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PERIODIC UNFOLDING METHOD FOR DOMAINS WITH VERY SMALL INCLUSIONS

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Abstract. This work creates a version of the periodic unfolding method suitable for domains with very small inclusions in $\mathbb{R}^N$ for $N \geq 3$. In the first part, we explore the properties of the associated operators. The second part involves the application of the method in obtaining the asymptotic behavior of a stationary heat dissipation problem depending on the parameter $\gamma < 0$. In particular, we consider the cases when $\gamma \in (-1, 0)$, $\gamma < -1$ and $\gamma = -1$. We also include here the corresponding corrector results for the solution of the problem, to complete the homogenization process.

1. Introduction

A recent and novel approach for homogenization theory is the periodic unfolding method originally introduced by Cioranescu, Damlamian, and Griso for fixed domains in [9, 10, 19, 20]. This method is favored because it gives an elementary proof for the classical periodic homogenization problem and due to the nature of technique which maps the oscillating domain to a fixed domain, it does not further require any extension operators. Later on, to take into account materials with periodic perforations, Cioranescu, Donato and Zaki extended the method in [14] to perforated domains, for more details we refer to [13, 15, 16]. Successively, when the size of the holes are smaller than the period, the technique was adapted by Cioranescu, Damlamian, Griso, and Onofrei in [12] (for domains with two small holes, see [3, 28]). For a general presentation of unfolding, we refer to the comprehensive book [11]. Meanwhile, extensions to time-dependent functions involved in the heat and wave equations in perforated domains are treated by Donato and Yang in [23, 24], and similarly in small holes by Cabarrubias and Donato in [5]. Moreover, Donato, Le Nguyen, and Tardieu constructed a variant for domains with two components in [21], and another type for highly oscillating boundary by Aiyappan, Nandakumaran, and Prakash in [1]. Since then, the adaptation of the method suitable for different domain configurations has been extensively explored and studied.

In this article we intend to develop a version of the periodic unfolding method suitable for domains with very small inclusions whose sizes are smaller than its

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period. Next, we apply this method by studying the asymptotic behavior of an elliptic problem where in the interface of the components, the jump of the solution is proportional to the flux. To complete the homogenization process, we also obtain some corrector results.

To this goal, for $N \geq 3$ we consider an open and bounded set $\Omega \subset \mathbb{R}^N$ with a Lipschitz continuous boundary $\partial\Omega$, where Ω is a union of the open sets $\Omega_1^{\delta,\varepsilon}$ and $\Omega_2^{\delta,\varepsilon}$ with a common boundary $\Gamma^{\delta,\varepsilon}$. The component $\Omega_2^{\delta,\varepsilon}$ is a disconnected union of ε-periodic very small inclusions of size $\delta(\varepsilon) \ll \varepsilon$ in Ω. Moreover, we let the component $\Omega_1^{\delta,\varepsilon} = \Omega \setminus \overline{\Omega_2^{\delta,\varepsilon}}$ be connected while $\Gamma^{\delta,\varepsilon} := \partial\Omega_2^{\delta,\varepsilon}$.

For the first part, we introduce two unfolding operators: $\mathcal{T}_1^{\delta,\varepsilon}$ acting on functions defined in $\Omega_1^{\delta,\varepsilon}$ and another operator $\mathcal{T}_2^{\delta,\varepsilon}$ for the functions defined in $\Omega_2^{\delta,\varepsilon}$. Here, we prove their corresponding properties and establish the relationship of these two operators and the behavior of their traces on the interface. To achieve the second goal of the paper, we apply this method to describe the asymptotic behavior and obtain corrector results for the elliptic problem given by

$$\begin{gathered} -\operatorname{div}(A^\varepsilon \nabla u_1^{\delta,\varepsilon}) = f \quad \text{in } \Omega_1^{\delta,\varepsilon}, \\ -\operatorname{div}(A^\varepsilon \nabla u_2^{\delta,\varepsilon}) = f \quad \text{in } \Omega_2^{\delta,\varepsilon}, \\ A^\varepsilon \nabla u_1^{\delta,\varepsilon} \cdot n_1^{\delta,\varepsilon} = -A^\varepsilon \nabla u_2^{\delta,\varepsilon} \cdot n_2^{\delta,\varepsilon} \quad \text{on } \Gamma^{\delta,\varepsilon}, \\ -A^\varepsilon \nabla u_1^{\delta,\varepsilon} \cdot n_1^{\delta,\varepsilon} = \varepsilon^\gamma h^{\delta,\varepsilon}(u_1^{\delta,\varepsilon} - u_2^{\delta,\varepsilon}) \quad \text{on } \Gamma^{\delta,\varepsilon}, \\ u_1^{\delta,\varepsilon} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

for $\gamma < 0$ where $n_i^{\delta,\varepsilon}$ is the unit outward normal to $\Omega_i^{\delta,\varepsilon}$, for $i = 1, 2$. We assume that f is a square integrable function in Ω, the matrix field A^ε is bounded and uniformly elliptic and $h^{\delta,\varepsilon}$ is a positive and bounded Y-periodic function.

The physical motivation of this problem concerns a stationary heat dissipation in a two-component composite with very small inclusions with a thermal barrier on the interface whose influence in the heat propagation varies with ε^γ. This condition on the interface can be observed in radiation phenomena. As discussed in [6], if there is no conduction due to the continuity of the temperature field when we traverse on the components due to imperfect bonding among the phases, we obtain an interfacial condition which relates the jump of the temperature to the heat flux across the interface.

The pioneer work in homogenization for two-component domains can be traced back to Auriault and Ene [2] using the multiple scale method. Meanwhile, for inclusions whose size is the same as the period via Tartar's method, one may consult the work of Monsurrò [26] for $\gamma \leq -1$, together with Donato [22] for $\gamma > -1$. For a related problem in domains with very small inclusions via Tartar's method, the reader is referred to the article [27] by Monsurrò. The first time where periodic unfolding method was used in a two-component domain for $\gamma \leq 1$ was due to the work of Donato, Le Nguyen, and Tardieu in [21]. In our case, we only consider $\gamma < 0$ because when $\gamma \in [0, 1]$ one cannot obtain the necessary trace convergences and when $\gamma > 1$ the solution becomes unbounded as investigated by Hummel [25].

Meanwhile, for works concerning the homogenization in domains with small holes, Tartar's method was used by Cioranescu and Murat in[17] (see also [29, 30] by Tartar) to obtain the limiting behavior of a Poisson equation with Dirichlet boundary condition in perforated domains where the critical size $\varepsilon^{N/(N-2)}$ gives rise to an additional zero-order "strange term" in the limit problem which depends on the capacity of the set of holes in the limit. Also, a related problem for the case of a nonhomogeneous Neumann problem for the Laplacian in the same geometric setting but with critical size of order $\varepsilon^{N/(N-1)}$ was done in [18] by Conca and Donato.

The primary novelty in this work is the introduction of another version of the Periodic Unfolding Method that is suited for domains with very small inclusions. Alongside is its application in finding the asymptotic behavior of a particular elliptic problem as well as in determining the corresponding corrector results. In fact, with the aid of this new version, we effectively reveal the contribution of the small inclusions on the homogenized problems by means of a zero order strange term at the limit. This key feature arising from the small scale in this type of domain has not been previously observed for instance in [27] where the limit is only the classical Dirichlet problem. Also, this is the first time for this class of problem where we obtain an additional corrector as a consequence of the strange term at the limit.

The main difficulties addressed in this work are the following: establishing the necessary conditions associated with the parameters ε and δ in order to describe the appropriate trace behaviors, demonstrating the contribution of the small inclusions in the limit problem using an appropriate class of test functions, and analyzing how the small scale manifests in the correctors.

This article organized as follows: Section 2 recalls the unfolding operators for two-component domains. Next, Section 3 develops the version of the unfolding method suitable for domains with very small inclusions. We first assume that $\gamma \leq 1$ and provide the properties of the associated operators and then due to some limitations, we shift to $\gamma < 0$ for the trace behaviors. In Section 4, we describe the asymptotic behavior of problem (1.1) by starting with the case $\gamma < -1$ followed by $\gamma \in (-1, 0)$ where no interface influence can be observed at the limits. We present lastly, the case $\gamma = -1$ as the integral term on the common boundary appears at the homogenized problem. At the end of this work is Section 5 which gives the convergence of the energies leading to the corrector results.

2. Unfolding operator for two-component domains

We start by recalling the periodic unfolding operator for two-component domains originally developed by Donato, Le Nguyen, and Tardieu in [21]. This operator is one of the key tools in the homogenization results later.

Let $\Omega \subset \mathbb{R}^N$ for $N \geq 2$ be an open bounded set with a Lipschitz continuous boundary $\partial\Omega$. Set $Y = \prod_{i=1}^{N}(0, \ell_i)$ to be a reference cell where each $\ell_i > 0$. Let Y_1 and Y_2 be two open connected subsets of Y such that $\overline{Y}_2 \subset Y$ and $Y = Y_1 \cup \overline{Y}_2$ and the boundary $\Gamma = \partial Y_2$ is also Lipschitz continuous.

Let $i = 1, 2$. For any $k \in \mathbb{Z}^N$, denote by $k_\ell = (k_1\ell_1, \dots, k_N\ell_N)$ and define the sets $Y^k = k_\ell + Y$, $\quad Y_i^k = k_\ell + Y_i$ and $K^\varepsilon = \{k \in \mathbb{Z}^N \mid \varepsilon\overline{Y_2^k} \subset \Omega\}$, $\Omega_2^\varepsilon = \operatorname{int} \cup_{k\in K_\varepsilon} \varepsilon Y_2^k$, $\quad \Gamma^\varepsilon = \partial\Omega_2^\varepsilon$ and $\Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}$.

We also consider the sets for $i = 1, 2$, $\widehat{K}_\varepsilon = \{k \in \mathbb{Z}^N \mid \varepsilon Y^k \subset \Omega\}$, $\widehat{\Omega}_\varepsilon = \operatorname{int} \cup_{k\in\widehat{K}_\varepsilon} \varepsilon(k_\ell + \overline{Y})$ and $\Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon$. $\widehat{\Omega}_i^\varepsilon = \cup_{k\in\widehat{K}_\varepsilon} \varepsilon Y_i^k$, $\Lambda_i^\varepsilon = \Omega_i^\varepsilon \setminus \widehat{\Omega}_i^\varepsilon$, and $\widehat{\Gamma}^\varepsilon = \partial\widehat{\Omega}_2^\varepsilon$.

In the sequel, we let ε take on values from a positive real sequence tending to zero and for $g \in L^1(\mathcal{O})$, where $\mathcal{O}$ is an open set in $\mathbb{R}^N$, we use the notations:

$$(i)\ \theta_i = \frac{|Y_i|}{|Y|} \quad \text{and} \quad (ii)\ \mathcal{M}_{\mathcal{O}}(g) = \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} g\,dy. \tag{2.1}$$

We also denote by $\widetilde{\varphi}$ the zero extension of the function φ defined on Ω_i^ε to the whole of Ω.

Definition 2.1. Let $i = 1, 2$. For any Lebesgue measurable function φ in Ω_i^ε the periodic unfolding operator $\mathcal{T}_i^\varepsilon$ is given by

$$\mathcal{T}_i^\varepsilon(\varphi)(x,y) = \begin{cases} \varphi\big(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon y\big) & \text{for a.e. } (x,y) \in \widehat{\Omega}_\varepsilon \times Y_i, \\ 0 & \text{for a.e. } (x,y) \in \Lambda_\varepsilon \times Y_i. \end{cases}$$

Remark 2.2. Notice that if we define the operator $\mathcal{T}_\varepsilon$ in $\Omega \times Y$ to be

$$\mathcal{T}_\varepsilon(\varphi) = \begin{cases} \mathcal{T}_1^\varepsilon(\varphi) & \text{in } \Omega \times Y_1, \\ \mathcal{T}_2^\varepsilon(\varphi) & \text{in } \Omega \times Y_2, \end{cases}$$

we obtain the unfolding operator for the fixed domain Ω given in [9].

Next, let us recall some properties of this unfolding operators. We only state here the necessary properties for this work.

Theorem 2.3. *Let $p \in [1, +\infty)$ and $i = 1, 2$. The operators $\mathcal{T}_i^\varepsilon(\varphi)$ are linear and continuous from $L^p(\Omega_i^\varepsilon)$ to $L^p(\Omega \times Y)$. Moreover,*

(1) *$\mathcal{T}_i^\varepsilon(\varphi\psi) = \mathcal{T}_i^\varepsilon(\varphi)\mathcal{T}_i^\varepsilon(\psi)$ for every Lebesgue measurable functions φ, ψ on Ω_i^ε;*
(2) *for every $\varphi \in L^1(\Omega_i^\varepsilon)$,*
$$\frac{1}{|Y|}\int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon(\varphi)(x,y)\,dxdy = \int_{\widehat{\Omega}_i^\varepsilon} \varphi(x)\,dx = \int_{\Omega_i^\varepsilon} \varphi(x)\,dx - \int_{\Lambda_i^\varepsilon} \varphi(x)\,dx;$$
(3) *$\mathcal{T}_i^\varepsilon(\varphi) \to \varphi$ strongly in $L^p(\Omega \times Y_i)$, for $\varphi \in L^p(\Omega)$;*
(4) *if $\varphi_\varepsilon \in L^p(\Omega_i^\varepsilon))$ satisfies $\|\varphi_\varepsilon\|_{L^p(\Omega_i^\varepsilon)} \le C$ and $\mathcal{T}_i^\varepsilon(\varphi_\varepsilon) \rightharpoonup \widehat{\varphi}$ weakly in $L^p(\Omega \times Y_i)$, then $\widetilde{\varphi}_\varepsilon \rightharpoonup \theta_i \mathcal{M}_{Y_i}(\widehat{\varphi})$ weakly in $L^p(\Omega)$.*

We now have the following adjoints of these unfolding operators together with properties that we will need later.

Definition 2.4. For $p \in [1, +\infty]$, the averaging operators $\mathcal{U}_i^\varepsilon : L^p(\Omega \times Y_i) \to L^p(\widehat{\Omega}_i^\varepsilon), i = 1, 2$, are defined as follows:

$$\mathcal{U}_i^\varepsilon(\varphi)(x) = \begin{cases} \frac{1}{|Y|}\int_Y \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y\right) dz & \text{for a.e. } x \in \widehat{\Omega}_i^\varepsilon, \\ 0 & \text{for a.e. } x \in \Lambda_i^\varepsilon. \end{cases}$$

Theorem 2.5. *Let $p \in [1, +\infty)$ and $i = 1, 2$. The averaging operators $\mathcal{U}_i^\varepsilon$ are linear and continuous. Moreover,*

(1) *$\|\mathcal{U}_i^\varepsilon(\varphi) - \varphi\|_{L^p(\Omega_i^\varepsilon)} \to 0$ for every $\varphi \in L^p(\Omega)$;*
(2) *if φ_ε belongs to $L^p(\Omega_i^\varepsilon)$, then the following assertions are equivalent:*

$$(a) \quad \mathcal{T}_i^\varepsilon(\varphi_\varepsilon) \to \widehat{\varphi} \quad \text{strongly in } L^p(\omega \times Y_i) \quad \text{and} \quad \int_{\Lambda_i^\varepsilon} |\varphi_\varepsilon|^p dx \to 0,$$

$$(b) \quad \|\varphi_\varepsilon - \mathcal{U}_i^\varepsilon(\widehat{\varphi})\|_{L^p(\Omega_i^\varepsilon)} \to 0.$$

Now, for $\gamma \le 1$ we define the space H^ε_γ (see [21] for the details regarding this space) as,

$$H^\varepsilon_\gamma = \{u = (u_1, u_2) \mid u_1 \in V^\varepsilon, u_2 \in H^1(\Omega_2^\varepsilon)\},$$

where $V^\varepsilon = \{v \in H^1(\Omega_1^\varepsilon) \mid v = 0 \text{ on } \partial\Omega\}$.

