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INVERSE NODAL PROBLEMS FOR DIRAC OPERATORS AND THEIR NUMERICAL APPROXIMATIONS

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Abstract. In this article, we consider an inverse nodal problem of Dirac operators and obtain approximate solution and its convergence based on the second kind Chebyshev wavelet and Bernstein methods. We establish a uniqueness theorem of this problem from parts of nodal points instead of a dense nodal set. Numerical examples are carried out to illustrate our method.

1. Introduction

We consider the matrix equation

$$Lu := Bu' + Q(x)u = \lambda u, \quad 0 \le x \le \pi, \tag{1.1}$$

with the boundary conditions

$$u_1(0,\lambda)\sin\alpha + u_2(0,\lambda)\cos\alpha = 0, \tag{1.2}$$

$$u_1(\pi,\lambda)\sin\beta + u_2(\pi,\lambda)\cos\beta = 0, \tag{1.3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} v(x)+m & 0 \\ 0 & v(x)-m \end{pmatrix}, \quad u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix},$$

and λ is the spectral parameter, $v(x)$ is a real-valued, absolutely continuous function, and $\alpha, \beta \in [0,\pi)$, $m > 0$ is constant.

$L := L(v, m, \alpha, \beta)$ is called the Dirac operator which is the relativistic Schrödinger operator in quantum physics [12]. Inverse nodal problems consist in recovering the potential $Q(x)$ and the coefficients α, β from the given subsets of the nodal points (zeros of eigenfunctions). This class of inverse nodal problems has been studied for the Sturm-Liouville operator [15], which showed that one set of nodal points can determine the Sturm-Liouville operators uniquely. The solution of inverse nodal problem was given by Hald and McLaughlin [10]. Some recent results on such problems can be found in [3, 4, 6, 7, 8, 20, 21, 23, 29]. The stability of the inverse nodal problem was explored in [5, 16]. Numerical solutions of the inverse nodal problem were presented in [17, 22].

Basic and comprehensive results about Dirac operators were introduced in [12]. Inverse nodal problems for Dirac operators were studied in [1, 9, 11, 24, 27, 30].

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Inverse nodal problems of reconstructing the Dirac operator on a finite interval were studied in [28], where it was proved that the operator L is determined uniquely by specifying a dense set of nodal points. This article investigates the inverse nodal problem of the Dirac operator (1.1)-(1.3). We establish a uniqueness theorem of the inverse nodal problem for the operator L from parts of nodal points. Meanwhile, we will show numerical solution of inverse nodal problem for the Dirac operator based on the second kind Chebyshev wavelet (SCW) [26, 31, 32] and Bernstein methods [14].

We first present preliminaries and give the asymptotic formulas of nodal points, which plays an important role in the following analysis. We denote by $u(x,\lambda)$ the solution of (1.1), satisfying the initial conditions

$$u_1(0,\lambda)=\cos\alpha,\ u_2(0,\lambda)=-\sin\alpha.$$

Then $u_1(x,\lambda)$, $u_2(x,\lambda)$ have the asymptotic formulas [25], respectively,

$$\begin{aligned} u_1(x,\lambda)&=\cos\big(\lambda x-\eta(x)-\alpha\big)+O\Big(\frac{\mathrm{e}^{\tau x}}{\lambda}\Big),\\ u_2(x,\lambda)&=\sin\big(\lambda x-\eta(x)-\alpha\big)+O\Big(\frac{\mathrm{e}^{\tau x}}{\lambda}\Big), \end{aligned}$$

where $\tau=|\operatorname{Im}\lambda|$, $\eta(x)=\int_0^x v(t)\mathrm{d}t$.

According to [12], the asymptotic forms of $u_1(x,\lambda)$, $u_2(x,\lambda)$ can be written by the method of successive approximations

$$u_1(x,\lambda)=\cos\big(\lambda x-\eta(x)-\alpha\big)+\frac{U_1(x,\lambda)}{\lambda}+\mathrm{o}\Big(\frac{\mathrm{e}^{\tau x}}{\lambda}\Big), \tag{1.4}$$

$$u_2(x,\lambda)=\sin\big(\lambda x-\eta(x)-\alpha\big)-\frac{U_2(x,\lambda)}{\lambda}+\mathrm{o}\Big(\frac{\mathrm{e}^{\tau x}}{\lambda}\Big) \tag{1.5}$$

for large $|\lambda|$ and

$$\begin{aligned} U_1(x,\lambda)&=m\sin\big(\lambda x-\eta(x)\big)\sin\alpha+\frac{m^2x}{2}\sin\big(\lambda x-\eta(x)-\alpha\big),\\ U_2(x,\lambda)&=m\sin\big(\lambda x-\eta(x)\big)\cos\alpha+\frac{m^2x}{2}\cos\big(\lambda x-\eta(x)-\alpha\big). \end{aligned}$$

All estimates are uniform with respect to x for $x\in[0,\pi]$.

The characteristic function $\Delta(\lambda)$ of (1.1)-(1.3) is defined by

$$\Delta(\lambda):=u_1(\pi,\lambda)\sin\beta+u_2(\pi,\lambda)\cos\beta,$$

and all zeros $\lambda_n, n\in\mathbb{Z}$ of $\Delta(\lambda)$ coincide with the eigenvalues of L.

