(x)| > t\})$ centered at the origin. Here, m is the Lebesgue measure. The following lemma shows that when f and g are replaced by $|f^*|$ and $|g^*|$ we have that $E(|f^*|, |g^*|)$ decreases.

Lemma 3.7. *If $(f, g) \in \mathcal{B}$, then $(|f|^*, |g|^*) \in \mathcal{B}$ and*

$$E(|f|^*, |g|^*) \le E(|f|, |g|) \le E(f, g).$$

The proof of Lemma 3.7 can be found in [7, Lemma 3.7].

Lemma 3.8. *Let $\eta, \zeta \geq 0$ given, then there exist a minimizing sequence $\{(f_n, g_n)\}$ for $J(\eta, \zeta)$ such that the functions f_n and g_n belong to $H^1(\mathbb{R})$, are even, non-increasing and non-negative for each n and $x \geq 0$, and satisfy the condition $\|f_n\|_{L^2}^2 = \eta$, $\|g_n\|_{L^2}^2 = \zeta$.*

The previous result can be proved in almost the same way as [7, Lemma 3.8]. We omit it here. The following lemma states the strict sub-additivity of $J(\eta, \zeta)$. For a proof see [2, Lemma 2.10].

Lemma 3.9. *Suppose that $u, v : \mathbb{R} \to [0, \infty)$ are even, C^∞ functions with compact support in $\mathbb{R}$, which are non-increasing on $\{x : x \geq 0\}$. Let a_1 and a_2 be numbers such that $u(x + a_1)$ and $v(x + a_2)$ have disjoint supports, and define*

$$w(x) = u(x + a_1) + v(x + a_2).$$

Let $w^\star : \mathbb{R} \to \mathbb{R}$ be the symmetric decreasing rearrangement of w. Then the distributional derivative $(w^\star)'$ of $w^\star$ is in L^2 and satisfies

$$\|(w^\star)'\|^2 \leq \|w'\|^2 - \frac{3}{4} \min\{\|u'\|^2, \|v'\|^2\}.$$

The following two lemmas are central to show that dichotomy cannot occur.

Lemma 3.10. *For each minimizing sequence $\{(f_n, g_n)\}$ of $J(\eta, \zeta)$, there exists a $(\mu_1, \mu_2) \in [0, \eta] \times [0, \zeta]$ such that $\gamma = \eta + \zeta$ and*

$$J(\mu_1, \mu_2) + J(\eta - \mu_1, \zeta - \mu_2) \leq J(\eta, \zeta). \tag{3.7}$$

Proof. Let $\phi \in C_0^\infty[-2, 2]$ be such that ϕ is identically 1 on $[-1, 1]$, and let $\psi \in C^\infty(\mathbb{R})$ be such that $\phi^2 + \psi^2 = 1$ on $\mathbb{R}$. Define $\phi_r(x) = \phi(\frac{x}{r})$ and $\psi_r(x) = \psi(\frac{x}{r})$, for $r > 0$. From the definition of γ it follows that for given $\epsilon > 0$ and for all sufficiently large r we obtain that

$$\gamma - \epsilon < \lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} (|f_n|^2 + |g_n|^2)\, dx \leq \lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-2r}^{y+2r} (|f_n|^2 + |g_n|^2)\, dx \leq \gamma.$$

We may assume that $1/r < \epsilon$, thus we may choose a positive integer N large enough so that for all $n \geq N$ we have

$$\gamma - \epsilon < \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} (|f_n|^2 + |g_n|^2)\, dx \leq \sup_{y\in\mathbb{R}} \int_{y-2r}^{y+2r} (|f_n|^2 + |g_n|^2)\, dx < \gamma + \epsilon.$$

Hence, for all $n \geq N$, we can find y_n such that

$$\int_{y_n-r}^{y_n+r} (|f_n|^2 + |g_n|^2) dx > \gamma - \epsilon \quad \text{and} \quad \int_{y_n-2r}^{y_n+2r} (|f_n|^2 + |g_n|^2) dx < \gamma + \epsilon. \tag{3.8}$$

We define

$$\begin{gathered} \big(h_n(x), l_n(x)\big) = (\phi_r(x - y_n) f_n(x), \phi_r(x - y_n) g_n(x)) \\ \big(\widetilde{h}_n(x), \widetilde{l}_n(x)\big) = (\psi_r(x - y_n) f_n(x), \psi_r(x - y_n) g_n(x)). \end{gathered}$$