Theorem 2.6. *Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) \in H^\varepsilon_\gamma$. Then*

$$\frac{1}{\varepsilon|Y|}\int_{\Omega\times\Gamma} |\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)|^2\, dx\, d\sigma_y \le \int_{\Gamma^\varepsilon} |u_1^\varepsilon - u_2^\varepsilon|^2\, d\sigma_x.$$

Theorem 2.7. *If $\varphi \in \mathcal{D}(\Omega)$ and $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) \in H^\varepsilon_\gamma$, then for ε small enough,*

$$\varepsilon\int_{\Gamma^\varepsilon} h^\varepsilon(u_1^\varepsilon - u_2^\varepsilon)\varphi\, d\sigma_x = \frac{1}{|Y|}\int_{\Omega\times\Gamma} h(y)\left(\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)\right)\varphi\, dx\, d\sigma_y.$$

Theorem 2.8. *If $u = (u_1^\varepsilon, u_2^\varepsilon)$ is a bounded sequence in H^ε_γ, then*

$$\begin{gathered}\|\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon)\|_{L^p(\Omega\times Y_1)} \le C,\\ \|\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon)\|_{L^p(\Omega\times Y_2)} \le C,\\ \|\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)\|_{L^p(\Omega\times\Gamma)} \le C\varepsilon^{\frac{1-\gamma}{2}}.\end{gathered}$$

Theorem 2.9. *Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a bounded sequence in H^ε_γ. Then there exists a subsequence (still denoted by ε), $u_1 \in H^1_0(\Omega)$ and $\widehat{u}_1 \in L^2(\Omega; H^1_{per}(Y_1))$ such that*

$$\mathcal{T}_1^\varepsilon(u_1^\varepsilon) \to u_1 \quad \text{strongly in } L^2(\Omega; H^1(Y_1)), \tag{2.2}$$

$$\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \quad \text{weakly in } L^2(\Omega\times Y_1), \tag{2.3}$$

with $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ for almost every $x \in \Omega$. Furthermore,

$$Z_1^\varepsilon = \frac{1}{\varepsilon}\left(\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{M}_\Gamma(\mathcal{T}_1^\varepsilon(u_1^\varepsilon))\right) \rightharpoonup y_\Gamma\nabla u_1 + \widehat{u}_1$$

weakly in $L^2(\Omega; H^1(Y_1))$, where

$$y_\Gamma = y - \mathcal{M}_\Gamma(y). \tag{2.4}$$

Theorem 2.10. *Let $\gamma \le 1$ and $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a bounded sequence in H^ε_γ. Then there exists a subsequence (still denoted by ε) and $u_2 \in L^2(\Omega)$ such that*

$$\begin{gathered}\mathcal{T}_2^\varepsilon(u_2^\varepsilon) \rightharpoonup u_2 \quad \text{weakly in } L^2(\Omega; H^1(Y_2)),\\ \varepsilon\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \to 0 \quad \text{strongly in } L^2(\Omega\times Y_2)).\end{gathered} \tag{2.5}$$

Moreover, if $\gamma < 1$ and (2.2) and (2.3) hold for a subsequence, then $u_2 = u_1$, i.e.,

$$\mathcal{T}_2^\varepsilon(u_2^\varepsilon) \rightharpoonup u_1 \quad \text{weakly in } L^2(\Omega; H^1(Y_2)).$$

Theorem 2.11. *Let u^ε be a bounded sequence in H^ε_γ. Then there exists a subsequence (still denoted by ε) and $\widehat{u}_2 \in L^2(\Omega; H^1(Y_2))$ such that*

$$Z_2^\varepsilon = \frac{1}{\varepsilon}\left(\mathcal{T}_2^\varepsilon(u_2^\varepsilon) - \mathcal{M}_\Gamma(\mathcal{T}_2^\varepsilon(u_2^\varepsilon))\right) \rightharpoonup \widehat{u}_2 \quad \text{weakly in } L^2(\Omega; H^1(Y_2)), \tag{2.6}$$

$$\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla_y \widehat{u}_2 \quad \text{weakly in } L^2(\Omega\times Y_2), \tag{2.7}$$

where $\mathcal{M}_\Gamma(\widehat{u}_2) = 0$ for almost every $x \in \Omega$.

Theorem 2.12. *If $\gamma \le 1$, and $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is a bounded sequence in H_γ^ε, then there exist a subsequence (still denoted by ε), $u_1 \in H_0^1(\Omega), u_2 \in L^2(\Omega), \widehat{u}_1 \in L^2(\Omega; H^1_{per}(Y_1))$, and $\widehat{u}_2 \in L^2(\Omega; H^1(Y_2))$ such that (2.2), (2.3), and (2.5)–(2.7) hold.*

Furthermore, if $\gamma < 1$, then $u_1 = u_2$ and

(i) *if $\gamma < -1$ then $\widehat{u}_1 = \widehat{u}_2 - y_\Gamma \nabla u_1$ on $\Omega \times \Gamma$,*
(ii) *if $\gamma = -1$, then for some function $\xi_\Gamma \in L^2(\Omega)$,*

$$\frac{\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)}{\varepsilon} \rightharpoonup \widehat{u}_1 - \widehat{u}_2 + y_\Gamma \nabla u_1 + \xi_\Gamma \quad \text{weakly in } L^2(\Omega \times \Gamma).$$

3. Unfolding operator for domains with very small inclusions

Let $\Omega \subset \mathbb{R}^N$ for $N \ge 3$ be an open bounded set with a Lipschitz continuous boundary $\partial\Omega$. Let ε be a positive real sequence that approaches zero and let $\delta = \delta(\varepsilon) < 1$ be such that $\delta \to 0$ as $\varepsilon \to 0$. Let $Y = \prod_{i=1}^N (0, \ell_i)$ be a reference cell where each $\ell_i > 0$. Let $B \subset Y$ and set $Y_2 = \delta B$, the δ-scaled version of B, with Lipschitz continuous boundary $\Gamma = \partial Y_2$. Moreover, we set $Y_1 = Y \setminus \overline{Y_2}$. From this construction, Y is the disjoint union $Y = Y_1 \cup Y_2 \cup \Gamma$. We assume that both Y_1 and Y_2 are connected

For any $\xi \in \mathbb{Z}^N$, denote by $\xi_\ell = (\xi_1 \ell_1, \dots, \xi_N \ell_N)$ and define the set $K^\varepsilon = \{\xi \in \mathbb{Z}^N \mid \varepsilon(\xi_\ell + Y_2) \cap \Omega \ne \emptyset\}$. From here, define the sets $\Omega_2^{\delta,\varepsilon} = \operatorname{int} \cup_{\xi \in K_\varepsilon} \varepsilon(\xi_\ell + Y_2)$, $\Gamma^{\delta,\varepsilon} = \partial\Omega_2^{\delta,\varepsilon}$ and $\Omega_1^{\delta,\varepsilon} = \Omega \setminus \overline{\Omega_2^{\delta,\varepsilon}}$. Thus, we have $\partial\Omega_1^{\delta,\varepsilon} = \partial\Omega \cup \Gamma^{\delta,\varepsilon}$ and Ω is the disjoint union $\Omega = \Omega_1^{\delta,\varepsilon} \cup \Omega_2^{\delta,\varepsilon} \cup \Gamma^{\delta,\varepsilon}$. We mention that $\Omega_1^{\delta,\varepsilon}$ is connected and assume that $\partial\Omega \cap \Gamma^{\delta,\varepsilon} = \emptyset$ so that $\Omega_2^{\delta,\varepsilon}$ is a collection of pairwise disjoint translated sets Y_2 distributed with period ε.

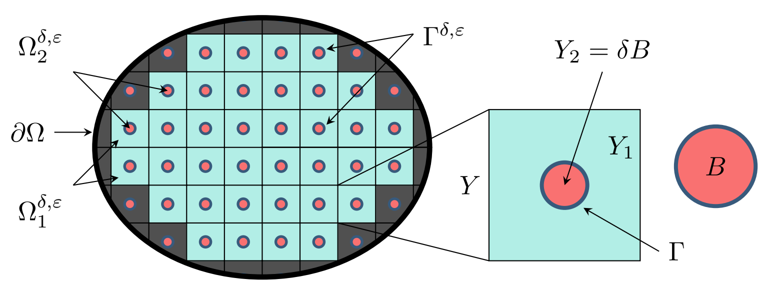

Figure 1. Two-component domain with very small inclusions

Furthermore, we define the sets

$$\widehat{K}_\varepsilon = \{\xi \in \mathbb{Z}^N \mid \varepsilon(\xi_\ell + Y) \subset \Omega\},$$

$$\widehat{\Omega}_\varepsilon = \operatorname{int} \cup_{\xi \in \widehat{K}_\varepsilon} \varepsilon(\xi_\ell + \overline{Y}), \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon, \tag{3.1}$$

$$\widehat{\Omega}_i^{\delta,\varepsilon} = \operatorname{int} \cup_{\xi \in \widehat{K}_\varepsilon} \varepsilon(\xi_\ell + Y_i), \quad \Lambda_i^{\delta,\varepsilon} = \Omega_i^{\delta,\varepsilon} \setminus \widehat{\Omega}_i^{\delta,\varepsilon}, \quad \widehat{\Gamma}^{\delta,\varepsilon} = \partial\widehat{\Omega}_2^{\delta,\varepsilon}. \tag{3.2}$$

Again, we use the notation $\widetilde{\cdot}$ to denote the zero extension as defined in the previous section.

Let $p \in [1, \infty)$. We define the functional space

$$V_p^{\delta,\varepsilon} = \{v \in W^{1,p}(\Omega_1^{\delta,\varepsilon}) \mid v = 0 \text{ on } \partial\Omega\}, \tag{3.3}$$

endowed with the norm $\|v\|_{V_p^{\delta,\varepsilon}} = \|\nabla v\|_{L^p(\Omega_1^{\delta,\varepsilon})}$ for every $v \in V_p^{\delta,\varepsilon}$.

Remark 3.1. A Poincaré inequality holds in the space $V_p^{\delta,\varepsilon}$. Consequently, the norms in $V_p^{\delta,\varepsilon}$ and $W^{1,p}(\Omega_1^{\delta,\varepsilon})$ are equivalent.

For each real number γ, we define the function space

$$H_{\gamma,p}^{\delta,\varepsilon} = \{u^{\delta,\varepsilon} = (u_1^{\delta,\varepsilon}, u_2^{\delta,\varepsilon}) \mid u_1^{\delta,\varepsilon} \in V_p^{\delta,\varepsilon}, u_2^{\delta,\varepsilon} \in W^{1,p}(\Omega_2^{\delta,\varepsilon})\}, \tag{3.4}$$

equipped with the norm,

$$\|u^{\delta,\varepsilon}\|^p_{H_{\gamma,p}^{\delta,\varepsilon}} = \|\nabla u_1^{\delta,\varepsilon}\|^p_{L^p(\Omega_1^{\delta,\varepsilon})} + \|\nabla u_2^{\delta,\varepsilon}\|^p_{L^p(\Omega_2^{\delta,\varepsilon})} + \varepsilon^\gamma \|u_1^{\delta,\varepsilon} - u_2^{\delta,\varepsilon}\|^p_{L^p(\Gamma^{\delta,\varepsilon})}. \tag{3.5}$$

Remark 3.2. The norms in $H_{\gamma,p}^{\delta,\varepsilon}$ and $V_p^{\delta,\varepsilon} \times W^{1,p}(\Omega_2^{\delta,\varepsilon})$ are equivalent (see, for instance, [27] for analogous developments on this equivalence).

Theorem 3.3. *If $u = (u_1^{\delta,\varepsilon}, u_2^{\delta,\varepsilon}) \in H_{\gamma,p}^{\delta,\varepsilon}$ is bounded, then there is a constant $C > 0$ independent of ε such that*

$$\|u_1^{\delta,\varepsilon}\|_{W^{1,p}(\Omega_1^{\delta,\varepsilon})} \le C, \tag{3.6}$$

$$\|\nabla u_2^{\delta,\varepsilon}\|_{L^p(\Omega_2^{\delta,\varepsilon})} \le C, \tag{3.7}$$

$$\|u_1^{\delta,\varepsilon} - u_2^{\delta,\varepsilon}\|_{L^p(\Gamma^{\delta,\varepsilon})} \le C\varepsilon^{-\gamma/p}. \tag{3.8}$$

Moreover, if $\gamma \le 1$, then

$$\|u_2^{\delta,\varepsilon}\|_{W^{1,p}(\Omega_2^{\delta,\varepsilon})} \le C. \tag{3.9}$$

Proof. Suppose that u in $H_{\gamma,p}^{\delta,\varepsilon}$ is bounded. Then (3.6) is immediate from Remark 3.1 and (3.5), while estimates (3.7) and (3.8) follow from (3.5). Finally, estimate (3.9) follows from Remark 3.2 and after minor computations when $\gamma \le 1$. □

3.1. **Unfolding operator.** Let us now introduce the unfolding operator suitable for our geometric setting and provide some properties. We also describe here the relationship of the operators and prove the behavior of its traces on the interface. From now on, we let $i = 1, 2$ unless otherwise stated.

Definition 3.4. Let $p \in [1, +\infty)$. For $\varphi \in L^p(\Omega_i^{\delta,\varepsilon})$, the unfolding operator $\mathcal{T}_i^{\delta,\varepsilon}$ from $L^p(\Omega_i^{\delta,\varepsilon})$ to $L^p(\Omega \times \mathbb{R}^N)$, is defined by

$$\mathcal{T}_i^{\delta,\varepsilon}(\varphi)(x,z) = \begin{cases} \varphi\big(\varepsilon\big[\frac{x}{\varepsilon}\big]_Y + \varepsilon\delta z\big) & \text{for a.e. } (x,z) \in \widehat{\Omega}_\varepsilon \times \frac{1}{\delta}Y_i, \\ 0 & \text{for a.e. } (x,z) \in \Lambda_\varepsilon \times \frac{1}{\delta}Y_i. \end{cases}$$

For ease of presentation, if φ is a function defined in Ω, we denote $\mathcal{T}_i^{\delta,\varepsilon}(\varphi) = \mathcal{T}_i^{\delta,\varepsilon}\big(\varphi|_{\Omega_i^{\delta,\varepsilon}}\big)$.

Remark 3.5. The operator $\mathcal{T}_1^{\delta,\varepsilon}$ is the unfolding operator "$\mathcal{T}_{\varepsilon,\delta}$" in [11, 12]. When $\delta = 1$, the operators $\mathcal{T}_i^{\delta,\varepsilon}$ are the unfolding operators "$\mathcal{T}_i^{\varepsilon}$" in [21]. Moreover, we also recover the unfolding operator "$\mathcal{T}_\varepsilon$" for fixed domains given in [9] if we set

$$\mathcal{T}_\varepsilon(\varphi) = \begin{cases} \mathcal{T}_1^{\delta,\varepsilon}(\varphi) & \text{in } \Omega \times \frac{1}{\delta}Y_1, \\ \mathcal{T}_2^{\delta,\varepsilon}(\varphi) & \text{in } \Omega \times \frac{1}{\delta}Y_2. \end{cases}$$

Before we proceed, let us first introduce the mean value and local average operators.

Definition 3.6. For $p \in [1, +\infty)$, the *mean value operator* $\mathcal{M}_{\frac{1}{\delta}Y_i}$ from $L^p(\Omega \times \frac{1}{\delta}Y_i)$ to $L^p(\Omega)$ is defined as

$$\mathcal{M}_{\frac{1}{\delta}Y_i}(\varphi)(x) = \frac{\delta^N}{|Y_i|}\int_{\frac{1}{\delta}Y_i} \varphi(x,z)\,dz, \quad \forall \varphi \in L^p(\Omega \times \frac{1}{\delta}Y_i).$$

An immediate consequence of this definition is the following proposition.

