

EXISTENCE OF GLOBAL SOLUTIONS FOR CROSS-DIFFUSION MODELS IN A FRACTIONAL SETTING

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ABSTRACT. This article is devoted to the analysis of a fractional chemotaxis model in \mathbb{R}^N with a time fractional variation in the Caputo sense and a fractional spatial diffusion. This model encompasses the fractional Keller-Segel system [9] which describes the movement of living organisms towards higher concentration regions of chemical attractants, and a fractional Lotka-Volterra competition model [16] describing the competition interspecies in which one of the competing species avoids encounters with rivals by means of chemorepulsion. We prove product estimates in Besov-Morrey spaces and derive global estimates for mild solutions of the fractional heat equation. We use these results to prove the existence and uniqueness of global mild solutions for the differential system in a framework of Besov-Morrey spaces.

1. INTRODUCTION

We consider a generic fractional chemotaxis model describing the evolution of two species subject to attraction and repulsion phenomena. This model includes the fractional Keller-Segel system which describes the movement of living organisms towards higher concentration regions of chemical attractants [9], and a fractional Lotka-Volterra competition model describing the competition interspecies to avoid encounters with rivals by means of chemorepulsion mechanism [16]. This model is composed of three coupled parabolic equations describing the interaction between the densities of competing species and the concentration of a chemical substance, which reads as follows

$$\begin{aligned} {}^c D_t^\alpha n + d_n (-\Delta)^{\theta/2} n &= \chi \nabla \cdot (nG(v)) + a_1 n^2 - b_1 n m, \\ {}^c D_t^\alpha m + d_m (-\Delta)^{\theta/2} m &= a_2 m^2 - b_2 n m, \\ {}^c D_t^\alpha v + d_v (-\Delta)^{\theta/2} v &= a_3 m + b_3 n - \gamma v, \\ n(x, 0) = n_0(x), \quad m(x, 0) &= m_0(x), \quad v(x, 0) = v_0(x), \end{aligned} \tag{1.1}$$

where the unknowns are $n = n(x, t)$ and $m = m(x, t)$, for $x \in \mathbb{R}^N$ and $t \in (0, \infty)$, which denote the densities of competing species, while $v = v(x, t)$ denotes the chemical signal. In (1.1), d_n, d_m , and d_v are positive parameters representing the

2020 *Mathematics Subject Classification.* 35K55, 35Q35, 35Q92, 92C17.

Key words and phrases. Keller-Segel system; Lotka-Volterra; cross-diffusion; Besov-Morrey spaces.

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Submitted August 6, 2022. Published November 10, 2023.

diffusion coefficients, $a_1, a_2 \in \mathbb{R}$ denote growth coefficients ($a_1, a_2 > 0$) or self-competition ($a_1, a_2 < 0$), a_3, b_1, b_2, b_3 and γ are positive parameters related to the population dynamics and the strength of competition, and χ denotes the chemotaxis coefficient. In (1.1) ${}^c D_t^\alpha$ denotes the time fractional derivative operator of order $\alpha \in (0, 1)$ in the Caputo sense. We recall that for $f \in C([0, T]; X)$, $0 < T \leq \infty$, such that $I_t^{1-\alpha} f \in W^{1,1}(0, T; X)$, the Caputo fractional derivative of order α of f is defined by

$${}^c D_t^\alpha f(t) := \frac{d}{dt} \{ I_t^{1-\alpha} [f(t) - f(0)] \} = \frac{d}{dt} \left\{ \int_0^t (t - \tau)^{-\alpha} [f(\tau) - f(0)] d\tau \right\},$$

where $I_t^\alpha f$ denotes the Riemann-Liouville fractional integral of order α of f , defined by

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t \in [0, T].$$

In addition, in (1.1), $(-\Delta)^{\theta/2}$, $\theta \in (0, 2]$, denotes the fractional Laplacian operator of order $\theta/2$ defined by $(-\Delta)^{\theta/2} f(x) = \mathcal{F}^{-1}(|\xi|^\theta \hat{f}(\xi))(x)$, where $\hat{f}(\xi) = \mathcal{F}(f)(\xi)$ and $\mathcal{F}^{-1}(f)(\xi)$ denote the Fourier transform and the inverse Fourier transform of f , respectively. Finally, $G(v)$ is also a nonlocal term defined by

$$G(v)(x) = \nabla((-\Delta)^{-\theta_1/2} v)(x), \quad x \in \mathbb{R}^N,$$

for $\theta_1 \in [0, N)$, which can be alternatively represented by $G(v) = K(x) * v$, $K(x) \sim \frac{x}{|x|^{N-\theta_1}}$.

If we take $\chi < 0$, $a_1 = 0$ and $m = 0$ in (1.1), we obtain the fractional version of the classical Keller-Segel system describing the movement of living organisms n towards higher concentration regions of chemical attractants v (cf. [1]). On the other hand, if we consider $\chi > 0$ and $b_3 = 0$ in (1.1), we obtain a fractional Lotka-Volterra competition model describing the competition interspecies n and m in which one of the competing species avoids encounters with rivals by means of chemorepulsion mechanism caused by the chemical signal v . In addition, taking $\alpha = 1$, $\theta = 2$ and $\theta_1 = 0$ in (1.1) we formally obtain, as particular cases, the classical Keller-Segel system (cf. [9]) and the non-fractional Lotka-Volterra model (cf. [16]).

The fractional population model (1.1) is justified by the nonlocal behavior of the dynamics of the organisms. In fact, in several situations found in nature, organisms develop alternative search strategies, particularly when chemoattractants, food, or other targets are sparse or rare. Then, as pointed out in [5, 10], a good description of the trajectories of the population of organisms can be performed by using the so called Lévy flights in place of Brownian motion. We recall that Lévy flights have been considered in numerous biological contexts, including immune cells, ecology and human populations (see [6] and references for a deeper discussion). This consideration motivates the substitution of the classical diffusion in system (1.1) by a fractional diffusion. On the other hand, regarding the flux by chemotaxis (both attractive and repulsive), it is also relevant to consider the case where the attraction-repulsion source is replaced by a less singular interaction kernel. This last consideration has been pointed out in the analysis of the propagation of chaos for some aggregation-diffusion models [15]. Finally, taking into account that the behavior of most biological systems has memory properties, which are neglected when an integer-order time derivative is assumed, it is justified to consider a time

variation in a fractional framework, which introduces a nonlocal delay in time for the moving population [7].

The aim of this article is to analyze the existence and uniqueness of global solutions for the space-time fractional system (1.1) in the framework of critical Besov-Morrey spaces. To show the existence of global solutions of system (1.1) we first prove some product estimates in Besov-Morrey spaces and derive global estimates for the solutions of the linear fractional heat equation, which are key to apply the iterative contraction method. To the best of our knowledge, the complete fractional model (1.1) has not been analyzed in the literature. Some global existence results and long time behavior of solutions for the particular case of (1.1) assuming $\theta = 2$, $\theta_1 = 0$, $\chi < 0$, $a_1 = 0$, $m = 0$, and $\alpha \in (0, 1)$, with small initial data in a different class of Besov-Morrey initial data, were obtained in [1]. Recently, some global existence results of the fractional chemotaxis-Navier-Stokes system with consumption, in the framework of Morrey spaces were obtained in [13]. The plan of this paper is as follows. In Section 2, we give preliminaries and state our main results. In Section 3, we prove some necessary lemmas and estimates in order to handle the system in our setting, and finally, in Section 4, we prove our existence and uniqueness result.

2. FUNCTIONAL SETTING AND MAIN RESULTS

In this section we recall some preliminary results related to Morrey and homogeneous Besov-Morrey spaces. Furthermore, we establish some essential linear and nonlinear estimates in our framework, including the continuity of the paraproduct and the Bony decomposition. For more details of Morrey spaces and homogeneous Besov-Morrey spaces the reader can see [8, 11, 14, 17]. Throughout this article, we denote by \mathcal{S} and \mathcal{S}' the Schwartz class and the set of tempered distributions over \mathbb{R}^N , respectively. As usual, \mathcal{F} and \mathcal{F}^{-1} denote the Fourier transform and its inverse, respectively. We also denote by $\mathcal{D}(\Omega)$ the set of C^∞ -functions with compact support defined in $\Omega \subseteq \mathbb{R}^N$.

Definition 2.1. Let $1 \leq p \leq \infty$ and $0 \leq \lambda < N$. The Morrey space $\mathcal{M}_{p,\lambda}(\mathbb{R}^N)$ is defined by the set

$$\mathcal{M}_{p,\lambda}(\mathbb{R}^N) = \{f \in L^p_{loc}(\mathbb{R}^N) : \|f\|_{\mathcal{M}_{p,\lambda}} := \sup_{x_0 \in \mathbb{R}^N} \sup_{R > 0} R^{-\lambda/p} \|f\|_{L^p(B(x_0, R))} < \infty\},$$

where $B(x_0, R)$ denotes the open ball in \mathbb{R}^N with center x_0 and radius R .