It follows from Lemma 3.1 that (h_n), (l_n), $(\widetilde{h}_n)$, and $(\widetilde{l}_n)$ are bounded sequences in $L^2(\mathbb{R})$. Then, passing to a subsequence if necessary, we have $\|h_n\|_{L^2}^2 \to \mu_1$, $\|l_n\|_{L^2}^2 \to \mu_2$, $\|\widetilde{h}_n\|_{L^2}^2 \to \eta - \mu_1$, and $\|\widetilde{l}_n\|_{L^2}^2 \to \zeta - \mu_2$ as $n \to \infty$, where $(\mu_1, \mu_2) \in [0, \eta] \times [0, \zeta]$. Also,

$$\mu_1 + \mu_2 = \lim_{n\to\infty} \int_{\mathbb{R}} (|h_n|^2 + |l_n|^2) dx = \lim_{n\to\infty} \int_{\mathbb{R}} \phi_r^2 (|f_n|^2 + |g_n|^2) dx.$$

Thus by (3.8) we have $|(\mu_1+\mu_2)-\gamma|<\epsilon$. Moreover, for all n, we infer that

$$E(h_n,l_n)+E(\widetilde{h}_n,\widetilde{l}_n)\le E(f_n,g_n)+C\epsilon. \tag{3.9}$$

Indeed, first notice that

(i) $\|\partial_x\phi_r\|_{L^\infty}=\frac{1}{r}\|\partial_x\phi\|_{L^\infty}\le C\epsilon$;

(ii) Since ϕ_r is identically 1 on $|x-y_n|\le r$ and ϕ_r vanishes on $|x-y_n|\ge 2r$, then $|\phi_r|^2\log|\phi_r|^2$ vanishes on both $|x-y_n|\le r$ and $|x-y_n|\ge 2r$. Next by (3.8) we see that

$$\int_{\mathbb{R}}(|f_n|^2+|g_n|^2)|\phi_r|^2\log|\phi_r|^2dx\le 2e^{-1}\int_{r\le|x-y_n|\le 2r}(|f_n|^2+|g_n|^2)dx\le C\epsilon;$$

(iii) Also, again using (3.8) and (2.6), we have inferred that

$$\int_{\mathbb{R}}\big((\phi_r^2-\phi_r^{2(p+1)})|f_n|^{p+1}|g_n|^{p+1}\big)dx\le C\int_{r\le|x-y_n|\le 2r}(|f_n|^2+|g_n|^2)dx\le C\epsilon.$$

And so, to prove (3.9), we write

$$\begin{aligned}
2E(h_n,l_n)&=\int_{\mathbb{R}}\phi_r^2\Big(|\partial_xf_n|^2+|\partial_xg_n|^2-|f_n|^2\log|f_n|^2-|g_n|^2\log|g_n|^2\\
&\quad-\frac{2}{p+1}|f_n|^{p+1}|g_n|^{p+1}\Big)dx+\int_{\mathbb{R}}2\phi_r\partial_x\phi_r\big(\mathrm{Re}f_n\overline{\partial_x}f_n+\mathrm{Re}g_n\overline{\partial_x}g_n\big)dx\\
&\quad+\int_{\mathbb{R}}(|f_n|^2+|g_n|^2)\big((\partial_x\phi_r)^2-|\phi_r|^2\log|\phi_r|^2\big)dx\\
&\quad+\frac{2}{p+1}\int_{\mathbb{R}}\big((\phi_r^2-\phi_r^{2(p+1)})|f_n|^{p+1}|g_n|^{p+1}\big)dx\\
&\le\int_{\mathbb{R}}\phi_r^2\Big(|\partial_xf_n|^2+|\partial_xg_n|^2-|f_n|^2\log|f_n|^2-|g_n|^2\log|g_n|^2\\
&\quad-\frac{2}{p+1}|f_n|^{p+1}|g_n|^{p+1}\Big)dx+C\epsilon.
\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}
2E(\widetilde{h}_n,\widetilde{l}_n)&\le\int_{\mathbb{R}}\psi_r^2\Big(|\partial_xf_n|^2+|\partial_xg_n|^2-|f_n|^2\log|f_n|^2-|g_n|^2\log|g_n|^2\\
&\quad-\frac{2}{p+1}|f_n|^{p+1}|g_n|^{p+1}\Big)dx+C\epsilon.
\end{aligned}$$