Proposition 3.7. *If $\varphi \in L^p(\Omega \times \frac{1}{\delta}Y_i)$, then*

$$\|\mathcal{M}_{\frac{1}{\delta}Y_i}(\varphi)\|_{L^p(\Omega)} \le \frac{\delta^N}{|Y_i|}\|\varphi\|_{L^p(\Omega\times\frac{1}{\delta}Y_i)}.$$

Definition 3.8. For $p \in [1, +\infty)$, the *local average operator* $\mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_i}$ from $L^p(\Omega_i^{\delta,\varepsilon})$ to $L^p(\Omega)$ is defined as

$$\mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_i}(\varphi)(x) = \frac{\delta^N}{|Y_i|}\int_{\frac{1}{\delta}Y_i} \mathcal{T}_i^{\delta,\varepsilon}(\varphi)(x,z)\,dz, \quad \forall \varphi \in L^p(\Omega_i^{\delta,\varepsilon}).$$

We are in a position to give some properties of $\mathcal{T}_i^{\delta,\varepsilon}$. In what follows, for $p \in [1,\infty)$ and $N \ge 3$, set p^* to be the associated Sobolev exponent to p given by

$$p^* = \frac{pN}{N-p}. \tag{3.10}$$

Theorem 3.9. *Let $p \in [1, +\infty)$. The unfolding operator $\mathcal{T}_i^{\delta,\varepsilon}$ is linear and continuous. Moreover, it has the following properties.*

(i) *For every $v_i^{\delta,\varepsilon}, w_i^{\delta,\varepsilon} \in L^p(\Omega_i^{\delta,\varepsilon})$, $\mathcal{T}_i^{\delta,\varepsilon}(v_i^{\delta,\varepsilon}w_i^{\delta,\varepsilon}) = \mathcal{T}_i^{\delta,\varepsilon}(v_i^{\delta,\varepsilon})\mathcal{T}_i^{\delta,\varepsilon}(w_i^{\delta,\varepsilon})$.*

(ii) *For every $u_i^{\delta,\varepsilon} \in L^p(\Omega_i^{\delta,\varepsilon})$,*

$$\frac{\delta^N}{|Y|}\int_{\Omega\times\frac{1}{\delta}Y_i} \mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})\,dx\,dz = \int_{\widehat{\Omega}_i^{\delta,\varepsilon}} u_i^{\delta,\varepsilon}\,dx = \int_{\Omega_i^{\delta,\varepsilon}} u_i^{\delta,\varepsilon}\,dx - \int_{\Lambda_i^{\delta,\varepsilon}} u_i^{\delta,\varepsilon}\,dx.$$

(iii) *For every $u_i^{\delta,\varepsilon} \in L^p(\Omega_i^{\delta,\varepsilon})$, $\|\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})\|_{L^p(\Omega\times\mathbb{R}^N)} \le \big(\frac{|Y|}{\delta^N}\big)^{1/p}\|u_i^{\delta,\varepsilon}\|_{L^p(\Omega_i^{\delta,\varepsilon})}$.*

(iv) *For every $u_i^{\delta,\varepsilon} \in L^1(\Omega_i^{\delta,\varepsilon})$,*

$$\Big|\int_{\Omega_i^{\delta,\varepsilon}} u_i^{\delta,\varepsilon}\,dx - \frac{\delta^N}{|Y|}\int_{\Omega\times\frac{1}{\delta}Y_i} \mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})\,dx\,dz\Big| \le \int_{\Lambda_i^{\delta,\varepsilon}} |u_i^{\delta,\varepsilon}|\,dx.$$

(v) *Let $u_i^{\delta,\varepsilon} \in W^{1,p}(\Omega_i^{\delta,\varepsilon})$. Then*

$$\mathcal{T}_i^{\delta,\varepsilon}(\nabla_x u_i^{\delta,\varepsilon}) = \frac{1}{\varepsilon\delta}\nabla_z \mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \quad \text{in } \Omega \times \frac{1}{\delta}Y_i, \tag{3.11}$$

$$\|\nabla_z \mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})\|_{L^p(\Omega\times\mathbb{R}^N)} \le \frac{\varepsilon|Y|^{1/p}}{\delta^{\frac{N}{p}-1}}\|\nabla u_i^{\delta,\varepsilon}\|_{L^p(\Omega_i^{\delta,\varepsilon})}. \tag{3.12}$$

(vi) *If $\{w_i^{\delta,\varepsilon}\}$ is a sequence in $L^p(\Omega_i^{\delta,\varepsilon})$ such that $\widetilde{w_i^{\delta,\varepsilon}} \to w_i$ strongly in $L^p(\Omega)$, then we have the convergence $\mathcal{T}_i^{\delta,\varepsilon}(w_i^{\delta,\varepsilon}) \to w_i$ strongly in $L^p(\Omega \times \mathbb{R}^N)$.*

(vii) *Let ω_i be a bounded open set in $\mathbb{R}^N$. For every $u_i^{\delta,\varepsilon} \in W^{1,p}(\Omega_i^{\delta,\varepsilon})$, the following estimates hold for $i = 1, 2$:*

$$\big\|\mathcal{T}_i^{\delta,\varepsilon}\big[u_i^{\delta,\varepsilon} - \mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_i}(u_i^{\delta,\varepsilon})\big]\big\|_{L^p(\Omega;L^{p^*}(\mathbb{R}^N))} \le C\frac{\varepsilon|Y|^{1/p}}{\delta^{\frac{N}{p}-1}}\|\nabla u_i^{\delta,\varepsilon}\|_{L^p(\Omega_i^{\delta,\varepsilon})}, \tag{3.13}$$

$$\begin{aligned}&\|\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})\|_{L^p(\Omega\times\omega_i)}\\&\le 2C\frac{\varepsilon|Y|^{1/p}}{\delta^{\frac{N}{p}-1}}\|\nabla u_i^{\delta,\varepsilon}\|_{L^p(\Omega_i^{\delta,\varepsilon})}+2|\omega_i|^{1/p}\frac{|Y|^{1/p}}{|Y_i|}\delta^{N(1-\frac{1}{p})}\|u_i^{\delta,\varepsilon}\|_{L^p(\Omega_i^{\delta,\varepsilon})},\end{aligned}\tag{3.14}$$

where C is the Sobolev-Poincaré-Wirtinger constant for $W^{1,p}(Y_i)$ and p^* as in (3.10).

(viii) *Let $\{w_i^{\delta,\varepsilon}\}$ be a sequence in $W^{1,p}(\Omega_i^{\delta,\varepsilon})$ which is uniformly bounded when both ε and δ approach zero. Then, up to a subsequence, there is $W_i\in L^p(\Omega;L^{p^*}(\mathbb{R}^N))$ with $\nabla_z W_i\in L^p(\Omega\times\mathbb{R}^N)^N$ such that*

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_i^{\delta,\varepsilon}\big[w_i^{\delta,\varepsilon}-\mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_i}(w_i^{\delta,\varepsilon})\big]1_{\frac{1}{\delta}Y_i}\rightharpoonup W_i\quad\text{weakly in } L^p(\Omega;L^{p^*}(\mathbb{R}^N)),\tag{3.15}$$

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z\mathcal{T}_i^{\delta,\varepsilon}(w_i^{\delta,\varepsilon})1_{\frac{1}{\delta}Y_i}\rightharpoonup\nabla_z W_i\quad\text{weakly in } L^p(\Omega\times\mathbb{R}^N)^N,\tag{3.16}$$

where p^* is given by (3.10). *Assuming furthermore that*

$$\limsup_{\varepsilon\to0}\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}<+\infty,\tag{3.17}$$

then we can choose the subsequence above and some $U_i\in L^p(\Omega;L^p_{\rm loc}(\mathbb{R}^N))$ such that

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_i^{\delta,\varepsilon}(w_i^{\delta,\varepsilon})\rightharpoonup U_i\quad\text{weakly in } L^p(\Omega;L^p_{\rm loc}(\mathbb{R}^N)).\tag{3.18}$$

Proof of (i)–(vi). The corresponding properties for $\mathcal{T}_1^{\delta,\varepsilon}$ follow by a change of variable $y=\delta z$ as similarly shown in [12] for the case $p=2$ and in [11] for $p\in[1,\infty)$ (for time-dependent functions, see [5]). The proofs for $\mathcal{T}_2^{\delta,\varepsilon}$ are analogously obtained. □

To prove Theorem 3.9 (vii) and (viii), we require the next result which describes the interplay between the mean value and local average operators.

Proposition 3.10. *Let $p\in[1,\infty)$.*

(i) *For $u_i^{\delta,\varepsilon}\in L^p(\Omega_i^{\delta,\varepsilon})$, we have*

$$\mathcal{T}_i^{\delta,\varepsilon}\big[\mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_i}(u_i^{\delta,\varepsilon})\big]=\mathcal{M}_{\frac{1}{\delta}Y_i}\big[\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})\big]=\mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_i}(u_i^{\delta,\varepsilon}).$$

(ii) *If $\{w_i^{\delta,\varepsilon}\}$ is a sequence in $L^p(\Omega_i^{\delta,\varepsilon})$ such that $\widetilde{w_i^{\delta,\varepsilon}}\to w_i$ strongly in $L^p(\Omega)$, then*

$$\mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_i}(w_i^{\delta,\varepsilon})\to\mathcal{M}_{\frac{1}{\delta}Y_i}(w_i)=w_i\quad\text{strongly in } L^p(\Omega).$$

(iii) *If $u_i^{\delta,\varepsilon}\in L^p(\Omega_i^{\delta,\varepsilon})$, then $\|\mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_i}(u_i^{\delta,\varepsilon})\|_{L^p(\Omega)}\le\frac{|Y|^{1/p}}{|Y_i|}\delta^{N(1-\frac{1}{p})}\|u_i^{\delta,\varepsilon}\|_{L^p(\Omega_i^{\delta,\varepsilon})}$.*

Proof. Let us show the case $i=1$; when $i=2$ the proofs are similar.

The identity in (i) uses Definitions 3.4, 3.6, 3.8, and the fact that $y=\delta z$.

(ii) If $\{w_1^{\delta,\varepsilon}\}$ is a sequence in $L^p(\Omega_1^{\delta,\varepsilon})$ such that $\widetilde{w_1^{\delta,\varepsilon}}\to w_1$ strongly in $L^p(\Omega)$, then by (i), linearity of $\mathcal{M}_{\frac{1}{\delta}Y_1}$, and using Proposition 3.7 and Theorem 3.9 (vi) we have

$$\|\mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_1}(w_1^{\delta,\varepsilon})-w_1\|_{L^p(\Omega)}=\|\mathcal{M}_{\frac{1}{\delta}Y_1}\big[\mathcal{T}_1^{\delta,\varepsilon}(w_1^{\delta,\varepsilon})\big]-\mathcal{M}_{\frac{1}{\delta}Y_1}(w_1)\|_{L^p(\Omega)}$$

$$\begin{aligned}&= \|\mathcal{M}_{\frac{1}{\delta}Y_1}\big[\mathcal{T}_1^{\delta,\varepsilon}(w_1^{\delta,\varepsilon}) - w_1\big]\|_{L^p(\Omega)}\\ &\le \frac{\delta^N}{|Y_1|}\|\mathcal{T}_1^{\delta,\varepsilon}(w_1^{\delta,\varepsilon}) - w_1\|_{L^p(\Omega\times\mathbb{R}^N)} \to 0.\end{aligned}$$

Finally, (iii) follows from Definition 3.8 and Theorem 3.9 (iii). □

Proof of Theorem 3.9 (vii) and (viii). We only show here the case $i = 1$; when $i = 2$ the proofs are essentially the same. Estimate (3.13) in (vii) is a direct consequence of the Sobolev-Poincaré-Wirtinger inequality. Meanwhile, the second estimate in (vii) follows from the fact that by Proposition 3.10 (i) one has

$$\begin{aligned}|\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon})|^p &= |\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - \mathcal{T}_1^{\delta,\varepsilon}(\mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_1}(u_1^{\delta,\varepsilon})) + \mathcal{T}_1^{\delta,\varepsilon}(\mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_1}(u_1^{\delta,\varepsilon}))|^p\\ &\le 2^p\big(|\mathcal{T}_1^{\delta,\varepsilon}\big[u_1^{\delta,\varepsilon} - \mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_1}(u_1^{\delta,\varepsilon})\big]|^p + |\mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_1}(u_1^{\delta,\varepsilon})|^p\big).\end{aligned} \tag{3.19}$$

This, since $\omega_1 \subset \mathbb{R}^N$, by (3.13) and Proposition 3.10 (iii), we obtain the desired estimate.

Let us now prove (viii). Using estimate (3.13), there exists $W_1 \in L^p(\Omega; L^{p^*}\mathbb{R}^N))$ such that (3.15) holds. Next, we show (3.16). From (3.12), there exists $\mathcal{S}_1 \in L^p(\Omega\times\mathbb{R}^N)^N$ such that

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z\mathcal{T}_1^{\delta,\varepsilon}(w_1^{\delta,\varepsilon})1_{\frac{1}{\delta}Y_1} \rightharpoonup \mathcal{S}_1 \quad \text{weakly in } L^p(\Omega\times\mathbb{R}^N)^N. \tag{3.20}$$

Meanwhile,from Proposition 3.10 (i) and Definition 3.8 we have $\nabla_z\mathcal{T}_1^{\delta,\varepsilon}\big[w_1^{\delta,\varepsilon} - \mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_1}(w_1^{\delta,\varepsilon})\big] = \nabla_z\mathcal{T}_1^{\delta,\varepsilon}(w_1^{\delta,\varepsilon})$. Using this, for $\varphi\in\mathcal{D}\big(\Omega\times\mathbb{R}^N\big)$ we have

$$\begin{aligned}&\int_{\Omega\times\frac{1}{\delta}Y_1}\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z\mathcal{T}_1^{\delta,\varepsilon}(w_1^{\delta,\varepsilon})\varphi\,dx\,dz\\ &= -\int_{\Omega\times\frac{1}{\delta}Y_1}\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_1^{\delta,\varepsilon}\big[w_1^{\delta,\varepsilon} - \mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_1}(w_1^{\delta,\varepsilon})\big]\nabla\varphi\,dx\,dz.\end{aligned}$$

Passing to the limit in this equation using (3.20) and (3.15), we obtain

$$\int_{\Omega\times\mathbb{R}^N}\mathcal{S}_1\varphi\,dx\,dz = -\int_{\Omega\times\mathbb{R}^N}W_1\nabla\varphi\,dx\,dz = \int_{\Omega\times\mathbb{R}^N}\nabla_z W_1\varphi\,dx\,dz,$$

and so $\mathcal{S}_1 = \nabla_z W_1$. In view of (3.20), then (3.16) holds. The last convergence in (3.18) is immediate from (3.14) with the aid of (3.17). □

Also, an unfolding criterion for integrals holds for this operator. This is an immediate consequence of Theorem 3.9 (iv).

Proposition 3.11. *If a sequence* $\{w_i^{\delta,\varepsilon}\}$ *in* $L^1(\Omega_i^{\delta,\varepsilon})$ *satisfies*

$$\int_{\Lambda_i^{\delta,\varepsilon}}|w_i^{\delta,\varepsilon}(x)|\,dx \to 0,$$

then

$$\int_{\Omega_i^{\delta,\varepsilon}}w_i^{\delta,\varepsilon}(x)\,dx - \frac{\delta^N}{|Y|}\int_{\Omega\times\frac{1}{\delta}Y_i}\mathcal{T}_i^{\delta,\varepsilon}(w_i^{\delta,\varepsilon})(x,z)\,dx\,dz \to 0.$$

Moreover, we write

$$\int_{\Omega_i^{\delta,\varepsilon}}w_i^{\delta,\varepsilon}(x)\,dx \overset{\mathcal{T}_i^{\delta,\varepsilon}}{\simeq} \frac{\delta^N}{|Y|}\int_{\Omega\times\frac{1}{\delta}Y_i}\mathcal{T}_i^{\delta,\varepsilon}(w_i^{\delta,\varepsilon})(x,z)\,dx\,dz.$$

Corollary 3.12. *Let $\{u_i^{\delta,\varepsilon}\}$ be a bounded sequence in $L^p(\Omega_i^{\delta,\varepsilon})$ and $v \in L^q(\Omega_i^{\delta,\varepsilon})$ such that $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_{\Omega_i^{\delta,\varepsilon}} u_i^{\delta,\varepsilon}\, v\, dx \overset{\mathcal{T}_i^{\delta,\varepsilon}}{\simeq} \frac{\delta^N}{|Y|}\int_{\Omega\times\frac{1}{\delta}Y_i} \mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})(x,z)\, \mathcal{T}_i^{\delta,\varepsilon}(v)(x,z)\, dx\, dz. \tag{3.21}$$

Furthermore, if $\{v_i^{\delta,\varepsilon}\}$ is a bounded sequence in $L^{p_0}(\Omega_i^{\delta,\varepsilon})$ such that $\frac{1}{p}+\frac{1}{p_0}<1$, then

$$\int_{\Omega_i^{\delta,\varepsilon}} u_i^{\delta,\varepsilon}\, v_i^{\delta,\varepsilon}\, dx \overset{\mathcal{T}_i^{\delta,\varepsilon}}{\simeq} \frac{\delta^N}{|Y|}\int_{\Omega\times\frac{1}{\delta}Y_i} \mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})(x,z)\, \mathcal{T}_i^{\delta,\varepsilon}(v_i^{\delta,\varepsilon})(x,z)\, dx\, dz. \tag{3.22}$$

Proof. For the relation in (3.21), we employ the Lebesgue Dominated Convergence Theorem. Meanwhile, (3.22) is straightforward from the boundedness of the boundary. □

3.2. **Trace behaviors.** Let us now proceed to some results concerning the jump on the interface. We investigate the relationship between the two operators and establish the behavior of their traces on the common boundary.