We use $\sigma(L):=\{\lambda_n: n\in\mathbb{Z}\}$ to denote all eigenvalues λ_n, which are real and simple. By applying (1.4)-(1.5), eigenvalues λ_n satisfy the asymptotic formula [25]

$$\lambda_n=n+\frac{\omega_1}{\pi}+\frac{\omega_2}{n}+\mathrm{o}\Big(\frac{1}{n}\Big),$$

where

$$\omega_1=\alpha-\beta+\eta(\pi),\quad \omega_2=\frac{m(\sin2\alpha-\sin2\beta)+m^2\pi}{2\pi\cos^2\omega_1}.$$

If $\cos\omega_1=0$, we replace ω_2 by 0. Let $u(x,\lambda_n)=(u_1(x,\lambda_n),u_2(x,\lambda_n))^T$ be the eigenfunction corresponding to the λ_n of L. x_n^j is called the nodal point of $u_1(x,\lambda_n)$, i.e. $u_1(x_n^j,\lambda_n)=0$. And for sufficiently large $|n|$, $u_1(x,\lambda_n)$ has $|n|$ nodal points in $(0,\pi)$ (see [28]),

$$0<x_n^1<x_n^2<\dots<x_n^n<\pi,\quad \text{for } n>0,$$

$$0 < x_n^{-1} < x_n^{-2} < \cdots < x_n^{n+1} < \pi, \quad \text{for } n < 0.$$

When $|n| \to \infty$, x_n^j satisfies the following asymptotic formula by using (1.4)-(1.5),

$$\begin{aligned} x_n^j = &\frac{(j-\frac{1}{2})\pi + \alpha + \eta(x_n^j)}{n} - \frac{\big((j-\frac{1}{2})\pi + \alpha + \eta(x_n^j)\big)\omega_1}{n^2\pi} \\ &+ \frac{(-1)^{j+1}\big(m\sin 2\alpha + m^2 x_n^j\big)}{2n^2} + O\Big(\frac{1}{n^3}\Big), \end{aligned} \tag{1.6}$$

which is uniform with respect to $j \in \mathbb{Z}$.

We denote the nodal set by $X_n := \{x_n^j\}_{j=j_0+1}^{n+j_0}$ for some j_0 and $n > 0$, $j_0 \in \mathbb{Z}$. Clearly, $X := \cup_n X_n$ is a dense nodal set on $[0,\pi]$. For simplicity, we assume $j_0 = 0$ in this paper.

This article is organized as follows. In Section 2, the approximation of solution and convergence of the inverse nodal problem of the Dirac operator are studied based on the second kind Chebyshev wavelet and Bernstein methods. In Section 3, we establish a uniqueness theorem of the inverse nodal problem for the Dirac operator from parts of nodal points instead of a dense nodal set. Section 4 is devoted to showing numerical examples to demonstrate the efficiency of our method. Conclusion is given in Section 5.

Approximate solution. For each fixed n, $n \gg 1$, we obtain a nodal set $\{x_n^j\}_{j=1}^n$ which satisfies (1.6) together with the coefficients $\big(m,\alpha,\beta,\eta(\pi)\big)$, and consequently give the numerical solution of $v(x)$. It follows from [28, Theorem 2.2].

Theorem 1.1 ([28, Theorem 2.2]). *The function $v(x)$ on $[0,\pi]$ and the coefficient m,α,β can be uniquely determined by the dense nodal set X.*

IP1 Inverse Problem 1: Reconstruct $v(x), m, \alpha, \beta$ by parts of nodal set X and $\eta(\pi)$.

2. Approximation of solution and its convergence

In this section, we give the approximate solution of $v(x)$ by the nodal points x_n^j for $n > 0$ and $j = \overline{1,n}$ together with coefficients $\big(m,\alpha,\beta,\eta(\pi)\big)$ for sufficiently large n. It follows from (1.4) that

$$\int_0^{x_n^j} v(t)\mathrm{d}t \cong (n + \frac{\omega_1}{\pi} + \frac{\omega_2}{n})x_n^j - \alpha - (j-\frac{1}{2})\pi + \frac{(-1)^j(m\sin 2\alpha + m^2 x_n^j)}{2n}. \tag{2.1}$$

The above integral equation is called the first type of Fredholm integral equation. By using the SCW and Bernstein methods, we convert the 1-st type of Fredholm integral equation into linear equation systems. Since the solution of (2.1) is also a solution of $v(t)$, we obtain an approximation of the potential $v(x)$ by the SCW and Bernstein methods.

Consider the first type of Fredholm integral equation

$$\int_0^{\pi} f(s)k(t,s)\mathrm{d}s = g(t), \quad 0 \le t \le \pi, \tag{2.2}$$

where the functions $g(t)$ is continuous on the intervals $[0,\pi]$, $k(t,s)$ is continuous on $[0,\pi]\times[0,\pi]$, and the function $f(s)$ is unknown.

In recent years, some numerical methods have been studied to obtain the approximate solution of the equation (2.2) (see for example [2, 18, 19]). In this article, we use the SCW and Bernstein methods to calculate the approximate solution of

1-st type of Fredholm integral equation and consequently compute the approximate solution of $v(x)$.

For the convenience of readers, we briefly present the SCW and Bernstein methods.