Then (3.9) follows by combining the two estimates and using $\phi_r^2+\psi_r^2=1$. Now we assume that μ_1, μ_2, $\eta-\mu_1$ and $\zeta-\mu_2$ are all positive. We define

$$\alpha_n=\frac{\sqrt{\mu_1}}{\|h_n\|_{L^2}},\quad\beta_n=\frac{\sqrt{\mu_2}}{\|l_n\|_{L^2}},\quad\nu_n=\frac{\sqrt{\eta-\mu_1}}{\|\widetilde{h}_n\|_{L^2}},\quad\vartheta_n=\frac{\sqrt{\zeta-\mu_2}}{\|\widetilde{l}_n\|_{L^2}}.$$

Then α_n, β_n ν_n, and ϑ_n tend to 1 as $n\to\infty$ and

$$\|\alpha_nh_n\|_{L^2}^2=\mu_1,\quad\|\beta_nl_n\|_{L^2}^2=\mu_2,\quad\|\nu_n\widetilde{h}_n\|_{L^2}^2=\eta-\mu_1,\quad\|\vartheta_n\widetilde{l}_n\|=\zeta-\mu_2.$$

Thus

$$\liminf_{n\to\infty}\big\{E(h_n,l_n)+E(\widetilde{h}_n,\widetilde{l}_n)\big\}\ge J(\mu_1,\mu_2)+J(\eta-\mu_1,\zeta-\mu_2).$$

On the other hand, if $\mu_1=0$ and $\mu_2>0$, then $\|h_n\|_{L^2}\to 0$ and from Gagliardo-Nirenberg inequality we infer that

$$\|h_n\|_{L^{p+1}}^{p+1}\le C\|\partial_xh_n\|_{L^2}^{\frac{p-1}{2}}\|h_n\|_{L^2}^{\frac{p+1}{2}}\to 0.$$

Thus $\int_{\mathbb{R}} |h_n|^{p+1}|l_n|^{p+1}dx \to 0$. By (2.3), $\int_{\mathbb{R}} \Psi(|h_n|)dx \to 0$. Since $\Phi \geq 0$ we have

$$\begin{aligned}\lim_{n\to\infty} E(h_n, l_n) &= \lim_{n\to\infty} \int_{\mathbb{R}} (|\partial_x h_n|^2 + |\partial_x l_n|^2 + \Phi(|h_n|) - |l_n|^2 \log |l_n|^2)dx \\ &\geq \liminf_{n\to\infty} (|\partial_x l_n|^2 - |l_n|^2 \log |l_n|^2) \\ &\geq J(0, \mu_2),\end{aligned}$$

and similar estimates hold if μ_2, $\eta - \mu_1$ or $\zeta - \mu_2$ are zero. Thus, in all cases we have that the limit inferior as $n \to \infty$ of the left-hand side of (3.9) is greater than or equal to $J(\mu_1, \mu_2) + J(\eta - \mu_1, \zeta - \mu_2)$. We now take the limit inferior of the left-hand side and the limit of the right-hand side of (3.9) as $n \to \infty$ to obtain

$$J(\mu_1, \mu_2) + J(\eta - \mu_1, \zeta - \mu_2) \leq J(\eta, \zeta) + C\epsilon,$$

hence $J(\mu_1, \mu_2) + J(\eta - \mu_1, \zeta - \mu_2) \leq J(\eta, \zeta)$, since ϵ is arbitrary. □

Lemma 3.11. *For any $\eta_1, \zeta_1, \eta_2, \zeta_2 \geq 0$ such that $(\eta_1, \zeta_1), (\eta_2, \zeta_2) \neq (0, 0)$, we have*

$$J(\eta_1 + \zeta_2, \eta_1 + \zeta_2) < J(\eta_1, \zeta_1) + J(\eta_2, \zeta_2).$$

Proof. Let $(f_n^{(k)}, g_n^{(k)})$ be the minimizing sequences of $J(\eta_k, \zeta_k)$ given by Lemma 3.8, with $k = 1, 2$. Then, for each n, we can choose x_n such that both $f_n^{(1)}$ and $h_n(x) := f_n^{(2)}(x + x_n)$ as $g_n^{(1)}$ and $\widetilde{h}_n(x) := g_n^{(2)}(x + x_n)$ are disjointly supported. We define

$$u_n := (f_n^1 + h_n)^* \quad \text{and} \quad v_n := (g_n^1 + \widetilde{h}_n)^*.$$

