TURRITTIN'S NORMAL FORMS FOR LINEAR SYSTEMS OF MEROMORPHIC ODES OVER THE REAL FIELD

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ABSTRACT. We establish a version of Turrittin's result on normal forms of linear systems of meromorphic ODEs when the base field K is real and closed. Both the proposed normal forms and the transformations used have coefficients in K. Our motivation comes from applications to the study of trajectories of real analytic vector fields (already treated in the literature in dimension three). For the sake of clarity and completeness, we first review Turrittin's theorem in the case of an algebraically closed base field.

1. PRELIMINARIES

Let K be a field of characteristic zero and let $L_K = K[[x]][x^{-1}]$ be the field of formal meromorphic series with coefficients in K, endowed with the usual derivation with respect to x (denoted only by a prime), and the usual valuation $\nu : L_K \to \mathbb{Z} \cup \{\infty\}$ defined as the minimum of the support of the series, also called the *order*. As a matter of notation, if R is any ring and $n \in \mathbb{N}_{\geq 1}$, $\mathcal{M}_n(R)$ denotes the ring of square matrices of size n with entries in R.

A matrix $A \in \mathcal{M}_n(L_K)$ is identified with the formal meromorphic linear system of ODEs

$$[A] \quad Y' = AY,$$

where $Y = (Y_1, \ldots, Y_n)^t$ is a column vector of n variables. Define the order of A to be $\nu(A) := \min\{\nu(a_{ij}) : 1 \leq i, j \leq n\}$, where $A = (a_{ij})$. Sometimes we use the notation $A = A(x)$ to make explicit that we are dealing with meromorphic series in the variable x. Correspondingly, we will usually write the system as a series of matrices in the form

$$A = x^{\nu(A)}(A_0 + xA_1 + \ldots), \tag{1.1}$$

where $A_i \in \mathcal{M}_n(K)$ for all i and $A_0 \neq 0$. Also, if N is a non-negative integer, the truncated system up to degree N is defined by

$$J_N A := x^{\nu(A)}(A_0 + xA_1 + \cdots + x^N A_N).$$

The system A is called *singular* (at $x = 0$) if $\nu(A) < 0$. The *Poincaré rank* of the system is defined as the non-negative integer $q = q(A) := \max\{-\nu(A) - 1, 0\}$. We

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usually rewrite A as the system of formal linear ODEs

$$x^{q+1}Y' = \widetilde{A}Y, \quad \text{where } \widetilde{A} = A_0 + xA_1 + \dots.$$

A singular system with Poincaré rank $q = 0$ (resp. $q > 0$) is usually referred to be of *first kind* (resp. of *second kind*).

We denote by I_n the identity matrix of size n. We define the *radiality index* of A as the non-negative integer

$$k = k(A) := \min(\{j : A_j \notin KI_n\} \cup \{q\}).$$

The truncation $J_{k-1}A = a(x)I_n$, where $a(x)$ is a polynomial with degree at most $k - 1$, is called the *radial part* of A.

We are interested in the problem of getting formal normal forms of a given singular system under transformations of one of the following types:

(i) Given $P \in GL_n(L_K)$, the linear change of variables $Y = PZ$ transforms the system $[A] : Y' = AY$ into the system $[B] : Z' = BZ$ where

$$B = P^{-1}AP - P^{-1}P'.$$

The map $\Psi_P : \mathcal{M}_n(L_K) \to \mathcal{M}_n(L_K)$ sending A to $\Psi_P[A] := P^{-1}AP - P^{-1}P'$ is bijective. It is called the gauge transformation associated with P. In particular, if $P \in GL_n(K)$ is a constant matrix, then Ψ_P is the conjugation by P.

(ii) Given $r \in \mathbb{N}_{\geq 1}$, the change of the independent variable $x = z^r$ transforms the system $\frac{dY}{dx} = A(x)Y$ into a system $\frac{dY}{dz} = B(z)Y$, where

$$B(z) = rz^{r-1}A(z^r).$$

Re-written with the same letter x, we define the map $R_r : \mathcal{M}_n(L_K) \to \mathcal{M}_n(L_K)$ given by $R_r[A] := rx^{r-1}A(x^r)$, called the *ramification of order r*. It is an injective map but not bijective for $r > 1$.

A gauge transformation Ψ_P will be called: *regular*, if $\nu(P) = 0$ and $\det P(0) \neq 0$; *polynomial*, if each entry of P belongs to $K[x]$ (in which case the *degree* of Ψ_P is defined as the maximum of the degrees of the entries of P); *diagonal monomial*, if $P = \operatorname{diag}(x^{k_1}, \dots, x^{k_n})$ with $k_j \in \mathbb{N}_{\geq 0}$ for each j. Notice that if $P, Q \in GL_n(L_K)$ then we have $\Psi_{PQ} = \Psi_Q \circ \Psi_P$. Also, if $r, s \in \mathbb{N}_{\geq 1}$ then $R_{rs} = R_r \circ R_s$.

In addition, we are interested in polynomial (truncated) normal forms obtained by means of polynomial gauge transformations and ramifications (so that, if the initial system is polynomial or convergent, we preserve this character).

The case $K = \mathbb{C}$ (or more generally, K algebraically closed) is classical and treated with different approaches in the literature (Birkhoff [7], Hukuhara [13, 14], Turrittin [21], Wasow [23], Moser [19], Levelt [15], Balser-Jurkart-Lutz [3], Babbitt-Varadarajan [1], Hsieh-Sibuya [12], Barkatou [4, 5], Barkatou-Pflügel [6]. The different avatars of the algorithms for obtaining normal forms are commonly referred (as we will do here) by the generic expression Turrittin's Theorem.

In this article, we extend Turrittin's Theorem to the real case $K = \mathbb{R}$, or more generally to the case where K is a real closed field (see [8, Ch1] for the definition and basic facts about real closed fields).

As far as we know, this case has not been treated yet (except, of course, in the situation of a constant system $A \in \mathcal{M}_n(K)$ for which the usual well known real Jordan canonical form of A was proposed by Turrittin himself in [22]). We present versions of real (formal and polynomial) normal forms for any system, in such a way

that they can be obtained by transformations written in the base field K, without passing through the algebraic closure $\overline{K} = K(\sqrt{-1})$.

Certain cases of our real Turrittin's result have already been considered in the study of trajectories of real analytic vector fields around a formal invariant curve, mainly in dimension three [9, 10]. Our motivation was to establish general statements to serve to this study in any dimension, as well as other possible applications where systems of ODEs with real coefficients are involved.

1.1. Complex case.

To make precise statements and expose some of the steps that are useful to treat the real case, we propose first a brief revision of the complex case. Despite of its prevalence in the literature, we are led ourselves to sketch the different steps of the corresponding proofs (in section 2), instead of simply addressing the reader to the references. There are additional reasons to justify this revision:

- Although there are other proofs (even better ones from the point of view of computational effectiveness, see [5]), maybe the most commonly used reference for the complex case is Wasow's book [23]. We decided to follow also this last reference here. However, in that proof, the final arguments concerning the induction on the Poincaré rank is perhaps not sufficiently clarified: it drops as long as we do not need to make a ramification, but it increases after ramifications, an operation which is unavoidable in general. The required modification, even its simplicity, is worth to be made, in any case.

- In existing proofs of Turrittin's Theorem, it is frequently allowed the use of *exponential shiftings* when the leading matrix has a single eigenvalue. Such transformations have not an algebraic or formal nature and are "strange" to the initial setting of the systems. Although they commute with the whole matrix of the system so that the resulting system has also formal meromorphic coefficients, the exponential shiftings may behave very badly with respect to non-linear terms in general systems. Thus, for applications, it is better to avoid these operations.

- The search of precise statements for polynomial normal forms of Turrittin's result make necessary to enter in some details of the proofs; such statements are not exactly pursued in the common references, mostly devoted to obtain expressions of a fundamental matrix of solutions (cf. Remark 1.5, (b) below).

We start by defining the normal forms that we expect to obtain.

Definition 1.1 (Turrittin-Ramis-Sibuya form)**.** Let $A \in \mathcal{M}_n(L_K)$ be a system with Poincaré rank $q = q(A)$ and let $\mu \in \mathbb{N}_{\geq 0}$. We say that A is in Turrittin-Ramis-Sibuya form of degree μ (and of rank q), or in $(TRS)_\mu^q$-form for short, if it is written as

$$A(x) = x^{-(q+1)} \left(D(x) + x^q C + O(x^{q+\mu+1}) \right),$$

where $D(x) = \mathrm{diag}(d_1(x), \ldots, d_n(x))$ is a diagonal matrix with polynomial entries $d_j(x) \in K[x]$ of degree at most $q-1$ (equal zero if and only if $q = 0$) and $C \in \mathcal{M}_n(K)$ is a constant matrix commuting with $D(x)$. In this case, the truncated system $J_q(A) = x^{-(q+1)}(D(x) + x^q C)$ is called the *principal part* of A, while $D(x)$, resp. C, is called the *exponential part*, resp. the *residual matrix*.

Notice that if the system A is singular of first kind (that is $q = 0$) then A is already in $(TRS)_0^0$-form, with exponential part equal to $D(x) = 0$ and residual matrix $C = A_0$. On the other hand, if A is in $(TRS)_\mu^q$-form for some μ and $q > 0$ then its exponential part $D(x)$ is not zero; in fact, it satisfies $D(0) \neq 0$.

The names Ramis and Sibuya in the definition above come from those authors paper [20], devoted to summability properties of formal solutions of systems of holomorphic ODEs where the linear part is in (TRS)-form of some degree. We have added the name Turrittin by obvious reasons. It is worth to notice that analogous expressions as the (TRS)-forms appear also in the context of germs of biholomorphisms in [16, 17] (where the name of "Ramis-Sibuya form" is used).

Definition 1.2. Let $C \in \mathcal{M}_n(K)$ be a constant square matrix with entries in K. We say that C is *non-resonant* if for any pair of distinct eigenvalues $\lambda, \lambda' \in \overline{K}$ of C we have $\lambda - \lambda' \notin \mathbb{Z}$. Other authors, for instance Balser in his book [2], use the terminology "C has good spectrum".

Now, Turrittin's results for the case where $K = \overline{K}$ can be stated in the two following theorems.

Theorem 1.3 (Complex polynomial normal form). *Suppose that K is an algebraically closed field and let $A \in \mathcal{M}_n(L_K)$ be a singular system with Poincaré rank equal to $q = q(A)$.*

- (i) *There exist some $r \in \mathbb{N}_{\geq 1}$ and finitely many polynomial gauge transformations $\varphi_1, \ldots, \varphi_m$, either regular or diagonal monomial, such that, denoting*
  $$\psi = \varphi_m \circ \cdots \circ \varphi_1 \circ R_r,$$
  the transformed system $\widetilde{A} = \psi[A]$ is in $(TRS)_0^{\tilde{q}}$-form, where $\tilde{q} = q(\widetilde{A})$. Moreover, if $B \in \mathcal{M}_n(L_K)$ is another singular system with $q(B) = q$ and $J_{nq}A = J_{nq}B$ then $\widetilde{B} = \psi[B]$ is also in $(TRS)_0^{\tilde{q}}$-form with the same principal part as $\widetilde{A}$; i.e, $q(\widetilde{B}) = \tilde{q}$ and $J_{\tilde{q}}\psi[A] = J_{\tilde{q}}\psi[B]$.
- (ii) *Assume that A is in $(TRS)_0^q$-form and that its residual matrix is non-resonant. Then, for any given $\mu \geq 0$, there exists a regular polynomial gauge transformation $\phi_\mu = \Psi_{P^\mu}$, where $P^\mu(0) = I_n$, such that $\phi_\mu[\widetilde{A}]$ is in $(TRS)_\mu^{\tilde{q}}$-form with the same principal part as the original system A. Moreover, the family $\{P^\mu\}_\mu$ can be chosen such that P^μ is of degree at most $q + \mu$ and satisfying that $J_{q+\mu}P^{\mu'} = J_{q+\mu}P^\mu$ for any $\mu' > \mu$.*
- (iii) *Assume that A is in $(TRS)_0^q$-form. Then there exists a gauge transformation ϕ, given by a finite composition of regular polynomial or diagonal monomial transformations, such that $\phi[A]$ is in $(TRS)_0^q$-form with non-resonant residual matrix (and the same exponential part as A).*

As a consequence of the theorem above, one obtains the following version of Turrittin's formal normal forms of complex meromorphic linear ODEs.