Second kind Chebyshev wavelet method (SCW method). The second kind Chebyshev wavelets $\psi_{l,m}(t) = \psi(k, l, m, t)$ are a family of four-parameters functions which are defined on the interval $[0, 1)$ (see [26, 31, 32]),

$$\psi_{l,m}(t) = \begin{cases} 2^{k/2}\widetilde{U}_m(2^k t - 2l + 1), & \frac{l-1}{2^{k-1}} \le t < \frac{l}{2^{k-1}}, \\ 0, & \text{otherwise,} \end{cases}$$

where k can be any positive integer, $l = 1, 2, \dots, 2^{k-1}$,

$$\widetilde{U}_m(t) := \Big(\frac{2}{\pi}\Big)^{1/2} U_m(t),$$

$U_m(t)$ is of the second kind Chebyshev polynomial for $m = 0, 1, 2, \dots$. We note that $\{U_m(t)\}$ is a sequence of orthogonal polynomial on the interval $[-1, 1]$ with respect to the weight function $w(t) = \sqrt{1 - t^2}$ and can be defined by the recursive formula

$$\begin{gathered} U_0(t) = 1, \quad U_1(t) = 2t, \\ U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \quad m = 1, 2, \dots. \end{gathered}$$

Then, the SCW defined on the interval $[0, \pi)$ can be written in the form

$$\psi_{l,m}(t) = \begin{cases} \frac{1}{\pi} 2^{k/2}\widetilde{U}_m(\frac{2^k}{\pi} t - 2l + 1), & \frac{(l-1)\pi}{2^{k-1}} \le t < \frac{l\pi}{2^{k-1}}, \\ 0, & \text{otherwise.} \end{cases}$$

When $v(t) \in L^2[0, \pi)$, it can be approximated by the SCW method,

$$v(t) = \sum_{l=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{l,m}\psi_{l,m}(t),$$

where

$$c_{l,m} = \langle v(t), \psi_{l,m}(t)\rangle_{L^2_w[0,\pi)} = \int_0^\pi v(t)\psi_{l,m}(t) w\Big(\frac{2^k}{\pi} t - 2l + 1\Big) \mathrm{d}t,$$

and $\langle \cdot, \cdot \rangle_{L^2_w[0,\pi)}$ is the inner product in $L^2_w[0, \pi)$.

By truncating the above infinite series, we obtain

$$v(t) \cong \sum_{l=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{l,m}\psi_{l,m}(t) = \mathbf{C}^T \mathbf{\Psi}(t), \tag{2.3}$$

where

$$\begin{aligned} \mathbf{C} &= [c_{1,0}, c_{1,1}, \dots, c_{1,M-1}, c_{2,0}, \dots, c_{2,M-1}, \dots, c_{2^{k-1},0}, \dots, c_{2^{k-1},M-1}]^T, \\ \Psi(t) &= \big[\psi_{10}(t), \dots, \psi_{1(M-1)}(t), \psi_{20}(t), \dots, \psi_{2(M-1)}(t), \dots, \psi_{2^{k-1}0}(t), \\ &\quad \dots, \psi_{2^{k-1}(M-1)}(t)\big]^T. \end{aligned}$$

We denote by $\|v\|_{2,w}$ the norm of $v(t)$ in $L^2_w[0, \pi)$. Modifying the proof in [31], we have the following theorem to investigate the convergence of SCW method.

Theorem 2.1. *For each fixed $n = 2^{k-1}M$, $2^{k-1}, M \gg 1$, we have*

(i) *The potential function* $v(x)$ *can be written as an infinite sum of the second kind Chebyshev wavelets and this series converges to* $v(x)$*, that is*

$$v(x)=\sum_{l=1}^{2^{k-1}}\sum_{m=0}^{\infty}c_{l,m}\psi_{l,m}(x)$$

and

$$\|v-v_n\|_{2,w}^2=\int_0^{\pi}(v(x)-v_m(x))^2w(x)\mathrm{d}x=\sum_{l=1}^{2^{k-1}}\sum_{m=M}^{\infty}c_{l,m}^2,$$

where the approximation of the potential function $v(x)$ *is*

$$v_n(x)=\sum_{l=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{l,m}\psi_{l,m}(x).$$

(ii) *Let* $v(x)$ *be a differentiable function on* $[0,\pi)$ *and* $v'(x)$ *satisfies the Lipschitz condition. Then,* $v(x)$ *can be expanded as an infinite sum of the second kind Chebyshev wavelets and the series converges to* $v(x)$ *uniformly on* $[0,\pi)$*. Moreover*

$$\|v-v_n\|_{2,w}^2=\sum_{l=1}^{2^{k-1}}\sum_{m=M}^{\infty}c_{l,m}^2\le\frac{B^2\pi}{2^{k+6}}\sum_{m=M}^{\infty}\frac{1}{m^4}$$

for some $B>0$*.*

When $n\gg1$, taking (2.3) into (2.1), we have

$$\begin{aligned}\sum_{l=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{l,m}\Big(\int_0^{x_n^j}\psi_{l,m}(t)\mathrm{d}t\Big)&\cong\Big(n+\frac{\omega_1}{\pi}+\frac{\omega_2}{n}\Big)x_n^j-\alpha-(j-\frac{1}{2})\pi\\&\quad+\frac{(-1)^j\left(m\sin2\alpha+m^2x_n^j\right)}{2n},\end{aligned}\tag{2.4}$$

for $j=\overline{1,n}$ and $n=2^{k-1}M$. Consequently, the potential function $v(t)$ can be created by using the following numerical algorithm 1.

Numerical algorithm 1: (1) For $n\gg1$, choose the values M, k, α, β, m and set $n=2^{k-1}M$. We obtain the numerical values of nodal points $\{x_n^j\}_{j=1}^n$ through formula (1.6) together with $\eta(\pi)$.