Then $\|u_n\|_{L^2}^2 = \eta_1 + \eta_2$ and $\|v_n\|_{L^2}^2 = \zeta_1 + \zeta_2$, so

$$J(\eta_1 + \zeta_2, \eta_1 + \zeta_2) \leq E(u_n, v_n). \tag{3.10}$$

Now, from Lemma 3.9 we have

$$\begin{aligned}&\int_{\mathbb{R}} (\partial_x u_n^2 + \partial_x v_n^2)dx \\ &\leq \int_{\mathbb{R}} \{(\partial_x f_n^{(1)})^2 + (\partial_x h_n)^2 + (\partial_x g_n^{(1)})^2 + (\partial_x \widetilde{h}_n)^2\}dx - T_n,\end{aligned} \tag{3.11}$$

where

$$T_n = \frac{3}{4}(\min\{\|\partial_x f_n^{(1)}\|_{L^2}^2, \|\partial_x h_n\|_{L^2}^2\} + \min\{\|\partial_x g_n^{(1)}\|_{L^2}^2, \|\partial_x \widetilde{h}_n\|_{L^2}^2\}). \tag{3.12}$$

Furthermore, from [17, Theorem 3.4] and the properties of rearrangements (see Lemma 3.7), we have

$$\begin{aligned}\int_{\mathbb{R}} |u_n|^{p+1}|v_n|^{p+1}dx &\geq \int_{\mathbb{R}} |f_n^{(1)}|^{p+1}|g_n^{(1)}|^{p+1}dx + \int_{\mathbb{R}} |h_n|^{p+1}|\widetilde{h}_n|^{p+1}dx \\ \int_{\mathbb{R}} |u_n|^2 \log |u_n|^2 dx &= \int_{\mathbb{R}} |f_n^{(1)}|^2 \log |f_n^{(1)}|^2 dx + \int_{\mathbb{R}} |h_n|^2 \log |h_n|^2 dx \\ \int_{\mathbb{R}} |v_n|^2 \log |v_n|^2 dx &= \int_{\mathbb{R}} |g_n^{(1)}|^2 \log |g_n^{(1)}|^2 dx + \int_{\mathbb{R}} |\widetilde{h}_n|^2 \log |\widetilde{h}_n|^2 dx,\end{aligned}$$

hence, combining this with (3.10) and (3.11), we have

$$J(\eta_1 + \eta_2, \zeta_1 + \zeta_2) \leq E(u_n, v_n) \leq E(f_n^{(1)}, g_n^{(1)}) + E(h_n, \widetilde{h}_n) - T_n$$

for every n, and taking the limit superior on the right-hand side, we obtain

$$J(\eta_1 + \eta_2, \zeta_1 + \zeta_2) \leq J(\eta_1, \zeta_1) + J(\eta_2, \zeta_2) - \liminf_{n\to\infty} T_n. \tag{3.13}$$

Since $\zeta_1+\zeta_2>0$, then either ζ_1 and ζ_2 are both positive, or $\zeta_1=0$ and $\zeta_2>0$ or $\zeta_1>0$ and $\zeta_2=0$. As noted in [2, Lemma 2.12], it is sufficient to consider three cases:

(1) Suppose that $\zeta_1>0$ and $\zeta_2>0$. By Lemma 3.5 there exist κ_1 and κ_2 such that $\|\partial_x g_n^{(1)}\|_{L^2}^2\ge\kappa_1$ and $\|\partial_x\widetilde{h}_n\|_{L^2}^2\ge\kappa_2$, for all sufficiently large n. Let $\kappa=\min\{\kappa_1,\kappa_2\}$, then, it follows from (3.12) and (3.13) that $T_n\ge 3\kappa/4$ for all sufficiently large n, and

$$J(\eta_1+\eta_2,\zeta_1+\zeta_2)\le J(\eta_1,\zeta_1)+J(\eta_2,\zeta_2)-\frac{3}{4}\kappa<J(\eta_1,\zeta_1)+J(\eta_2,\zeta_2).$$

(2) Suppose that $\zeta_1=0$, $\zeta_2>0$ and $\eta_2>0$. Since $\eta_1+\zeta_1>0$, $\eta_1>0$, then $\eta_1>0$. By Lemma 3.5 there exist δ_1 and δ_2 such that $\|\partial_x f_n^{(1)}\|_{L^2}^2\ge\delta_1$ and $\|\partial_x h_n\|_{L^2}^2\ge\delta_2$, for all sufficiently large n. Let $\delta=\min\{\delta_1,\delta_2\}$, then, again from (3.12) and (3.13), we obtain that $T_n\ge 3\delta/4$ for all sufficiently large n, and

$$J(\eta_1+\eta_2,\zeta_1+\zeta_2)\le J(\eta_1,\zeta_1)+J(\eta_2,\zeta_2)-\frac{3}{4}\delta<J(\eta_1,\zeta_1)+J(\eta_2,\zeta_2).$$