Theorem 1.4 (Complex formal normal form). *Suppose that K is an algebraically closed field and let $A \in \mathcal{M}_n(L_K)$ be a singular system with Poincaré rank equal to $q = q(A)$. There exists a formal gauge transformation Ψ_P and a ramification R_r such that the transformed system $F = (\Psi_P \circ R_r)[A]$ has Poinaré rank equal to $\tilde{q} = q(F)$ and can be written in a* Formal Normal Form
$$[F] \quad Y' = x^{-(\tilde{q}+1)}(D(x) + x^{\tilde{q}}C)Y,$$
where $D(x)$ and C satisfy the requirements in Definition 1.1; i.e., F is in $(TRS)_0^{\tilde{q}}$-form and $J_{\tilde{q}}F = F$. Moreover, the transformation Ψ_P can be chosen to be equal to $\Psi_P = \Psi_Q \circ \psi$, where $Q \in \mathcal{M}_n(K[[x]])$ with $Q(0) = I_n$ and ψ is a finite composition of regular polynomial or diagonal monomial gauge transformations.

Remark 1.5. Concerning the statements in Theorems 1.3 and 1.4, we have the following statements.

(a) The sufficient truncation order nq in the second sentence of item (i) is already obtained for instance by Babbit-Varadarajan [1] or Lutz-Schäfke [18]. Below, we propose a proof with the slightly improved order $N := n(q - k) + k$, where k is the radial index of the initial system A.

(b) To obtain the formal normal form $[F]$ for the system A in Theorem 1.4 is equivalent to say that there exists a matrix $P(t) \in \mathcal{M}_n(K[[t]])$ and some $r \in \mathbb{N}_{\geq 1}$ such that

$$Z(x) = P(x^{1/r}) \exp \Big(\int \frac{D(x^{1/r})}{x^{(\tilde{q}+1))/r}} \Big) x^{C/r} \tag{1.2}$$

is a fundamental matrix of formal solutions of the system A.

(c) Another consequence of the expression (1.2) is that the ratio $\tilde{q}/r$ and the exponential part $D(x)$, modulo ramification of x, are both invariant under formal meromorphic gauge transformations. More precisely, if we have two systems A and B such that $B = \Psi_T[A]$ with $T \in GL_n(L_K)$ and we obtain (TRS)-normal forms of A and B as in item (i) of Theorem 1.3 with resulting Poincaré ranks $\tilde{q}_A$ and $\tilde{q}_B$ and exponential parts $D_A(x)$ and $D_B(x)$, respectively, then there are integers r_1, r_2 such that $\tilde{q}_A/r_1 = \tilde{q}_B/r_2$ and $D_A(x^{1/r_1}) = D_B(x^{1/r_2})$. In particular, $\tilde{q} = 0$ iff the system A is equivalent to a system of first kind under a formal gauge transformation.

(d) In the proof proposed below, one could see that the sequence of gauge transformations used in items (i) or (iii) can be chosen so that any one of them, individually, do not increase the Poincaré rank of the system it applies to in the process. As we know, this observation only concerns the diagonal monomial gauge transformations, since a regular gauge transformations always preserves the Poincaré rank.

(e) Below, we propose a bound, in terms of the eigenvalues of the residual matrix C, for the degree of the polynomial gauge transformation ϕ in item (iii).

1.2. **Real case.** Suppose that K is a real closed field, i.e., $K \subsetneq K(i) = \overline{K}$, where $i = \sqrt{-1}$. Given $\lambda = a + bi \in \overline{K}$, denote by

$$\Lambda_\lambda = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Recall that the characteristic polynomial of Λ_λ has roots $a \pm bi$ and is irreducible if and only if $\lambda \notin K$, i.e., $b \neq 0$. For any $m \in \mathbb{N}_{\geq 1}$, define the monomorphism of K-algebras

$$\Theta_m : \mathcal{M}_m(\overline{K}) \to \mathcal{M}_{2m}(K),$$

sending a matrix $C = (c_{uv}) \in \mathcal{M}_m(\overline{K})$ to the (2×2)-block matrix $(\Lambda_{c_{uv}}) \in \mathcal{M}_{2m}(K)$. A square matrix in the image of Θ_m will be called a *complex matrix over K*, or a $\mathbb{C}$-*matrix*, for short.

We extend Θ_m to a monomorphism of K-algebras, denoted with the same letter, from $\mathcal{M}_m(L_{\overline{K}})$ into $\mathcal{M}_{2m}(L_K)$; that is, from formal meromorphic linear systems over $\overline{K}$ to formal meromorphic linear systems over K of double dimension. A system in the image of this map will be called a *complex system (over K)* or a $\mathbb{C}$-*system*.

In what follows, if U, V are two square matrices of sizes k, l, respectively, we denote by $U \oplus V$ the square matrix of size $k + l$ given in blocks

$$U \oplus V := \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

Definition 1.6 (Real Turrittin-Ramis-Sibuya form). Suppose that K is a real and closed field. Let $A \in \mathcal{M}_n(L_K)$ be a system with Poincaré rank $q = q(A)$ and let $\mu \in \mathbb{N}_{\geq 0}$. We say that A is in *Real Turrittin-Ramis-Sibuya form of degree μ (and of rank q)*, or *in $(\mathbb{R}TRS)^q_\mu$-form* for short, if it can be written in the form

$$A(x) = x^{-(q+1)} \left(D_1(x) \oplus D_2(x) + x^q(C_1 \oplus C_2) + O(x^{q+\mu+1}) \right),$$

where

- $D_1(x) = \mathrm{diag}(e_1(x), \ldots, e_{n_1}(x))$ is diagonal polynomial with entries $e_j(x) \in K[x]$ of degree at most $q - 1$ (equal to zero if $q = 0$).
- $D_2(x) = \Theta_{n_2}(\mathrm{diag}(d_1(x), \ldots, d_{n_2}(x)))$ is a diagonal 2×2-block complex matrix such that the entries $d_j(x)$ belong to $\overline{K}[x] \setminus K[x]$ and are of degree at most $q - 1$ (equal to zero if $q = 0$).
- C_1 and C_2 are constant matrices with entries in K of sizes n_1 and $2n_2$, respectively, and C_2 is a $\mathbb{C}$-matrix.
- $[D_1(x), C_1] = 0$ and $[D_2(x), C_2] = 0$.

In this case, the truncated system $J_q(A) = x^{-(q+1)}(D_1(x) \oplus D_2(x) + x^q(C_1 \oplus C_2))$ is called the *principal part* of A, while $D_1(x) \oplus D_2(x)$, resp. $C_1 \oplus C_2$, is called the *exponential part*, resp. the *residual matrix*.

Our main result in the paper is the following real version of Theorems 1.3 and 1.4.

Theorem 1.7 (Real polynomial normal forms). *Suppose that K is a real closed field and let $A \in \mathcal{M}_n(L_K)$ be a singular system with Poincaré rank equal to $q = q(A)$.*

(i) *There exists $r \in \mathbb{N}_{\geq 1}$ and there exists finitely many polynomial gauge transformations $\varphi_1, \ldots, \varphi_m$ (with coefficients in K), either regular or diagonal monomial, such that, denoting*

$$\psi = \varphi_m \circ \cdots \circ \varphi_1 \circ R_r,$$

the transformed system $\widetilde{A} = \psi[A]$ is in $(\mathbb{R}TRS)^{\tilde{q}}_0$-form. Moreover, if $B \in \mathcal{M}_n(L_K)$ is another singular system with $q(B) = q$ and $J_{nq}A = J_{nq}B$ then, with the same transformation ψ, the transformed system $\widetilde{B} = \psi[B]$ is also in $(\mathbb{R}TRS)^{\tilde{q}}_0$ with the same principal part as $\widetilde{A}$, i.e., $J_{\tilde{q}}\psi[A] = J_{\tilde{q}}\psi[B]$.

(ii) *Assume that A is in $(\mathbb{R}TRS)^q_0$-form and that its residual matrix is non-resonant. Then, for any $\mu \geq 0$ there exists a regular polynomial gauge transformation $\phi_\mu = \Psi_{P^\mu}$ where $P^\mu(0) = I_n$ such that $\phi_\mu[A]$ is in $(\mathbb{R}TRS)^q_\mu$-form and with the same principal part as the system A. Moreover, the family $\{P^\mu\}_\mu$ can be chosen such that P^μ is of degree at most $q + \mu$ and satisfying that $J_\mu P^{\mu'} = J_\mu P^\mu$ for any $\mu' > \mu$.*

(iii) *Assume that A is in $(\mathbb{R}TRS)^q_0$-form. Then there exists a polynomial gauge transformation ϕ, given by a finite composition of regular or diagonal monomial transformations, such that $\phi[A]$ is in $(\mathbb{R}TRS)^q_0$-form with non-resonant residual matrix (and the same exponential part as A).*

Theorem 1.8. *Let K be a real closed field and let $A \in \mathcal{M}_n(L_K)$ be a singular system. Then there exists a formal gauge transformation ψ_P, with $P \in L_K$, and a ramification R_r such that the transformed system $F^{\mathbb{R}} = (\psi_P \circ R_r)[A]$ has Poincaré rank equal to $\tilde{q}$ and is written as:*

$$[F^{\mathbb{R}}] \quad Y' = x^{-(\tilde{q}+1)}(D_1(x) \oplus D_2(x) + x^{\tilde{q}}(C_1 \oplus C_2))Y,$$

where D_1, D_2, C_1, C_2 satisfy the conditions in Definition 1.6; that is $F^{\mathbb{R}}$ is in $(\mathbb{R}TRS)$-form and $F^{\mathbb{R}} = J_{\tilde{q}}F^{\mathbb{R}}$. Moreover, the transformation ψ_P can be chosen to be equal to $\Psi_P = \Psi_Q \circ \psi$, where $Q \in \mathcal{M}_n(K[[x]])$ satisfies $Q(0) = I_n$ and ψ is a finite composition of regular polynomial or diagonal monomial gauge transformations (with coefficients in K).

2. Proof of the complex Turrittin's theorem

Let $A(x) \in \mathcal{M}_n(L_K)$ be a system with Poincaré rank $q = q(A)$, written as in (1.1). Let $k = k(A)$ be the radiality index. Since the radial part $A_0 + xA_1 + \cdots + x^{k-1}A_{k-1}$ is preserved by any gauge transformation, the coefficient A_k is considered as the *first significant* matrix of the system. This must be compared with the usual proofs of Turrittin's theorem, where the radial part is ruled out by an *exponential shifting* so that A_k becomes the new leading coefficient (and q drops to $q - k$). In our approach, where we stress the finitely determined nature of the transformations, we do not allow the use of exponential shifting, so that the radial part is carried all along the procedure.