Lemma 3.13. *Let $p \in [1,\infty)$. If $u^{\delta,\varepsilon} = (u_1^{\delta,\varepsilon}, u_2^{\delta,\varepsilon}) \in H_{\gamma,p}^{\delta,\varepsilon}$, then*

$$\frac{\delta^{N-1}}{\varepsilon|Y|}\int_{\Omega\times\Gamma}\Big|\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - \mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon})\Big|^p\, dx\, d\sigma_z \le \int_{\Gamma^{\delta,\varepsilon}} |u_1^{\delta,\varepsilon} - u_2^{\delta,\varepsilon}|^p\, d\sigma_x.$$

Proof. In view of (3.11), all traces are well-defined. By the definition of $\widehat{\Omega}_\varepsilon$ given in (3.1), and Definition 3.4 of the unfolding operator,

$$\begin{aligned}
&\frac{\delta^{N-1}}{\varepsilon|Y|}\int_{\Omega\times\Gamma}\Big|\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - \mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon})\Big|^p\, dx\, d\sigma_z\\
&= \frac{\delta^{N-1}}{\varepsilon|Y|}\sum_{\xi\in\widehat{K}_\varepsilon}\int_{\varepsilon(\xi_\ell+Y)\times\Gamma}\Big|u_1^{\delta,\varepsilon}\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y+\varepsilon\delta z\Big) - u_2^{\delta,\varepsilon}\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y+\varepsilon\delta z\Big)\Big|^p\, dx\, d\sigma_z.
\end{aligned}$$

Note that if $x \in \varepsilon(\xi_\ell+Y)$, then $x = \varepsilon(\xi_\ell+y_1)$ for some $y_1 \in Y$. This, the periodicity in Y, and by a change of variable $x = \varepsilon\xi_\ell + \varepsilon\delta z$, along with the definition of $\widehat{\Gamma}^{\delta,\varepsilon}$ in (3.2), the above equation becomes

$$\begin{aligned}
&\frac{\delta^{N-1}}{\varepsilon|Y|}\int_{\Omega\times\Gamma}\Big|\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - \mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon})\Big|^p\, dx\, d\sigma_z\\
&= \frac{\delta^{N-1}}{\varepsilon|Y|}\sum_{\xi\in\widehat{K}_\varepsilon}\int_{\varepsilon(\xi_\ell+Y)\times\Gamma}\Big|u_1^{\delta,\varepsilon}(\varepsilon\xi_\ell+\varepsilon\delta z) - u_2^{\delta,\varepsilon}(\varepsilon\xi_\ell+\varepsilon\delta z)\Big|^p\, dx\, d\sigma_z\\
&= \frac{\delta^{N-1}}{\varepsilon|Y|}\varepsilon^N|Y|\sum_{\xi\in\widehat{K}_\varepsilon}\int_{\Gamma}\Big|u_1^{\delta,\varepsilon}(\varepsilon\xi_\ell+\varepsilon\delta z) - u_2^{\delta,\varepsilon}(\varepsilon\xi_\ell+\varepsilon\delta z)\Big|^p\, d\sigma_z\\
&= \varepsilon^{N-1}\delta^{N-1}\sum_{\xi\in\widehat{K}_\varepsilon}\int_{\Gamma}\Big|u_1^{\delta,\varepsilon}(\varepsilon\xi_\ell+\varepsilon\delta z) - u_2^{\delta,\varepsilon}(\varepsilon\xi_\ell+\varepsilon\delta z)\Big|^p\, d\sigma_z\\
&= \sum_{\xi\in\widehat{K}_\varepsilon}\int_{\partial[\varepsilon(\xi_\ell+Y_2)]}\Big|u_1^{\delta,\varepsilon}(x) - u_2^{\delta,\varepsilon}(x)\Big|^p\, d\sigma_x\\
&= \int_{\widehat{\Gamma}^{\delta,\varepsilon}}\Big|u_1^{\delta,\varepsilon} - u_2^{\delta,\varepsilon}\Big|^p\, d\sigma_x
\end{aligned}$$

$$\leq \int_{\Gamma^{\delta,\varepsilon}} \left| u_1^{\delta,\varepsilon} - u_2^{\delta,\varepsilon} \right|^p d\sigma_x,$$

which yields the desired inequality. □

Now, we prove some estimates for the unfolding of the gradients and the jump on the unfolded functions along the boundary.

Theorem 3.14. *Let $p \in [1, \infty)$ and $\gamma \leq 1$. If $u = (u_1^{\delta,\varepsilon}, u_2^{\delta,\varepsilon}) \in H_{\gamma,p}^{\delta,\varepsilon}$ is bounded, then there is a $C > 0$ independent of ε and δ such that*

$$\|\mathcal{T}_1^{\delta,\varepsilon}(\nabla u_1^{\delta,\varepsilon})\|_{L^p(\Omega\times\mathbb{R}^N)} \leq C\delta^{-\frac{N}{p}}, \tag{3.23}$$

$$\|\mathcal{T}_2^{\delta,\varepsilon}(\nabla u_2^{\delta,\varepsilon})\|_{L^p(\Omega\times\mathbb{R}^N)} \leq C\delta^{-\frac{N}{p}}, \tag{3.24}$$

$$\|\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - \mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon})\|_{L^p(\Omega\times\Gamma)} \leq C\varepsilon^{\frac{1-\gamma}{p}}\delta^{\frac{1-N}{p}}. \tag{3.25}$$

Proof. Estimates (3.23) and (3.24) are immediate from the boundedness of u in $H_{\gamma,p}^{\delta,\varepsilon}$ along with Theorem 3.9 (iii), (3.6), and (3.7). Meanwhile, one obtains (3.25) using Lemma 3.13, (3.8), and the boundedness hypothesis. □

To accurately describe the trace behaviors, let us introduce the following mean value operator acting on the interface.

Definition 3.15. For $p \in [1, +\infty)$, the *mean value operator* $\mathcal{M}_\Gamma$ is a function from $L^p(\Omega \times \Gamma)$ to $L^p(\Omega)$ and is defined as

$$\mathcal{M}_\Gamma(\varphi)(x) = \frac{1}{|\Gamma|}\int_\Gamma \varphi(x,z)\, d\sigma_z, \quad \forall \varphi \in L^p(\Omega \times \Gamma).$$

Remark 3.16. The properties obtained under Theorem 3.9 (vii) and (viii) also hold when formulated for this operator.

In the sequel, we suppose that (3.17) holds and we further assume that

$$\lim_{\varepsilon\to 0} \frac{\delta^{N-1}}{\varepsilon} \quad \text{exists in } \mathbb{R}^+. \tag{3.26}$$

We are now ready to give the trace behaviors. In what follows, we assume that $\gamma < 0$ since some of the properties do not hold when $\gamma \in [0, 1]$.

Theorem 3.17. *Let $p \in [1, \infty)$ and $\gamma < 0$. If $u^{\delta,\varepsilon} = (u_1^{\delta,\varepsilon}, u_2^{\delta,\varepsilon}) \in H_{\gamma,p}^{\delta,\varepsilon}$ is bounded, then up to a subsequence, there exist $U_1 \in L^p(\Omega; L^p_{\rm loc}(\mathbb{R}^N))$ and $W_1 \in L^p(\Omega; W^{1,p}(\mathbb{R}^N))$ such that*

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) \rightharpoonup U_1 \quad \textit{weakly in } L^p(\Omega; L^p_{\rm loc}(\mathbb{R}^N)), \tag{3.27}$$

$$\delta^{\frac{N}{p}}\mathcal{T}_1^{\delta,\varepsilon}(\nabla u_1^{\delta,\varepsilon}) \rightharpoonup \nabla_z W_1 \quad \textit{weakly in } L^p(\Omega \times \mathbb{R}^N)^N. \tag{3.28}$$

Moreover, we have

$$Z_1^{\delta,\varepsilon} := \frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\big[\mathcal{T}_1^{\delta,\varepsilon}(w_1^{\delta,\varepsilon}) - \mathcal{M}_\Gamma(\mathcal{T}_1^{\delta,\varepsilon}(w_1^{\delta,\varepsilon}))\big]1_{\frac{1}{\delta}Y_1} \rightharpoonup W_1 \tag{3.29}$$

weakly in $L^p(\Omega; L^{p^*}(\mathbb{R}^N))$, where $\mathcal{M}_\Gamma(W_1) = 0$.

Proof. The existence of $U_1 \in L^p(\Omega; L^p_{\rm loc}(\mathbb{R}^N))$ and $W_1 \in L^p(\Omega; W^{1,p}(\mathbb{R}^N))$ such that convergences (3.27) and (3.29) hold are immediate consequences of (3.18), (3.15) and Remark 3.16, respectively. Furthermore, by the linearity and Definition 3.15 of $\mathcal{M}_\Gamma$, in view of (3.29), we have $\mathcal{M}_\Gamma(W_1) = 0$ since

$$\mathcal{M}_\Gamma(Z_1^{\delta,\varepsilon}) = \frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\Big(\mathcal{M}_\Gamma(\mathcal{T}_1^{\delta,\varepsilon}(w_1^{\delta,\varepsilon})) - \big(\frac{1}{|\Gamma|}\big)|\Gamma|\mathcal{M}_\Gamma(\mathcal{T}_1^{\delta,\varepsilon}(w_1^{\delta,\varepsilon}))\Big) = 0. \tag{3.30}$$

On the other hand, (3.28) follows from (3.11) and (3.16) with Remark 3.16. □

Theorem 3.18. *Let $p \in [1,\infty)$ and $\gamma < 0$. If $u^{\delta,\varepsilon} = (u_1^{\delta,\varepsilon}, u_2^{\delta,\varepsilon}) \in H^{\delta,\varepsilon}_{\gamma,p}$ is uniformly bounded, then up to a subsequence, there exists $U_2 \in L^p(\Omega; L^p_{\rm loc}(\mathbb{R}^N))$ such that*

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon}) \rightharpoonup U_2 \quad \text{weakly in } L^p(\Omega; L^p_{\rm loc}(\mathbb{R}^N)), \tag{3.31}$$

$$\varepsilon\delta\mathcal{T}_2^{\delta,\varepsilon}(\nabla u_2^{\delta,\varepsilon}) \to 0 \quad \text{strongly in } L^p(\Omega\times\mathbb{R}^N)^N. \tag{3.32}$$

If we further assume that (3.26), *and* (3.27) *and* (3.28) *hold up to subsequences, then $U_2 = U_1$. That is,*

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon}) \rightharpoonup U_1 \quad \text{weakly in } L^p(\Omega; L^p_{\rm loc}(\mathbb{R}^N)). \tag{3.33}$$

Proof. Using (3.18), we have the existence of $U_2 \in L^p(\Omega; L^p_{\rm loc}(\frac{1}{\delta}Y_2))$ such that (3.31) holds. Moreover, (3.32) follows from using (3.11) and (3.24) which yield

$$\varepsilon\delta\mathcal{T}_2^{\delta,\varepsilon}(\nabla u_2^{\delta,\varepsilon}) = \nabla_z\mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon}) \to 0 \quad \text{strongly in } L^p(\Omega\times\mathbb{R}^N)^N.$$

By triangle inequality and (3.25),

$$\begin{aligned}
&\Big\|\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon}) - U_1\Big\|_{L^p(\Omega\times\Gamma)}\\
&\le \frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\|\mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon}) - \mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon})\|_{L^p(\Omega\times\Gamma)} + \Big\|\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - U_1\Big\|_{L^p(\Omega\times\Gamma)}\\
&\le \Big(\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\Big)C\varepsilon^{\frac{1-\gamma}{p}}\delta^{\frac{1-N}{p}} + \Big\|\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - U_1\Big\|_{L^p(\Omega\times\Gamma)}\\
&= C\Big(\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\Big)\Big(\frac{\delta^{N-1}}{\varepsilon}\Big)^{-1/p}\varepsilon^{-\gamma/p} + \Big\|\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - U_1\Big\|_{L^p(\Omega\times\Gamma)}.
\end{aligned} \tag{3.34}$$

The first term in the right-hand side of (3.34) approaches zero by (3.17), (3.26) and since $\gamma < 0$. Meanwhile, for the second term, by the trace theorem and (3.27),

$$\Big\|\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - U_1\Big\|_{L^p(\Omega\times\Gamma)} \to 0.$$

From these observations and (3.34), we obtain

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon}) \to U_1 \quad \text{strongly in } L^p(\Omega\times\Gamma).$$

From (3.31), uniqueness of the limit implies that

$$U_2 = U_1 \quad \text{a.e in } \Omega\times\Gamma, \tag{3.35}$$

which gives (3.33). □

Remark 3.19. It is important to notice that unlike the trace behaviors in [21] which hold for $\gamma \le 1$, Theorem 3.18 only permits the case $\gamma < 0$. This is due to the fact that when $\gamma \in [0,1]$, one cannot have convergence (3.33).

Theorem 3.20. *Let $u^{\delta,\varepsilon} = (u_1^{\delta,\varepsilon}, u_2^{\delta,\varepsilon}) \in H_{\gamma,p}^{\delta,\varepsilon}$ be bounded. Then up to subsequences, there exists $W_2 \in L^p(\Omega; W^{1,p}(\mathbb{R}^N))$ such that*

$$Z_2^{\delta,\varepsilon} := \frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\big[\mathcal{T}_2^{\delta,\varepsilon}(w_2^{\delta,\varepsilon}) - \mathcal{M}_\Gamma(\mathcal{T}_2^{\delta,\varepsilon}(w_2^{\delta,\varepsilon}))\big]1_{\frac{1}{\delta}Y_2} \rightharpoonup W_2 \tag{3.36}$$

weakly in $L^p(\Omega; L^{p^*}(\mathbb{R}^N))$, where $\mathcal{M}_\Gamma(W_2) = 0$, and

$$\delta^{\frac{N}{p}}\mathcal{T}_2^{\delta,\varepsilon}(\nabla u_2^{\delta,\varepsilon}) \rightharpoonup \nabla_z W_2 \quad \text{weakly in } L^p(\Omega \times \mathbb{R}^N)^N. \tag{3.37}$$

Proof. Convergence (3.36) is an immediate consequence of (3.15), where $W_2 \in L^p(\Omega; W^{1,p}(\mathbb{R}^N))$. A similar computation to (3.30) shows that $\mathcal{M}_\Gamma(Z_2^{\delta,\varepsilon}) = 0$ which by (3.36) yields $\mathcal{M}_\Gamma(W_2) = 0$.

To see (3.37), we just use (3.11) and (3.16) so that

$$\delta^{\frac{N}{p}}\mathcal{T}_2^{\delta,\varepsilon}(\nabla u_2^{\delta,\varepsilon}) = \delta^{\frac{N}{p}}\big[\frac{1}{\varepsilon\delta}\nabla_z\mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon})\big] = \frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z\mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon})1_{\frac{1}{\delta}Y_2} \rightharpoonup \nabla_z W_2,$$

weakly in $L^p(\Omega \times \mathbb{R}^N)^N$. □

We end this section with a theorem that summarizes our results so far.

Theorem 3.21. *Let $p \in [1,\infty), \gamma < 0$, and $u^{\delta,\varepsilon} = (u_1^{\delta,\varepsilon}, u_2^{\delta,\varepsilon}) \in H_{\gamma,p}^{\delta,\varepsilon}$ be bounded. Then, up to subsequences, there exist $U_i \in L^p(\Omega; L^p_{\rm loc}(\mathbb{R}^N))$ and $W_i \in L^p(\Omega; W^{1,p}(\mathbb{R}^N))$, such that (3.27), (3.28), (3.31), (3.32), (3.36), and (3.37) hold.*

Furthermore, if (3.26) holds, then $U_1 = U_2$ and

$$W_1 = W_2 \quad \text{on } \Omega \times \Gamma. \tag{3.38}$$

Proof. Convergences (3.27), (3.28), (3.31), (3.32), (3.36), and (3.37) hold by Theorems 3.17–3.20. Moreover, as in (3.35) we have $U_1 = U_2$.