(3) Suppose that $\zeta_1=0$, $\zeta_2>0$ and $\eta_2=0$. By Lemma 3.6 we have

$$J(\eta_1,0)=\inf\Big\{\frac{1}{2}\int_{\mathbb{R}}\big(|\partial_x u|+|u|^2-|u|^2\log|u|^2\big)dx: u\in W(\mathbb{R}),\ \|u\|_{L^2}^2=\eta_1>0\Big\},$$

where the minimum is achieved at $\varphi_{\eta_1}(x)=(\frac{1}{\pi})^{1/4}(\eta_1)^{1/2}e^{\frac{-x^2}{2}}$. Also,

$$J(0,\zeta_2)=\inf\Big\{\frac{1}{2}\int_{\mathbb{R}}(|\partial_x v|+|v|^2-|v|^2\log|v|^2)dx: v\in W(\mathbb{R}),\ \|v\|_{L^2}^2=\zeta_2>0\Big\},$$

where the minimum is achieved at $\varphi_{\zeta_2}(x)=(\frac{1}{\pi})^{1/4}(\zeta_2)^{1/2}e^{\frac{-x^2}{2}}$. Therefore,

$$\begin{aligned}
J(\eta_1+\eta_2,\zeta_1+\zeta_2)&\le E(\varphi_{\eta_1},\varphi_{\zeta_2})+Q(\varphi_{\eta_1},\varphi_{\zeta_2})\\
&=\frac{1}{2}\int_{\mathbb{R}}\Big(|\partial_x\varphi_{\eta_1}|^2+|\varphi_{\eta_1}|^2-|\varphi_{\eta_1}|^2\log|\varphi_{\eta_1}|^2\Big)dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}}\Big(|\partial_x\varphi_{\zeta_2}|^2+|\varphi_{\zeta_2}|^2-|\varphi_{\zeta_2}|^2\log|\varphi_{\zeta_2}|^2\Big)dx\\
&\quad-\frac{2}{p+1}\int_{\mathbb{R}}|\varphi_{\eta_1}|^{p+1}|\varphi_{\zeta_2}|^{p+1}dx\\
&=J(\eta_1,0)+J(0,\zeta_2)-\frac{2}{p+1}\int_{\mathbb{R}}|\varphi_{\eta_1}|^{p+1}|\varphi_{\zeta_2}|^{p+1}dx\\
&<J(\eta_1,0)+J(0,\zeta_2).
\end{aligned}$$

The proof is complete. □

We are now ready to prove that dichotomy of minimizing sequences does not occur.

Lemma 3.12. *Let $\eta>0$ and $\zeta>0$. Then for any minimizing sequence $\{(f_n,g_n)\}$ of $J(\eta,\zeta)$, we have $\gamma=\eta+\zeta$.*

Proof. Suppose by way of contradiction that $\gamma\in(0,\eta+\zeta)$. Let η_1, ζ_2 be definite as in Lemma 3.10, and let $\eta_2=\eta-\eta_1$ and $\zeta_2=\zeta-\zeta_2$. Then $\eta_2+\zeta_2=\eta+\zeta-\gamma>0$

and $\eta_1+\zeta_1=\gamma>0$. Since $\eta_1+\eta_2=\eta>0$ and $\zeta_1+\zeta_2=\zeta>0$, in view of Lemma 3.11, we have

$$J(\eta_1+\eta_2,\zeta_1+\zeta_2)<J(\eta_1,\zeta_1)+J(\eta_2,\zeta_2),$$

but this contradicts (3.7). □

Thus, we eliminated the cases $\gamma=0$ and $0<\gamma<\eta+\zeta$, it follows that $\gamma=\eta+\zeta$. Then we can prove that, up to translation, any minimizing sequence is precompact in $\mathcal{B}$ and problem (1.5) has at least one minimizer.