We denote $N(n, q, k) = n(q - k) + k$ as in Remark 1.5, (a) and consider the statement (i)' to be the same as item (i) in Theorem 1.3 but substituting in the second part the truncation order nq by $N(n, q, k)$.

For the proof of items (i)'-(iii) of Theorem 1.3, we perform, a priori in arbitrary ordering, several ramifications, or regular polynomial or diagonal monomial gauge transformations. The desired expression of the composition of those transformations required in the different items of the statement will be a consequence of the following lemma, whose proof is straightforward.

Lemma 2.1. *Fix a field κ. Let $r \in \mathbb{N}_{\geq 1}$, $P(x) \in \mathcal{M}_n(\kappa[x])$, and Ψ_P be the associated polynomial gauge transformation. Then there exists another polynomial gauge transformation $\Psi_{\tilde{P}}$ satisfying $R_r \circ \Psi_P = \Psi_{\tilde{P}} \circ R_r$. In fact, we can take $\tilde{P}(x) := P(x^r)$. In particular, Ψ_P is regular or diagonal monomial if and only if $\Psi_{\tilde{P}}$ is so.*

Another important tool is the following result, known with the name of Splitting Lemma, which is valid for any given base field, algebraically closed or not. It permits to reduce the dimension of the system when the first significant matrix has two disjoint subsets of eigenvalues. It is usually stated in the formal setting (see for instance [23, 2, 5]), but it has a finitely determined nature in terms of truncations of the system.

Lemma 2.2 (Splitting lemma). *Let K be a field. With the same notations as above, if k is the radiality index of the system A, assume that $k < q$ and that A_k is conjugated to $A_k^{11} \oplus A_k^{22}$, where the characteristic polynomials $\chi_{A_k^{11}}(\lambda)$ and $\chi_{A_k^{22}}(\lambda)$ are coprime, both of positive degrees, say n_1 and n_2, respectively. Then there exists a formal regular gauge transformation Ψ_T, where $T \in \mathcal{M}_n(K[[x]])$ satisfies $T(0) = I_n$, such that the transformed system $B = \Psi_T[A]$ writes as $B = B^{11} \oplus B^{22}$, where*

$B^{ii} \in \mathcal{M}_{n_i}(L_K)$ *is a system of dimension* n_i *for* $i = 1, 2$. *Moreover,* $q(B) = q$, $k(B) = k$ *and, writing* $B(x) = x^{-(q+1)}(\sum_{j \geq 0} x^j B_j)$, *we have* $A_j = B_j$ *for* $j = 0, 1, \ldots, k$ *and for any* $m > k$, *the truncation* $J_m B$ *only depends on* $J_{m-k}T$ *and* $J_m A$. *In other words, if* $\widetilde{A}$ *is another system with the same Poincaré rank* $q(\widetilde{A}) = q$ *and satisfying* $J_m\widetilde{A} = J_m A$ *then* $J_m \Psi_{T^{(m-k)}}[\widetilde{A}] = J_m B$, *where* $T^{(m-k)} = J_{m-k}T$.

2.1. **Proof of Theorem 1.3, (i)'.** Getting a (TRS)-form of degree 0 and some rank $\tilde{q}$. First, notice that the cases $q = 0$ and $q = k$ (the former being a particular case of the later, by definition) are trivial: in these cases, the system is already in $(TRS)_0^q$-form.

We proceed by induction on the dimension n of the system. The starting case $n = 1$ is also trivial. Assume then that $n > 1$.

For readability, let us summarize the proof that follows. We consider first the case where A_k has more than one eigenvalue, in which case, Splitting Lemma permits to decompose a truncation of the system into two systems of lower dimension and apply induction. Then we consider the case where A_k has a single eigenvalue. In this case, we define a tuple $I(A) = (\gamma(A_k), q - k) \in \mathbb{N}^{n+1}$ associated with the system, where $\gamma(A_k)$ only depends on the conjugation class of A_k. The aim is to define a series of transformations so that the resulting system B is in the precedent cases or else verifies $I(B) < I(A)$ in the lexicographical order. To choose the transformations, we define an "exponent" $g = g(A)$, a positive rational number that depends only on the truncation $J_N A$ of the system, where $N = N(n, q, k)$, after a finer preparation of A by a regular polynomial gauge transformation. If g is an integer, it determines a special type of monomial diagonal transformation to be done to win (a shearing transformation). If $g = h/r$ is the irreducible expression for g and $r > 1$, we perform first the ramification R_r and then the shearing associated with the numerator h does the work.

2.1.1. *Case with different eigenvalues.* Suppose that we are in the case where A_k has at least two different eigenvalues. Then we can reduce to a smaller dimension as follows. First, up to a constant regular gauge transformation we can assume that $A_k = A_k^{11} \oplus A_k^{22}$ where A_k^{11} and A_k^{22} are matrices of respective sizes n_1, n_2, both smaller than n, and having no common eigenvalue. Using Lemma 2.2 for $N = N(n, q, k)$, there exists a regular polynomial gauge transformation Ψ_P such that the N-truncation of $B = \Psi_P(A)$ is written as $J_N B = B^{11} \oplus B^{22}$, where, for $i = 1, 2$, B^{ii} is a system of dimension n_i. Moreover, $J_N B$ has the same Poincaré rank, the same radiality index and the same k-truncation than A. In particular, if $q_i = q(B^{ii})$ and $k_i = k(B^{ii})$ then $q_i \leq q$ and $q_i - k_i \leq q - k$. Taking into account that n_1 and n_2 are both positive and hence strictly smaller than n, we obtain for $i = 1, 2$ that

$$N(n, q, k) \geq n_i(q - k) + q \geq n_i(q_i - k_i) + q_i \geq n_i(q_i - k_i) + k_i = N(n_i, q_i, k_i).$$

Using the induction hypothesis to system B^{ii} for $i = 1, 2$, there is a finite composition ψ^{ii} of transformations in dimension n_i (as in statement (i)) such that $\widetilde{B}^{ii} = \psi^{ii}(B^{ii})$ is in $(TRS)_0^{q_i}$-form and such that the second part of statement (i)' holds for the truncation order $N(n_i, q_i, k_i)$ in the place of $n_i q_i$. Moreover, in the composition ψ^{ii} there is but a single ramification R_{r_i} with $r_i \in \mathbb{N}_{\geq 1}$ (including the case $R_1 = id$). Now, for $\{i, j\} = \{1, 2\}$ write, using Lemma 2.1,

$$R_{r_j} \circ \psi^{ii} = \widetilde{\varphi}^{ii} \circ R_{r_1 r_2},$$

where $\widetilde{\varphi}^{ii}$ is a composition of transformations in dimension n_i, either regular polynomial or diagonal monomial (that is, no ramification). Notice that the composition $R_{r_j} \circ \psi^{ii}$ satisfies the requirements of Theorem 1.3, (i)' for the system $R_{r_j}(B^{ii})$, for which the Poincaré rank and radiality index are equal to $r_j q_i$ and $r_j k_i$, respectively. Writting $\widetilde{\varphi}^{ii} = \Psi_{Q_i}$, where $Q_i \in \mathcal{M}_{n_i}(K[x])$, we put $r = r_1 r_2$ and define

$$\psi = \Psi_{Q_1 \oplus Q_2} \circ R_r \circ \Psi_P = \varphi \circ R_r,$$

where $\varphi = \Psi_{Q_1 \oplus Q_2} \circ \Psi_{P(x^r)}$ is a composition of gauge transformations, either regular polynomial or diagonal monomial. We check that ψ satisfies the requirements of statement (i)' for the initial system A. To be convinced, we need to observe two facts. On one hand, for $\{i, j\} = \{1, 2\}$, the composition $R_{r_j} \circ \psi^{ii}$ satisfies all the requirements of (i)' for the system B^{ii}, since so does ψ^{ii} by construction (notice that if B is any system then we have $q(R_r(B)) = rq(B)$, $k(R_r(B)) = rk(B)$ and for any $M \geq 0$, the truncation $J_{rM}R_r(B)$ is univocally determined by $J_M B$). On the other hand, use the second part of Lemma 2.2 to conclude that $J_N \Psi_P(A)$ only depends on $J_N A$ for $N = N(n, q, k)$ and the inequality $N(n_i, q_i, k_i) \leq N$ for $i = 1, 2$ proved above.

2.1.2. *Case with a single eigenvalue.* Suppose now that A_k has a single eigenvalue $\lambda_k \in \overline{K}$. Recall that we are assuming that $K = \overline{K}$. So, up to a constant gauge transformation, we may suppose that A_k is in Jordan normal form. Explicitly, there exists a (unique) sequence $1 \leq n_1 \leq n_2 \leq \cdots \leq n_\ell \leq n$ with $n = n_1 + \cdots + n_\ell$ such that

$$A_k = (\lambda_k I_{n_1} + H^{(n_1)}) \oplus \cdots \oplus (\lambda_k I_{n_\ell} + H^{(n_\ell)}), \tag{2.1}$$

where

$$H^{(n_j)} = \begin{pmatrix} 0 & 1 & & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & & \vdots & & \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}_{n_j \times n_j}$$

(each such matrix will be called in the sequel a *shifting matrix*). Notice also that, since A_k is not a radial matrix, we have $n_j > 1$ for at least one index j. We divide the proof in different steps.

Step 1. The tuple $I(A)$. In the situation above, for $i = 1, \ldots, n$, denote by $\gamma_i(A_k) \in \mathbb{N}_{\geq 0}$ the degree, as a polynomial in λ, of the g.c.d. of the family of all $i \times i$ minors of the characteristic matrix $A_k - \lambda I_n$. I n particular $\gamma_n(A_k) = n$. For such a system A (when A_k has a single eigenvalue), we define the following tuple of non-negative integer numbers

$$I(A) := (\gamma_1(A_k), \ldots, \gamma_n(A_k), q - k).$$

We remark that each $\gamma(A_k)$ depends only on the conjugation class of A_k. We need to recall also the following result on Linear Algebra (see in Wasow [23, Lemma 19.4] for a proof) concerning the behaviour of the values γ_i for a perturbation of the matrix A_k in the case where A_k has at least two blocks.

Lemma 2.3. *Consider A_k as a block-diagonal Jordan matrix $A_k = \left(A_k^{(ij)} \right)$, where the diagonal blocks are given by $A_k^{(ii)} = \lambda_k I_{n_i} + H^{(n_i)}$. Assume that $\ell \geq 2$. With the same block structure, let $G = \left(G^{(ij)} \right) \in \mathcal{M}_n(K)$ be a block-lower-triangular matrix*

with the same diagonal blocks as A_k (that is, $G^{(ii)} = A_k^{(ii)}$ and $G^{(ij)} = 0$ if $i < j$). Then we have

$$\gamma_t(G) \le \gamma_t(A_k) \quad \text{for each } t \in \{1, 2, \ldots, n\}$$

and the inequality is strict for at least one t if $G \ne A_k$ (that is, $G^{(ij)} \ne 0$ for at least a pair (i, j) with $i > j$).