Note that $\mathcal{M}_{p,0} = L^p$ and $\mathcal{M}_{\infty,\lambda} = L^\infty$. Also, $L^p \subset L^{(p,\infty)} \subset \mathcal{M}_{l,\lambda}$ for all $0 < \lambda < N$ and $1 \leq l \leq p \leq \infty$ with $\frac{N}{p} = \frac{N-\lambda}{l}$, where $L^{(p,\infty)}$ denotes the weak- L^p space. In addition, Hölder and Young inequalities hold true in Morrey spaces. That is, if $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, and $0 \leq \lambda_i < N$, $i = 1, 2, 3$, then

$$\|fg\|_{\mathcal{M}_{p_3,\lambda_3}} \leq \|f\|_{\mathcal{M}_{p_1,\lambda_1}} \|g\|_{\mathcal{M}_{p_2,\lambda_2}},$$

and

$$\|f * g\|_{\mathcal{M}_{p,\lambda}} \leq \|f\|_{L^1} \|g\|_{\mathcal{M}_{p,\lambda}}, \quad \text{for } 0 \leq \lambda < N \text{ and } 1 \leq p \leq \infty. \quad (2.1)$$

Next, we recall the Bernstein inequality in $\mathcal{M}_{p,\lambda}$ -spaces. Let $\mathcal{C} = \mathcal{C}(0, R_1, R_2) = \{x \in \mathbb{R}^N : R_1 \leq |x| \leq R_2\}$ and $B = B(0; R) = \{x \in \mathbb{R}^N : |x| \leq R\}$, for $0 < R_1 < R_2$ and $R > 0$. From now on, we denote $\text{supp } f$ the support of f .

Lemma 2.2 ([18, Lemma 2.1]). *Let k be a nonnegative integer, $1 \leq p \leq \infty$ and $0 \leq \lambda < N$.*

(i) *If $f \in \mathcal{M}_{p,\lambda}(\mathbb{R}^N)$ and $\text{supp } \mathcal{F}f \subset \tau B(0; R)$ for some $\tau > 0$ and $R > 0$, then*

$$\sup_{|\alpha|=k} \|\partial^\alpha f\|_{\mathcal{M}_{p,\lambda}} \leq C^{k+1} \tau^k \|f\|_{\mathcal{M}_{p,\lambda}}, \tag{2.2}$$

for some positive constant C depending only on p, λ, R and N .

(ii) *If $f \in \mathcal{M}_{p,\lambda}(\mathbb{R}^N)$ and $\text{supp } \mathcal{F}f \subset \tau \mathcal{C}$ for some $\tau > 0$, then*

$$C^{-k-1} \tau^k \|f\|_{\mathcal{M}_{p,\lambda}} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{\mathcal{M}_{p,\lambda}} \leq C^{k+1} \tau^k \|f\|_{\mathcal{M}_{p,\lambda}}, \tag{2.3}$$

for some positive constant C depending only on p, λ, R_1, R_2 and N .

Definition 2.3. Let \mathcal{C} be the annulus $\{\xi \in \mathbb{R}^N : 3/4 \leq |\xi| \leq 8/3\}$. Consider the radial functions $\phi_1 \in \mathcal{D}(B(0, 4/3))$ and $\phi_2 \in \mathcal{D}(\mathcal{C})$ valued in the interval $[0, 1]$ and such that

$$\phi_1(\xi) + \sum_{j \geq 0} \phi_2(2^{-j}\xi) = 1, \forall \xi \in \mathbb{R}^N, \tag{2.4}$$

$$\sum_{j \in \mathbb{Z}} \phi_2(2^{-j}\xi) = 1, \forall \xi \in \mathbb{R}^N \setminus \{0\}. \tag{2.5}$$

For each $j \in \mathbb{Z}$, the homogeneous dyadic block $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operators \dot{S}_j are defined as

$$\dot{\Delta}_j f(x) = \mathcal{F}^{-1}(\phi_2(2^{-j}\xi)\mathcal{F}(f)(\xi)), \quad \dot{S}_j f(x) = \mathcal{F}^{-1}(\phi_1(2^{-j}\xi)\mathcal{F}(f)(\xi)).$$

We also denote $\tilde{\varphi}_j(\xi) = \phi_2(2^{-(j-1)}\xi) + \phi_2(2^{-j}\xi) + \phi_2(2^{-(j+1)}\xi)$ and $\tilde{\mathcal{C}}_j = \mathcal{C}_{j-1} \cup \mathcal{C}_j \cup \mathcal{C}_{j+1}$, where $j \in \mathbb{Z}$ and $\mathcal{C}_j = 2^j \mathcal{C}$. Note that $\tilde{\varphi}_j = 1$ in \mathcal{C}_j .

Definition 2.4. Let $\mathcal{S}'_h = \{f \in \mathcal{S}' : \lim_{j \rightarrow -\infty} \dot{S}_j f = 0\}$. For s a real number, $1 \leq p, q \leq \infty$ and $0 \leq \lambda < N$, the homogeneous Besov-Morrey space $\mathcal{N}^s_{p,\lambda,q}$ is the Banach space of all distributions f in \mathcal{S}'_h such that

$$\|f\|_{\mathcal{N}^s_{p,\lambda,q}} := \|2^{js} \|\dot{\Delta}_j f\|_{\mathcal{M}_{p,\lambda}}\|_{\ell^q(\mathbb{Z})} < \infty.$$

Definition 2.5. For $s \in \mathbb{R}$, $1 \leq p, q, r \leq \infty$ and $0 \leq \lambda < N$, the Banach space $\widetilde{L}^r([0, \infty); \mathcal{N}^s_{p,\lambda,q})$ is defined as the set of all tempered distributions f over $\mathbb{R}^N \times (0, \infty)$ with $\lim_{j \rightarrow -\infty} \dot{S}_j f = 0$ in $L^r([0, \infty); L^\infty(\mathbb{R}^N))$ and such that

$$\|f\|_{\widetilde{L}^r([0, \infty); \mathcal{N}^s_{p,\lambda,q})} := \|2^{js} \|\dot{\Delta}_j f\|_{L^r([0, \infty); \mathcal{M}_{p,\lambda})}\|_{\ell^q} < \infty. \tag{2.6}$$

The following lemma corresponds to a Bernstein-type inequality in Morrey spaces. We omit its proof here because it is similar to the proof of Lemma 2.6 in [12]. In fact, it is sufficient to note that for $f \in \mathcal{S}'$ with $\text{supp } \hat{f} \subset 2^j \mathcal{C}$ we have $\mathcal{F}(f) = \tilde{\varphi}_j(\xi)\mathcal{F}(f)$, this is, $f = 2^{jn}(\tilde{\varphi}_0)(2^j \cdot) * f$, and the function $2^{jn}(\tilde{\varphi}_0)(2^j \cdot)$ and heat kernel $g(x, 2^{-j})$ have a similar behavior.

Lemma 2.6 (Bernstein-type inequality). *If $1 \leq p \leq r \leq \infty$ and $0 \leq \lambda < N$, then*

$$\|f\|_{\mathcal{M}_{r,\lambda}} \leq C 2^{j(\frac{N-\lambda}{p} - \frac{N-\lambda}{r})} \|f\|_{\mathcal{M}_{p,\lambda}},$$

for all $f \in \mathcal{M}_{p,\lambda}$ such that $\text{supp } \hat{f} \subset 2^j \mathcal{C}$.

To establish the main existence result, we start by recalling the mild formulation of (1.1) in the fractional setting. System (1.1) is formally equivalent to the following integral formulation (from now on, without loss of generality we assume $d_n, d_m, d_v = 1$):

$$\begin{aligned}
 n(t) &= E_\alpha(-t^\alpha(-\Delta)^{\theta/2})n_0 \\
 &\quad + \chi \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-(t-\tau)^\alpha(-\Delta)^{\theta/2}) \nabla \cdot (nG(v))(\tau) d\tau \\
 &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-(t-\tau)^\alpha(-\Delta)^{\theta/2})(a_1n^2 - b_1nm)(\tau) d\tau, \\
 m(t) &= E_\alpha(-t^\alpha(-\Delta)^{\theta/2})m_0 \\
 &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-(t-\tau)^\alpha(-\Delta)^{\theta/2})(a_2m^2 - b_2mn)(\tau) d\tau, \\
 v(t) &= E_\alpha(-t^\alpha((-\Delta)^{\theta/2} - \gamma))v_0 \\
 &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-(t-\tau)^\alpha((-\Delta)^{\theta/2} - \gamma))(a_3m + b_3n)(\tau) d\tau.
 \end{aligned}
 \tag{2.7}$$