(2) Use formula (2.4) to compute the vector $\mathbf{Y}$ by the linear system

$$\mathbf{AY}=\mathbf{B},$$

$$\mathbf{A}=[\mathbf{A}_1,\mathbf{A}_2,\dots,\mathbf{A}_{2^{k-1}}],$$

$$\mathbf{A}_1=\begin{pmatrix}\int_0^{x_n^1}\psi_{l,0}(t)\mathrm{d}t & \int_0^{x_n^1}\psi_{l,1}(t)\mathrm{d}t & \dots & \int_0^{x_n^1}\psi_{l,M-1}(t)\mathrm{d}t\\ \int_0^{x_n^2}\psi_{l,0}(t)\mathrm{d}t & \int_0^{x_n^2}\psi_{l,1}(t)\mathrm{d}t & \dots & \int_0^{x_n^2}\psi_{l,M-1}(t)\mathrm{d}t\\ \dots & \dots & \dots & \\ \int_0^{x_n^n}\psi_{l,0}(t)\mathrm{d}t & \int_0^{x_n^n}\psi_{l,1}(t)\mathrm{d}t & \dots & \int_0^{x_n^n}\psi_{l,M-1}(t)\mathrm{d}t\end{pmatrix}\quad l=\overline{1,2^{k-1}}.$$

Matrix $\mathbf{B}$ is written in the form

$$\mathbf{B} = \begin{pmatrix} (n+\frac{\omega_1}{\pi}+\frac{\omega_2}{n})x_n^1 - \alpha - \frac{\pi}{2} - \frac{(m\sin 2\alpha + m^2 x_n^1)}{2n} \\ (n+\frac{\omega_1}{\pi}+\frac{\omega_2}{n})x_n^2 - \alpha - \frac{3\pi}{2} + \frac{(m\sin 2\alpha + m^2 x_n^2)}{2n} \\ \dots \\ (n+\frac{\omega_1}{\pi}+\frac{\omega_2}{n})x_n^n - \alpha - (n-\frac{1}{2})\pi + \frac{(-1)^n(m\sin 2\alpha + m^2 x_n^n)}{2n} \end{pmatrix}$$

and $\mathbf{Y} = [y_1, y_2, \dots, y_n]^T$.

(3) Calculate the values $v(t_i)$ by the formula

$$[v(t_i)] = \mathbf{Y}^T\mathbf{\Phi},$$

where

$$t_i = \frac{(2i-1)\pi}{2^k M}, \quad i = \overline{1, 2^{k-1}M}, \quad \mathbf{\Phi} = \big[\mathbf{\Psi}(\frac{\pi}{2n}), \mathbf{\Psi}(\frac{3\pi}{2n}), \dots, \mathbf{\Psi}(\frac{(2n-1)\pi}{2n})\big].$$

Bernstein method. The $N-$th degree Bernstein basis polynomials on $[0,1]$ are defined as (see [14])

$$B_{k,N}(t) = \binom{N}{k} t^k (1-t)^{N-k}, \quad k = \overline{0, N}.$$

An arbitrary function $f(t)$ defined on $[0,1]$ can be approximated by Bernstein polynomials $B_N^f(t)$ as

$$f(t) \cong B_N^f(t) := \sum_{k=0}^{N} f\big(\frac{k}{N}\big) B_{k,N}(t).$$

Suppose that the function $f(t)$ is defined on $[0,\pi]$. Thus, using the variable $\frac{t}{\pi}$ instead of t, we can write

$$B_{k,N}\big(\frac{t}{\pi}\big) = \binom{N}{k} \big(\frac{t}{\pi}\big)^k (1-\frac{t}{\pi})^{N-k}, \quad k = \overline{0, N}.$$

And then

$$f(t) \cong B_N^f(t) := \sum_{k=0}^{N} f\big(\frac{k\pi}{N}\big) B_{k,N}\big(\frac{t}{\pi}\big).$$

Theorem 2.2 ([14]). *For the function $f(t)$ bounded on $[0,1]$, the relation*

$$\lim_{N\to\infty} B_N^f(t) = f(t) \tag{2.5}$$

holds at each point of continuity t of $f(t)$; and the relation holds uniformly on $[0,1]$ if $f(t)$ is continuous on this interval.

Consequently, if $f(t)$ is continuous on interval $[0,\pi]$, then the relation (2.5) holds uniformly on $[0,\pi]$.

Now, let the function $f(t)$ be integrable on $[0,\pi]$ and take $F(t) = \int_0^t f(s)ds$. Then, by using the same process in [14], we can write

$$\begin{aligned} P_N^f(t) &= \frac{d}{dt} B_{N+1}^F(t) \\ &= \sum_{k=0}^{N} \frac{N+1}{\pi} \binom{N}{k} \big(\frac{t}{\pi}\big)^k \big(1-\frac{t}{\pi}\big)^{N-k} \int_{\frac{k\pi}{N+1}}^{\frac{(k+1)\pi}{N+1}} f(s)ds \\ &= \sum_{k=0}^{N} c_k B_{k,N}\big(\frac{t}{\pi}\big), \end{aligned} \tag{2.6}$$

where $c_k = \frac{N+1}{\pi}\int_{\frac{k\pi}{N+1}}^{\frac{(k+1)\pi}{N+1}} f(s)ds$. Therefore, if the function $f(t)$ is integrable on $[0,\pi]$, we have

$$P_N^f(t) = \sum_{k=0}^{N} c_k B_{k,N}\big(\frac{t}{\pi}\big). \tag{2.7}$$

Theorem 2.3 ([14]). *For $t \in [0,\pi]$, if $f(t)$ is the derivative of its indefinite integral, then*

$$\lim_{N\to\infty} P_N^f(t) = f(t)$$

holds almost everywhere.