Step 2. Special matrices and choice of $g \in \mathbb{Q}$. Given an n-tuple $\sigma = (n_1, n_2, \ldots, n_\ell)$ of positive integers as above, denote $m_j := \sum_{u=1}^{j} n_u$ for $j = 1, \ldots, \ell$. A matrix $T = (t_{uv}) \in \mathcal{M}_n(K)$ will be called a *special matrix of type* σ if $t_{uv} = 0$ for every (u, v) such that $u \notin \{m_1, m_2, \ldots, m_\ell\}$.

A result in Wasow ([23, Lemma 19.2]) assures that for any $N \ge k + 1$ there exists a regular polynomial gauge transformation Ψ_{P_N}, where P_N is of degree N and $P_N(0) = I_n$, such that the transformed system $B = \Psi_{P_N}(A)$ satisfies that all coefficients of the truncation $J_N(B)$ are special matrices of type $\sigma = (n_1, n_2, \ldots, n_\ell)$, where the n_j are the sizes of Jordan blocks of the matrix A_k. We will put $N = N(n, q, k)$ and, renaming B again as the system A, we may assume the following assumption

For each $k + 1 \le j \le N$, the coefficient A_j is a special matrix of type σ.   (2.2)

Now we write

$$x^{q+1} A(x) = A_0 + x A_1 + \cdots = \sum_{i=0}^{k} \lambda_i x^i I_n + x^k \overline{A},$$

with $\overline{A} = A_k - \lambda_k I_n + O(x) = (\overline{a}_{uv}(x))$. We put $\alpha_{uv} := \nu(\overline{a}_{uv}(x))$. Notice that $\alpha_{uv} \in \mathbb{N}_{\ge 0}$ for any u, v, that $\alpha_{uu} > 0$ for any u and that $\alpha_{u,u+1} = 0$ for at least one index $u \in \{1, \ldots, n\}$.

Definition 2.4. With the conditions above, we define the *shearing order (of A)* as the rational number

$$g = g(A) := \min \left(\{ \frac{\alpha_{uv}}{1 + u - v} : u > v \} \cup \{ \alpha_{uu} : 1 \le u \le n \} \cup \{ q - k \} \right).$$

Geometrically, as discussed in Wasow [23, pp.104-105], the shearing order is the smallest abscissa in which a line of the family

$$\{ y = (v - u)x + \alpha_{uv} \}_{v < u} \cup \{ y = \alpha_{uu} \}_u \cup \{ y = q - k \}$$

cuts the diagonal $y = x$.

Notice also that $g \in \mathbb{Q}$ and that $0 < g \le q - k$. Write $g = \frac{h}{r}$ where h, r are positive integers with no common factor.

The name "shearing" comes from Wasow's name for the gauge "ramified" transformation with matrix $S_g = \text{diag}(1, x^g, x^{2g}, \ldots, x^{(n-1)g})$. In our definitions, such a transformation has only sense if $g = h$ is an integer. Otherwise, we perform instead the composition $\Psi_{S_h} \circ R_r (=: \text{``}\Psi_{S_g}\text{''})$.

Step 3. The case where $g = h$ is integer. We consider the monomial diagonal gauge transformation Ψ_{S_h} where $S_h = \text{diag}(1, x^h, x^{2h}, \ldots, x^{(n-1)h})$. If $A = (a_{uv}(x))$ then the transformed system $B := \Psi_{S_h}[A] = S_h^{-1} A S_h - S_h^{-1} S_h'$ writes as

$$B = \left(x^{h(j-i)} a_{ij}(x) \right)_{i,j} - x^{-1} K, \tag{2.3}$$

where K is a constant diagonal matrix (in fact $K = \text{diag}(0, h, 2h, \ldots, (n-1)h)$).

By the choice of the shearing order $g = h$, one can see that we can write $B(x) = x^{-(q+1)}(\widetilde{B}_0 + x\widetilde{B}_1 + \dots)$, where the $\widetilde{B}_j$ are constant matrices satisfying

(a) $\widetilde{B}_j$ is diagonal for $j \in \{0, 1, \dots, k+h-1\}$, with $\widetilde{B}_j = A_j$ if $j < k$, $\widetilde{B}_k = \lambda_k I_n$ and $\widetilde{B}_j = 0$ if $k < j < k + h$.

(b) The entries of $\widetilde{B}_{k+h}$ above the principal diagonal coincide with those of the matrix $H^{(n_1)} \oplus \cdots \oplus H^{(n_\ell)}$. Moreover, $\widetilde{B}_{k+h}$ has a non-zero entry on or above the principal diagonal, except, possibly, if $h = q - k$.

(c) For $M \geq k$, the matrix $\widetilde{B}_M$ depends only on the truncation $J_{M+(n-1)h}(A)$.

As a consequence, the Poincaré rank after the shearing transformation does not increase. That is,

$$q(B) \leq q. \tag{2.4}$$

Moreover, using (a) and (b), and denoting $q' = q(B)$, $k' = k(B)$, we have that if $q' = q$ then $k' = k + h$, while, if $q' < q$ then $k = 0$, $q' = q - h$ and $k' = 0$. In any case, we deduce that

$$q' - k' = q - k - h \tag{2.5}$$

and, since $k' \leq k + h$, that

$$N(n, q', k') + (n - 1)h \leq N(n, q, k). \tag{2.6}$$

Thus, using the property (c) and inequality (2.6), it will suffice to prove item (i)' of Theorem 1.3 for the system B.

Write, accordingly to our main notations, $B(x) = x^{-(q'+1)}(B_0 + xB_1 + \dots)$. We will be done if B is in the trivial case $q' - k' = 0$ or if $q' - k' > 0$ but $B_{k'}$ has at least two different eigenvalues. Thus, assume that $q' - k' > 0$ and that $B_{k'}$ has a single eigenvalue, say equal to $\lambda'_{k'}$. It is enough, by recurrence, to prove in this case that the tuple $I(B)$ satisfies $I(B) < I(A)$, for the lexicographical order.

Notice that $B_{k'} = \widetilde{B}_{k+h}$, after the discussion above concerning the value of q' and k'. Consider the matrix $B_{k'}$ written in blocks according to the Jordan structure of A_k:

$$B_{k'} = \left(B_{k'}^{(ij)}\right)_{1 \leq i,j \leq \ell}, \quad B_{k'}^{(ij)} \in \mathcal{M}_{n_j \times n_j}(K).$$

Using the property (b), we have that $B_{k'} = H^{(n_1)} \oplus \cdots \oplus H^{(n_1)} + T$ where T is a lower triangular matrix. In particular, for $i \in \{1, \dots, \ell\}$, since $\mathrm{Spec}(B_{k'}^{(ii)}) = \{\lambda'_{k'}\}$, the matrix $B_{k'}^{(ii)} - \lambda'_{k'} I_{n_i}$ is nilpotent and of rank $n_i - 1$. This implies that the Jordan decomposition of $B_{k'}^{(ii)}$ has a single block and hence $B_{k'}^{(ii)}$ is conjugated to $H^{n_i} + \lambda'_{k'} I_{n_i}$. Consequently, $B_{k'}$ is conjugated to a block-lower-triangular matrix with diagonal blocks equal to $H^{n_i} + \lambda'_{k'} I_{n_i}$, those of the matrix A_k. Thus, either $B_{k'}$ is conjugated to A_k (when $\ell = 1$) or we are in the situation where we can apply Lemma 2.3. From this lemma, and taking into account the equation (2.5), we deduce that $I(B) < I(A)$, as wanted.

Step 4. The case where g is not an integer. Assume now that the shearing order g is not an integer and put $g = \frac{h}{r}$ where h, r are positive integers without common factor and such that $r \geq 2$. We remark that the condition that g is not an integer implies that g is given by one of the quotients $\alpha_{uv}/(1 - u - v)$ in Definition 2.4, while $\alpha_{uu} > g$ for any u, as well as $q - k > g$.

Consider the ramification R_r and put $\widetilde{A} = R_r(A)$. This system has Poincaré rank and radiality index equal to $q(\widetilde{A}) = rq$ and $k(\widetilde{A}) = rk$, respectively. Moreover, if

we denote $\widetilde{N} = N(n, rq, rk)$ and $N = N(n, q, r)$, then $\widetilde{N} = rN$ but the truncation $J_{\widetilde{N}}(\widetilde{A})$ only depends on $J_N(A)$, showing that if item (i)' of Theorem 1.3 holds for $\widetilde{A}$ then it also holds for A.

We notice moreover that the first non-radial coefficient of $\widetilde{A}$ is equal to A_k and that $\widetilde{A}$ also satisfies assumption (2.2) with respect to the same type $\sigma = (n_1, \ldots, n_\ell)$ given by the Jordan structure of A_k. On the other hand, the shearing order of $\widetilde{A}$ is given by $g(\widetilde{A}) = rg = h$, a natural number (the new valuations α_{uv} for $\widetilde{A}$ are all multiplied by r). Hence we are, for $\widetilde{A}$, in the situation of step 3. However, the last component $q - k$ of the tuple $I(\widetilde{A})$ has increased and we can not conclude automatically. To finish, we put $B = \Psi_{S_h}(\widetilde{A})$, we denote by q', r' the Poincaré rank and the radiality index of B, respectively, and, writing $B = x^{-(q'+1)}(B_0 + xB_1 + \ldots)$, we show again that assuming that $q' - k' > 0$ and that $B_{k'}$ has a single eigenvalue, we have $I(B) < I(A)$.

Write $B_{k'} = \left(B_{k'}^{(ij)}\right)$ as a block matrix in the same block structure as A_k. We have (cf. property (b) in step 3 above) that $B_{k'}$ is block-lower triangular with diagonal blocks given by $B_{k'}^{(ii)} = H^{(n_i)} + T_i$, where T_i is a lower triangular matrix of size n_i. Moreover, we must have that all elements in the diagonal of T_i are zero (since all values α_{uu} for $\widetilde{A}$ are greater than $h = rg$, as mentioned above). On the other hand, by assumption (2.2), the entries of T_i on any row except possibly the last one are also zero. On the other hand, since we have assumed that each block $B_{k'}^{(ii)}$, as the entire matrix $B_{k'}$, has a single eigenvalue, being the trace of $B_{k'}^{(ii)}$ equal to zero, such eigenvalue is equal to zero. That is, each $B_{k'}^{(ii)}$ is nilpotent. But this implies that $T_i = 0$ and we conclude that

$$B_{k'}^{(ii)} = H^{(n_i)}, \quad \text{for each } i \in \{1, 2, \ldots, \ell\}. \tag{2.7}$$

Furthermore, by the definition of the shearing order of $\widetilde{A}$, we must have at least one non-zero entry below the principal diagonal of the matrix $B_{k'}$. Together with the equation (2.7), this implies that $\ell > 1$ and that $B_{k'}$ is a matrix G in the situation of Lemma 2.3 with a non-diagonal block $B_{k'}^{(ij)} \neq 0$ for at least one pair (i, j) with $i > j$. We conclude that $\gamma_t(B_{k'}) < \gamma_t(A_k)$ for at least one index t and thus $I(B) < I(A)$, as wanted.

Let us show now the statement in Remark 1.5, (c) concerning this item (i). In other words, we have to justify that all along the above process for obtaining a TRS-form of degree 0, the Poincaré rank can only increase after a ramification and never after a shearing transformation Ψ_{S_g} with $g \in \mathbb{N}$.