Here $\{E_\alpha(\cdot)\}_{t \geq 0}$ and $\{E_{\alpha,\alpha}(\cdot)\}_{t \geq 0}$ denote the Mittag-Leffler families defined by

$$\begin{aligned}
 E_\alpha(-t^\alpha(-\Delta)^{\theta/2}) &= \int_0^\infty M_\alpha(\tau) U_\theta(\tau t^\alpha) d\tau, \\
 E_\alpha(-t^\alpha((-\Delta)^{\theta/2} + a)) &= \int_0^\infty M_\alpha(\tau) U_{\theta,a}(\tau t^\alpha) d\tau, \\
 E_{\alpha,\alpha}(-t^\alpha(-\Delta)^{\theta/2}) &= \int_0^\infty \alpha\tau M_\alpha(\tau) U_\theta(\tau t^\alpha) d\tau, \\
 E_{\alpha,\alpha}(-t^\alpha((-\Delta)^{\theta/2} + a)) &= \int_0^\infty \alpha\tau M_\alpha(\tau) U_{\theta,a}(\tau t^\alpha) d\tau,
 \end{aligned}$$

where $U_\theta(t)$ and $U_{\theta,a}(t)$ are the fractional heat semigroup and the fractional damped heat semigroup, defined as $\widehat{U_\theta(t)f} = e^{-t|\xi|^\theta} \widehat{f}$ and $U_{\theta,a}(t)f = e^{at}U_\theta(t)f$, respectively. The function $M_\alpha : \mathbb{C} \rightarrow \mathbb{C}$ is the Mainardi function defined by

$$M_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{n! \Gamma(1 - \alpha(1 + n))},$$

which verifies that $M_\alpha(\tau) \geq 0$ for all $\tau \geq 0$ and

$$\int_0^\infty \tau^r M_\alpha(\tau) d\tau = \frac{\Gamma(r + 1)}{\Gamma(1 + \alpha r)},$$

for $-1 < r < \infty$.

To analyze the initial value problem (2.7), and motivated by the intrinsic scaling of (1.1), we impose the condition $\theta_1 = 2 - \theta$ and consider the following time-dependent functional spaces. For $1 \leq p < N - \lambda$, $\frac{1}{\alpha} < \kappa$ and $s^* = \frac{N-\lambda}{p}$, we define the Banach spaces \mathcal{X}_1 and \mathcal{X}_2 by

$$\begin{aligned}
 \mathcal{X}_1 &= \widetilde{L^\infty}([0, \infty); \mathcal{N}_{p,\lambda,\infty}^{s^*-\theta}) \cap \widetilde{L^\kappa}([0, \infty); \mathcal{N}_{p,\lambda,\infty}^{s^*-\theta(1-\frac{1}{\alpha\kappa})}), \\
 \mathcal{X}_2 &= \widetilde{L^\infty}([0, \infty); \mathcal{N}_{p,\lambda,\infty}^{s^*}),
 \end{aligned}
 \tag{2.8}$$

endowed with the corresponding norms

$$\begin{aligned} \|x\|_{\mathcal{X}_1} &= \|x\|_{\widetilde{L^\infty}([0,\infty); \mathcal{N}_{p,\lambda,\infty}^{s^*-\theta})} + \|x\|_{\widetilde{L^\kappa}([0,\infty); \mathcal{N}_{p,\lambda,\infty}^{s^*-\theta(1-\frac{1}{\alpha\kappa})})}, \\ \|x\|_{\mathcal{X}_2} &= \|x\|_{\widetilde{L^\infty}([0,\infty); \mathcal{N}_{p,\lambda,\infty}^{s^*}}}. \end{aligned} \tag{2.9}$$

With this notation, we establish the notion of solution that we use.

Definition 2.7. Let $1 \leq p < N - \lambda$, $s^* = \frac{N-\lambda}{p}$, and $[n_0, m_0, v_0] \in \mathcal{N}_{p,\lambda,\infty}^{s^*-\theta} \times \mathcal{N}_{p,\lambda,\infty}^{s^*-\theta} \times \mathcal{N}_{p,\lambda,\infty}^{s^*}$. A mild solution for (1.1) is a triple $[n, m, v] \in \mathcal{X}_1 \times \mathcal{X}_1 \times \mathcal{X}_2$, satisfying the integral system (2.7).

Next, we state our main existence result.

Theorem 2.8. Let $0 \leq \lambda < N$, $\frac{1}{\alpha} < \kappa \leq \infty$, $1 < \theta$, $1 \leq p < \frac{N-\lambda}{\theta(2-\frac{1}{\alpha\kappa})}$, $s^* = \frac{N-\lambda}{p}$, $[n_0, m_0, v_0] \in \mathcal{N}_{p,\lambda,\infty}^{s^*-\theta} \times \mathcal{N}_{p,\lambda,\infty}^{s^*-\theta} \times \mathcal{N}_{p,\lambda,\infty}^{s^*}$. Then, there exists $\delta > 0$ such that if

$$\|n_0\|_{\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta}} + \|m_0\|_{\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta}} + \|v_0\|_{\mathcal{N}_{p,\lambda,\infty}^{s^*}} < \delta,$$

then problem (1.1) has a unique mild solution $[n, m, v]$ in the class

$$(C(\mathbb{R}^+; \mathcal{N}_{p,\lambda,\infty}^{s^*-\theta}) \cap \mathcal{X}_1) \times (C(\mathbb{R}^+; \mathcal{N}_{p,\lambda,\infty}^{s^*-\theta}) \cap \mathcal{X}_1) \times C(\mathbb{R}^+; \mathcal{N}_{p,\lambda,\infty}^{s^*}).$$

Remark 2.9. (1) If in (1.1) we consider $\chi < 0$, $a_1 = 0$ and $m = 0$, we obtain the fractional Keller-Segel system. Thus, Theorem 2.8 provides the existence of global solution for the fractional Keller-Segel system in the class $(C(\mathbb{R}^+; \mathcal{N}_{p,\lambda,\infty}^{s^*-\theta}) \cap \mathcal{X}_1) \times C(\mathbb{R}^+; \mathcal{N}_{p,\lambda,\infty}^{s^*})$ for small initial data $[n_0, v_0]$ in $\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta} \times \mathcal{N}_{p,\lambda,\infty}^{s^*}$.

(2) If in (1.1) we consider $\chi > 0$ and $b_3 = 0$, we obtain a fractional Lotka-Volterra competition model describing the competition interspecies n and m under a regime of chemorepulsion mechanism caused by the chemical signal v . Thus, Theorem 2.8 provides the existence of global solution for the fractional Lotka-Volterra system (under small initial data) in the class $(C(\mathbb{R}^+; \mathcal{N}_{p,\lambda,\infty}^{s^*-\theta}) \cap \mathcal{X}_1) \times (C(\mathbb{R}^+; \mathcal{N}_{p,\lambda,\infty}^{s^*-\theta}) \cap \mathcal{X}_1) \times C(\mathbb{R}^+; \mathcal{N}_{p,\lambda,\infty}^{s^*})$.

3. LINEAR AND NONLINEAR ESTIMATES

The aim of this section is to derive estimates to be used in proving the existence of mild solutions. First we establish estimates for fractional heat semigroups, in Morrey spaces, acting on distributions whose Fourier transforms have support in an annulus.

Lemma 3.1. *There exist positive constants C_1 and C_2 such that for any $1 \leq p \leq \infty$, $0 \leq \lambda < N$, $a \in \mathbb{R}$, $0 < \theta$, and any positive real numbers t, τ , if $\text{supp } \mathcal{F}f \subset \tau\mathcal{C}$, then*

$$\|E_\alpha(-t^\alpha((-\Delta)^{\theta/2} + a))f\|_{\mathcal{M}_{p,\lambda}} \leq C_2 \int_0^\infty M_\alpha(s)e^{ast^\alpha} e^{-C_1st^\alpha\tau^\theta} ds \|f\|_{\mathcal{M}_{p,\lambda}}, \tag{3.1}$$

and if $\text{supp } \mathcal{F}f \subset \tau\mathcal{C}$, then

$$\|E_{\alpha,\alpha}(-t^\alpha((-\Delta)^{\theta/2} + a))f\|_{\mathcal{M}_{p,\lambda}} \leq C_2 \int_0^\infty sM_\alpha(s)e^{ast^\alpha} e^{-C_1st^\alpha\tau^\theta} ds \|f\|_{\mathcal{M}_{p,\lambda}}. \tag{3.2}$$

Proof. The proof follows the ideas in [2, Lemma 2.4] but adjusted to the context of Morrey spaces. Let $\phi \in \mathcal{D}(\mathbb{R}^N \setminus \{0\})$ be a function such that $\phi = 1$ near the annulus \mathcal{C} . Then,

$$\begin{aligned} & E_\alpha(-t^\alpha((-\Delta)^{\theta/2} + a))f \\ &= \int_0^\infty M_\alpha(s)e^{ast^\alpha} \mathcal{F}^{-1}\left(\phi(\xi/\tau)e^{-st^\alpha|\xi|^\theta} \mathcal{F}(f)(\xi)\right) ds \\ &= \int_0^\infty M_\alpha(s)e^{ast^\alpha} \left(g(\cdot, s^{\frac{1}{\alpha}}t) * f\right) ds, \end{aligned} \tag{3.3}$$

where $g(x, t) = \mathcal{F}^{-1}(\phi(\xi/\tau)e^{-t^\alpha|\xi|^\theta})$. The hypotheses on ϕ imply that

$$\|g(\cdot, t)\|_{L^1} \leq C_2 e^{-C_1 t^\alpha \tau^\theta}, \quad \forall t > 0. \tag{3.4}$$

Thus, by applying the Young inequality (2.1) in (3.3), and using (3.4), we obtain

$$\|E_\alpha(-t^\alpha((-\Delta)^{\theta/2} + a))f\|_{\mathcal{M}_{p,\lambda}} \leq C_2 \int_0^\infty M_\alpha(s)e^{ast^\alpha} e^{-C_1 st^\alpha \tau^\theta} ds \|f\|_{\mathcal{M}_{p,\lambda}},$$

which proves (3.1). A similar argument proves (3.2). □

The next lemma provides some inclusions involving Besov-Morrey spaces (see [11, 14]).