Theorem 2.4 ([14]). *Suppose the kernel $K_N(t,s)$ is measurable in the square $a \le t \le b$, $a \le s \le b$ and*

$$\int_a^b |K_N(t,s)|ds \le M, \quad \int_a^b |K_N(t,s)|dt \le M,$$

with a constant M for all $N = 1,2,\dots$ and almost all t or s, respectively. Then for $f \in L^p, p > 1$, the singular integral

$$F_n(t) = \int_a^b K_N(t,s) f(s)ds$$

exists for almost all $t \in [a,b]$ and belongs to the class L^p. In addition, for an everywhere dense H in L^p, if $F_n \to f \in H$ strongly, it is also true for any $f \in L^p$:

$$\|f - F_n\| := \Big(\int_a^b |f(t) - F_n(t)|^p dt\Big)^{1/p} \to 0.$$

We have the following convergence theorem for the Bernstein method.

Theorem 2.5. *Suppose that $f \in L^p[0,\pi]$ and*

$$P_N^f(t) = \sum_{k=0}^{N} c_k B_{k,N}\big(\frac{t}{\pi}\big),$$

where $c_k = \frac{N+1}{\pi}\int_{\frac{k\pi}{N+1}}^{\frac{(k+1)\pi}{N+1}} f(s)ds$. Then it holds almost everywhere that

$$\lim_{N\to\infty} P_N^f(t) = f(t).$$

Proof. According to [14], the formula (2.6) can be regarded as

$$P_N^f(t) = \sum_{k=0}^{N} \int_{\frac{k\pi}{N+1}}^{\frac{(k+1)\pi}{N+1}} K_N(t,s) f(s) ds = \int_0^{\pi} K_N(t,s) f(s) ds, \tag{2.8}$$

where for $0 \le t \le \pi$,

$$K_N(t,s) = \frac{N+1}{\pi} \binom{N}{k} (\frac{t}{\pi})^k (1-\frac{t}{\pi})^{N-k}, \quad \frac{k\pi}{N+1} < s \le \frac{(k+1)\pi}{N+1},$$

for $k = \overline{0,N}$. Then, we have

$$\begin{aligned}
\int_0^{\pi} |K_N(t,s)| ds &= \sum_{k=0}^{N} \int_{\frac{k\pi}{N+1}}^{\frac{(k+1)\pi}{N+1}} \frac{N+1}{\pi} \binom{N}{k} (\frac{t}{\pi})^k (1-\frac{t}{\pi})^{N-k} ds \\
&= \sum_{k=0}^{N} \binom{N}{k} (\frac{t}{\pi})^k (1-\frac{t}{\pi})^{N-k} \\
&= \sum_{k=0}^{N} B_{k,N}(\frac{t}{\pi}) = 1,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{\pi} |K_N(t,s)| dt &= \int_0^{\pi} \frac{N+1}{\pi} \binom{N}{k} (\frac{t}{\pi})^k (1-\frac{t}{\pi})^{N-k} dt \\
&= \frac{N+1}{\pi} \binom{N}{k} \frac{(N-k)!}{N(N-1)\dots(k+2)(k+1)} \int_0^{\pi} (\frac{t}{\pi})^N dt \\
&= \big[(\frac{t}{\pi})^{N+1}\big]_0^{\pi} dt = 1.
\end{aligned}$$

Take $M = 1$. Assume that $f \in H$ and H is the set of continuous functions in $L^p, p > 1$. According to Theorem 2.1 and Theorem 2.3, it can be written that $P_N^f \to f$. Since H is an everywhere dense set in L^p (see [14]), $P_N^f \to f$ in L^p by using Theorem 2.4. Thus, for any $f \in L^p$, the polynomial $P_N^f(t)$ is strongly convergent to f. □

Therefore, the approximate solution of potential function $v(t) \in L^2([0,\pi])$ and the solution of integral equation (2.1) can be obtained by the Bernstein method. Indeed, according to (2.7), we have

$$v(t) \cong \sum_{k=0}^{N} c_k B_{k,N}(\frac{t}{\pi}) = \mathbf{C_1}^T \phi(t), \tag{2.9}$$

where

$$\mathbf{C}_1 = [c_0, c_1, \dots, c_N]^T, \quad \boldsymbol{\phi}(t) = [B_{0,N}(\frac{t}{\pi}), B_{1,N}(\frac{t}{\pi}), \dots, B_{N,N}(\frac{t}{\pi})]^T.$$

For $n \gg 1$, taking (2.9) into (2.1), we arrive at

$$\begin{aligned}
&\sum_{k=0}^{N} c_k \Big(\int_0^{x_n^j} B_{k,N}(\frac{t}{\pi}) \mathrm{d}t \Big) \\
&\cong (n + \frac{\omega_1}{\pi} + \frac{\omega_2}{n}) x_n^j - \alpha - (j - \frac{1}{2})\pi + \frac{(-1)^j (m \sin 2\alpha + m^2 x_n^j)}{2n},
\end{aligned} \tag{2.10}$$

for $j=\overline{1,n}$ and $n=N+1$. Consequently, the potential function $v(t)$ can be created by using the following numerical algorithm.

Numerical algorithm 2: (1) For $n\gg 1$, choose the values N, α, β, m and set $n=N+1$. We obtain the numerical values of nodal points $\{x_n^j\}_{j=1}^n$ through formula (1.6) together with $\eta(\pi)$.