Now, notice that using the convergences in (3.29) and (3.36) together with the trace theorem for $W^{1,p}(\frac{1}{\delta}Y_i)$ for $i = 1,2$, we obtain

$$Z_1^{\delta,\varepsilon} - Z_2^{\delta,\varepsilon} \rightharpoonup W_1 - W_2 \quad \text{weakly in } L^p(\Omega \times \Gamma). \tag{3.39}$$

Meanwhile, by Definition 3.15, we have

$$\|\mathcal{M}_\Gamma(\varphi)\|_{L^p(\Omega\times\Gamma)} \le \frac{1}{|\Gamma|^{1-\frac{1}{p}}}\Big(\int_{\Omega\times\Gamma}|\varphi|^p\,dx\,d\sigma_y\Big)^{1/p} \le \|\varphi\|_{L^p(\Omega\times\Gamma)}.$$

This, along with the definitions of $Z_1^{\delta,\varepsilon}$ and $Z_2^{\delta,\varepsilon}$ in (3.29) and (3.36), triangle inequality, and (3.25), we obtain

$$\begin{aligned}
\|Z_1^{\delta,\varepsilon} - Z_2^{\delta,\varepsilon}\|_{L^p(\Omega\times\Gamma)} &\le \frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\Big[\|\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - \mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon})\|_{L^p(\Omega\times\Gamma)} \\
&\quad + \Big\|\mathcal{M}_\Gamma\Big(\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - \mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon})\Big)\Big\|_{L^p(\Omega\times\Gamma)}\Big] \\
&\le \Big(\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\Big)2\|\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - \mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon})\|_{L^p(\Omega\times\Gamma)} \\
&\le \Big(\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\Big)2C\varepsilon^{\frac{1-\gamma}{p}}\delta^{\frac{1-N}{p}}
\end{aligned}$$

$$= 2C\Big(\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\Big)\Big(\frac{\delta^{N-1}}{\varepsilon}\Big)^{-1/p}\varepsilon^{-\gamma/p}.$$

Then (3.17), (3.26) and since $\gamma < 0$ imply that

$$Z_1^{\delta,\varepsilon} - Z_2^{\delta,\varepsilon} \to 0 \quad \text{strongly in } L^p(\Omega\times\Gamma).$$

Comparing this with (3.39), we obtain (3.38). □

4. Homogenization results

Let us now obtain the asymptotic behavior of our dissipation problem given in (1.1) for $\gamma < 0$ as $(\varepsilon,\delta)\to(0,0)$. First, we denote by $\mathcal{M}(\alpha,\beta,\mathcal{O})$ the set of matrix fields $A \in L^\infty(\mathcal{O})^{N\times N}$ satisfying

$$(A(y)\xi,\xi) \geq \alpha|\xi|^2 \quad \text{and} \quad |A(y)\xi| \leq \beta|\xi|, \quad \forall \xi\in\mathbb{R}^N, \forall y\in\mathcal{O},$$

where $\mathcal{O}$ is an open set in $\mathbb{R}^N$ and $\alpha,\beta\in\mathbb{R}$ such that $0<\alpha<\beta$.

We define the functions

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad h^{\delta,\varepsilon}(x) = h\Big(\frac{1}{\delta}\Big\{\frac{x}{\varepsilon}\Big\}\Big), \tag{4.1}$$

and consider the following assumptions on our data:

- (A1) $A \in M(\alpha,\beta,Y)$ and $f\in L^2(\Omega)$;
- (A2) $h\in L^\infty(\Gamma)$ be periodic in Y_2 and there exists $h_0\in\mathbb{R}$ such that $0<h_0<h(z)$ a.e. in Γ;
- (A3) Suppose that $\delta=\delta(\varepsilon)$ is such that (3.26) holds and

$$k_1 = \lim_{\varepsilon\to 0}\frac{\delta^{\frac{N}{2}-1}}{\varepsilon} \quad \text{exists in } \mathbb{R}^+.$$

Remark 4.1. The number k_1 corresponds to the critical size $\varepsilon^{N/(N-2)}$ of Dirichlet small holes first observed in [17] (this is also (3.17) for the case $p=2$). Meanwhile, (3.26) corresponds to the critical size $\varepsilon^{N/(N-1)}$ of the Neumann small holes from [18].

From (3.3) and (3.4), when $p=2$, we write $V^{\delta,\varepsilon} := V_2^{\delta,\varepsilon}$ and $H_\gamma^{\delta,\varepsilon} := H_{\gamma,2}^{\delta,\varepsilon}$. The variational formulation of (1.1) is: Find $u^{\delta,\varepsilon} = (u_1^{\delta,\varepsilon}, u_2^{\delta,\varepsilon}) \in H_\gamma^{\delta,\varepsilon}$ such that for all $v=(v_1,v_2)\in H_\gamma^{\delta,\varepsilon}$,

$$\begin{aligned} &\int_{\Omega_1^{\delta,\varepsilon}} A^\varepsilon \nabla u_1^{\delta,\varepsilon}\nabla v_1\,dx + \int_{\Omega_2^{\delta,\varepsilon}} A^\varepsilon\nabla u_2^{\delta,\varepsilon}\nabla v_2\,dx \\ &+ \varepsilon^\gamma\int_{\Gamma^{\delta,\varepsilon}} h^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}-u_2^{\delta,\varepsilon})(v_1-v_2)\,d\sigma_x \\ &= \sum_{i=1}^2\int_{\Omega_i^{\delta,\varepsilon}} f v_i\,dx. \end{aligned} \tag{4.2}$$

Theorem 4.2. *Under assumptions* (A1) *and* (A2)*, problem* (4.2) *admits a unique solution* $u^{\delta,\varepsilon}\in H_\gamma^{\delta,\varepsilon}$ *such that for some constant* $C>0$,

$$\|u^{\delta,\varepsilon}\|_{H_\gamma^{\delta,\varepsilon}} \leq C. \tag{4.3}$$

Proof. The existence and uniqueness of the solution of (4.2) follows from invoking the Lax-Milgram Theorem together with (A1) and (A2).

On the other hand, taking the unique solution $u^{\delta,\varepsilon}$ as a test function in (4.2) and then applying (A1), (A2), triangle and Cauchy-Schwarz inequalities, and Remark 3.2, one obtains (4.3). □

All throughout, let $N \geq 3$ and p^* as in (3.10). Let us start by recalling some spaces and results from [11] for homogenization in domains with small holes via the periodic unfolding method.

For $p \in [1, N)$, we recall the homogeneous space $\dot{W}^{1,p}(\mathbb{R}^N) = \{\varphi \in L^{p^*}(\mathbb{R}^N) \mid \nabla\varphi \in L^p(\mathbb{R}^N)^N\}$. In what follows, for $p = 2$, we write $\dot{H}^1(\mathbb{R}^N) := \dot{W}^{1,2}(\mathbb{R}^N)$. Moreover, we define the functional space

$$\mathcal{W}^{1,p}(\mathbb{R}^N) := \{W^{1,p}_{\text{loc}}(\mathbb{R}^N) \mid \nabla\varphi \in L^p(\mathbb{R}^N)^N\}$$

equipped with the norm

$$\|\varphi\|^p_{\mathcal{W}^{1,p}(\mathbb{R}^N)} = \|\nabla\varphi\|^p_{L^p(\mathbb{R}^N)} + |\varphi(\infty)|^p, \quad \forall\varphi \in \mathcal{W}^{1,p}(\mathbb{R}^N).$$

Proposition 4.3 ([11])**.** *Let $p \in [1, N)$.*

(i) *The space $\mathcal{W}^{1,p}(\mathbb{R}^N)$ is isomorphic to $\dot{W}^{1,p}(\mathbb{R}^N) \oplus \mathbb{R}$. That is, for every $\varphi \in \mathcal{W}^{1,p}(\mathbb{R}^N)$, there exists a real number $\varphi(\infty)$ (called the* weak limit *of φ at infinity) such that $\varphi - \varphi(\infty) \in \dot{W}^{1,p}(\mathbb{R}^N)$.*

(ii) *We have the estimate $\|\varphi - \varphi(\infty)\|_{L^{p^*}(\mathbb{R}^N)} \leq C\|\nabla\varphi\|_{L^p(\mathbb{R}^N)} \quad \forall\varphi \in \mathcal{W}^{1,p}(\mathbb{R}^N)$, where $C > 0$ is the constant for the Sobolev embedding of $\dot{W}^{1,p}(\mathbb{R}^N)$ to $L^{p^*}(\mathbb{R}^N)$.*

For an open set $B \subset \mathbb{R}^N$, we define the subspace $\mathbf{K}_B$ of $\dot{H}^1(\mathbb{R}^N) \oplus \mathbb{R}$ by $\mathbf{K}_B = \{\varphi \in H^1_{\text{loc}}(\mathbb{R}^N) \mid \nabla\varphi \in L^2(\mathbb{R}^N)^N$ and $\varphi = 0$ on $B\}$, equipped with the norm

$$\|\varphi\|_{\mathbf{K}_B} = \|\nabla\varphi\|_{L^2(\mathbb{R}^N)}, \quad \forall\varphi \in \mathbf{K}_B.$$

Furthermore, we define the space

$$\mathbf{L}_B = \{V \in L^2(\Omega; \dot{H}^1(\mathbb{R}^N)) \oplus H^1_0(\Omega) \mid V = 0 \quad \text{a.e. in } \Omega \times B\},$$

equipped with the norm

$$\|V\|^2_{\mathbf{L}_B} = \|\nabla V(\cdot, \infty)\|^2_{L^2(\Omega)} + \|\nabla_y V\|^2_{L^2(\Omega\times\mathbb{R}^N)}, \quad \forall V \in \mathbf{L}_B.$$

Next we recall some density results from [11] that are essential in reaching our goal.

Lemma 4.4 ([11])**.**

(i) *For $p \in [1, \infty)$, the set $\cup_{\delta\in(0,\delta_0]}\{\varphi \in W^{1,p}(Y) \mid \varphi$ constant on $\partial B\}$ is dense in $W^{1,p}(Y)$.*

(ii) *For $p \in [1, N)$, the set $\cup_{\delta\in(0,\delta_0]}\{\varphi \in W^{1,p}(Y) \mid \varphi = 0$ on $\partial B\}$ is dense in $W^{1,p}(Y)$.*

Remark 4.5 ([11])**.** Lemma 4.4 holds also true in the space $W^{1,p}_{\text{per}}(Y)$ (in place of $W^{1,p}(Y)$).

We now introduce the important class of test functions which will aid us in revealing the contribution of the small scale of the inclusions to the homogenized problems.

Lemma 4.6 ([11])**.** *Let $p \in [1,\infty)$ and $v \in W^{1,p}_{\rm loc}(\mathbb{R}^N)$ such that $\nabla_z v \in L^p(\mathbb{R}^N)^N$ and has compact support. Set*

$$v^{\delta,\varepsilon}(x) = v\Big(\frac{1}{\delta}\Big\{\frac{x}{\varepsilon}\Big\}_Y\Big) \quad \text{for a.e. } x \in \Omega. \tag{4.4}$$

By Proposition 4.3(i), v has a limit at infinity denoted by $v(\infty)$. If δ is small enough, the function $v^{\delta,\varepsilon}$ belongs to $W^{1,p}(\Omega)$ and

$$v^{\delta,\varepsilon} \to v(\infty) \quad \text{strongly in } L^p(\Omega). \tag{4.5}$$

Moreover, if $\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}$ is uniformly bounded, then $v^{\delta,\varepsilon} \rightharpoonup v(\infty)$ weakly in $W^{1,p}(\Omega)$.

Remark 4.7. For $v^{\delta,\varepsilon}$ defined in (4.4), in view of (3.11) for $i = 1,2$, $\mathcal{T}_i^{\delta,\varepsilon}(\nabla v^{\delta,\varepsilon}) = \frac{1}{\varepsilon\delta}\nabla_z v$.

Theorem 4.8. *Let $p \in [1,\infty)$, $u^{\delta,\varepsilon} = (u_1^{\delta,\varepsilon}, u_2^{\delta,\varepsilon}) \in H^{\delta,\varepsilon}_{\gamma,p}$ and $h^{\delta,\varepsilon}$ be as in (4.1) such that h satisfies* (A2)*. Then, for $\varphi \in \mathcal{D}(\Omega)$ we have*

$$\begin{aligned} &\frac{\delta^{N-1}}{\varepsilon|Y|}\int_{\Omega\times\Gamma} h(z)\Big(\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - \mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon})\Big)\mathcal{T}_1^{\delta,\varepsilon}(\varphi)\,dx\,d\sigma_z \\ &= \int_{\Gamma^{\delta,\varepsilon}} h^{\delta,\varepsilon}(u_1^{\delta,\varepsilon} - u_2^{\delta,\varepsilon})\varphi\,d\sigma_x. \end{aligned}$$

Proof. By similar arguments as in the proof of Lemma 3.13, we have

$$\begin{aligned} &\frac{\delta^{N-1}}{\varepsilon|Y|}\int_{\Omega\times\Gamma} h(z)\Big(\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - \mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon})\Big)\mathcal{T}_1^{\delta,\varepsilon}(\varphi)\,dx\,d\sigma_z \\ &= \varepsilon^{N-1}\delta^{N-1}\sum_{\xi\in\widehat{K}_\varepsilon}\int_\Gamma h(z)\Big(u_1^{\delta,\varepsilon}(\varepsilon\xi_\ell+\varepsilon\delta z) - u_2^{\delta,\varepsilon}(\varepsilon\xi_\ell+\varepsilon\delta z)\Big)\varphi(\varepsilon\xi_\ell+\varepsilon\delta z)\,d\sigma_z. \end{aligned}$$

By the change of variable $x = \varepsilon\xi_\ell + \varepsilon\delta z$, (A2), and since the support of φ is a compact subset of Ω, the integral on $\Gamma^{\delta,\varepsilon}$ is the same over $\widehat{\Gamma}^{\delta,\varepsilon}$ so that the above equation is transformed into

$$\begin{aligned} &\frac{\delta^{N-1}}{\varepsilon|Y|}\int_{\Omega\times\Gamma} h(z)\Big(\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) - \mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon})\Big)\mathcal{T}_1^{\delta,\varepsilon}(\varphi)\,dx\,d\sigma_z \\ &= \varepsilon^{N-1}\delta^{N-1}\sum_{\xi\in\widehat{K}_\varepsilon}\int_{\partial[\varepsilon(\xi_\ell+Y_2)]} h\Big(\frac{x}{\varepsilon\delta} - \frac{\xi_\ell}{\delta}\Big)(u_1^{\delta,\varepsilon}(x) - u_2^{\delta,\varepsilon}(x))\varphi(x)\,\frac{d\sigma_x}{\varepsilon^{N-1}\delta^{N-1}} \\ &= \sum_{\xi\in\widehat{K}_\varepsilon}\int_{\partial[\varepsilon(\xi_\ell+Y_2)]} h^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}(x) - u_2^{\delta,\varepsilon}(x))\varphi(x)\,d\sigma_x \\ &= \int_{\partial[\cup_{\xi\in\widehat{K}_\varepsilon}\varepsilon(\xi_\ell+Y_2)]} h^{\delta,\varepsilon}(u_1^{\delta,\varepsilon} - u_2^{\delta,\varepsilon})\varphi\,d\sigma_x \\ &= \int_{\widehat{\Gamma}^{\delta,\varepsilon}} h^{\delta,\varepsilon}(u_1^{\delta,\varepsilon} - u_2^{\delta,\varepsilon})\varphi\,d\sigma_x \\ &= \int_{\Gamma^{\delta,\varepsilon}} h^{\delta,\varepsilon}(u_1^{\delta,\varepsilon} - u_2^{\delta,\varepsilon})\varphi\,d\sigma_x, \end{aligned}$$

which completes the proof. □

We are now in a position to describe the asymptotic behavior of (4.2) for different values of $\gamma < 0$. To aid us in passing to the limit, we further assume that, (A4) there exists a matrix $A \in \mathcal{M}(\alpha, \beta, \Omega \times Y)$ such that $\mathcal{T}_i^\varepsilon(A^\varepsilon)(x,y) \to A(x,y)$ for a.e. $(x,y) \in \Omega \times Y$; and

(A5) there exists a matrix $F \in \mathcal{M}(\alpha, \beta, \Omega \times \mathbb{R}^N)$ such that $\mathcal{T}_i^{\delta,\varepsilon}(A^\varepsilon)(x,z) \to F(x,z)$ for a.e. $(x,z) \in \Omega \times \mathbb{R}^N$.