Lemma 3.2. *Suppose that $1 \leq r \leq \infty$, $s, s_1, s_2 \in \mathbb{R}$, $1 \leq p, p_1, p_2 \leq \infty$, and $0 \leq \lambda < N$.*

(i) *If $1 \leq q_1 \leq q_2 \leq \infty$, then*

$$\widetilde{L}^r([0, \infty); \mathcal{N}_{p,\lambda,q_1}^s) \subset \widetilde{L}^r([0, \infty); \mathcal{N}_{p,\lambda,q_2}^s).$$

(ii) *If $p_1 \leq p_2$ and $s_2 - \frac{N-\lambda}{p_2} = s_1 - \frac{N-\lambda}{p_1}$, then*

$$\widetilde{L}^r([0, \infty); \mathcal{N}_{p_1,\lambda,q}^{s_1}) \subset \widetilde{L}^r([0, \infty); \mathcal{N}_{p_2,\lambda,q}^{s_2}).$$

Proof. Part (i) follows from the definition of the norm (2.6) and the fact that $l^{q_1} \subset l^{q_2}$ if $q_1 \leq q_2$. To prove part (ii), from Lemma 2.6 we have

$$\|\dot{\Delta}_j f\|_{\mathcal{M}_{p_2,\lambda}} \leq C 2^{j\left(\frac{N-\lambda}{p_1} - \frac{N-\lambda}{p_2}\right)} \|\dot{\Delta}_j f\|_{\mathcal{M}_{p_1,\lambda}}.$$

Taking the L^r -norm, multiplying by 2^{js_2} and taking the l^q -norm we obtain the result. □

Next we recall estimates for multiplier operators with polynomial growth in the context Besov-Morrey spaces (see [11, 14]).

Lemma 3.3. *Let $m \in \mathbb{R}$, $1 \leq p < \infty$, $0 \leq \lambda < N$, and $P(\xi) \in C^{[N/2]+1}(\mathbb{R}^N \setminus \{0\})$. Suppose that there exists $A > 0$ such that*

$$\left| \frac{\partial^k P}{\partial \xi^k}(\xi) \right| \leq A |\xi|^{m-|k|},$$

for all $k \in (\mathbb{N} \cup \{0\})^N$ with $|k| \leq [N/2] + 1$ and $|\xi| \neq 0$. Then, for every g such that $\text{supp } \mathcal{F}(g) \subset 2^j \mathcal{C}$ we have that

$$\|\mathcal{F}^{-1}(P(\xi)\mathcal{F}(g))\|_{\mathcal{M}_{p,\lambda}} \leq CA 2^{mj} \|g\|_{\mathcal{M}_{p,\lambda}}.$$

The following lemma is a consequence of the one above.

Lemma 3.4. *Let $m, s \in \mathbb{R}$, $1 \leq p < \infty$, $0 \leq \lambda < N$, $1 \leq q, r \leq \infty$ and $P(\xi) \in C^{[N/2]+1}(\mathbb{R}^N \setminus \{0\})$. Suppose that there exists $A > 0$ such that*

$$\left| \frac{\partial^k P}{\partial \xi^k}(\xi) \right| \leq A |\xi|^{m-|k|},$$

for all $k \in (\mathbb{N} \cup \{0\})^N$ with $|k| \leq [N/2] + 1$ and $|\xi| \neq 0$. Then the multiplier operator $P(D)$ defined by $P(D)g = \mathcal{F}^{-1}(P(\xi)\mathcal{F}(g)(\xi))$ is bounded from $\mathcal{N}_{p,\lambda,q}^s$ to $\mathcal{N}_{p,\lambda,q}^{s-m}$. Moreover,

$$\|P(D)g\|_{\widetilde{L}^r([0,\infty);\mathcal{N}_{p,\lambda,q}^{s-m}}} \leq CA \|g\|_{\widetilde{L}^r([0,\infty);\mathcal{N}_{p,\lambda,q}^s}}.$$

Lemma 3.5. *If f be a distribution in \mathcal{S}'_h , $s < 0$, $1 \leq p, q \leq \infty$, $0 \leq \lambda < N$ and $1 \leq r \leq \infty$, then f belongs to the Besov-Morrey space $\widetilde{L}^r([0,\infty);\mathcal{N}_{p,\lambda,q}^s)$ if and only if*

$$(2^{js} \|\dot{S}_j f\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda}}})_{j \in \mathbb{Z}} \in \ell^q.$$

Moreover,

$$C_1 \|f\|_{\widetilde{L}^r([0,\infty);\mathcal{N}_{p,\lambda,q}^s}} \leq \|(2^{js} \|\dot{S}_j f\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda}}})_{j \in \mathbb{Z}}\|_{\ell^r} \leq C_2 \|f\|_{\widetilde{L}^r([0,\infty);\mathcal{N}_{p,\lambda,q}^s}},$$

where the positive constants C_1 and C_2 depend only on N, s .

Proof. This proof is inspired in the proof of [2, Proposition 2.33] adapted to the context of $\widetilde{L}^r([0,\infty);\mathcal{N}_{p,\lambda,q}^s)$ spaces. After taking the norm $L^r([0,\infty);\mathcal{M}_{p,\lambda})$ in the localization operator, we obtain

$$\begin{aligned} 2^{js} \|\dot{\Delta}_j f\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda}}} &\leq 2^{js} \left(\|\dot{S}_j f\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda}}} + \|\dot{S}_{j+1} f\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda}}} \right) \\ &= 2^{js} \|\dot{S}_j f\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda}}} + 2^{-s} 2^{(j+1)s} \|\dot{S}_{j+1} f\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda}}}, \end{aligned}$$

which implies the left-hand side inequality. On the other hand, we have

$$\begin{aligned} 2^{js} \|\dot{S}_j f\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda}}} &\leq 2^{js} \sum_{j' \leq j-1} \|\dot{\Delta}_{j'} f\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda}}} \\ &= \sum_{j' \leq j-1} 2^{(j-j')s} 2^{j's} \|\dot{\Delta}_{j'} f\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda}}}, \end{aligned}$$

and thus, using that $s < 0$ and applying the Young inequality in the framework of ℓ^r -spaces, we conclude the result. \square

The lemma below contains a criterion for the limit of a series to belong to a class of homogeneous Besov-Morrey space. Its proof follows the ideas in [2, Lemma 2.23].

Lemma 3.6. *Let \mathcal{C}' be an annulus, $s \in \mathbb{R}$, $0 \leq \lambda < N$, and $1 \leq p, q, r \leq \infty$ such that $s < N/p$, or $s = N/p$ with $q = 1$. Let $(f_j)_{j \in \mathbb{Z}}$ be a sequence of smooth functions such that*

$$\text{supp } \mathcal{F}f_j \subset 2^j \mathcal{C}' \quad \text{and} \quad \|(2^{js} \|f_j\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda}}})_{j \in \mathbb{Z}}\|_{\ell^q} < \infty.$$

Then $\sum_{j \in \mathbb{Z}} f_j$ converges in \mathcal{S}' to some f in $\widetilde{L}^r([0,\infty);\mathcal{N}_{p,\lambda,q}^s)$; moreover, there exists a constant $C = C(s) > 0$ such that

$$\|f\|_{\widetilde{L}^r([0,\infty);\mathcal{N}_{p,\lambda,q}^s}} \leq C \|(2^{js} \|f_j\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda}}})_{j \in \mathbb{Z}}\|_{\ell^q}.$$

Proof. Considering the annulus \mathcal{C} in Definition 2.3, there exists $k_0 \in \mathbb{N}$ such that $2^{j'}\mathcal{C} \cap 2^j\mathcal{C}' = \emptyset$ provided that $|j' - j| \geq k_0$. Therefore, $\dot{\Delta}_{j'}f_j = 0$ for $|j' - j| \geq k_0$, and

$$\begin{aligned} \|\dot{\Delta}_{j'}f\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda})} &= \left\| \sum_{|j'-j| < k_0} \dot{\Delta}_{j'}f_j \right\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda})} \\ &\leq C \sum_{|j'-j| < k_0} \|f_j\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda})}. \end{aligned}$$

Hence, we obtain that

$$\begin{aligned} 2^{j's} \|\dot{\Delta}_{j'}f\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda})} &\leq C \sum_{|j'-j| \leq k_0} 2^{j's} \|f_j\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda})} \\ &\leq C \sum_{|j'-j| \leq k_0} 2^{j's} \|f_j\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda})}. \end{aligned}$$

Taking the ℓ^q -norm and using the Young inequality for discrete convolutions, we conclude the proof. \square

The next lemma is similar to the previous one but now with distributions supported in balls. Therefore we omit its proof.