Lemma 3.13. *Let $\eta>0$ and $\zeta>0$. Then, for any minimizing sequence $\{(f_n,g_n)\}$ of $J(\eta,\zeta)$, there exists a sequence of real numbers (y_n) such that, by passing to subsequence if necessary, $\{f_n(\cdot+y_n),g_n(\cdot+y_n)\}$ converges strongly in $\mathcal{B}$ to some minimizer (f,g) of $J(\eta,\zeta)$.*

Proof. By Lions' concentration compactness lemma [18, 19], since $\gamma=\eta+\zeta$ we have that, for every $k\in\mathbb{N}$, there exists λ_k such that

$$\int_{-\lambda_k}^{\lambda_k}(|f_n(x+y_n)|^2+|g_n(x+y_n)|^2)\,dx>\eta+\zeta-\frac{1}{k},$$

for all sufficiently large n. Hence, in view of the embedding $H^1_{\rm loc}(\mathbb{R})\hookrightarrow L^2_{\rm loc}(\mathbb{R})$, it follows that some further subsequence of $\{f_n(\cdot+y_n),g_n(\cdot+y_n)\}$ converges in $L^2[-\lambda_k,\lambda_k]$-norm to a limit $(f,g)\in L^2[-\lambda_k,\lambda_k]\times L^2[-\lambda_k,\lambda_k]$. Moreover,

$$\int_{-\lambda_k}^{\lambda_k}(|f|^2+|g|^2)\,dx>\eta+\zeta-\frac{1}{k}.$$

Then we apply Cantor's diagonalization process, and we obtain that $\{f_n(\cdot+y_n),g_n(\cdot+y_n)\}$ converges in $L^2(\mathbb{R})$-norm to a limit $(f,g)\in L^2(\mathbb{R})\times L^2(\mathbb{R})$ and

$$\int_{\mathbb{R}}(|f|^2+|g|^2)\,dx=\eta+\zeta.$$

Thus, $(f_n(\cdot+y_n),g_n(\cdot+y_n))\rightharpoonup(f,g)$ weakly in $\mathcal{B}$, since $\mathcal{B}$ is reflexive. Now, since the sequences $\{f_n(\cdot+y_n)\}$ and $\{g_n(\cdot+y_n)\}$ are bounded in $H^1(\mathbb{R})$, from (2.3) we have

$$\lim_{n\to\infty}\int_{\mathbb{R}}\Psi(|f_n(x+y_n)|)\,dx=\int_{\mathbb{R}}\Psi(|f(x)|)\,dx, \tag{3.14}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}}\Psi(|g_n(x+y_n)|)\,dx=\int_{\mathbb{R}}\Psi(|g(x)|)\,dx. \tag{3.15}$$

Also we have

$$\begin{aligned}&\int_{\mathbb{R}}|\partial_x f|^2\,dx+\int_{\mathbb{R}}\Phi(|f|)\,dx\\&\le\liminf_{n\to\infty}\Big\{\int_{\mathbb{R}}|\partial_x f_n(x+y_n)|^2\,dx+\int_{\mathbb{R}}\Phi(|f_n(x+y_n)|)\,dx\Big\},\end{aligned}\tag{3.16}$$

$$\begin{aligned}&\int_{\mathbb{R}}|\partial_x g|^2\,dx+\int_{\mathbb{R}}\Phi(|g|)\,dx\\&\le\liminf_{n\to\infty}\Big\{\int_{\mathbb{R}}|\partial_x g_n(x+y_n)|^2\,dx+\int_{\mathbb{R}}\Phi(|g_n(x+y_n)|)\,dx\Big\},\end{aligned}\tag{3.17}$$

because the functional $h \mapsto \int_{\mathbb{R}} \Phi(|h|)dx$, $\Phi(|h|) = \Psi(|h|) - |h|^2 \log |h|^2$, is convex and continuous on $W(\mathbb{R})$, thus weakly l.s.c on $W(\mathbb{R})$. Then, by (3.14), (3.15), (3.16), and (3.17), we have

$$E(f,g) \le \liminf_{n\to\infty} E(f_n, g_n) = J(\eta,\zeta),$$

hence $E(f,g) = J(\eta,\zeta)$ and $(f,g) \in \mathcal{G}(\eta,\zeta)$. Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} \Big\{ \int_{\mathbb{R}} (|\partial_x f_n(x+y_n)|^2 + |\partial_x g_n(x+y_n)|^2) + \Phi(|f_n(x+y_n)|) \\
&+ \Phi(|g_n(x+y_n)|)dx \Big\} \\
&= \int_{\mathbb{R}} \left(|\partial_x f|^2 + |\partial_x g|^2\right) + \Phi(|f|) + \Phi(|g|)dx.
\end{aligned}$$