(2) Use formula (2.10) to compute the vector $\mathbf{Y_2}$ by the linear system

$$\mathbf{A}_0\mathbf{C}_1=\mathbf{B}, \tag{2.11}$$

where

$$\mathbf{A}_0=\begin{pmatrix}\int_0^{x_n^1}B_{0,N}(\frac{t}{\pi})\mathrm{d}t & \int_0^{x_n^1}B_{1,N}(\frac{t}{\pi})\mathrm{d}t & \cdots & \int_0^{x_n^1}B_{N,N}(\frac{t}{\pi})\mathrm{d}t\\ \int_0^{x_n^2}B_{0,N}(\frac{t}{\pi})\mathrm{d}t & \int_0^{x_n^2}B_{1,N}(\frac{t}{\pi})\mathrm{d}t & \cdots & \int_0^{x_n^2}B_{N,N}(\frac{t}{\pi})\mathrm{d}t\\ & \cdots & & \\ \int_0^{x_n^n}B_{0,N}(\frac{t}{\pi})\mathrm{d}t & \int_0^{x_n^n}B_{1,N}(\frac{t}{\pi})\mathrm{d}t & \cdots & \int_0^{x_n^n}B_{N,N}(\frac{t}{\pi})\mathrm{d}t\end{pmatrix},$$

and the matrix $\mathbf{B}$ and $\mathbf{C}_1$ are as defined above.

(3) Calculate the values $v(t_i)$ by the formula

$$[v(t_i)]=\mathbf{C}_1^T\mathbf{\Phi},$$

where

$$t_i=\frac{i\pi}{N},\quad i=\overline{0,N},\quad \mathbf{\Phi}=[\boldsymbol{\phi}(t_0),\boldsymbol{\phi}(t_1),\dots,\boldsymbol{\phi}(t_N)].$$

3. Uniqueness

In this section, we present a solution for IP1 and give the uniqueness theorem of the inverse nodal problem for the Dirac operator. For sufficiently large k, let $n_r=2^{k-1}M_r$, where M_r is a strictly increasing sequence and M_1 is sufficiently large.

We denote by $\delta\gamma$ the error of γ, then $\gamma+\delta\gamma$ is the approximate solution of γ. Equation (2.11) can be rewritten as

$$(\mathbf{A}_0+\delta\mathbf{A}_0)(\mathbf{C}_1+\delta\mathbf{C}_1)=\mathbf{B}+\delta\mathbf{B}. \tag{3.1}$$

Now, we study the absolute error between the approximate solution $\mathbf{C}_1+\delta\mathbf{C}_1$ and exact solution $\mathbf{C_1}$ of (3.1). Similar to proof of [13, Theorems 23] on error estimates (pages 206-207), one can easily prove the following theorem.

Theorem 3.1. *The absolute error between the approximate solution $\mathbf{C_1}+\delta\mathbf{C_1}$ and exact solution $\mathbf{C}_1$ of (3.1) satisfies*

$$\|\delta\mathbf{C_1}\|=O\big(\frac{1}{n}\big) \tag{3.2}$$

for sufficiently large n .

Theorem 3.2 (Uniqueness)**.** *Given the nodal point subset $\bigcup_{r=1}^{\infty}X_{n_r}$ of L, where $x_{n_r}^j$ satisfies (1.6) for each fixed n_r with its mean value $\eta(\pi)$, then $\big(v(x),m,\alpha,\beta\big)$ can be uniquely determined by $\bigcup_{r=1}^{\infty}X_{n_r}$.*

Proof. (1) Find α and β. Choose $x_{n_r}^1$ and $x_{n_r}^{n_r}$, then

$$\lim_{n_r\to\infty}x_{n_r}^1=0\quad\text{and}\quad\lim_{n_r\to\infty}x_{n_r}^{n_r}=\pi. \tag{3.3}$$

It follows from (1.6) and (3.3) that

$$\alpha = \lim_{n_r\to\infty} \big(n_r x^1_{n_r} - \frac{\pi}{2}\big) \quad \text{and} \quad \beta = \lim_{n_r\to\infty} \big[n_r x^{n_r}_{n_r} - (n_r - \frac{1}{2})\pi\big]. \tag{3.4}$$

(2) Reconstruct m. If $\alpha \neq 0$, then

$$m = \frac{1}{\sin 2\alpha} \lim_{n_r\to\infty} \Big(n_r^2 x^1_{n_r} - n_r(\frac{\pi}{2} + \alpha) + \frac{\omega_1}{\pi}(\frac{\pi}{2} + \alpha)\Big). \tag{3.5}$$

If $\alpha = 0$, then

$$m^2 = -\frac{1}{\pi} \lim_{n_r\to\infty} \Big(n_r^2 x^{n_r}_{n_r} - n_r(\frac{\pi}{2} + \alpha + \eta(\pi)) + \frac{\omega_1}{\pi}(\frac{\pi}{2} + \alpha + \eta(\pi))\Big). \tag{3.6}$$

We reconstruct m from (3.5) or (3.6).

(3) Find $v(x)$. For each n_r, from (2.9), we have the approximation of solution $v_{n_r}(x)$ of $v_{n_r}(x)$,

$$v_{n_r,0}(x) = \sum_{k=0}^{n_r} c_{k,0} B_{k,n_r}(\frac{x}{\pi}). \tag{3.7}$$

It follows from (3.2) and (3.7) that

$$\|v_{n_r,0} - v_{n_r}\| = O(\frac{1}{n}). \tag{3.8}$$

Using Theorem 2.5 and (3.8), we find

$$v(x) := \lim_{n_r\to\infty} v_{n_r}(x) = \lim_{n_r\to\infty} v_{n_r,0}(x). \tag{3.9}$$

The proof is complete. □

From Theorem 3.1, we obtain the following theorem by parts of nodal points instead of dense nodal set.