Lemma 3.7. *Let B be a ball, $s > 0$, $0 \leq \lambda < N$, and $1 \leq p, q, r \leq \infty$ be such that $s < N/p$, or $s = N/p$ with $q = 1$. If $(f_j)_{j \in \mathbb{Z}}$ be a sequence of smooth functions such that*

$$\text{supp } \mathcal{F}f_j \subset 2^j B \quad \text{and} \quad \|(2^{js} \|f_j\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda})})_{j \in \mathbb{Z}}\|_{\ell^q} < \infty,$$

then $\sum_{j \in \mathbb{Z}} f_j$ converges in \mathcal{S}' to some f in $\widetilde{L}^r([0,\infty);\mathcal{N}_{p,\lambda,q}^s)$. Moreover, there exists a constant $C = C(s) > 0$ such that

$$\|f\|_{\widetilde{L}^r([0,\infty);\mathcal{N}_{p,\lambda,q}^s)} \leq C \|(2^{js} \|f_j\|_{L^r([0,\infty);\mathcal{M}_{p,\lambda})})_{j \in \mathbb{Z}}\|_{\ell^q}.$$

The paraproduct of v by u denoted by $T_u v$ is the bilinear operator

$$T_u v := \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v.$$

The remainder of u and v , denoted by $R(u, v)$, is the bilinear operator

$$R(u, v) := \sum_{|i-j| \leq 1} \dot{\Delta}_i u \dot{\Delta}_j v.$$

Formally, using the operators $T_u v$ and $R(u, v)$, we can express the product uv by means of the Bony decomposition $uv = T_u v + T_v u + R(u, v)$.

In the next two lemmas we provide continuity properties for the paraproduct and remainder term on Besov-Morrey spaces, which can be seen as extensions from the corresponding ones in Besov spaces found in [2, Theorems 2.47 and 2.52].

Lemma 3.8. *Let $s, \sigma \in \mathbb{R}$ with $\sigma > 0$, $0 \leq \lambda < N$, and let $1 \leq p, q, q_1, q_2, r, r_1, r_2 \leq \infty$ with $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$.*

- (i) *If $s < \frac{N}{p}$, or $s = \frac{N}{p}$ with $q = 1$, then, there exists a positive constant $C > 0$ such that*

$$\|T_u v\|_{\widetilde{L}^r([0,\infty);\mathcal{N}_{p,\lambda,q}^s)} \leq C \|u\|_{\widetilde{L}^{r_1}([0,\infty);L^\infty)} \|v\|_{\widetilde{L}^{r_2}([0,\infty);\mathcal{N}_{p,\lambda,q}^s)}.$$

(ii) If $s - \sigma < \frac{N}{p}$, or $s - \sigma = \frac{N}{p}$ with $q = 1$, then, there exists a positive constant $C > 0$ such that

$$\|T_u v\|_{\widetilde{L}^r([0, \infty); \mathcal{N}_{p, \lambda, q}^{s-\sigma})} \leq C \|u\|_{\widetilde{L}^{r_1}([0, \infty); \mathcal{N}_{\infty, \lambda, q_1}^{-\sigma})} \|v\|_{\widetilde{L}^{r_2}([0, \infty); \mathcal{N}_{p, \lambda, q_2}^s)}.$$

Proof. From Definition 2.3, the support of $\mathcal{F}(\dot{S}_{j-1} u \dot{\Delta}_j v)$ is contained in $2^j(C + B)$. In addition, there is an integer k_0 such that $\dot{\Delta}_{j'}(\dot{S}_{j-1} u \dot{\Delta}_j v) = 0$ provided that $j' > j + k_0$. Thus, it is sufficient to estimate $\|\dot{S}_{j-1} u \dot{\Delta}_j v\|_{L^r([0, \infty); \mathcal{M}_{p, \lambda})}$. From the definition of the operators \dot{S}_j , it holds

$$\|\dot{S}_{j-1} u\|_{L^\infty} \leq C \|u\|_{L^\infty}. \tag{3.5}$$

Thus, using Lemma 3.6 and the Hölder inequality in L^r , we conclude that $T_u v \in \widetilde{L}^r([0, \infty); \mathcal{N}_{p, \lambda, q}^s)$, and the inequality in the part (i) of the lemma. On the other hand, from Lemma 3.5 it follows that

$$\|\dot{S}_{j-1} u\|_{L^{r_1}([0, \infty); L^\infty)} \leq \frac{C}{-z} c_{jq_1} 2^{-jz} \|u\|_{L^{r_1}([0, \infty); \mathcal{N}_{\infty, \lambda, q_1}^z)}, \tag{3.6}$$

for all $j \in \mathbb{Z}$ and $z < 0$, where $(c_{j, q_1})_{j \in \mathbb{Z}}$ has norm 1 in $\ell^{q_1}(\mathbb{Z})$. Again, Lemma 3.6 and the Hölder inequality in L^r -spaces imply that $T_u v \in \widetilde{L}^r([0, \infty); \mathcal{N}_{p, \lambda, q}^{s-\sigma})$; moreover, by using (3.6) we obtain the inequality in the part (ii) of the lemma. \square

Lemma 3.9 ([4]). *Let $0 \leq \lambda, \lambda_1, \lambda_2 < N$, $s_1, s_2 \in \mathbb{R}$ and $p, p_1, p_2, q, q_1, q_2, r, r_1, r_2 \in [1, \infty]$ satisfying $s_1 + s_2 > 0$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. If $s_1 + s_2 < \frac{N}{p}$, or $s_1 + s_2 = \frac{N}{p}$ with $q = 1$, then there exists a constant $C > 0$ such that*

$$\|R(u, v)\|_{\widetilde{L}^r([0, \infty); \mathcal{N}_{p, \lambda, q}^{s_1+s_2})} \leq C \|u\|_{\widetilde{L}^{r_1}([0, \infty); \mathcal{N}_{p_1, \lambda_1, q_1}^{s_1})} \|v\|_{\widetilde{L}^{r_2}([0, \infty); \mathcal{N}_{p_2, \lambda_2, q_2}^{s_2})}.$$

Proof. From Definition 2.3, we have the support of $\mathcal{F}(\sum_{|\nu| \leq 1} \dot{\Delta}_{j-\nu} u \dot{\Delta}_j v)$ is contained in $2^j B$. Moreover, there is an integer N_0 such that $\dot{\Delta}_{j'}(\sum_{|\nu| \leq 1} \dot{\Delta}_{j-\nu} u \dot{\Delta}_j v) = 0$ for all $j' > j + N_0$. Applying Hölder's inequality we have

$$\begin{aligned} & 2^{j'(s_1+s_2)} \|\dot{\Delta}_j R(u, v)\|_{L^r([0, \infty); \mathcal{M}_{p, \lambda})} \\ & \leq C 2^{j'(s_1+s_2)} \sum_{|\nu| \leq 1, j \geq j' - N_0} \|\dot{\Delta}_{j-\nu} u \dot{\Delta}_j v\|_{L^r([0, \infty); \mathcal{M}_{p, \lambda})} \\ & \leq C 2^{j'(s_1+s_2)} \sum_{|\nu| \leq 1, j \geq j' - N_0} \left(\|\dot{\Delta}_{j-\nu} u\|_{L^{r_1}([0, \infty); \mathcal{M}_{p_1, \lambda_1})} \right. \\ & \quad \left. \times \|\dot{\Delta}_j v\|_{L^{r_2}([0, \infty); \mathcal{M}_{p_2, \lambda_2})} \right) \\ & \leq C \sum_{|\nu| \leq 1, j \geq j' - N_0} \left(2^{-(j-j')(s_1+s_2)} 2^{(j-\nu)s_1} \right. \\ & \quad \left. \times \|\dot{\Delta}_{j-\nu} u\|_{L^{r_1}([0, \infty); \mathcal{M}_{p_1, \lambda_1})} 2^{js_2} \|\dot{\Delta}_j v\|_{L^{r_2}([0, \infty); \mathcal{M}_{p_2, \lambda_2})} \right). \end{aligned}$$

Note that $s_1 + s_2 > 0$. From Lemma 3.7, $R(u, v) \in \widetilde{L}^r([0, \infty); \mathcal{N}_{p, \lambda, q}^{s_1+s_2})$. Applying the discrete Hölder and Young inequalities, we conclude the proof of the lemma. \square