Hence and in view of [10, Lemma 2.4.4] we have

$$\begin{gathered}
\lim_{n\to\infty} \int_{\mathbb{R}} |\partial_x f_n(x+y_n)|^2\, dx = \int_{\mathbb{R}} |\partial_x f|^2 dx, \\
\lim_{n\to\infty} \int_{\mathbb{R}} |\partial_x g_n(x+y_n)|^2\, dx = \int_{\mathbb{R}} |\partial_x g|^2 dx, \\
\lim_{n\to\infty} \int_{\mathbb{R}} \Phi(|f_n(x+y_n)|)dx = \int_{\mathbb{R}} \Phi(|f|)dx, \\
\lim_{n\to\infty} \int_{\mathbb{R}} \Phi(|g_n(x+y_n)|)dx = \int_{\mathbb{R}} \Phi(|g|)dx.
\end{gathered}$$

Hence and in view of Lemma 2.1 we have $f_n \to f$ and $g_n \to g$ in $L^\Phi(\mathbb{R})$. Thus, by definition of the $\mathcal{B}$-norm, we infer that $\{f_n(\cdot + y_n), g_n(\cdot + y_n)\}$ converges in the norm of $\mathcal{B}$. That is, $(f_n, g_n) \to (f,g)$ strongly in $\mathcal{B}$ as $n \to \infty$. $\square$

Based on the above lemmas we are ready for the following proof.

Proof of Theorem 1.2. (i) is an immediate consequence of Lemma 3.13.

(ii) By the Lagrange multiplier principle, if $(f,g) \in \mathcal{G}(\gamma,\mu)$, there exist real numbers ρ and ϑ such that

$$E'(f,g) = \rho f + \vartheta g,$$

where the prime denotes the Fréchet derivative. Since

$$E'(f,g) = \begin{bmatrix} -\partial_x^2 f - f\log|f|^2 - f|f|^{p-1}|g|^{p+1} \\ -\partial_x^2 g - g\log|g|^2 - g|g|^{p-1}|f|^{p+1} \end{bmatrix},$$

the equations

$$\begin{aligned}
-\partial_x^2 f + \rho f &= f\log|f|^2 + f|f|^{p-1}|g|^{p+1} \\
-\partial_x^2 g - \vartheta g &= g\log|g|^2 + g|g|^{p-1}|f|^{p+1}
\end{aligned} \tag{3.18}$$

hold at least in the sense of distributions. By a standard bootstrap argument, we see that f and g are in $C^2(\mathbb{R})$ (see [9, Chapter 8]). Thus, we may write

$$f(x) = e^{i\theta(x)}\phi(x), \quad g(x) = e^{i\omega(x)}\varphi(x),$$

where $\theta, \omega, \phi, \varphi \in C^2(\mathbb{R})$ and $\phi, \varphi \ge 0$. It remains to prove that both θ and ω are constants. By Lemma 3.7 (ϕ,φ) is a minimizer for $J(\eta,\zeta)$, then (ϕ,φ) satisfies

(3.18). On the other hand, multiplying the first equation by $\overline{f}$ and the second by $\overline{g}$ in (3.18) and integrating by parts over $\mathbb{R}$ we have

$$\begin{aligned}\rho &= -\frac{1}{\eta}\int_{\mathbb{R}}(|\partial_x f|^2 - |f|^2\log|f|^2 - |f|^{p+1}|g|^{p+1})dx\\ \vartheta &= -\frac{1}{\zeta}\int_{\mathbb{R}}(|\partial_x g|^2 - |g|^2\log|f|^2 - |f|^{p+1}|g|^{p+1})dx.\end{aligned}$$

Hence, and $E(f,g)=E(\phi,\varphi)$, it follows that

$$\begin{aligned}&\int_{\mathbb{R}}(|\partial_x\phi|^2 - |\phi|^2\log|\phi|^2 - |\phi|^{p+1}|g|^{p+1})dx\\ &= \int_{\mathbb{R}}(|\partial_x f|^2 - |f|^2\log|f|^2 - |f|^{p+1}|g|^{p+1})dx.\end{aligned} \tag{3.19}$$

Then computing the second derivative of f we have

$$\begin{aligned}\partial_x^2 f = e^{i\theta(x)}\Big(&\rho\phi - |\phi|^2\log|\phi|^2 - |\phi|^{p+1}|g|^{p+1}\Big)\\ &- (\partial_x\theta)^2\phi + 2i\partial_x\theta\partial_x\phi + i\phi\partial_x^2\theta,\end{aligned} \tag{3.20}$$

and by the first equality in (3.18) we infer that

$$\partial_x^2 f = \rho e^{i\theta}\phi - e^{i\theta}|\phi|^2\log|\phi|^2 - e^{i\theta}|\phi|^{p+1}|g|^{p+1}. \tag{3.21}$$