4.1. **Case $\gamma < -1$.**

Theorem 4.9. *Let $\gamma < -1$. Under assumptions* (A1)–(A3)*, let $u^{\delta,\varepsilon} = (u_1^{\delta,\varepsilon}, u_2^{\delta,\varepsilon}) \in H_\gamma^{\delta,\varepsilon}$ be the solution of* (4.2)*. Then there exist $u_1 \in H_0^1(\Omega)$ and $G_1 \in \mathbf{L}_B$ such that*

$$\widetilde{u_i^{\delta,\varepsilon}} \rightharpoonup \theta_i u_1 \quad \text{weakly in } L^2(\Omega), \tag{4.6}$$

$$\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \rightharpoonup G_1 \quad \text{weakly in } L^2(\Omega; L^2_{\rm loc}(\mathbb{R}^N)) \text{ with } G_1(\cdot,\infty) = u_1, \tag{4.7}$$

$$\nabla_z \mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \rightharpoonup \nabla_z G_1 \quad \text{weakly in } L^2(\Omega \times \mathbb{R}^N)^N, \tag{4.8}$$

where θ_i is given by (2.1) *(i), and $\widehat{u}_i \in L^2(\Omega; H^1_{per,0}(Y_i))$ such that*

$$\mathcal{T}_1^\varepsilon(\nabla u_1^{\delta,\varepsilon}) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \quad \text{weakly in } L^2(\Omega \times Y_1)^N, \tag{4.9}$$

$$\mathcal{T}_2^\varepsilon(\nabla u_2^{\delta,\varepsilon}) \rightharpoonup \nabla_y \widehat{u}_2 \quad \text{weakly in } L^2(\Omega \times Y_2)^N, \tag{4.10}$$

for $i = 1, 2$. Moreover, under assumptions (A4) *and* (A5)*, the pair $(G_1, \widehat{u})$ is the unique solution of the unfolded limit problem*

$$\begin{aligned} &\frac{k_1^2}{|Y|} \int_{\Omega \times (\mathbb{R}^N \setminus \overline{B})} F(x,z) \nabla_z G_1(x,z) \nabla_z V \, dx \, dz \\ &+ \frac{1}{|Y|} \int_{\Omega \times Y} A(x,y)(\nabla G_1(\cdot,\infty) + \nabla_y \widehat{u})(\nabla V(\cdot,\infty) + \nabla_y \Psi) \, dx \, dy \\ &= \int_\Omega f V(\cdot,\infty) \, dx, \quad \forall \Psi \in L^2(\Omega; H^1_{\rm per}(Y)), \quad \forall V \in \mathbf{L}_B, \end{aligned} \tag{4.11}$$

where $\widehat{u} \in L^2(\Omega; H^1_{per,0}(Y))$ is the extension by periodicity of the function

$$\widehat{u}(\cdot,y) = \begin{cases} \widehat{u}_1(\cdot,y) & y \in Y_1, \\ \widehat{u}_2(\cdot,y) - y_\gamma \nabla u_1 & y \in Y_2, \end{cases} \tag{4.12}$$

where

$$y_\Gamma = y - \mathcal{M}_\Gamma(y). \tag{4.13}$$

Proof. We divide the proof in four steps.

Step 1. Using (4.3), (2.2) and (2.5) in Theorem 2.3 (iv), we obtain

$$\widetilde{u_i^{\delta,\varepsilon}} \rightharpoonup \theta_i u_i \quad \text{weakly in } L^2(\Omega), \tag{4.14}$$

proving (4.6) for $i = 1$. We will show later that $u_2 = u_1$ so that $u_2 \in H_0^1(\Omega)$ which will prove the case $i = 2$ in (4.6). Furthermore, the existence of $\widehat{u}_i \in L^2(\Omega; H^1_{per,0}(Y_i))$ for $i = 1, 2$ such that convergences (4.9) and (4.10) are true, come from (2.3) and (2.7). We also delay the proof of (4.7) and (4.8).

Step 2. To capture the effect of the periodic oscillation of the coefficients in (4.2), for $\varphi \in \mathcal{D}(\Omega)$ and $\psi \in H^1_{\rm per}(Y)$ vanishing in a neighborhood of the origin, we let

$v_1 = v_2 = \varphi = \varepsilon\varphi(x)\psi\left(\frac{x}{\varepsilon}\right)$ be the test functions in (4.2). Since $v_1 = v_2$, the third term in (4.2) becomes zero and so we obtain

$$\int_{\Omega_1^{\delta,\varepsilon}} A^\varepsilon \nabla u_1^{\delta,\varepsilon} \nabla\varphi\,dx + \int_{\Omega_2^{\delta,\varepsilon}} A^\varepsilon \nabla u_2^{\delta,\varepsilon} \nabla\varphi\,dx = \sum_{i=1}^{2} \int_{\Omega_i^{\delta,\varepsilon}} f\varphi\,dx. \tag{4.15}$$

Concerning the test functions, direct computations show that

$$\mathcal{T}_i^\varepsilon(\varphi) \to 0 \quad \text{strongly in } L^2(\Omega \times Y_i), \tag{4.16}$$

$$\mathcal{T}_i^\varepsilon(\nabla\varphi) \to \nabla_y \Psi \quad \text{strongly in } L^2(\Omega \times Y_i)^N, \tag{4.17}$$

where $\Psi(x,y) = \varphi(x)\psi(y)$. Unfolding the right-hand side of (4.15) using $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_2^\varepsilon$, respectively, using Theorem 2.3 (i) (iii), and passing to the limit using (4.16) yield

$$\lim_{\varepsilon\to 0} \sum_{i=1}^{2} \int_{\Omega_i^{\delta,\varepsilon}} f\varphi\,dx \overset{\mathcal{T}_1^\varepsilon}{\simeq} \lim_{\varepsilon\to 0} \sum_{i=1}^{2} \frac{1}{|Y|} \int_{\Omega\times Y_i} \mathcal{T}_i^\varepsilon(f)\mathcal{T}_i^\varepsilon(\varphi)\,dx\,dy = 0. \tag{4.18}$$

For the first term on the left-hand side of (4.15), we apply $\mathcal{T}_1^\varepsilon$ and Theorem 2.3 (i), then pass to the limit using (A4), (4.9), and (4.17) yield

$$\lim_{\varepsilon\to 0} \int_{\Omega_1^{\delta,\varepsilon}} A^\varepsilon \nabla u_1^{\delta,\varepsilon} \nabla\varphi\,dx = \frac{1}{|Y|} \int_{\Omega\times Y_1} A(x,y)(\nabla u_1 + \nabla_y \widehat{u}_1)\nabla_y \Psi\,dx\,dy. \tag{4.19}$$

Now we consider the second term on the left-hand side of (4.15). By using $\mathcal{T}_2^\varepsilon$ and Theorem 2.3 (i), passing to the limit with (A4), (4.10), and (4.17), we obtain

$$\lim_{\varepsilon\to 0} \int_{\Omega_2^{\delta,\varepsilon}} A^\varepsilon \nabla u_2^{\delta,\varepsilon} \nabla\varphi\,dx = \frac{1}{|Y|} \int_{\Omega\times Y_2} A(x,y)\nabla_y \widehat{u}_2 \nabla_y \Psi\,dx\,dy. \tag{4.20}$$

Therefore, via (4.18), (4.19), and (4.20), we obtain the limit equation

$$\begin{aligned} &\frac{1}{|Y|} \int_{\Omega\times Y_1} A(x,y)(\nabla u_1 + \nabla_y \widehat{u}_1)\nabla_y \Psi\,dx\,dy \\ &+ \frac{1}{|Y|} \int_{\Omega\times Y_2} A(x,y)\nabla_y \widehat{u}_2 \nabla_y \Psi\,dx\,dy = 0. \end{aligned} \tag{4.21}$$

Finally, observe that using Theorem **??** (i) as $\gamma < -1$, we can define a function $\widehat{u}$ (extended by periodicity) in $L^2(\Omega; H^1_{\rm per}(Y))$ given by (4.12), where y_Γ is similarly defined as in (2.4). Using this along with the density of $\mathcal{D}(\Omega) \times H^1_{\rm per}(Y)$ in $L^2(\Omega; H^1_{\rm per}(Y))$, then equation (4.21) becomes

$$\int_{\Omega\times Y} A(x,y)(\nabla u_1 + \nabla_y \widehat{u})\nabla_y \Psi\,dx\,dy = 0, \quad \forall \Psi \in L^2(\Omega; H^1_{\rm per}(Y)). \tag{4.22}$$

Step 3. Let us now show the existence of $G_1 \in \mathbf{L}_B$ such that (4.7) and (4.8) hold, and that $u_2 = u_1$ as postponed in Step 1.

For $i = 1, 2$, let us have the following results. From (A3) and (3.18) with Remark 3.16, there exists $G_i \in L^2(\Omega; L^2_{\rm loc}(\mathbb{R}^N))$ such that up to subsequences,

$$\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \rightharpoonup G_i \quad \text{weakly in } L^2(\Omega; L^2_{\rm loc}(\mathbb{R}^N)). \tag{4.23}$$

Theorem 3.21 then implies that $G_1 = G_2$ so that the above becomes

$$\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \rightharpoonup G_1 \quad \text{weakly in } L^2(\Omega; L^2_{\rm loc}(\mathbb{R}^N)). \tag{4.24}$$

From estimates (3.6) and (3.9) and Theorem 3.10 (ii), we have

$$\mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_i}(u_i^{\delta,\varepsilon})1_{\frac{1}{\delta}Y_i} \to u_i \quad \text{strongly in } L^2(\Omega; L^2_{\rm loc}(\mathbb{R}^N)). \tag{4.25}$$

Meanwhile, by (3.13), there exists $J_i \in L^2(\Omega; L^{2^*}(\mathbb{R}^N))$ with $\nabla_z J_i \in L^2(\Omega \times \mathbb{R}^N)^N$ such that

$$\mathcal{T}_i^{\delta,\varepsilon}\big[u_i^{\delta,\varepsilon} - \mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_i}(u_i^{\delta,\varepsilon})\big]1_{\frac{1}{\delta}Y_i} \rightharpoonup J_i \quad \text{weakly in } L^2(\Omega; L^{2^*}(\mathbb{R}^N)). \tag{4.26}$$

Now, the linearity of $\mathcal{T}_i^{\delta,\varepsilon}$ together with (4.24), (4.25), and (4.26) imply

$$G_1 - u_i = J_i \quad \text{and} \quad \nabla_z G_1 = \nabla_z J_i.$$

By (3.38), $J_1 = J_2$ and since u_i is independent of z, we obtain $u_1 = u_2$ in Ω concluding the proof of (4.6) from Step 1.

Meanwhile, by (A3) and (3.16) with Remark 3.16,

$$\nabla_z \mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \rightharpoonup \nabla_z G_1 \quad \text{weakly in } L^2(\Omega \times \mathbb{R}^N)^N. \tag{4.27}$$

For the case $i = 1$, Definition 3.4 gives

$$\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) = 0 \quad \text{in } \Omega \times B, \tag{4.28}$$

and with (4.24), we obtain

$$G_1 = 0 \quad \text{in } \Omega \times B. \tag{4.29}$$

Thanks to (4.26) - (4.29), we have $G_1 \in \mathbf{L}_B$ and

$$G_1(\cdot, \infty) = u_1. \tag{4.30}$$

This along with (4.24) and (4.27) prove (4.7) and (4.8).

Step 4. In this part, we will prove the rest of the limit equations. To see the effect of the very small inclusions, let $\varphi \in \mathcal{D}(\Omega)$ and $v \in \mathbf{K}_B$ such that $\nabla_z v$ has compact support. Take $v_1 = v_2 = v^{\delta,\varepsilon}(x)\varphi(x)$ as a test function in (4.2), where $v^{\delta,\varepsilon}$ is defined in (4.4).

Since $v_1 = v_2$, the third term in (4.2) vanishes and we obtain

$$\int_{\Omega_1^{\delta,\varepsilon}} A^\varepsilon \nabla u_1^{\delta,\varepsilon} \nabla(v^{\delta,\varepsilon}\varphi)\,dx + \int_{\Omega_2^{\delta,\varepsilon}} A^\varepsilon \nabla u_2^{\delta,\varepsilon} \nabla(v^{\delta,\varepsilon}\varphi)\,dx = \sum_{i=1}^{2} \int_{\Omega_i^{\delta,\varepsilon}} f v^{\delta,\varepsilon}\varphi\,dx. \tag{4.31}$$

Before we proceed, observe that

$$\mathcal{T}_1^{\delta,\varepsilon}(\varphi)\nabla_z v \to \varphi \nabla_z v \quad \text{strongly in } L^2(\Omega \times \mathbb{R}^N). \tag{4.32}$$

The first term in the left-hand side of (4.31) becomes

$$\begin{aligned}
&\int_{\Omega_1^{\delta,\varepsilon}} A^\varepsilon \nabla u_1^{\delta,\varepsilon} \nabla(v^{\delta,\varepsilon}\varphi)\,dx \\
&= \int_{\Omega_1^{\delta,\varepsilon}} A^\varepsilon \nabla u_1^{\delta,\varepsilon} \nabla v^{\delta,\varepsilon} \varphi\,dx + \int_{\Omega_1^{\delta,\varepsilon}} A^\varepsilon \nabla u_1^{\delta,\varepsilon} v^{\delta,\varepsilon} \nabla\varphi\,dx.
\end{aligned} \tag{4.33}$$

For the first term on the right-hand side of (4.33), we unfold using $\mathcal{T}_1^{\delta,\varepsilon}$ due to the factor $\nabla v^{\delta,\varepsilon}$. Using Theorem 3.11, (3.11), Remark 4.7, and passing to the limit using (A3), (A5), (4.8), and (4.32),

$$\lim_{\varepsilon \to 0} \int_{\Omega_1^{\delta,\varepsilon}} A^\varepsilon \nabla u_1^{\delta,\varepsilon} \nabla v^{\delta,\varepsilon}\varphi\,dx = \frac{k_1^2}{|Y|} \int_{\Omega \times (\mathbb{R}^N \setminus \overline{B})} F(x,z) \nabla_z G_1(x,z) \varphi \nabla_z v\,dx\,dz. \tag{4.34}$$

Unfolding the second integral in (4.33) using $\mathcal{T}_1^\varepsilon$ and passing to the limit using (A4), (4.9), (4.5), Theorem 2.3 (iii) we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_1^{\delta,\varepsilon}} A^\varepsilon \nabla u_1^{\delta,\varepsilon} v^{\delta,\varepsilon}\nabla\varphi\,dx = \frac{v(\infty)}{|Y|}\int_{\Omega\times Y_1} A(x,y)(\nabla u_1 + \nabla_y \widehat{u}_1)\nabla\varphi\,dx\,dy. \quad (4.35)$$

Similarly, for the second term in the left-hand side of (4.31),

$$\begin{aligned}&\int_{\Omega_2^{\delta,\varepsilon}} A^\varepsilon \nabla u_2^{\delta,\varepsilon}\nabla(v^{\delta,\varepsilon}\varphi)\,dx\\ &= \int_{\Omega_2^{\delta,\varepsilon}} A^\varepsilon \nabla u_2^{\delta,\varepsilon}\nabla v^{\delta,\varepsilon}\varphi\,dx + \int_{\Omega_2^{\delta,\varepsilon}} A^\varepsilon \nabla u_2^{\delta,\varepsilon} v^{\delta,\varepsilon}\nabla\varphi\,dx.\end{aligned} \quad (4.36)$$

For the first term in the right-hand side of (4.36), we unfold using $\mathcal{T}_2^{\delta,\varepsilon}$ in a similar fashion in getting (4.34). Using Theorem 3.11, (3.11), and Remark 4.7, and then passing to the limit using (A3), (A5), (4.8), and (4.32), because of (4.29), we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_2^{\delta,\varepsilon}} A^\varepsilon \nabla u_2^{\delta,\varepsilon}\nabla v^{\delta,\varepsilon}\varphi\,dx = \frac{k_1^2}{|Y|}\int_{\Omega\times B} F(x,z)\nabla_z G_1(x,z)\varphi\nabla_z v\,dx\,dz = 0. \quad (4.37)$$

Unfolding the second term in the right-hand side of (4.36) using $\mathcal{T}_2^\varepsilon$ and invoking (A4), (2.7), (4.5), and Theorem 2.3 (iii), we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_2^{\delta,\varepsilon}} A^\varepsilon \nabla u_2^{\delta,\varepsilon} v^{\delta,\varepsilon}\nabla\varphi\,dx = \frac{v(\infty)}{|Y|}\int_{\Omega\times Y_2} A(x,y)\nabla_y \widehat{u}_2\nabla\varphi\,dx\,dy. \quad (4.38)$$

For the right-hand side of (4.31), we unfold the terms using $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_2^\varepsilon$, respectively,

$$\sum_{i=1}^{2}\int_{\Omega_i^{\delta,\varepsilon}} f v^{\delta,\varepsilon}\varphi\,dx \overset{\mathcal{T}_i^\varepsilon}{\simeq} \sum_{i=1}^{2}\frac{1}{|Y|}\int_{\Omega\times Y_i}\mathcal{T}_i^\varepsilon(f)\mathcal{T}_i^\varepsilon(v^{\delta,\varepsilon})\mathcal{T}_i^\varepsilon(\varphi)\,dx\,dy,$$

and so by (4.5) and Theorem 2.3 (iii), and since f and φ are independent of y,

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{2}\int_{\Omega_i^{\delta,\varepsilon}} f v^{\delta,\varepsilon}\varphi\,dx = v(\infty)\int_\Omega f\varphi\,dx. \quad (4.39)$$

Therefore, (4.35), (4.34), (4.38), (4.37), and (4.39) allow us to pass to the limit as $\varepsilon\to 0$ in (4.31), by (4.30), and the density of $\mathcal{D}(\Omega)\times\mathbf{K}_B$ in $\mathbf{L}_B$, using (4.12), we obtain

$$\begin{aligned}&\frac{k_1^2}{|Y|}\int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} F(x,z)\nabla_z G_1(x,z)\nabla_z V\,dx\,dz\\ &+\frac{1}{|Y|}\int_{\Omega\times Y} A(x,y)(\nabla G_1(\cdot,\infty)+\nabla_y\widehat{u})\nabla V(\cdot,\infty)\,dx\,dy\\ &=\int_\Omega fV(\cdot,\infty)\,dx,\quad \forall V\in\mathbf{L}_B.\end{aligned} \quad (4.40)$$

Taking into account (4.22) into (4.40) gives (4.11). Furthermore, the existence, uniqueness, and stability of the solution $(G_1,\widehat{u})$ to problem (4.11) follows from Lax-Milgram Theorem, and so the convergences mentioned in the theorem holds for the whole sequence. □

The next result gives the classical form of the homogenized system given in (4.11).