First, notice that this property goes through the induction arguments discussed in paragraph 2.1.1. Thus, we may assume that we are in the case where A_k has a unique eigenvalue. If the shearing order $g = g(A)$ is an integer, the required property is already established by equation (2.4). On the contrary, if the shearing order is $g = h/r$, with $r > 1$ and h not divisible by r, the procedure consists in the shearing Ψ_{S_h} after the ramification R_r. We conclude using the same equation (2.4) once we observe that the system $\widetilde{A} := R_r[A]$ has as first non-radial term the same matrix A_k and satisfies $g(\widetilde{A}) = h \in \mathbb{N}$ (just check that $\widetilde{A}$ satisfies already the property (2.2) and that the new values α_{uv} in Definition 2.4 are the old ones multiplied by r).

2.2. **Proof of Theorem 1.3, (ii).** Getting (TRS)-form of higher degree μ when C is non-resonant.

A proof of item (ii) when $q = 0$ can be found for instance in Wasow [23, Thm. 5.1], Coddington-Levinson [11, Thm. 4.1, Ch IV] or Balser [2, Thm. 5]. The general case $q > 0$ is not really different from those references because the exponential part $D(x)$, being diagonal and commuting with C will play no essential role to achieve (ii). For the sake of completeness, let us indicate here the steps of the general proof.

Assume that the system A is already in $(TRS)_0^q$-form with principal part $D(x) + x^q C$ and write it in the form $x^{q+1} Y' = (A_0 + x A_1 + \dots) Y$, so that

$$x^{q+1} J_q A = A_0 + \dots + x^q A_q = D(x) + x^q C.$$

We will need the following result from Linear Algebra (which is actually the core of the proof of the Splitting Lemma 2.2). See [23, Thm. 4.1] for a proof.

Lemma 2.5. *Fix a field K and let R, S be two square matrices with coefficients in K and of sizes $n \times n$ and $m \times m$, respectively. Assume that R, S have no common eigenvalue in the algebraic closure $\overline{K}$ of K. Then the linear map $X \mapsto RX - XS$ from $\mathcal{M}_{n \times m}(K)$ to itself is an isomorphism.*

Up to reorder the variables of Y, we write

$$D(x) = D^{11}(x) \oplus D^{22}(x) \oplus \dots \oplus D^{\ell\ell}(x), \qquad (2.8)$$

where $D^{jj}(x)$ is a radial matrix of size n_j (that is $D^{jj}(x) = Q_j(x) I_{n_j}$, where $Q_j(x) \in K[x]_{q-1}$) for each $j = 1, \dots, \ell$, and such that $D^{ii}(x) \neq D^{jj}(x)$ if $i \neq j$. Write also the residual matrix C, as well as any coefficient matrix A_j with $j \geq q + 1$, as block matrices $C = (C^{uv})_{1 \leq u,v \leq \ell}$ and $A_j = (A_j^{uv})_{1 \leq u,v \leq \ell}$, where $C^{uv}, A_j^{uv} \in \mathcal{M}_{n_u \times n_v}(K)$. Using the commutativity property $[D(x), C] = 0$, the assumption $D^{ii}(x) \neq D^{jj}(x)$ for $i \neq j$ and Lemma 2.5, we obtain that

$$C^{uv} = 0 \quad \text{if } u \neq v.$$

We eliminate all coefficients A_j for $j \geq q + 1$ by means of a regular formal transformation in two steps.

First step. We eliminate all non-diagonal blocks $\{A_j^{uv}\}_{u \neq v}$. We proceed by induction with respect to $q = q(A)$. If $q = 0$ (that is, $D(x) = 0$), the block structure is the trivial one and there are no non-diagonal blocks, so that there is nothing to prove.

Suppose that $q > 0$. We consider a coarser block structure

$$D(x) = \overline{D}^{11}(x) \oplus \dots \oplus \overline{D}^{\ell_1 \ell_1}(x) \qquad (2.9)$$

in such a way that $\overline{D}^{jj}(0)$ is a radial constant matrix for every $j = 1, \dots, \ell_1$ and $\overline{D}^{jj}(0) \neq \overline{D}^{ii}(0)$ if $i \neq j$. Notice that, up to reordering, each block $\overline{D}^{jj}(x)$ is formed by several of the diagonal blocks of the decomposition (2.8). We consider a matrix $\overline{T}(x) \in \mathcal{M}_n(K[[x]])$ of the form $\overline{T}(x) = I_n + x^{q+1} \overline{T}_{q+1} + x^{q+2} \overline{T}_{q+2} + \dots$ such that, writing each coefficient $\overline{T}_j = (\overline{T}_j^{uv})$ in the same block structure as the one in (2.9), we have $\overline{T}_j^{uu} = 0$ for any $u \in \{1, \dots, \ell_1\}$. The regular gauge transformation $\psi_{\overline{T}}$ transforms the system A into a system $\overline{B} := \psi_{\overline{T}}[A]$ which is also in $(TRS)_0^q$-form with the same principal part $D(x) + x^q C$. Write $x^{q+1} \overline{B} = D(x) + x^q C + \sum_{j \geq q+1} x^j \overline{B}_j$, and, for any j, consider each coefficient of A or $\overline{B}$ written in the

block structure (2.9) as $A_j = (\overline{A}_j^{uv})$ and $\overline{B}_j = (\overline{B}_j^{uv})$, respectively. We obtain recursively, for any couple of indices $u, v \in \{1, \ldots, \ell_1\}$ with $u \neq v$:

$$\overline{B}_{q+1}^{uv} = \overline{A}_{q+1}^{uv} + \overline{D}_0^{uu}\overline{T}_{q+1}^{uv} - \overline{T}_{q+1}^{uv}\overline{D}_0^{vv}$$

$$\overline{B}_{q+2}^{uv} = \overline{A}_{q+2}^{uv} + \overline{D}_0^{uu}\overline{T}_{q+2}^{uv} - \overline{T}_{q+2}^{uv}\overline{D}_0^{vv} + \theta_2^{uv}(A_0, A_1, \overline{T}_{q+1})$$

$$\cdots \tag{2.10}$$

$$\overline{B}_{q+j}^{uv} = \overline{A}_{q+j}^{uv} + \overline{D}_0^{uu}\overline{T}_{q+j}^{uv} - \overline{T}_{q+j}^{uv}\overline{D}_0^{vv} + \theta_j^{uv}(\{A_s, \overline{T}_{q+s}\}_{s<j})$$

$$\cdots$$

where each θ_j^{uv} is a polynomial matrix acting on the block entries of the explicitly indicated matrices. Using Lemma 2.5 with $R = \overline{D}_0^{uu}$ and $S = \overline{D}_0^{vv}$, we obtain recursively blocks $\overline{T}_{q+1}^{uv}, \overline{T}_{q+2}^{uv}, \ldots$ (i.e. the entire matrix $\overline{T}(x)$) so that $\overline{B}_j^{uv} = 0$ for every $j \geq q + 1$ and for every $u, v \in \{1, \ldots, \ell_1\}$ with $u \neq v$. In other words, $\overline{B} = \psi_{\overline{T}}[A]$ is a block-diagonal formal system $\overline{B} = \overline{B}^{11} \oplus \cdots \oplus \overline{B}^{\ell_1\ell_1}$. Now, we consider separately each one of the subsystems $\overline{B}^{jj}$, for $j = 1, \ldots, \ell_1$. By construction, this system is in (TRS)-form of degree 0 and with exponential part equal to $\overline{D}^{jj}(x) = \overline{D}_0^{jj} + \widetilde{D}^{jj}(x)$. Put $\widetilde{B}^{jj} := \overline{B}^{jj} - x^{-(q+1)}\overline{D}_0^{jj}$, a system with Poincaré rank equal to $q(\widetilde{B}^{jj}) = q - \nu(\widetilde{D}^{jj}(x)) < q$, and also in (TRS)-form of degree 0 and exponential part equal to $x^{-\nu_j}\widetilde{D}^{jj}(x)$, where $\nu_j := \nu(\widetilde{D}^{jj}(x))$. Consider a block decomposition

$$\widetilde{D}^{jj}(x) = \widetilde{D}^{jj,1}(x) \oplus \cdots \oplus \widetilde{D}^{jj,\ell^j}(x) \tag{2.11}$$

analogous to that of $D(x)$ in equation (2.8); that is, each block in (2.11) is a radial matrix and two such blocks are different. By recurrence on q, there is a formal regular gauge transformation with matrix $T^j(x) = I + \ldots$ (with size equal to the size of the system $\overline{B}^{jj}$) such that $\Psi_{T^j}[\widetilde{B}^{jj}]$ is block-diagonal in the same block structure as (2.11). Taking into account that $\overline{D}_0^{jj}$ is a radial matrix, the same is true for $\Psi_{T^j}[\overline{B}^{jj}]$. Put

$$T(x) = (T^1(x) \oplus \cdots \oplus T^{\ell_1}) \cdot \overline{T}(x) = I + \cdots \in \mathcal{M}_{n \times n}(K[[x]]).$$

Then, $B := \Psi_T[A]$ is a system in $(TRS)_0^q$-form with the same principal part as A and block-diagonal with respect to the concatenation of the different block-structures given by (2.11); i.e., $D(x) = \oplus_{j=1}^{\ell_1}\widetilde{D}^{jj}(x) = \oplus_{j=1}^{\ell_1} \oplus_{i=1}^{\ell^j} \widetilde{D}^{jj,i}$. We are done, since this last decomposition is, up to reordering, the same as the initial one (2.8).

Step 2. Eliminating the diagonal terms. Consider the block-diagonal system $B = \Psi_T[A] = B^{11} \oplus \cdots \oplus B^{\ell\ell}$ as in the step 1, in the block structure given by (2.8). We propose a second formal regular gauge transformation associated with $U \in \mathcal{M}_n(K[[x]])$ of the form

$$U = (I_{n_1} + U^{11}) \oplus \cdots \oplus (I_{n_\ell} + U^{\ell\ell}),$$

where U^{uu} has size n_u and $U^{uu}(0) = 0$ for any u. Write $U^{uu} = xU_1^{uu} + x^2U_2^{uu} + \ldots$. The transformed system $E := \Psi_U[B]$ is again block-diagonal and $x^{q+1}J_qE = D(x) + x^qC$. We write $E = E^{11} \oplus \cdots \oplus E^{\ell\ell}$ and $x^{q+1}E^{uu} = D^{uu} + x^qC^{uu} + x^{q+1}E_{q+1}^{uu} + \ldots$ for each block E^{uu}. Taking into account that each block D^{uu} is radial and hence commutes with any matrix, we obtain recursively for any $u \in \{1, \ldots, \ell\}$ and for any $j \geq 1$

$$E_{q+j}^{uu} = B_{q+j}^{uu} + C^{uu}U_j^{uu} - U_j^{uu}(C^{uu} + jI_{n_u}) + \eta_j^{uu}(\{U_s^{uu}, B_{q+s}^{uu}\}_{s<j}), \tag{2.12}$$

where, as in equation (2.10), η_j^{uu} is a polynomial acting over the expressed matrices, with $\eta_1^{uu} = 0$. Using again Lemma 2.5 with matrices $R = C^{uu}$ and $S = C^{uu} + jI_{n_u}$ (notice that these two matrices have no common eigenvalue by the non resonance condition for C), we can construct recursively from (2.12) the coefficients $U_{q+1}^{uu}, U_{q+2}^{uu}, \ldots$ such that $E_{q+j}^{uu} = 0$ for any $u \in \{1, \ldots, \ell\}$ and for any $j \geq 1$.