Theorem 3.3. *If $X_{n_r} = \widetilde{X}_{n_r}$ for all n_r, $r \in \mathbb{N}$ and $\eta(\pi) = \widetilde{\eta}(\pi)$, then*

$$v(x) = \widetilde{v}(x) \quad on \quad [0,\pi], \quad \alpha = \widetilde{\alpha}, \quad \beta = \widetilde{\beta} \quad m = \widetilde{m}.$$

Proof. From Theorem 3.1, we reconstruct α, β and m by (3.3), (3.4) and (3.5), or (3.6), respectively. It follows from (2.10), $X_{n_r} = \widetilde{X}_{n_r}$ for each $r \in \mathbb{N}$ and $\eta(\pi) = \widetilde{\eta}(\pi)$ that

$$A_l = \widetilde{A}_l + O(\frac{1}{n}) \quad \text{and} \quad B_l = \widetilde{B}_l + O(\frac{1}{n}). \tag{3.10}$$

Using (3.2) and (3.10), we have

$$v_{n_r,0} - \widetilde{v}_{n_r,0} = \mathrm{o}(1) \tag{3.11}$$

for $r > N$. Therefore, we obtain two sequences of functions $\{v_{n_r,0}(x)\}_{r=1}^{\infty}$ and $\{\widetilde{v}_{n_r,0}(x)\}_{r=1}^{\infty}$.

From Theorem 3.1 and (3.11), we obtain

$$\lim_{r\to\infty} v_{n_r,0}(x) = \lim_{r\to\infty} \widetilde{v}_{n_r,0}(x). \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\widetilde{v}(x) = v(x) \quad \text{on} \quad [0,\pi].$$

The proof is complete. □

4. Numerical examples

In this section, the SCW and Bernstein methods are used to compute the approximate solution of the inverse problem and the accuracy of presented methods is shown by providing two numerical examples.

Example 4.1. For the function $v(x) = \cos(3x) + \sin(x)$, we take $M = k = 3$ in the SCW method and $N = 11$ in the Bernstein method. Set $\alpha = \pi/6$, $\beta = \pi/3$, $m = 0.4$, then we obtain

$$\eta(\pi) = \int_0^\pi v(t)\mathrm{d}t = \int_0^\pi (\cos(3t) + \sin(t))\mathrm{d}t = 2.$$

The numerical values of nodal points x_n^j, $j = \overline{1,n} = \overline{1,12}$ from formula (1.6) are shown in Table 1.

Table 1. Numerical values of x_n^j with $\alpha = \pi/6$, $\beta = \pi/3$ and $m = 0.4$ obtained by (1.6) in Example 4.1.

j	1	2	3	4	5	6
x_n^j	0.16853351	0.42000759	0.66835934	0.90097215	1.13794007	1.37503758
j	7	8	9	10	11	12
x_n^j	1.63434058	1.89791120	2.17430407	2.43222644	2.68290530	2.90441560

The above nodal points x_n^j and the values of α, β and m are used to calculate the approximation of the function v. We draw the exact solution and the numerical approximations with $n = 12, 24, 48, 96$ by using the SCW and Bernstein methods for $\alpha = \pi/6$, $\beta = \pi/3$ and $m = 0.4$, which are shown in Figure 1 and Figure 2.

Example 4.2. Suppose that the function $v(x) = (t+1)/\sqrt{t^2+1}$. Take $M = k = 3$ in the SCW method and $N = 11$ in the Bernstein method. Set $\alpha = \pi/10$, $\beta = \pi/4$, $m = 0.01$, then we have

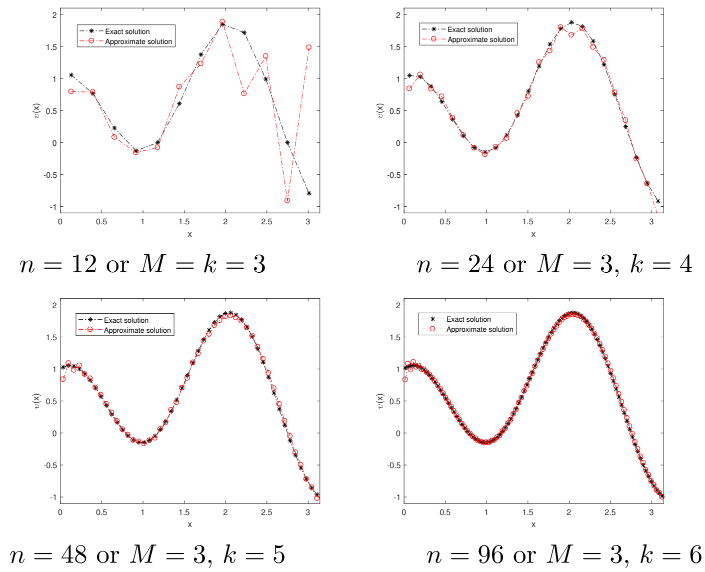
$$\eta(\pi) = \int_0^\pi v(t)\mathrm{d}t = \int_0^\pi \frac{t+1}{\sqrt{t^2+1}}\mathrm{d}t = \operatorname{arcsinh}(\pi) + \sqrt{\pi^2+1} - 1 \cong 4.15920405.$$

Using formula (1.6), we calculate the numerical values of nodal points x_n^j, $j = \overline{1,n} = \overline{1,12}$ are shown in Table 2.