Corollary 4.10. *Under the assumptions of Theorem 4.9, the limit function* $u_1 \in H_0^1(\Omega)$ *is the unique solution of the homogenized problem*

$$\begin{gathered} -\operatorname{div}(A^{\mathrm{hom},1}\nabla u_1) + k_1^2\Theta u_1 = f \quad \text{in } \Omega, \\ u_1 = 0 \quad \text{in } \partial\Omega, \end{gathered} \tag{4.41}$$

where the homogenized matrix $A^{\mathrm{hom},1} = (a_{ij}^{\mathrm{hom},1})_{N\times N} \in \mathcal{M}(\alpha,\beta,\Omega)$ *has entries*

$$a_{ij}^{\mathrm{hom},1} = \mathcal{M}_Y\Big(a_{ij} + \sum_{k=1}^N a_{ik}\frac{\partial\widehat{\chi}_j}{\partial y_k}\Big) \tag{4.42}$$

such that the correctors $\widehat{\chi}_j$ *for* $j = 1,\dots,N$ *solve the cell problems*

$$\begin{gathered} -\operatorname{div}(A(x,y)\nabla(\widehat{\chi}_j + y_j)) = 0 \quad \text{in } Y, \\ \widehat{\chi}_j \text{ is } Y\text{-periodic}, \quad \mathcal{M}_Y(\widehat{\chi}_j) = 0, \end{gathered} \tag{4.43}$$

and where for a.e. $x \in \Omega$,

$$\Theta(x) := \frac{1}{|Y|}\int_{\mathbb{R}^N\setminus\overline{B}} F(x,z)\nabla_z\theta(x,z)\nabla_z\theta(x,z)\,dz. \tag{4.44}$$

Proof. We first introduce the classical correctors $\widehat{\chi}_j$ for $j = 1,\dots,N$ for the homogenization in fixed domains (for more details, see [4]) that solve (4.43) which is equivalently given by

$$\begin{gathered} \widehat{\chi}_j \in L^\infty(\Omega; H^1_{\mathrm{per}}(Y)), \\ \int_Y A(x,y)\nabla(\widehat{\chi}_j + y_j)\nabla\varphi\,dy = 0 \quad \text{a.e. in } \Omega, \quad \forall\varphi \in H^1_{\mathrm{per}}(Y). \end{gathered}$$

From (4.30), we also have that

$$\nabla G_1(\cdot,\infty) = \nabla u_1. \tag{4.45}$$

Using this along with (4.22) implies that $\widehat{u}$ can be expressed as

$$\widehat{u}(x,y) = \sum_{j=1}^N \frac{\partial u_1}{\partial x_j}(x)\widehat{\chi}_j(x,y) = \sum_{j=1}^N \frac{\partial G_1}{\partial x_j}(x,\infty)\widehat{\chi}_j(x,y) \quad \text{a.e. in } \Omega\times Y. \tag{4.46}$$

Using this expression for $\widehat{u}$ in (4.40), we have for all $V \in \mathbf{L}_B$,

$$\begin{aligned} &\int_\Omega A^{\mathrm{hom},1}\nabla G_1(\cdot,\infty)\nabla V(\cdot,\infty)\,dx\,dy \\ &+ \frac{k_1^2}{|Y|}\int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} F(x,z)\nabla_z G_1(x,z)\nabla_z V\,dx\,dz \\ &= \int_\Omega fV(\cdot,\infty)\,dx, \end{aligned} \tag{4.47}$$

where for a.e. $x \in \Omega$, and by (4.45),

$$\begin{aligned} A^{\mathrm{hom},1}(x)\nabla G_1(x,\infty) &:= \frac{1}{|Y|}\int_Y A(x,y)\big[\nabla G_1(x,\infty) + \nabla_y\widehat{u}(x,y)\big]\,dy \\ &= \frac{1}{|Y|}\int_Y A(x,y)\big[\nabla u_1 + \nabla_y\widehat{u}(x,y)\big]\,dy, \end{aligned}$$

from which $A^{\mathrm{hom},1}$ given by (4.42) follows (see, for instance, [11]). Now, let us introduce θ to be the solution of the following cell problem corresponding to component B given by

$$
\begin{gathered}
\theta \in L^\infty(\Omega;\mathbf{K}_B), \quad \theta(x,\infty)\equiv 1,\\
\int_{\mathbb{R}^N\setminus\overline{B}} F(x,z)\nabla_z\theta(x,z)\nabla_z\Psi(z)\,dz = 0 \quad \text{a.e. in } \Omega,\\
\forall\Psi\in \dot{H}^1(\mathbb{R}^N) \quad \text{with } \Psi = 0 \text{ on } B.
\end{gathered}
\tag{4.48}
$$

Multiplying the equation in (4.48) by $u_1(x)$ which is independent of z and setting $\Psi(z)=V$, we have a.e. in Ω that

$$
\int_{\mathbb{R}^N\setminus\overline{B}} F(x,z)\nabla_z\big(u_1(x)\theta(x,z)\big)\nabla_z V\,dz = 0. \tag{4.49}
$$

On the other hand, in view of (4.47), setting $V(\cdot,\infty)=0$ we have

$$
\int_{\mathbb{R}^N\setminus\overline{B}} F(x,z)\nabla_z G_1(x,z)\nabla_z V\,dz = 0 \quad \text{a.e. in } \Omega. \tag{4.50}
$$

By the Lax-Milgram Theorem, (4.48) admits a unique solution. Therefore, in view of (4.49) and (4.50), since $G_1(x,\cdot)$ and $u_1(x)\theta(x,\cdot)$ are both in $\boldsymbol{K}_B$ and are solutions of the same problem which admits a unique solution, we have that $G_1(x,\cdot) = u_1(x)\theta(x,\cdot)$. Indeed, this coincides with (4.30) since $G_1(\cdot,\infty) = u_1(x)\theta(x,\infty) = u_1(x)(1) = u_1(x)$. Now, if we set

$$
V(x,z) = \Upsilon(x)\theta(x,z) \quad \text{a.e. in } \Omega\times\mathbb{R}^N
$$

in (4.47), where $\Upsilon\in H_0^1(\Omega)$ and θ as in (4.48), then (4.47) becomes

$$
\begin{aligned}
&\int_\Omega A^{\mathrm{hom},1}\nabla G_1(\cdot,\infty)\nabla\Upsilon\,dx\,dy + k_1^2\int_\Omega \Theta u_1\Upsilon\,dx\,dz\\
&= \int_\Omega f\Upsilon\,dx, \quad \forall\Upsilon\in H_0^1(\Omega),
\end{aligned}
\tag{4.51}
$$

where $\Theta(x)$ is given by (4.44), a nonnegative function and can be interpreted as the local capacity of the set B. Finally, we have (4.51) as the variational formulation of (4.41). ☐

4.2. Case $\gamma\in(-1,0)$.

Theorem 4.11. *Let $\gamma\in(-1,0)$. Under assumptions (A1)–(A3), let $u^{\delta,\varepsilon} = (u_1^{\delta,\varepsilon},u_2^{\delta,\varepsilon})\in H_\gamma^{\delta,\varepsilon}$ be the solution of (4.2). Then there exist $u_1\in H_0^1(\Omega)$ and $G_1\in\mathbf{L}_B$ such that*

$$
\widetilde{u_i^{\delta,\varepsilon}} \rightharpoonup \theta_i u_1 \quad \text{weakly in } L^2(\Omega), \tag{4.52}
$$

$$
\mathcal{T}_i^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) \rightharpoonup G_1 \quad \text{weakly in } L^2(\Omega;L^2_{\mathrm{loc}}(\mathbb{R}^N)), \text{ with } G_1(\cdot,\infty)=u_1 \tag{4.53}
$$

$$
\nabla_z\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) \rightharpoonup \nabla_z G_1 \quad \text{weakly in } L^2(\Omega\times\mathbb{R}^N)^N, \tag{4.54}
$$

$$
\nabla_z\mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon}) \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega\times\mathbb{R}^N)^N. \tag{4.55}
$$

and $\widehat{u}_1\in L^2(\Omega;H^1_{per,0}(Y_1))$ such that

$$
\mathcal{T}_1^\varepsilon(\nabla u_1^{\delta,\varepsilon}) \rightharpoonup \nabla u_1 + \nabla_y\widehat{u}_1 \quad \text{weakly in } L^2(\Omega\times Y_1)^N, \tag{4.56}
$$

$$
\mathcal{T}_2^\varepsilon(\nabla u_2^{\delta,\varepsilon}) \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega\times Y_2)^N. \tag{4.57}
$$

Moreover, under assumptions (A4) *and* (A5)*, the pair* $(G_1, \widehat{u}_1)$ *satisfies the unfolded limit problem*

$$\begin{aligned} &\frac{k_1^2}{|Y|}\int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} F(x,z)\nabla_z G_1(x,z)\nabla_z V\,dx\,dz \\ &+\frac{1}{|Y|}\int_{\Omega\times Y_1} A(x,y)(\nabla G_1(\cdot,\infty)+\nabla_y\widehat{u}_1)(\nabla V(\cdot,\infty)+\nabla_y\Psi_1)\,dx\,dy \\ &=\int_\Omega fV(\cdot,\infty)\,dx,\quad \forall\Psi_1\in L^2(\Omega;H^1_{\rm per}(Y)),\quad \forall V\in\mathbf{L}_B. \end{aligned} \tag{4.58}$$

Proof. We divide the proof in five steps.

Step 1. As in Step 1 from the proof of Theorem 4.9, by a Poincaré inequality and using the boundedness of the solution from Theorem 4.2, there exists $u_1 \in H^1_0(\Omega)$ such that (4.52) holds for $i = 1$. Also, as in (4.14), we have

$$\widetilde{u_2^{\delta,\varepsilon}} \rightharpoonup \theta_2 u_2 \quad \text{weakly in } L^2(\Omega). \tag{4.59}$$

Moreover, the existence of $\widehat{u}_1 \in L^2(\Omega; H^1_{per,0}(Y_1))$ such that convergence (4.56) is true, come from (2.3).

Step 2. To capture the effect of the periodic oscillation of the coefficients in (4.2), for $\varphi_i \in \mathcal{D}(\Omega)$ and $\psi_i \in H^1_{\rm per}(Y)$ vanishing in a neighborhood of the origin, we let $v_i = \varphi_i = \varepsilon\varphi_i(x)\psi_i\left(\frac{x}{\varepsilon}\right)$ be a test function in (4.2) for $i = 1, 2$. Following the arguments in Step 2 in obtaining (4.21), we obtain the limit equation

$$\begin{aligned} &\frac{1}{|Y|}\int_{\Omega\times Y_1} A(x,y)(\nabla u_1+\nabla_y\widehat{u}_1)\nabla_y\Psi_1\,dx\,dy \\ &+\frac{1}{|Y|}\int_{\Omega\times Y_2} A(x,y)\nabla_y\widehat{u}_2\nabla_y\Psi_2\,dx\,dy=0. \end{aligned} \tag{4.60}$$

This along with the density of $\mathcal{D}(\Omega)\times H^1_{\rm per}(Y)$ in $L^2(\Omega; H^1_{\rm per}(Y))$, then equation (4.60) holds for every $\Psi_i \in L^2(\Omega; H^1_{\rm per}(Y_i))$.

Step 3. As argued in Step 3 from the proof of Theorem 4.9, we have the existence of $G_i \in L^2(\Omega; L^2_{\rm loc}(\mathbb{R}^N))$ such that

$$\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \rightharpoonup G_i \quad \text{weakly in } L^2(\Omega; L^2_{\rm loc}(\mathbb{R}^N)). \tag{4.61}$$

In view of (4.52) and (4.59), we obtain

$$G_i - u_i = J_i, \quad \nabla_z G_i = \nabla_z J_i, \tag{4.62}$$

$$\nabla_z\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \rightharpoonup \nabla_z G_i \quad \text{weakly in } L^2(\Omega\times\mathbb{R}^N)^N, \tag{4.63}$$

for $i = 1, 2$. We also have here that $G_1 \in \mathbf{L}_B$ and

$$G_1(\cdot,\infty) = u_1. \tag{4.64}$$

From these statements, when $i = 1$ in (4.62) and (4.63) prove (4.53) and (4.54).

Step 4. Let us now show the contribution of the inclusions to the limit equations. As in Step 4 of the proof of Theorem 4.9, to capture the effect of the very small inclusions, let $\varphi \in \mathcal{D}(\Omega)$ and $v \in \mathbf{K}_B$ such that $\nabla_z v$ has compact support. Take $v_1 = v_2 = v^{\delta,\varepsilon}(x)\varphi(x)$ as test functions in (4.2) where $v^{\delta,\varepsilon}$ is defined in (4.4).