Thus by (3.20) and (3.21) we have

$$(\partial_x\theta)^2\phi + 2i\partial_x\theta\partial_x\phi + i\phi\partial_x^2\theta = 0, \quad \text{for all } x\in\mathbb{R}. \tag{3.22}$$

Similarly, we obtain that

$$(\partial_x\omega)^2\varphi + 2i\partial_x\omega\partial_x\varphi + i\varphi\partial_x^2\omega = 0, \quad \text{for all } x\in\mathbb{R}. \tag{3.23}$$

Thus since $\theta,\omega\in C^2(\mathbb{R})$, then, taking the real part of (3.22) and (3.23), we have $\partial_x\theta(x)=0$ and $\partial_x\omega(x)=0$, Hence θ and ω are constant, say $\theta=\theta_0$ and $\omega=\omega_0$, thus $f(x)=e^{i\theta_0}\phi(x)$ and $g(x)=e^{i\omega_0}\varphi(x)$ for all $x\in\mathbb{R}$. Finally, defining $\tau(s)=\rho s - s\log s^2$, we have

$$\partial_x^2\phi = \rho\phi - \phi\log\phi^2 - \phi^{p+1}|g|^{p+1} \le \tau(\phi) \quad \text{on } \mathbb{R}.$$

Notice that τ is positive, continues, non-decreasing near zero, $\tau(0)=0$ and $\tau(e^{\frac{\rho}{2}})=0$. Thus in view of [24, Theorem 1] we have $\phi=0$ or $\phi>0$. But $\|f\|_{L^2}^2=\eta>0$, then $\phi>0$. Similarly, we obtain that $\varphi>0$ for all $x\in\mathbb{R}$.

(iii) It is trivial from the definition of true two-parameter family. This completes of proof. □

Proof of Corollary 1.3. Let us suppose that $\mathcal{G}(\eta,\zeta)$ is not $\mathcal{B}$-stable. For each $k\in\mathbb{N}$ there exist initial data $(f_{0,k},g_{0,k})$ and $(t_k)\subset[0,+\infty)$ such that for some $\epsilon>0$ and all k we have

$$\inf\{\|(f_{0,k},g_{0,k})-(\varphi,\phi)\|_{\mathcal{B}} : (\varphi,\phi)\in\mathcal{G}(\eta,\zeta)\} < \frac{1}{k}$$

and

$$\inf\{\|(f_k(t_k),g_k(t_k))-(\varphi,\phi)\|_{\mathcal{B}} : (\varphi,\phi)\in\mathcal{G}(\eta,\zeta)\} \ge \epsilon, \tag{3.24}$$

where $(f_k(t),g_k(t))$ denotes the solution of (1.1) with initial data $(f_{0,k},g_{0,k})$. Since $f_{0,k}\xrightarrow{k}\varphi$ and $g_{0,k}\xrightarrow{k}\phi$ in $\mathcal{B}$ and $Q(\varphi,\phi)=\eta+\zeta$, we have

$$\lim_{k\to\infty}Q(f_{0,k},g_{0,k})=\eta+\zeta \quad\text{and}\quad \lim_{k\to\infty}E(f_{0,k},g_{0,k})=J(\eta,\zeta). \tag{3.25}$$

Since E and Q are independent of t, by (3.25) we have

$$\lim_{k\to\infty} Q(f_k(t_k), g_k(t_k)) = \eta + \zeta \quad \text{and} \quad \lim_{k\to\infty} E(f_k(t_k), g_k(t_k)) = J(\eta, \zeta).$$

Hence, $\{(f_k(t_k), g_k(t_k))\}$ is a minimizing sequence for $J(\eta, \zeta)$ and by Theorem 1.2, up to a subsequence, there exist a sequence $\{y_k\}$ in $\mathbb{R}$ and functions $(\nu, \psi) \in \mathcal{G}(\eta, \zeta)$ such that

$$\lim_{k\to\infty} \|(f_k(\cdot + y_k, t_k), g_k(\cdot + y_k, t_k)) - (\nu, \psi)\|_{\mathcal{B}} = 0$$

Since $(\nu(\cdot - y_k), \psi(\cdot - y_k)) \in \mathcal{G}$, we have

$$\inf\{\|(f_k(t_k), g_k(t_k)) - (\varphi, \phi) : (\varphi, \phi) \in \mathcal{G}(\eta, \zeta)\|_{\mathcal{B}}\} < \epsilon$$

for all sufficiently large k, which is a contradiction with (3.24). This completes the proof. $\square$

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