4. EXISTENCE RESULTS

First we derive global estimates for mild solutions of the fractional heat equation and then obtain a solution to (2.7) using an iterative scheme. Let us rewrite equations (2.7) in the form

$$\begin{aligned}
 n(x, t) &= E_\alpha(-t^\alpha(-\Delta)^{\theta/2})n_0 \\
 &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-(t-\tau)^\alpha(-\Delta)^{\theta/2})(f_1 + f_2 + f_3)(\tau) d\tau, \\
 m(x, t) &= E_\alpha(-t^\alpha(-\Delta)^{\theta/2})m_0 \\
 &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-(t-\tau)^\alpha(-\Delta)^{\theta/2})(f_4 + f_5)(\tau) d\tau, \\
 v(x, t) &= E_\alpha(-t^\alpha((-\Delta)^{\theta/2} - \gamma))v_0 \\
 &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-(t-\tau)^\alpha((-\Delta)^{\theta/2} - \gamma))(f_6 + f_7)(\tau) d\tau,
 \end{aligned}
 \tag{4.1}$$

where

$$\begin{aligned}
 f_1 &= \chi \nabla \cdot (nG(v)), \quad f_2 = a_1 n^2, \quad f_3 = -b_1 nm, \\
 f_4 &= a_2 m^2, \quad f_5 = -b_2 mn, \quad f_6 = a_3 m, \quad f_7 = b_3 n.
 \end{aligned}
 \tag{4.2}$$

Proposition 4.1. *Let $s \in \mathbb{R}$, $0 < \theta, 1 \leq p, q \leq \infty$, $1 \leq \kappa \leq r \leq \infty$, $\frac{1}{\alpha} < r \leq \infty$, $a \geq 0$, and $0 \leq \lambda < N$. If $v_0 \in \mathcal{N}_{p,\lambda,q}^s$ and $f(x, t) \in \widetilde{L}^\kappa([0, \infty); \mathcal{N}_{p,\lambda,q}^{s-\theta(1-\frac{1}{\alpha\kappa})})$, then*

$$\begin{aligned}
 v(x, t) &= E_\alpha(-t^\alpha((-\Delta)^{\theta/2} - a))v_0 \\
 &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-(t-\tau)^\alpha((-\Delta)^{\theta/2} - a))f(x, \tau) d\tau
 \end{aligned}$$

satisfies

$$\|v(x, t)\|_{\widetilde{L}^r([0, \infty); \mathcal{N}_{p,\lambda,q}^{s+\frac{\theta}{\alpha r}})} \leq C \left(\|v_0\|_{\mathcal{N}_{p,\lambda,q}^s} + \|f\|_{\widetilde{L}^\kappa([0, \infty); \mathcal{N}_{p,\lambda,q}^{s-\theta(1-\frac{1}{\alpha\kappa})}} \right),$$

for some $C > 0$.

Proof. Applying $\dot{\Delta}_j$ to the fractional heat equation ${}^c D_t^\alpha v + (-\Delta)^{\theta/2} v = -av + f$ with initial data $v(0) = v_0$, we obtain

$${}^c D_t^\alpha \dot{\Delta}_j v + (-\Delta)^{\theta/2} \dot{\Delta}_j v = -a \dot{\Delta}_j v + \dot{\Delta}_j f, \quad \dot{\Delta}_j v|_{t=0} = \dot{\Delta}_j v_0.
 \tag{4.3}$$

The solution to this equation is given by

$$\begin{aligned}
 \dot{\Delta}_j v &= E_\alpha(-t^\alpha((-\Delta)^{\theta/2} - a))\dot{\Delta}_j v_0 \\
 &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-(t-\tau)^\alpha((-\Delta)^{\theta/2} - a))\dot{\Delta}_j f d\tau, \quad t > 0.
 \end{aligned}
 \tag{4.4}$$

Since the Fourier transform of $\dot{\Delta}_j v_0$ and $\dot{\Delta}_j f$ are supported in the annulus $2^j \mathcal{C}$, using Lemma 3.1 and the Young inequality in L^r , we have

$$\begin{aligned}
 & \|E_\alpha(-t^\alpha(-\Delta)^{\theta/2} - a)\dot{\Delta}_j v_0\|_{L^r_{[0,\infty)}(\mathcal{M}_{p,\lambda})} \\
 & \leq C_2 \left\| \int_0^\infty M_\alpha(s) e^{-ast^\alpha} e^{-C_1 st^\alpha 2^{\theta j}} \|\dot{\Delta}_j v_0\|_{\mathcal{M}_{p,\lambda}} ds \right\|_{L^r(0,\infty)} \\
 & \leq C_2 \int_0^\infty M_\alpha(s) \|e^{-C_1 st^\alpha 2^{\theta j}} \|\dot{\Delta}_j v_0\|_{\mathcal{M}_{p,\lambda}}\|_{L^r(0,\infty)} ds \\
 & \leq C_2 \int_0^\infty M_\alpha(s) \left(\int_0^\infty e^{-C_1 st^\alpha 2^{\theta j}} \right)^{1/r} \|\dot{\Delta}_j v_0\|_{\mathcal{M}_{p,\lambda}} ds \\
 & \leq C_2 2^{-\frac{\theta j}{\alpha r}} \int_0^\infty s^{-\frac{1}{\alpha r}} M_\alpha(s) ds \|\dot{\Delta}_j v_0\|_{\mathcal{M}_{p,\lambda}} \\
 & \leq C 2^{-\frac{\theta j}{\alpha r}} \|\dot{\Delta}_j v_0\|_{\mathcal{M}_{p,\lambda}},
 \end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
 & \left\| \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha(-\Delta)^{\theta/2} - a)\dot{\Delta}_j f d\tau \right\|_{L^r_{[0,\infty)}(\mathcal{M}_{p,\lambda})} \\
 & \leq C_2 \int_0^\infty s M_\alpha(s) \left\| \int_0^t (t-\tau)^{\alpha-1} e^{-as(t-\tau)^\alpha} e^{-C_1 s(t-\tau)^\alpha 2^{\theta j}} \|\dot{\Delta}_j f\|_{\mathcal{M}_{p,\lambda}} d\tau \right\|_{L^r(0,\infty)} ds \\
 & \leq C_2 \int_0^\infty s M_\alpha(s) \left(\int_0^\infty e^{-C_1 st^\alpha 2^{\theta j} \kappa^*} t^{(\alpha-1)\kappa^*} \right)^{1/\kappa^*} \|\dot{\Delta}_j f\|_{L^\kappa_{[0,\infty)}(\mathcal{M}_{p,\lambda})} ds \\
 & \leq C \int_0^\infty s M_\alpha(s) (s 2^{\theta j})^{-\frac{1}{\alpha \kappa^*} - \frac{(\alpha-1)}{\alpha}} ds \|\dot{\Delta}_j f\|_{L^\kappa_{[0,\infty)}(\mathcal{M}_{p,\lambda})} \\
 & \leq C \int_0^\infty s M_\alpha(s) s^{-\frac{1}{\alpha r} - 1 + \frac{1}{\alpha \kappa}} 2^{-\frac{\theta j}{\alpha r} - \theta j(1 - \frac{1}{\alpha \kappa})} ds \|\dot{\Delta}_j f\|_{L^\kappa_{[0,\infty)}(\mathcal{M}_{p,\lambda})} \\
 & \leq C 2^{-\frac{\theta j}{\alpha r} - \theta j(1 - \frac{1}{\alpha \kappa})} \int_0^\infty s^{-\frac{1}{\alpha r} + \frac{1}{\alpha \kappa}} M_\alpha(s) ds \|\dot{\Delta}_j f\|_{L^\kappa_{[0,\infty)}(\mathcal{M}_{p,\lambda})} \\
 & \leq C 2^{-\frac{\theta j}{\alpha r} - \theta j(1 - \frac{1}{\alpha \kappa})} \|\dot{\Delta}_j f\|_{L^\kappa_{[0,\infty)}(\mathcal{M}_{p,\lambda})},
 \end{aligned} \tag{4.6}$$

where κ^* is such that $1 + 1/r = 1/\kappa^* + 1/\kappa$. Thus, from (4.5)-(4.6) we obtain

$$\|\dot{\Delta}_j v\|_{L^r_{[0,\infty)}(\mathcal{M}_{p,\lambda})} \leq C \left(2^{-j\frac{\theta}{\alpha r}} \|\dot{\Delta}_j v_0\|_{\mathcal{M}_{p,\lambda}} + 2^{-j\frac{\theta}{\alpha r}} 2^{-j\theta(1 - \frac{1}{\alpha \kappa})} \|\dot{\Delta}_j f\|_{L^\kappa_{[0,\infty)}(\mathcal{M}_{p,\lambda})} \right). \tag{4.7}$$

Therefore, multiplying both sides of (4.7) by $2^{j\frac{\theta}{\alpha r}} 2^{j\theta}$ and taking the ℓ^q -summation, we obtain

$$\|v\|_{\widetilde{L^r}([0,\infty); \mathcal{N}_{p,\lambda,q}^{s+\frac{\theta}{\alpha r}})} \leq C \left(\|v_0\|_{\mathcal{N}_{p,\lambda,q}^s} + \|f\|_{\widetilde{L^\kappa}([0,\infty); \mathcal{N}_{p,\lambda,q}^{s-\theta(1 - \frac{1}{\alpha \kappa})}} \right),$$

where $C > 0$ is a constant. The proof is complete. □

Lemma 4.2. *If $0 \leq \lambda < N$, $1 \leq p < \infty$, $\frac{1}{\alpha} < \kappa \leq \infty$, $1 < \theta$, $p < \frac{N-\lambda}{\theta(2 - \frac{1}{\alpha \kappa})}$, and $f_j = 1, \dots, 7$ are defined as in (4.2), then the following estimates hold*