The conclusion from the two steps above is that the composition $\Psi_{TU} = \Psi_U \circ \Psi_T$ satisfies

$$\Psi_{TU}[A] = x^{-(q+1)}(D(x) + x^q C).$$

To finish, notice from the expression of the transform of a system by a regular gauge transformation that, for every $\mu \geq 0$, the $(q + \mu)$-jet of $\Psi_{TU}[A]$ only depends on the $(q + \mu)$-jet of the product TU (and on $J_{q+\mu}A$). Thus, we put $P^\mu := J_{q+\mu}(TU)$ and Theorem 1.3, (ii) follows.

2.3. Proof of Theorem 1.3, (iii). Getting a non-resonant residual matrix. A proof of this item, for $q = 0$, can be found in the same references cited in paragraph 2.2 above.

We assume that A is in $(TRS)_0^q$ form with exponential part equal to $D(x)$ and residual part equal to C.

Radial case. We consider first the case where $D(x)$ is a radial matrix; i.e., $D(x) = Q(x)I_n$, where $Q(x)$ is a polynomial (including the case $q = 0$ for which $D(x) = 0$).

Take the partition $\mathrm{Spec}(C) = \Omega_1 \cup \cdots \cup \Omega_r$ of the spectrum of C in such a way that two eigenvalues differ by a non-zero integer number if and only if they belong both to some Ω_j. Denote $\Omega_j = \{\lambda_1^j, \ldots, \lambda_{s_j}^j\}$ and assume that the indices are chosen so that $\mathrm{Re}(\lambda_1^j) > \mathrm{Re}(\lambda_2^j) > \cdots > \mathrm{Re}(\lambda_{s_j}^j)$ in case $s_j > 1$. Let

$$m(C) := \sum_{\{j : s_j > 1\}} \lambda_1^j - \lambda_{s_j}^j \in \mathbb{N}. \tag{2.13}$$

Notice that $m(C) > 0$ if and only if C is resonant. In order to prove item (iii) in this case, we show that, if $m(C) > 0$, there is a constant regular transformation Ψ_{P_0} and a monomial diagonal transformation Ψ_S such that the transformed system $\Psi_S \circ \Psi_{P_0}[A]$ has Poincaré rank equal to q, the same exponential part $D(x)$ and a residual matrix C' satisfying $m(C') < m(C)$.

The matrix P_0 is chosen so that $\overline{C} := P_0^{-1}CP_0$ is block diagonal of the form

$$\overline{C} = (C^{11} \oplus \cdots \oplus C^{1s_1}) \oplus \cdots \oplus (C^{r1} \oplus \cdots \oplus C^{rs_r}), \tag{2.14}$$

where $\mathrm{Spec}(C^{ji}) = \{\lambda_i^j\}$. The transformed system $\overline{A} = \Psi_{P_0}[A] = P_0^{-1}AP_0$ has the same exponential part $D(x)$ (since this is a radial matrix) and residual matrix equal to $\overline{C}$. Put $x^{q+1}\overline{A} = \sum_{l \geq 0} \overline{A}_l x^l$ and use the block structure given by equation (2.14) for each coefficient $\overline{A}_l = (\overline{A}_l^{uv})$, where (uv) run in the set $\{(ji) : j = 1, \ldots, r, i = 1, \ldots, s_j\}$. Assume for instance that Ω_1 has at least two elements. Then we consider the diagonal monomial matrix

$$S := xI_t \oplus I_{n-t}, \tag{2.15}$$

where t is equal to the size of C^{11}. Consider $B := \Psi_S[\overline{A}]$ and write $x^{q+1}B = \sum_{l \geq 0} B_l x^l$ and $B_l = (B_l^{uv})$ with the same block structure as in (2.14). A calculation

shows that

$$B_l^{uu} = \overline{A}_l^{uu}, \quad \text{if } l \neq q,$$
$$B_l^{uv} = \overline{A}_l^{uv}, \quad \text{if } u \neq (11) \text{ and } v \neq (11),$$
$$B_l^{(11)v} = \overline{A}_{l+1}^{(11)v}, \quad \text{if } v \neq (11), \tag{2.16}$$
$$B_l^{u(11)} = \overline{A}_{l-1}^{u(11)}, \quad \text{if } u \neq (11)$$
$$B_q^{(11)(11)} = \overline{A}_q^{(11)(11)} - I_t.$$

In particular, we obtain $q(B) = q$, B is in $(TRS)_0^q$-form with the same exponential part $D(x)$ and the residual matrix $C' = B_q$ is upper triangular with respect to the block structure (B_q^{uv}) and with diagonal equal to

$$\text{diag}(C') = ((C^{11} - I_t) \oplus \cdots \oplus C^{1s_1}) \oplus \cdots \oplus (C^{r1} \oplus \cdots \oplus C^{rs_r}).$$

We deduce that $m(C') = m(C) - 1$ and we are done.

General case. Notice that, in preceding case, the degree of a polynomial gauge transformation needed to obtain a non-resonant residual matrix can be bounded by $m(C)$ (cf. Remark 1.5, (e)). Moreover, such transformation depends only on the truncation $J_{m(C)+q}(A)$. We use this remark and Step 1 in paragraph 2.2 to reduce the general case to the precedent case. More precisely, consider the decomposition of the exponential part in radial matrices $D(x) = D^{11}(x) \oplus \cdots \oplus D^{\ell\ell}(x)$, as in equation (2.8). Consider also $C = C^{11} \oplus \cdots \oplus C^{\ell\ell}$, decomposed into the same block structure. Put

$$m := \max\{m(C^{ii}), \quad i = 1, \ldots, \ell\}.$$

Taking into account Step 1 in the proof of (ii), there is regular formal gauge transformation ψ such that $\psi[A]$ decomposes as

$$\psi[A] = A^{11}(x) \oplus \cdots \oplus A^{\ell\ell}(x),$$

where the principal part of $A^{jj}(x)$ is $x^{-\nu(D^{jj})}(D^{jj}(x) + x^q C^{jj})$. If $\psi = \Psi_P$ and we put $\overline{P} := J_{m+q}P$, $\overline{\psi} := \Psi_{\overline{P}}$, we have that $\overline{\psi}$ is a polynomial regular transformation of degree at most $m + q$ that satisfies

$$J_{m+q}(\overline{\psi}[A]) = J_{m+q}(A^{11}(x)) \oplus \cdots \oplus J_{m+q}(A^{\ell\ell}(x))$$

The system $A^{jj}(x)$ is in the radial case treated above, so that there is a finite composition of constant regular and monomial diagonal transformations φ_j such that $\varphi_j[A^{jj}]$ has the same exponential part as $A^{jj}(x)$ and a non-resonant residual matrix. Moreover, as already noticed, the degree of each φ_j is bounded by $m(C^{jj})$ and the principal part of $\varphi_j[A^{jj}]$ only depends on $J_{m(C^{jj})+q}(A^{jj})$. We conclude that the composition $\phi = (\varphi_1 \oplus \cdots \oplus \varphi_\ell) \circ \overline{\psi}$ is a polynomial gauge transformation so that $\phi[A]$ is a system in $(TRS)_0^q$-form with non-resonant residual matrix. Moreover, the degree of ϕ can be bounded by $2m + q$. This ends the proof of item (iii) of Theorem 1.3 and completes the statement in Remark 1.5, (e).   □

One final comment on how to conclude Remark 1.5, (d) in what it concerns for this item (iii). We need to take into account that the diagonal monomial transformations used in the process are only those of the form Ψ_S, where S is as in equation (2.15). As we have already observed from equations (2.16), such transformations preserve the Poincaré rank.

3. Proof of the real Turrittin's theorem

In this section we fix a real closed field K and we prove Theorem 1.7, the real version of Turrittin's theorem on polynomial normal forms. As mentioned, the formal statement Theorem 1.8 will be a consequence of it.

We use the monomorphism of K-algebras defined in paragraph 1.2. That is, for any $m \in \mathbb{N}$, we consider

$$\Theta_m : \mathcal{M}_m(\overline{K}) \to \mathcal{M}_{2m}(K), \ (c_{uv}) \mapsto (\Lambda_{c_{uv}}).$$

and (with the same name), its extension to a morphism of K-algebras from $\mathcal{M}_m(L_{\overline{K}})$ to $\mathcal{M}_{2m}(L_K)$ sending an m-dimensional system $B = x^{-(q+1)} \sum_{j\geq 0} x^j B_j$ with coefficients in $\overline{K}$ to the system $\Theta_m(B) = x^{-(q+1)} \sum_{j\geq 0} x^j \Theta_m(B_j)$. Notice that Θ_m preserves the Poincaré rank but not necessarily the radiality index of the system. On the other hand, one can check easily that Θ_m commutes with the gauge transformations and with ramifications. To be precise, if $B, P \in \mathcal{M}_m(L_{\overline{K}})$ with $\det(P) \neq 0$, we have

$$\Theta_m(\Psi_P[B]) = \Psi_{\Theta_m(P)}[\Theta_m(B)], \tag{3.1}$$

and, if r is a natural non-zero number, then

$$\Theta_m \circ R_r = R_r \circ \Theta_m. \tag{3.2}$$

3.1. Propagating a $\mathbb{C}$-matrix to higher order coefficients.
The key result for the proof of Theorem 1.7 is the following proposition.

Proposition 3.1. *Consider a system $A \in \mathcal{M}_n(L_K)$ with Poincaré rank equal to q and written as $A = x^{-(q+1)} (A_0 + xA_1 + \dots)$. Let k be the radiality index of A and assume that $k < q$ and that the spectrum of A_k in $\overline{K}$ consists in a pair of conjugated values $a \pm ib$ with $a, b \in K$ and $b \neq 0$ (thus in particular $n = 2m$ is even). Then there exists a formal regular gauge transformation Ψ_T, where $T \in \mathcal{M}_n(K[[x]])$, such that the transformed system $B = \Psi_T[A]$ is a $\mathbb{C}$-system. Moreover, writing $B = x^{-(q+1)}(\sum_{j\geq 0} x^j B_j)$, we have $A_j = B_j \in KI_n$ for $j = 0, 1, \dots, k-1$ and for any $\mu \geq k$, the truncation $J_\mu B$ only depends on $J_{\mu-k}T$ and $J_\mu A$. In other words, if $\widetilde{A}$ is another system with $q(\widetilde{A}) = q$ and satisfying $J_\mu \widetilde{A} = J_\mu A$, then $J_\mu(\Psi_{J_{\mu-k}T}[\widetilde{A}]) = J_\mu B$.*

The proof of Proposition 3.1 is similar to the one of the Splitting Lemma (cf. Lemma 2.2) or of Theorem 1.3, (ii). This time, it is based on the following result for $\mathbb{C}$-matrices of size two.