Table 2. Numerical values of x_n^j with $\alpha = \pi/10$, $\beta = \pi/4$ and $m = 0.01$ obtained by (1.6) in Example 4.2.

j	1	2	3	4	5	6
x_n^j	0.15380336	0.41665952	0.68218220	0.94872939	1.21557194	1.48210054
j	7	8	9	10	11	12
x_n^j	1.74829765	2.01394010	2.27919981	2.54393599	2.80835527	3.0723250

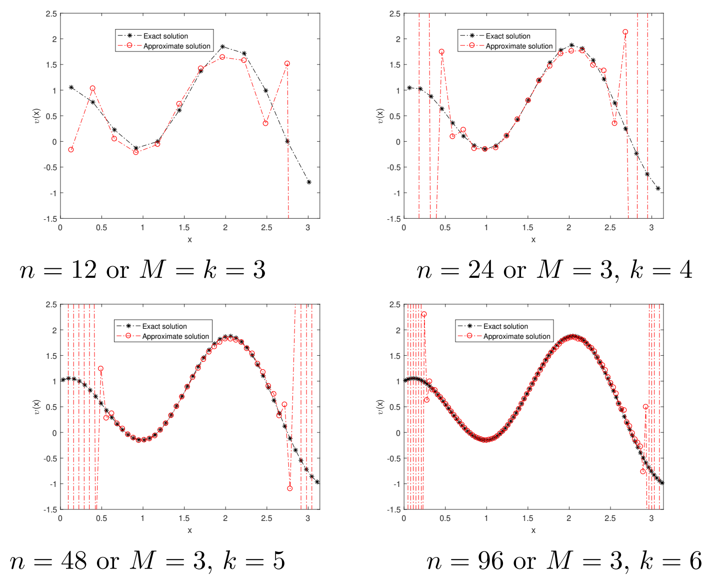
The above nodal points x_n^j and the values of α, β and m are used to compute the approximation of function v. We draw the exact solution and the numerical approximations with $n = 12, 24, 48, 96$ by using the SCW and Bernstein methods for $\alpha = \pi/10$, $\beta = \pi/4$ and $m = 0.01$, which are shown in Figure 3 and Figure 4.



$n = 12$ or $M = k = 3$ $n = 24$ or $M = 3$, $k = 4$

$n = 48$ or $M = 3$, $k = 5$ $n = 96$ or $M = 3$, $k = 6$

FIGURE 1. Exact and approximate solutions with $\alpha = \pi/6$, $\beta = \pi/3$ and $m = 0.4$ obtained by SCW method in Example 4.1.

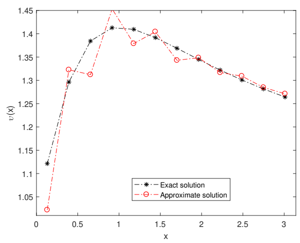


$n = 12$ or $M = k = 3$ $n = 24$ or $M = 3$, $k = 4$

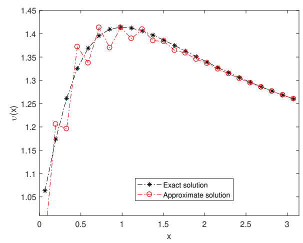
$n = 48$ or $M = 3$, $k = 5$ $n = 96$ or $M = 3$, $k = 6$

FIGURE 2. Exact and approximate solutions with $\alpha = \pi/6$, $\beta = \pi/3$ and $m = 0.4$ obtained by Bernstein method in Example 4.1.

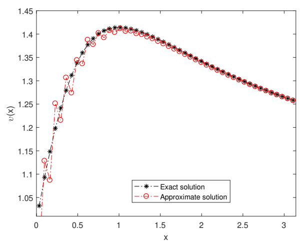
For the given examples, it can be seen that the SCW method gives a better approximation than Bernstein method for the large values of n. Meanwhile, we can see the approximation result obtained by Bernstein method is also well except at the beginning and end of the interval.



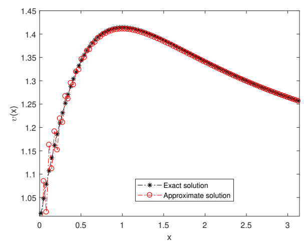
$n = 12$ or $M = k = 3$



$n = 24$ or $M = 3,\ k = 4$

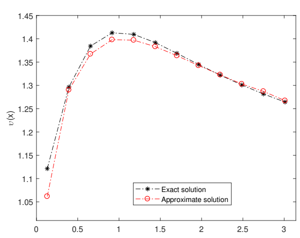


$n = 48$ or $M = 3,\ k = 5$

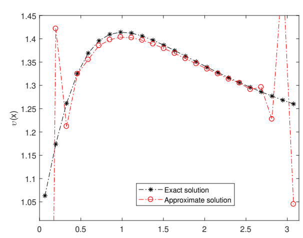


$n = 96$ or $M = 3,\ k = 6$

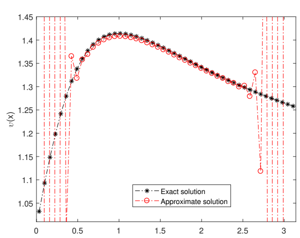
FIGURE 3. Exact and approximate solutions with $\alpha = \pi/10$, $\beta = \pi/4$ and $m = 0.01$ obtained by SCW method in Example 4.2.



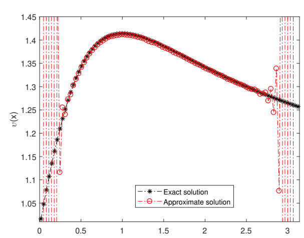
$n = 12$ or $M = k = 3$



$n = 24$ or $M = 3,\ k = 4$



$n = 48$ or $M = 3,\ k = 5$



$n = 96$ or $M = 3,\ k = 6$

FIGURE 4. Exact and approximate solutions with $\alpha = \pi/10$, $\beta = \pi/4$ and $m = 0.01$ obtained by Bernstein method in Example 4.2.

5. Conclusion

In this article, we consider the inverse nodal problem for Dirac operators. The asymptotic form of the eigenfunctions and nodal points are presented. The uniqueness theorem for solution of inverse problem by a dense subset of nodal points is proved. We offer the second kind Chebyshev wavelets and Bernstein collocation methods to obtain the approximate solution and finally give two numerical examples to demonstrate their efficiency.

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