- (i) $\|f_1\|_{\widetilde{L^\kappa}([0,\infty); \mathcal{N}_{p,\lambda,\infty}^{s^* - \theta(2 - \frac{1}{\alpha \kappa})}} \leq C \|n\|_{\mathcal{X}_1} \|v\|_{\mathcal{X}_2}$,
- (ii) $\|f_2\|_{\widetilde{L^\kappa}([0,\infty); \mathcal{N}_{p,\lambda,\infty}^{s^* - \theta(2 - \frac{1}{\alpha \kappa})}} \leq C \|n\|_{\mathcal{X}_1}^2$,

$$\begin{aligned}
 (iii) \quad & \|f_3 + f_5\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta(2-\frac{1}{\alpha\kappa})}}} \leq C\|n\|_{\mathcal{X}_1}\|m\|_{\mathcal{X}_1}, \\
 (iv) \quad & \|f_4\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta(2-\frac{1}{\alpha\kappa})}}} \leq C\|m\|_{\mathcal{X}_1}^2, \\
 (v) \quad & \|f_6 + f_7\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta(1-\frac{1}{\alpha\kappa})}}} \leq C(\|m\|_{\mathcal{X}_1} + \|n\|_{\mathcal{X}_1}).
 \end{aligned}$$

Proof. We start by analyzing (i). From Lemma 3.4 we obtain

$$\|\partial_j([G(v)]_j n)\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta(2-\frac{1}{\alpha\kappa})}}} \leq C\|n[G(v)]_j\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*+1-\theta(2-\frac{1}{\alpha\kappa})}}}, \quad j = 1, 2, 3.$$

Now, from the Bony decomposition we have

$$[G(v)]_j n = T_{[G(v)]_j} n + T_n[G(v)]_j + R([G(v)]_j, n), \quad j = 1, 2, 3.$$

Taking into account that $s^* = \frac{N-\lambda}{p}$, using Lemmas 3.8, 3.2 and 3.4, we estimate

$$\begin{aligned}
 & \|T_n[G(v)]_j\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta+1-\theta(1-\frac{1}{\alpha\kappa})}}} \\
 & \leq C\|n\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{\infty,\lambda,\infty}^{-\theta(1-\frac{1}{\alpha\kappa})}}} \| [G(v)]_j \|_{\widetilde{L}^\infty([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta+1})} \\
 & \leq C\|n\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta(1-\frac{1}{\alpha\kappa})}}} \|v\|_{\widetilde{L}^\infty([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*})} \\
 & \leq C\|n\|_{\mathcal{X}_1} \|v\|_{\mathcal{X}_2}, \quad (\text{by the condition } \kappa > \frac{1}{\alpha})
 \end{aligned}$$

and

$$\begin{aligned}
 & \|T_{[G(v)]_j} n\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta+1-\theta(1-\frac{1}{\alpha\kappa})}}} \\
 & \leq C\| [G(v)]_j \|_{\widetilde{L}^\infty([0,\infty);\mathcal{N}_{\infty,\lambda,\infty}^{-(\theta-1)}}} \|n\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta(1-\frac{1}{\alpha\kappa})}}} \\
 & \leq C\| [G(v)]_j \|_{\widetilde{L}^\infty([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta(1-\frac{1}{\alpha\kappa})}}} \|n\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta(1-\frac{1}{\alpha\kappa})}}} \\
 & \leq C\|v\|_{\widetilde{L}^\infty([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*})} \|n\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta(1-\frac{1}{\alpha\kappa})}}} \\
 & \leq C\|n\|_{\mathcal{X}_1} \|v\|_{\mathcal{X}_2}, \quad (\text{by the condition } \theta > 1).
 \end{aligned}$$

Moreover, from Lemmas 3.9 and 3.2 we obtain

$$\begin{aligned}
 & \|R([G(v)]_j, n)\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta+1-\theta(1-\frac{1}{\alpha\kappa})}}} \\
 & \leq C\|n\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{\infty,\lambda,\infty}^{-\theta(1-\frac{1}{\alpha\kappa})}}} \| [G(v)]_j \|_{\widetilde{L}^\infty([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta+1})} \\
 & \leq C\|n\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta(1-\frac{1}{\alpha\kappa})}}} \|v\|_{\widetilde{L}^\infty([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*})} \\
 & \leq C\|n\|_{\mathcal{X}_1} \|v\|_{\mathcal{X}_2}.
 \end{aligned}$$

Therefore,

$$\|f_1\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta(2-\frac{1}{\alpha\kappa})}}} \leq C\|n\|_{\mathcal{X}_1} \|v\|_{\mathcal{X}_2}. \tag{4.8}$$

Now we obtain the estimate for f_2 . From Lemmas 3.8 and 3.2 we have

$$\begin{aligned}
 \|T_n n\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta-\theta(1-\frac{1}{\alpha\kappa})}}} & \leq C\|n\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{\infty,\lambda,\infty}^{-\theta(1-\frac{1}{\alpha\kappa})}}} \|n\|_{\widetilde{L}^\infty([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta})} \\
 & \leq C\|n\|_{\widetilde{L}^\kappa([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta(1-\frac{1}{\alpha\kappa})}}} \|n\|_{\widetilde{L}^\infty([0,\infty);\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta})} \\
 & \leq C\|n\|_{\mathcal{X}_1} \|n\|_{\mathcal{X}_1}.
 \end{aligned}$$

On the other hand, from Lemmas 3.9 and 3.2, and using the condition $s^* - \theta - \theta(1 - \frac{1}{\alpha\kappa}) > 0$, we obtain

$$\begin{aligned} \|R(n, n)\|_{\widetilde{L}^\kappa([0, \infty); \mathcal{N}_{p, \lambda, \infty}^{s^* - \theta - \theta(1 - \frac{1}{\alpha\kappa})})} &\leq C \|n\|_{\widetilde{L}^\kappa([0, \infty); \mathcal{N}_{\infty, \lambda, \infty}^{-\theta(1 - \frac{1}{\alpha\kappa})})} \|n\|_{\widetilde{L}^\infty([0, \infty); \mathcal{N}_{p, \lambda, \infty}^{s^* - \theta})} \\ &\leq C \|n\|_{\widetilde{L}^\kappa([0, \infty); \mathcal{N}_{p, \lambda, \infty}^{s^* - \theta(1 - \frac{1}{\alpha\kappa})})} \|n\|_{\widetilde{L}^\infty([0, \infty); \mathcal{N}_{p, \lambda, \infty}^{s^* - \theta})} \\ &\leq C \|n\|_{\mathcal{X}_1} \|n\|_{\mathcal{X}_1}. \end{aligned}$$

Here we have used that $p < \frac{N-\lambda}{\theta(2-\frac{1}{\alpha\kappa})}$ to guarantee that $s^* - \theta - \theta(1 - \frac{1}{\alpha\kappa}) > 0$. Note that the conditions for κ and θ imply that $1 - \theta(2 - \frac{1}{\alpha\kappa}) < 0$ and therefore $s^* - \theta + 1 - \theta(1 - \frac{1}{\alpha\kappa}) < \frac{N}{p}$. In conclusion, there exists a positive constant C such that

$$\|f_2\|_{\widetilde{L}^\kappa([0, \infty); \mathcal{N}_{p, \lambda, \infty}^{s^* - \theta(2 - \frac{1}{\alpha\kappa})})} \leq C \|n\|_{\mathcal{X}_1} \|n\|_{\mathcal{X}_1}. \tag{4.9}$$

The proof of (iii) and (iv) for f_3, f_4 , and f_5 follows similarly. Finally, for f_6 and f_7 , we have

$$\|f_6 + f_7\|_{\widetilde{L}^\kappa([0, \infty); \mathcal{N}_{p, \lambda, \infty}^{s^* - \theta(1 - \frac{1}{\alpha\kappa})})} \leq C (\|n\|_{\mathcal{X}_1} + \|m\|_{\mathcal{X}_1}). \tag{4.10}$$