Lemma 3.2. *Let $\lambda \in \overline{K} \setminus K$ and let $\Lambda_\lambda = \Theta_1(\lambda)$ be the corresponding $\mathbb{C}$-matrix of size 2. Given $S \in \mathcal{M}_2(K)$ an arbitrary matrix with coefficients in K, there exists a matrix $X \in \mathcal{M}_2(K)$ such that $\Lambda_\lambda X - X\Lambda_\lambda + S$ is a $\mathbb{C}$-matrix.*

Proof. Put $\lambda = a + ib$ with $a, b \in K$ and $b \neq 0$ and write $S = (s_{ij})$ and $X = (x_{ij})$ with $1 \leq i, j \leq 2$. Computing we have

$$\Lambda_\lambda X - X\Lambda_\lambda + S = \begin{pmatrix} -u + s_{11} & v + s_{12} \\ v + s_{21} & u + s_{22} \end{pmatrix}, \tag{3.3}$$

where $u = b(x_{12} + x_{21})$ and $v = b(x_{11} - x_{22})$. The matrix in (3.3) is a $\mathbb{C}$-matrix iff we have $-u + s_{11} = u + s_{22}$ and $v + s_{12} = -(v + s_{21})$. These two last equations have solutions in u, v once we are given the entries s_{ij} of S and, taking into account that $b \neq 0$, we conclude the lemma. $\square$

Proof of Proposition 3.1. First, using a real canonical form of A_k, there is a non-singular matrix T_0 with entries in K such that $T_0^{-1}A_kT_0 = \Lambda + H$ where

$$\Lambda = \Lambda_\lambda \oplus \Lambda_\lambda \oplus \cdots \oplus \Lambda_\lambda, \ H = \begin{pmatrix} 0 & \epsilon_1 I_2 & & & \\ & 0 & \epsilon_2 I_2 & & \\ & & \ddots & & \\ & & & & \epsilon_{m-1}I_2 \\ & & & & 0 \end{pmatrix},$$

with $\epsilon_j \in \{0,1\}$. Up to replacing A by $\Psi_{T_0}[A]$, we may assume that the original coefficient A_k has already the form above $A_k = \Lambda + H$, a $\mathbb{C}$-matrix. We look for a regular formal gauge transformation Ψ_T with $T = I_n + xT_1 + x^2T_2 + \dots$ satisfying the required property. For that, we compute the coefficients of the transformed system $B := \Psi_T[A]$ in terms of the coefficients of A and of T. With similar computations as already done in the preceding section, if we write $B = x^{-(q+1)}(B_0 + xB_1 + \dots)$ then we obtain:
- The radial part does not change; i.e., $B_\ell = A_\ell$ for $\ell = 0, 1, \dots, k-1$.
- $B_k = A_k = \Lambda + H$.
- For $j \geq k+1$, we obtain

$$B_j = [A_k, T_{j-k}] + A_j + Q_j, \tag{3.4}$$

where Q_j is a matrix which depends polynomially only on the matrices of the family $\{A_{k+s}, T_s\}_{s<j-k}$.

Let us show that we can choose recursively $T_1, T_2, \dots$ such that each B_j in equation (3.4) is a $\mathbb{C}$-matrix for any $j \geq k$. This will complete the proof of Proposition 3.1.

The starting case $j = k$ is done since $B_k = A_k$ is already a $\mathbb{C}$-matrix. Suppose that for $j > k$ we have already constructed $T_1, \dots, T_{j-k-1}$ such that B_ℓ is a $\mathbb{C}$-matrix for $\ell < j$. For each value of the letter $Y \in \{A, T, B, Q\}$ and for each $\ell \leq j$, we write $Y_\ell = (Y_\ell^{uv})_{1 \leq u,v \leq m}$ in a block structure of 2×2 matrices. We construct the different blocks T_{j-k}^{uv} in the following order. We start by the bottom of the first column: the block T_{j-k}^{m1} satisfies, after equation (3.4),

$$[\Lambda_\lambda, T_{j-k}^{m1}] + A_j^{m1} + Q_j^{m1} = B_j^{m1}.$$

Using Lemma 3.2, we choose T_{j-k}^{m1} in such a way that B_j^{m1} is a $\mathbb{C}$-matrix. Then we continue with the block $T_{j-k}^{m-1,1}$ which satisfies

$$[\Lambda_\lambda, T_{j-k}^{m-1,1}] + \epsilon_{m-1}T_{j-k}^{m1} + A_j^{m-1,1} + Q_j^{m-1,1} = B_j^{m-1,1}.$$

Taking into account that T_{j-k}^{m1} has already been chosen and using Lemma 3.2, we choose $T_{j-k}^{m-1,1}$ such that $B_j^{m-1,1}$ is a $\mathbb{C}$-matrix. The process can be repeated in this way until we construct all blocks in the first column, that is, those of the form T_{j-k}^{u1} in inverse order for u from $u = m$ to $u = 1$. After that, we construct the blocks in the second column T_{j-k}^{u2}, again from $u = m$ to $u = 1$: by (3.4) we obtain

$$[\Lambda_\lambda, T_{j-k}^{u,2}] + \epsilon_u T_{j-k}^{u+1,2} - \epsilon_1 T_{j-k}^{u,1} + A_j^{u2} + Q_j^{u2} = B_j^{u2}$$

(with $\epsilon_m = 0$), and we choose T_{j-k}^{u2} such that B_j^{u2} is a $\mathbb{C}$-matrix, once the blocks $T_{j-k}^{u+1,2}, T_{j-k}^{u1}$ in the above equation are already known. We continue in this way in

order to complete the construction of all blocks T_{j-k}^{uv} so that any block B_j^{uv} (and hence the whole matrix B_j) is a $\mathbb{C}$-matrix. $\qquad\square$

3.2. **Proof of Theorem 1.7, (i).** Getting a $(\mathbb{R}TRS)$-form of degree 0. Fix a singular system $A \in \mathcal{M}_n(L_K)$ with Poincaré rank $q = q(A)$ and write $A = x^{-(q+1)}(A_0 + xA_1 + \dots)$ as in (1.1) with $A_0 \neq 0$. Denote by $k = k(A)$ the radiality index of A. As in the complex case, we prove a slightly improvement of item (i) in Theorem 1.7 (called item (i') in what follows), where the sufficient jet order nq to obtain the same real Turrittin-Ramis-Sibuya form is replaced by the order $N = N(n, q, k) := n(q - k) + k$.

We start with the trivial case where $q = k$ (this includes the case $q = 0$). Using the real Jordan canonical form of $A_k = A_q$, there is a non-singular matrix $T_0 \in \mathcal{M}_n(K)$ such that $T_0^{-1}A_0T_0 = C_1 \oplus C_2$, where C_1 is a matrix with eigenvalues in K and C_2 is a $\mathbb{C}$-matrix. The radial part $A_0 + xA_1 + \dots + x^{q-1}A_{q-1}$ is preserved by Ψ_{T_0} and can be written in the form $D_1(x) \oplus \Theta_{n_2}(D_2(x))$ where $D_1(x) \in \mathcal{M}_{n_1}(K[X])$ and $D_2(x) \in \mathcal{M}_{n_2}(K[x])$ are both diagonal polynomial. Hence $\Psi_{T_0}[A]$ is in $(\mathbb{R}TRS)_0^q$-form and (i') follows (notice that $N = q$ in this case).

Assume that $0 \leq k < q$. We proceed by induction with respect to the size n of the system. The case $n = 1$ is also trivial: A is already in $(\mathbb{R}TRS)_0^q$-form with $n_1 = n = 1$ and $n_2 = 0$ and $N = q$ in this case. Suppose then that $n > 1$.

Suppose first that the first non-radial term A_k has at least two non-conjugated eigenvalues in $\overline{K}$. In this case, after a constant regular gauge transformation Ψ_{T_0}, with $T_0 \in GL_n(K)$, we can assume that $A_k = A_k^{11} \oplus A_k^{22}$, where each A_k^{ii} is a square matrix with positive size n_i with entries in K and $\mathrm{Spec}(A_k^{11}) \cap \mathrm{Spec}(A_k^{22}) = \emptyset$. Apply the Splitting Lemma to A up to order $N = N(n, q, k)$. That is, there exists a regular polynomial gauge transformation Ψ_T where $T = I + xT_1 + \dots + x^{N-k}T_{N-k}$ such that $J_N(\Psi_T[A]) = B^{11} \oplus B^{22}$, where B^{ii} is a (polynomial) system of size $n_i < n$ with coefficients in K. By induction on the size, item (i') holds for both systems B_k^{ii}. In a way completely analogous as we did for the complex case in paragraph 2.1, we use this to conclude item (i') for the original system A.

Suppose now that $\mathrm{Spec}(A_k) = \{\lambda_k, \overline{\lambda_k}\}$ for some $\lambda_k \in \overline{K}$. We consider the two possible situations:

Case 1: $\lambda_k \neq \overline{\lambda_k}$. Notice that $n = 2m$ is even in this case. We apply Proposition 3.1 to the system A. Notably, let Ψ_T be a formal regular gauge transformation with $T \in \mathcal{M}_n(K[[x]])$ such that $B = \Psi_T[A]$ is a $\mathbb{C}$-system. Let $\overline{B}$ be the system with coefficients in $\overline{K}$ satisfying $B = \Theta_m(\overline{B})$. Apply Theorem 1.3, (i) to $\overline{B}$: we obtain a natural number $r \geq 1$ and gauge transformations $\varphi_1, \dots, \varphi_s$, either regular polynomial or monomial diagonal (with coefficients in $\overline{K}$) such that, putting $\psi = \varphi_s \circ \dots \circ \varphi_1 \circ R_r$, we have

   (a) The system $\psi[\overline{B}]$ is in $(TRS)_0^{\tilde{q}}$-form for some $\tilde{q} \geq 0$.
   (b) Being $N = N(n, q, k)$, if $\overline{E}$ is another system with $q(\overline{E}) = q$ and $J_N\overline{E} = J_N\overline{B}$, the system $\psi[\overline{E}]$ is also in $(TRS)_0^{\tilde{q}}$-form with the same principal part as $\psi[\overline{B}]$.

Now, for any $i = 1, \dots, s$, if $\varphi_i = \Psi_{P_i}$ with $P_i \in \mathcal{M}_n(\overline{K}[[x]])$, we put $\widetilde{\varphi}_i := \Psi_{\Theta_m(P_i)}$. Notice that $\widetilde{\varphi}_i$ is a gauge transformation with coefficients in K, either regular polynomial or diagonal monomial. Denote by $\widetilde{\psi} = \widetilde{\varphi}_s \circ \dots \circ \widetilde{\varphi}_1 \circ R_r$ and let

us see that the composition

$$\widetilde{\psi} \circ \Psi_{J_N T} = \widetilde{\varphi}_s \circ \cdots \circ \widetilde{\varphi}_1 \circ \Psi_{J_N T(x^r)} \circ R_r$$

satisfies the requirements of Theorem 1.7, (i'). Using the property (b) above and equations (3.1) and (3.2) we have

$$\Theta_m(\psi[J_N(\overline{B})]) = \widetilde{\psi}[J_N(\Theta_m(\overline{B}))] = \widetilde{\psi}[J_N(B)]$$

is in $(\mathbb{R}TRS)_0^{\tilde{q}}$-form. On the other hand, we have the following

Claim. If $E \in \mathcal{M}_n(L_K)$ is any system with Poincaré rank equal to q and $J_N E = J_N B$ then $J_{\tilde{q}}(\widetilde{\psi}[E]) = J_{\tilde{q}}(\widetilde{\psi}[J_N B])$.

Applying this claim to $E = \Psi_{J_N T}[A]$, taking into account that $J_N(\Psi_{J_N T}[A]) = J_N B$, we conclude Theorem 1.7, (i') in this Case 1.

It remains to show the Claim. It is a consequence of the property (b) above (using (3.1) and (3.2)) in the case that E is a $\mathbb{C}$-system). To be convinced that it is true for any system E in the hypothesis of the statement, we notice, using the description of a general gauge transformation or a ramification, that there exists some integer $M > 0$ such that $J_{\tilde{q}}(\widetilde{\psi}[E]) = J_{\tilde{q}}(\widetilde{\psi}[J_M E])$ for such systems and that the map $H_M : J_M E \mapsto J_{\tilde{q}}(\widetilde{\psi}[J_M E])$ is a polynomial map in the entries of the coefficient matrices $E_0, \ldots, E_M$ of $J_M E$. Necessarily the minimum M with this property must be greater or equal than N. But, as we have just said, property (b) implies that the value of H_M does not depend on the entries of $E_{N+1}, \ldots, E_M$ if E is a $\mathbb{C}$-system. Since the set of J_M-jets of $\mathbb{C}$-systems with a fixed Poincaré rank q has non-empty interior in the space of J_M-jets of all systems (with that fixed Poincaré rank), we conclude that $M = N$ satisfies the property above. The Claim follows.

Case 2: $\lambda_k = \overline{\lambda_k}$. In this case, A_k has a single eigenvalue $\lambda_k \in K$. After a constant gauge transformation with entries in K, we can write A_k in its Jordan canonical form as in equation (2.1). We can define in this case the tuple $I(A) = (\gamma_1(A_k), \ldots, \gamma_n(A_k), q - k)$ and proceed exactly as in the proof of the complex Turrittin theorem in paragraph 2.1 from the step in which A_k has the Jordan form (2.1). Notably, the terms in the truncation $J_N A$, with $N = N(n, q, k)$, determine a shearing order $g = h/r \in \mathbb{Q}$ (cf. Definition 2.4). Then, we consider the transformed system $B = \Psi_{S_h} \circ R_r[A]$, where R_r is the ramification of index r and $S_h = \mathrm{diag}(1, x^h, \ldots, x^{(n-1)h})$. Denoting by q' and k' the Poincaré rank and the radiality index of B respectively, one of the following situations occurs:

- $k' = q$ (including the case $q' = 0$): we finish since this the trivial case.
- $0 \le k' < q'$ and $B_{k'}$ has at least two non-conjugated eigenvalues in $\overline{K}$: we finish using splitting lemma and induction on n as above.
- $0 \le k' < q'$ and $B_{k'}$ has a unique pair of conjugated eigenvalues that do not belong to K: we finish since we are in Case 1 above.
- $0 \le k' < q'$ and $B_{k'}$ has a unique eigenvalue that belongs to K: in this case, the arguments in Steps 3 and 4 in paragraph 2.1 are valid for the real closed field K and they permit to conclude $I(B) < I(A)$ (in lexicographical order). We finish again since this tuple of non-negative integers number cannot decrease indefinitely.

This completes the proof of Theorem 1.7, (i'). □

3.3. **Proof of Theorem 1.7, (ii):** getting $(\mathbb{R}TRS)$-form of higher degree. The proof can be done similarly to the case where K is algebraically closed in paragraph 2.2, with only some minor changes. Let us indicated them.

Suppose that the system A is in $(\mathbb{R}TRS)_0^q$-form with exponential part equal to $D(x) = D_1(x) \oplus D_2(x)$ and residual matrix $C = C_1 \oplus C_2$ in the conditions of Definition 1.6. In particular, $D_2(x) = \Theta_{n_2}(E_2(x))$ and $C_2 = \Theta_{n_2}(G_2)$, where $G_2 \in \mathcal{M}_{n_2}(\overline{K})$ and $E_2(x) = \mathrm{diag}(d_1(x), \ldots, d_{n_2}(x))$ with $d_j(x) \in \overline{K}[x]_{q-1} \setminus K[x]$. We also assume that C is non-resonant, which is equivalent to say that both C_1 and G_2 are non-resonant matrices (of sizes n_1 and n_2, respectively).

We consider a block structure to write our system, similar to the one given in (2.8), but compatible with the fact that $D_2(x)$ is a $\mathbb{C}$-matrix. Notably, we write

$$D(x) = D^{11}(x) \oplus \cdots \oplus D^{ss}(x) \oplus \Phi_{k_1}(E^{11}(x)) \oplus \cdots \oplus \Phi_{k_t}(E^{tt}(x)), \qquad (3.5)$$

where

- Each $D^{jj}(x) = f_j(x) I_{m_j}$ is a radial matrix in $\mathcal{M}_{m_j}(K[x])$ (i.e., the coefficients of $f_j(x)$ are in K).
- Each $E^{jj}(x) = g_j(x) I_{k_j}$ is a radial matrix in $\mathcal{M}_{k_j}(\overline{K}[x])$ (i.e., the coefficients of $g_j(x)$ are in $\overline{K}$).
- $D_1(x) = D^{11}(x) \oplus \cdots \oplus D^{ss}(x)$ and $D_2(x) = \Phi_{n_2}(E^{11}(x) \oplus \cdots \oplus E^{tt}(x))$.

In accordance with the notation of equation (2.8), we denote $\ell := s+t$ and $D^{jj}(x) := \Phi_{k_{j-s}}(E^{j-s,j-s}(x))$ for $j = s+1, \ldots, \ell$. We write also $A = x^{-(q+1)} \sum_l x^l A_l$ as in (1.1) and each coefficient $A_l = (A_l^{uv})_{1 \le u,v \le \ell}$ in the block structure of (3.5).

Notice that the residual matrix $C = C_1 \oplus C_2$ is block-diagonal in this structure, since it commutes with $D(x) = D_1(x) \oplus D_2(x)$ and C_2 is a $\mathbb{C}$-matrix. Thus, $A_{q+1}^{uv} = C^{uv} = 0$ if $u \ne v$.

We want to eliminate all coefficients $A_{q+1}, A_{q+2}, \ldots$ by means of a formal regular gauge transformation Ψ_P with $P \in \mathcal{M}_n(K[[x]])$ and $P(0) = I_n$. We proceed as in paragraph 2.2 in two steps: first we eliminate the non-diagonal blocks A_l^{uv}, for $u \ne v$ and $l \ge q+1$, and then the diagonal blocks A_l^{uu}, for $u = 1, \ldots, \ell$ and $l \ge q+1$.

The first step is proved by induction on q, as in the mentioned paragraph. The case $q = 0$ is trivial. If $q > 0$, we consider a coarser block structure than the one given by (2.8) More precisely, we consider a similar block structure as the one in (2.9), where each block $\overline{D}^{jj}(x)$ with $j \in \{1, \ldots, \ell_1\}$ is of maximal size such that its value $\overline{D}_0^{jj} := \overline{D}^{jj}(0)$ at zero is:

(a) Either a radial matrix (i.e., $\overline{D}_0^{jj} = a_j I_{h_j}$ with some $a_j \in K$).

(b) Or a radial $\mathbb{C}$-matrix (i.e., $\overline{D}_0^{jj} = \Phi_{h_j}((a_j + ib_j) I_{h_j})$ for some $a_j + ib_j \in \overline{K} \setminus K$).

Notice that a block $\overline{D}^{jj}(x)$ in the case (a) may contain several of the blocks in the decomposition (3.5), even of the two different types $\{D^{ll}(x)\}_{l \le s}$ and $\{D^{ll}(x)\}_{l > s}$. In any case, using the same equations (2.10) and Lemma 2.5, we can construct a formal matrix $\overline{T} = I_n + x^{q+1} \overline{T}_{q+1} + \cdots \in K[[x]]$ such that the system $\overline{B} = \Psi_{\overline{T}}(A)$ has zero non-diagonal blocks with respect to this last structure $D(x) = \oplus_{1 \le j \le \ell_1} \overline{D}^{jj}(x)$. Hence, $\overline{B} = \oplus_{1 \le j \le \ell_1} \overline{B}^{jj}$, where $\overline{B}^{jj}$ is of size h_j when $\overline{D}_0^{jj}$ is in the case (a), or $\overline{B}^{jj}$ is of size $2h_j$ when $\overline{D}_0^{jj}$ is in the case (b). In this last case, using Proposition 3.1,

we can assume that $\overline{B}^{jj}$ is a $\mathbb{C}$-system (notice that $b_j \neq 0$ in this case, so that $k(\overline{B}^{jj}) = 0$ and $q(\overline{B}^{jj}) = q > 0$).

Put $\widetilde{B}^{jj} := \overline{B}^{jj} - x^{-(q+1)}\overline{D}_0^{jj}$ for $j = 1, \ldots, \ell_1$, a system with Poincaré rank strictly smaller than q. At this point, the proof continues as the one in Step 1 of paragraph 2.2 by constructing, using the induction hypothesis, a regular transformation Ψ_{T^j} that applies and transform the subsystem $\widetilde{B}^{jj}$ into a block-diagonal one with respect to the structure induced on the block $\overline{B}^{jj}$ by (3.5). We only have to take care about the following: for any index $j \in \{1, \ldots, \ell_1\}$ such that $\overline{D}_0^{jj}$ in the case (b) above, the matrix T^j must be chosen to be a $\mathbb{C}$-matrix, so that Ψ_{T^j} preserves the system $x^{-(q+1)}\overline{D}_0^{jj}$ and thus this transformation applied to $\overline{B}^{jj}$ produces the same result.

Finally, the second step (eliminating the diagonal blocks A_l^{uu} for $l \geq q+1$) is obtained exactly in the same way as in Step 2 of the proof of Theorem 1.3, (ii) in paragraph 2.2: we have to solve recursively the same equations (2.12) for the blocks U^{jj}, and this can be done independently of the base field K, since we only need Lemma 2.5 (valid for any field) and the hypothesis that C is non-resonant.

3.4. **Proof of Theorem 1.7, (iii).** Getting a non-resonant matrix. The proof of this item is made entirely equal to the corresponding complex case (cf. Theorem 1.3, (iii)) in paragraph 2.3. The only difference is that the first case treated there, called the "radial case", must be treated here in two different cases: either we are in the similar "radial case" with coefficients in K (that is $D(x) = f(x)I_n$ with some polynomial $f(x) \in K[x]_{q-1}$), or we are in the $\mathbb{C}$-radial case (that is $D(x) = \Theta_{n/2}(g(x)I_{n/2})$, where $g(x) \in \overline{K}[x]_{q-1} \setminus K[x]$). In the second of these two cases, we have that $D(x) \neq 0$ and $0 \leq k(A) < q(A)$, so that we can apply Proposition 3.1 and assume, after a regular polynomial gauge transformation of degree $m = m(C)$ (cf. equation (2.13)), that the truncation $J_{q+m}(A)$ is a $\mathbb{C}$-system, image by $\Theta_{n/2}$ of some system $\widetilde{A} \in \mathcal{M}_{n/2}(L_{\overline{K}})$ with exponential part equal to $\widetilde{D}(x) = g(x)I_{n/2}$. By Theorem 1.3, (iii), we transform $\widetilde{A}$ into another one with the same exponential part and non-resonant residual matrix by a regular gauge transformation Ψ_S, where $S = I_{n/2} + xS_1 + \cdots \in \mathcal{M}_{n/2}(\overline{K}[x]_m)$. In this case, the regular transformation $\Psi_{\Theta_{n/2}(S)}$ proves item (iii) for the real system A.

The general case is done as in paragraph 2.3 by means of the decomposition (3.5) of $D(x)$ and using step 1 of the proof of Theorem 1.7, (ii), already discussed in the previous paragraph 3.3. This completes the proof of Theorem 1.7. $\qquad\square$

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