□

4.1. Proof of Theorem 2.8. Motivated by [3], we consider the following iterative scheme whose limit will give the global mild solution for the system (1.1) in the sense of Definition 2.7:

$$\begin{aligned} n^{(1)} &= E_\alpha(-t^\alpha(-\Delta)^{\theta/2})n_0, \quad m^{(1)} = E_\alpha(-t^\alpha(-\Delta)^{\theta/2})m_0, \\ v^{(1)} &= E_\alpha(-t^\alpha((-\Delta)^{\theta/2} - \gamma))v_0, \\ n^{(k+1)} &= n^{(1)} + \chi \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(-(t-\tau)^\alpha(-\Delta)^{\theta/2}) \nabla \cdot (n^{(k)} G(v^{(k)}))(\tau) d\tau \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(-(t-\tau)^\alpha(-\Delta)^{\theta/2}) (a_1 n^{(k)} n^{(k)} - b_1 n^{(k)} m^{(k)})(\tau) d\tau, \\ m^{(k+1)} &= m^{(1)} + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(-(t-\tau)^\alpha(-\Delta)^{\theta/2}) \\ &\quad \times (a_2 m^{(k)} m^{(k)} - b_1 m^{(k)} n^{(k)})(\tau) d\tau, \\ v^{(k+1)} &= v^{(1)} + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(-(t-\tau)^\alpha((-\Delta)^{\theta/2} - \gamma)) \\ &\quad \times (a_3 m^{(k+1)} + b_3 n^{(k+1)})(\tau) d\tau. \end{aligned}$$

Applying Proposition 4.1 and Lemma 4.2 to the above equality, we obtain the following estimates:

$$\begin{aligned} \|n^{(k+1)}\|_{\mathcal{X}_1} &\leq C \left(\|n_0\|_{\mathcal{N}_{p, \lambda, \infty}^{s^* - \theta}} + \|n^{(k)}\|_{\mathcal{X}_1} \|v^{(k)}\|_{\mathcal{X}_2} + \|n^{(k)}\|_{\mathcal{X}_1}^2 + \|n^{(k)}\|_{\mathcal{X}_1} \|m^{(k)}\|_{\mathcal{X}_1} \right), \\ \|m^{(k+1)}\|_{\mathcal{X}_1} &\leq C \left(\|m_0\|_{\mathcal{N}_{p, \lambda, \infty}^{s^* - \theta}} + \|m^{(k)}\|_{\mathcal{X}_1}^2 + \|n^{(k)}\|_{\mathcal{X}_1} \|m^{(k)}\|_{\mathcal{X}_1} \right), \\ \|v^{(k+1)}\|_{\mathcal{X}_2} &\leq C \left(\|v_0\|_{\mathcal{N}_{p, \lambda, \infty}^{s^*}} + \|n^{(k+1)}\|_{\mathcal{X}_1} + \|m^{(k+1)}\|_{\mathcal{X}_1} \right). \end{aligned} \tag{4.11}$$

For small initial data, the sequence $[n^{(k)}, m^{(k)}, v^{(k)}]$ is uniformly bounded in the space $\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_1 \times \mathcal{X}_2$ with the norm

$$\|[n^{(k)}, m^{(k)}, v^{(k)}]\|_{\mathcal{X}} = \|n^{(k)}\|_{\mathcal{X}_1} + \|m^{(k)}\|_{\mathcal{X}_1} + \|v^{(k)}\|_{\mathcal{X}_2}.$$

In fact, from (4.11) we obtain

$$\begin{aligned} \|[n^{(k+1)}, m^{(k+1)}, v^{(k+1)}]\|_{\mathcal{X}} &= \|n^{(k+1)}\|_{\mathcal{X}_1} + \|m^{(k+1)}\|_{\mathcal{X}_1} + \|v^{(k+1)}\|_{\mathcal{X}_2} \\ &\leq C(A_0 + 4\|[n^{(k)}, m^{(k)}, v^{(k)}]\|_{\mathcal{X}}^2), \end{aligned} \tag{4.12}$$

where $A_0 = \|n_0\|_{\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta}} + \|m_0\|_{\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta}} + \|v_0\|_{\mathcal{N}_{p,\lambda,\infty}^{s^*}}$ and C is a positive constant. Let $\delta > 0$ be small enough such that if $\|n_0\|_{\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta}} + \|m_0\|_{\mathcal{N}_{p,\lambda,\infty}^{s^*-\theta}} + \|v_0\|_{\mathcal{N}_{p,\lambda,\infty}^{s^*}} < \delta$, then $1 - 16A_0C^2 > 0$, and consider the smallest root R of $4CR^2 - R + CA_0 = 0$; that is,

$$R = \frac{1 - \sqrt{1 - 16A_0C^2}}{8C}. \tag{4.13}$$

Thus, if

$$\|[n^{(k)}, m^{(k)}, v^{(k)}]\|_{\mathcal{X}} \leq R, \tag{4.14}$$

then

$$\|[n^{(k+1)}, m^{(k+1)}, v^{(k+1)}]\|_{\mathcal{X}} \leq R, \tag{4.15}$$

which implies that $[n^{(k)}, m^{(k)}, v^{(k)}]$, $k \in \mathbb{N}$, is uniformly bounded in \mathcal{X} . Next, we bound the difference vector

$$[n^{(k+1)} - n^{(k)}, m^{(k+1)} - m^{(k)}, v^{(k+1)} - v^{(k)}].$$

Noting that

$$\begin{aligned} &n^{(k+1)} - n^{(k)} \\ &= \chi \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-(t - \tau)^\alpha (-\Delta)^{\theta/2}) \nabla \cdot (n^{(k)} G(v^{(k)}))(\tau) d\tau \\ &\quad - \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-(t - \tau)^\alpha (-\Delta)^{\theta/2}) \nabla \cdot (n^{(k-1)} G(v^{(k-1)}))(\tau) d\tau \\ &\quad + \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-(t - \tau)^\alpha (-\Delta)^{\theta/2}) (a_1 n^{(k)} n^{(k)} - b_1 n^{(k)} m^{(k)})(\tau) d\tau \\ &\quad - \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-(t - \tau)^\alpha (-\Delta)^{\theta/2}) \\ &\quad \times (a_1 n^{(k-1)} n^{(k-1)} - b_1 n^{(k-1)} m^{(k-1)})(\tau) d\tau, \end{aligned} \tag{4.16}$$

by using Proposition 4.1 and Lemma 4.2, we obtain

$$\begin{aligned} &\|n^{(k+1)} - n^{(k)}\|_{\mathcal{X}_1} \\ &\leq C \left(\|n^{(k-1)}\|_{\mathcal{X}_1} \|v^{(k-1)} - v^{(k)}\|_{\mathcal{X}_2} + \|n^{(k-1)} - n^{(k)}\|_{\mathcal{X}_1} \|v^{(k)}\|_{\mathcal{X}_2} \right. \\ &\quad + \|n^{(k-1)}\|_{\mathcal{X}_1} \|n^{(k-1)} - n^{(k)}\|_{\mathcal{X}_1} + \|n^{(k-1)} - n^{(k)}\|_{\mathcal{X}_1} \|n^{(k)}\|_{\mathcal{X}_1} \\ &\quad \left. + \|m^{(k-1)}\|_{\mathcal{X}_1} \|n^{(k-1)} - n^{(k)}\|_{\mathcal{X}_1} + \|m^{(k-1)} - m^{(k)}\|_{\mathcal{X}_1} \|n^{(k)}\|_{\mathcal{X}_1} \right) \\ &\leq CR \|[n^{(k-1)} - n^{(k)}, m^{(k-1)} - m^{(k)}, v^{(k-1)} - v^{(k)}]\|_{\mathcal{X}}. \end{aligned} \tag{4.17}$$

Similarly we have

$$\|m^{(k+1)} - m^{(k)}\|_{\mathcal{X}_1} \leq CR \|[n^{(k-1)} - n^{(k)}, m^{(k-1)} - m^{(k)}, v^{(k-1)} - v^{(k)}]\|_{\mathcal{X}}. \tag{4.18}$$

On the other hand,

$$\|v^{(k+1)} - v^{(k)}\|_{\mathcal{X}_2} \leq C \left(\|n^{(k+1)} - n^{(k)}\|_{\mathcal{X}_1} + \|m^{(k+1)} - m^{(k)}\|_{\mathcal{X}_1} \right). \quad (4.19)$$

From estimates (4.17) and (4.19) we obtain

$$\begin{aligned} & \| [n^{(k+1)} - n^{(k)}, m^{(k+1)} - m^{(k)}, v^{(k+1)} - v^{(k)}] \|_{\mathcal{X}} \\ & \leq 4CR \| [n^{(k-1)} - n^{(k)}, m^{(k-1)} - m^{(k)}, v^{(k-1)} - v^{(k)}] \|_{\mathcal{X}}. \end{aligned} \quad (4.20)$$

Thus, choosing R as in (4.13) such that $R < \frac{1}{4C}$ (reducing δ , if necessary), we see that $[n^{(k)}, m^{(k)}, v^{(k)}]$ is a Cauchy sequence in \mathcal{X} . The limit $[n, m, v]$ is a mild solution in \mathcal{X} for system (4.1). The proof of uniqueness follows by using arguments similar to those in the proof of inequality (4.20), and therefore we omit the details. This proves Theorem 2.8.

Acknowledgments. The authors were supported by Vicerrectoría de Investigación y Extensión of Universidad Industrial de Santander, Colombia, project “Análisis teórico y numérico de sistemas diferenciales que modelan la dinámica de células tumorales en el cerebro”, code 3704. The authors would like to thank the anonymous referees for their useful comments and suggestions.

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