Since $v_1 = v_2$, the third term in (4.2) vanishes and we obtain

$$\int_{\Omega_1^{\delta,\varepsilon}} A^\varepsilon \nabla u_1^{\delta,\varepsilon} \nabla(v^{\delta,\varepsilon}\varphi)\,dx + \int_{\Omega_2^{\delta,\varepsilon}} A^\varepsilon \nabla u_2^{\delta,\varepsilon} \nabla(v^{\delta,\varepsilon}\varphi)\,dx = \sum_{i=1}^{2} \int_{\Omega_i^{\delta,\varepsilon}} f v^{\delta,\varepsilon}\varphi\,dx. \tag{4.65}$$

For the first and second terms in the left-hand side of (4.65), similar computations used in (4.34)–(4.38) yield

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{\Omega_1^{\delta,\varepsilon}} A^\varepsilon \nabla u_1^{\delta,\varepsilon} \nabla(v^{\delta,\varepsilon}\varphi)\,dx \\ &= \frac{k_1^2}{|Y|} \int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} F(x,z)\nabla_z G_1(x,z)\varphi\nabla_z v\,dx\,dz \\ &\quad + \frac{v(\infty)}{|Y|} \int_{\Omega\times Y_1} A(x,y)(\nabla u_1 + \nabla_y \widehat{u}_1)\nabla\varphi\,dx\,dy, \end{aligned} \tag{4.66}$$

and

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_{\Omega_2^{\delta,\varepsilon}} A^\varepsilon \nabla u_2^{\delta,\varepsilon} \nabla(v^{\delta,\varepsilon}\varphi)\,dx &= \frac{k_1^2}{|Y|} \int_{\Omega\times B} F(x,z)\nabla_z G_2(x,z)\varphi\nabla_z v\,dx\,dz \\ &\quad + \frac{v(\infty)}{|Y|} \int_{\Omega\times Y_2} A(x,y)\nabla_y \widehat{u}_2 \nabla\varphi\,dx\,dy. \end{aligned} \tag{4.67}$$

Meanwhile, for the right-hand side of (4.65), computations to those used in (4.39) give

$$\lim_{\varepsilon\to 0} \sum_{i=1}^{2} \int_{\Omega_i^{\delta,\varepsilon}} f v_i^{\delta,\varepsilon}\varphi_i\,dx = v(\infty)\int_\Omega f\varphi\,dx. \tag{4.68}$$

Using equations (4.66)–(4.68) when passing to the limit as $\varepsilon \to 0$ in (4.65), by (4.64) and the density of $\mathcal{D}(\Omega)\times \mathbf{K}_B$ in $\mathbf{L}_B$, and taking into account (4.60), we obtain the limit equation

$$\begin{aligned} &\frac{k_1^2}{|Y|} \int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} F(x,z)\nabla_z U(x,z)\nabla_z V\,dx\,dz \\ &+ \frac{1}{|Y|} \int_{\Omega\times Y_1} A(x,y)(\nabla G_1(\cdot,\infty) + \nabla_y \widehat{u}_1)(\nabla V(\cdot,\infty) + \nabla_y \Psi_1)\,dx\,dy \\ &+ \frac{k_1^2}{|Y|} \int_{\Omega\times B} F(x,z)\nabla_z G_2(x,z)\nabla_z V\,dx\,dz \\ &+ \frac{1}{|Y|} \int_{\Omega\times Y_2} A(x,y)\nabla_y \widehat{u}_2(\nabla V(\cdot,\infty) + \nabla_y \Psi_2)\,dx\,dy \\ &= \int_\Omega f V(\cdot,\infty)\,dx, \quad \forall \Psi_i \in L^2(\Omega; H^1_{\rm per}(Y_i)) \quad \text{for } i=1,2, \quad \forall V \in \mathbf{L}_B. \end{aligned} \tag{4.69}$$

Let us now have an insight about the explicit form of $\nabla_z G_2$ and $\nabla_y \widehat{u}_2$. To this aim, choose $V \equiv 0$, $\Psi_1 \equiv 0$, and $\Psi_2 \equiv \widehat{u}_2$, then (4.69) becomes

$$\frac{1}{|Y|} \int_{\Omega\times Y_2} A(x,y)\nabla_y \widehat{u}_2 \nabla_y \widehat{u}_2 = 0. \tag{4.70}$$

However, as $A(x,y) \in \mathcal{M}(\alpha,\beta,\Omega\times Y)$, we have by ellipticity and (4.70) that $\nabla_y \widehat{u}_2 \equiv 0$, which by (2.7) gives (4.57). Furthermore, in view of (3.11) for $i = 2$, and (3.32), we have

$$\nabla_z \mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon}) = \varepsilon\delta \mathcal{T}_2^{\delta,\varepsilon}(\nabla u_2^{\delta,\varepsilon}) \to 0.$$

Comparing this with (4.63) for $i = 2$, we have $\nabla_z G_2 \equiv 0$ giving (4.55).

Therefore, since $\nabla_y \widehat{u}_2 \equiv 0$ and $\nabla_z G_2 \equiv 0$, equation (4.69) simplifies to (4.58). Furthermore, the existence, uniqueness, and stability of the solution $(G_1, \widehat{u}_1)$ to problem (4.58) follows from Lax-Milgram Theorem, and so the convergences mentioned in the theorem holds for the whole sequence.

Step 5. Finally, since $\gamma < 0$, the second part of Theorem 3.21 is applicable; and consequently, $G_1 = G_2$. In view of (4.61), we obtain (4.53) for $i = 2$. Moreover, using (3.38) in (4.62), we have $J_1 = J_2$ and since u_i is independent of z, we have $u_1 = u_2$. This along with (4.59) yield (4.52) for $i = 2$. □

Let us now have the classical form of the homogenized system given in (4.58). The arguments of the proof are similar to that of Corollary 4.10, however instead of integrating over Y at some parts, here the integrals are over Y_1 only.

Corollary 4.12. *Under the assumptions of Theorem 4.11, the limit function $u_1 \in H_0^1(\Omega)$ is the unique solution of the homogenized problem*

$$\begin{gathered} -\operatorname{div}(A^{\mathrm{hom},2}\nabla u_1) + k_1^2 \Theta u_1 = f \quad \text{in } \Omega, \\ u_1 = 0 \quad \text{in } \partial\Omega, \end{gathered} \tag{4.71}$$

where the homogenized matrix $A^{\mathrm{hom},2} = (a_{ij}^{\mathrm{hom},2})_{N\times N} \in \mathcal{M}(\alpha,\beta,\Omega)$ has entries

$$a_{ij}^{\mathrm{hom},2} = \mathcal{M}_{Y_1}\Big(a_{ij} + \sum_{k=1}^{N} a_{ik}\frac{\partial \overline{\chi}_j}{\partial y_k}\Big) \tag{4.72}$$

such that the correctors $\overline{\chi}_j$ for $j = 1, \dots, N$ solve the cell problems

$$\begin{gathered} -\operatorname{div}(A(x,y)\nabla(\overline{\chi}_j + y_j)) = 0 \quad \text{in } Y_1, \\ A(x,y)\nabla(\overline{\chi}_j + y_j)\cdot n_1 = 0 \quad \text{on } \Gamma, \\ \overline{\chi}_j \text{ is } Y\text{-periodic}, \quad \mathcal{M}_Y(\overline{\chi}_j) = 0, \end{gathered}$$

and where Θ is given by (4.44).

4.3. **Case $\gamma = -1$.** As presented in the next theorem, the asymptotic behavior for this case is more delicate as the limit problem contains a jump term on the common boundary.

Theorem 4.13. *Let $\gamma = -1$. Under assumptions (A1)–(A3), let $u^{\delta,\varepsilon} = (u_1^{\delta,\varepsilon}, u_2^{\delta,\varepsilon}) \in H_\gamma^{\delta,\varepsilon}$ be the solution of (4.2). Then there exist $u_1 \in H_0^1(\Omega)$ and $G_1 \in \mathbf{L}_B$ such that*

$$\widetilde{u_i^{\delta,\varepsilon}} \rightharpoonup \theta_i u_1 \quad \text{weakly in } L^2(\Omega), \tag{4.73}$$

$$\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \rightharpoonup G_1 \quad \text{weakly in } L^2(\Omega; L^2_{\mathrm{loc}}(\mathbb{R}^N)), \tag{4.74}$$

$$\nabla_z \mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \rightharpoonup \nabla_z G_1 \quad \text{weakly in } L^2(\Omega\times\mathbb{R}^N)^N, \tag{4.75}$$

and $\widehat{u}_i \in L^2(\Omega; H^1_{per,0}(Y_i))$ such that

$$\mathcal{T}_1^{\varepsilon}(\nabla u_1^{\delta,\varepsilon}) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \quad \text{weakly in } L^2(\Omega\times Y_1)^N, \tag{4.76}$$

$$\mathcal{T}_2^{\varepsilon}(\nabla u_2^{\varepsilon}) \rightharpoonup \nabla_y \widehat{u}_2 \quad \text{weakly in } L^2(\Omega\times Y_2)^N, \tag{4.77}$$

for $i=1,2$, Moreover, under assumptions (A4) *and* (A5)*, the triple $(G_1,\widehat{u}_1,\bar{u}_2)$ is the unique solution of the unfolded limit problem*

$$\begin{aligned} &\frac{k_1^2}{|Y|}\int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} F(x,z)\nabla_z G_1(x,z)\nabla_z V\,dx\,dz \\ &+\frac{1}{|Y|}\int_{\Omega\times Y_1} A(x,y)(\nabla G_1(\cdot,\infty)+\nabla_y\widehat{u}_1)(\nabla V(\cdot,\infty)+\nabla_y\Psi_1)\,dx\,dy \\ &+\frac{1}{|Y|}\int_{\Omega\times Y_2} A(x,y)(\nabla G_1(\cdot,\infty)+\nabla_y\bar{u}_2)(\nabla V(\cdot,\infty)+\nabla_y\Psi_2)\,dx\,dy \\ &+\frac{1}{|Y|}\int_{\Omega\times\Gamma} h(y)(\widehat{u}_1-\bar{u}_2)(\Psi_1-\Psi_2)\,dx\,d\sigma_y \\ &=\int_\Omega fV(\cdot,\infty)\,dx,\quad \forall\Psi_i\in L^2(\Omega;H^1_{\rm per}(Y))\ \text{for } i=1,2,\ \forall V\in\mathbf{L}_B, \end{aligned}\tag{4.78}$$

where $\bar{u}_2\in L^2(\Omega;H^1(Y_2))$ is the extension by periodicity of the function

$$\bar{u}_2=\widehat{u}_2-y_\Gamma\nabla u_1-\xi_\Gamma,\tag{4.79}$$

for some $\xi_\Gamma\in L^2(\Omega)$ and where $y_\gamma=y-\mathcal{M}_\Gamma(y)$.

Proof. We divide the proof in three steps.

Step 1. The existence of $u_1\in H^1_0(\Omega)$, $G_1\in\mathbf{L}_B$, and $\widehat{u}_i\in L^2(\Omega;H^1_{per,0}(Y_i))$ such that convergences (4.73)–(4.77) hold follow from the same arguments as in the proofs of (4.6)–(4.10) from Theorem 4.9.

Step 2. To capture the effect of the periodic oscillation of the coefficients, for $\varphi_i\in\mathcal{D}(\Omega)$ and $\psi_i\in H^1_{\rm per}(Y)$ vanishing in a neighborhood of the origin, we let $v_i=\varphi_i=\varepsilon\varphi_i(x)\psi_i\left(\frac{x}{\varepsilon}\right)$ be a test function in (4.2) for $i=1,2$. Proceeding as in Step 2 of Theorem 4.11 and employing similar arguments in [21] for the interfacial term, we obtain the following limit equation

$$\begin{aligned} &\int_{\Omega\times Y_1} A(x,y)(\nabla u_1+\nabla_y\widehat{u}_1)\nabla_y\Psi_1\,dx\,dy+\int_{\Omega\times Y_2} A(x,y)\nabla_y\widehat{u}_2\nabla_y\Psi_2\,dx\,dy \\ &+\int_\Gamma h(y)(\widehat{u}_1-\bar{u}_2)(\Psi_1-\Psi_2)\,dx\,d\sigma_y=0, \end{aligned}\tag{4.80}$$

for all $\Psi_i\in L^2(\Omega;H^1_{\rm per}(Y))$ and $i=1,2$.

Step 3. Let us now capture the effect of the very small inclusions in (4.2). To this goal, for $\varphi\in\mathcal{D}(\Omega)$ and $v\in\mathbf{K}_B$ such that $\nabla_z v$ has compact support, we use again the function $v^{\delta,\varepsilon}$ as given in (4.4) and let $v_1=v_2=v^{\delta,\varepsilon}(x)\varphi(x)$ be test functions in (4.2).

Following the arguments in Step 4 of Theorem 4.9, we obtain

$$\begin{aligned} &\lim_{\varepsilon\to0}\int_{\Omega_1^{\delta,\varepsilon}} A^\varepsilon\nabla u_1^{\delta,\varepsilon}\nabla(v^{\delta,\varepsilon})\varphi\,dx \\ &=\frac{k_1^2}{|Y|}\int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} F(x,z)\nabla_z G_1(x,z)\varphi\nabla_z v\,dx\,dz \\ &\quad+\frac{v(\infty)}{|Y|}\int_{\Omega\times Y_1} A(x,y)(\nabla u_1+\nabla_y\widehat{u}_1)\nabla\varphi\,dx\,dy, \end{aligned}\tag{4.81}$$

$$\lim_{\varepsilon\to0}\int_{\Omega_2^{\delta,\varepsilon}} A^\varepsilon\nabla u_2^{\delta,\varepsilon}\nabla(v^{\delta,\varepsilon}\varphi)\,dx=\frac{v(\infty)}{|Y|}\int_{\Omega\times Y_2} A(x,y)\nabla_y\widehat{u}_2\nabla\varphi\,dx\,dy,\tag{4.82}$$

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{2}\int_{\Omega_i^{\delta,\varepsilon}} f v_i^{\delta,\varepsilon}\varphi_i\,dx. = v(\infty)\int_\Omega f\varphi\,dx. \tag{4.83}$$

Therefore, in view of (4.81) - (4.83), passing to the limit, by (4.30) and the density of $\mathcal{D}(\Omega)\times\mathbf{K}_B$ in $\mathbf{L}_B$, we obtain

$$\begin{aligned}
&\frac{k_1^2}{|Y|}\int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} F(x,z)\nabla_z G_1(x,z)\nabla_z V\,dx\,dz\\
&+\frac{1}{|Y|}\int_{\Omega\times Y_1} A(x,y)(\nabla G_1(\cdot,\infty)+\nabla_y\widehat{u}_1)\nabla V(\cdot,\infty)\,dx\,dy\\
&+\frac{1}{|Y|}\int_{\Omega\times Y_2} A(x,y)\nabla_y\widehat{u}_2\nabla V(\cdot,\infty)\,dx\,dy\\
&=\int_\Omega fV(\cdot,\infty)\,dx,\quad \forall V\in\mathbf{L}_B.
\end{aligned}\tag{4.84}$$

Finally, taking into account (4.80) in (4.84) gives (4.78). Furthermore, the existence, uniqueness, and stability of the solution $(G_1,\widehat{u}_1,\bar{u}_2)$ to problem (4.78) follows from Lax-Milgram Theorem, and so the convergences mentioned in the theorem holds for the whole sequence. □

For the classical form of the homogenized system given in (4.78), the arguments are similar to the the proof of Corollary 4.10 with additional straightforward computations on the interface.

Corollary 4.14. *Under the assumptions of Theorem 4.13, the limit function $u_1\in H_0^1(\Omega)$ is the unique solution of the homogenized problem*

$$\begin{gathered}
-\operatorname{div}(A^{\mathrm{hom},3}\nabla u_1)+k_1^2\Theta u_1=f\quad \text{in } \Omega,\\
u_1=0\quad \text{in } \partial\Omega,
\end{gathered}\tag{4.85}$$

where the homogenized matrix $A^{\mathrm{hom},3}=(a_{ij}^{\mathrm{hom},3})_{N\times N}\in\mathcal{M}(\alpha,\beta,\Omega)$ has entries

$$a_{ij}^{\mathrm{hom},3}=\mathcal{M}_{Y_1}\Big(a_{ij}+\sum_{k=1}^N a_{ik}\frac{\partial\widehat{\chi}_j^1}{\partial y_k}\Big)+\mathcal{M}_{Y_2}\Big(a_{ij}+\sum_{k=1}^N a_{ik}\frac{\partial\widehat{\chi}_j^2}{\partial y_k}\Big) \tag{4.86}$$

such that the correctors $(\widehat{\chi}_j^1,\widehat{\chi}_j^2)$ for $j=1,\dots,N$ solve the cell problems

$$\begin{gathered}
-\operatorname{div}(A(x,y)\nabla(\widehat{\chi}_j^1+y_j))=0\quad \text{in } Y_1,\\
-\operatorname{div}(A(x,y)\nabla(\widehat{\chi}_j^2+y_j))=0\quad \text{in } Y_2,\\
A(x,y)\nabla(\widehat{\chi}_j^1+y_j)\cdot n_1=-A(x,y)\nabla(\widehat{\chi}_j^2+y_j)\cdot n_2\quad \text{on } \Gamma,\\
-A(x,y)\nabla(\widehat{\chi}_j^1+y_j)\cdot n_1=h(\widehat{\chi}_j^1-\widehat{\chi}_j^2)\quad \text{on } \Gamma,\\
\widehat{\chi}_j^1 \text{ is } Y\text{-periodic},\quad \mathcal{M}_Y(\widehat{\chi}_j^1)=0.
\end{gathered}$$

and where Θ is still of the form (4.44).

Remark 4.15. Let us have the following observations:

(1) From Theorems 4.10, 4.12, and 4.14, the contribution of the coefficient matrix A^ε in the corresponding components Y_1 and Y_2 is taken into account by the term $A^{\mathrm{hom},j}$, for $j=1,2,3$, which also recovers the homogenized matrices in [21].

(2) The appearance of the zero order strange term $k_1^2\Theta u_1$ in (4.41), (4.71), and (4.85) is brought about by the effect of the very small inclusions.
(3) In contrast to the limit problem in [27] being the classical Dirichlet problem which does not account for the presence of a strange term, the results of this paper provide an information on the contribution of the small scale in the homogenized problem.

To end this section, let us further investigate what happens to the limit function and the contribution of the very small inclusions to the limit problem if instead of (A3), we have the assumptions $k_1 = +\infty$ or $k_1 = 0$.

4.4. **Case $k_1 = +\infty$.** If k_1 in (A3) is infinite, together with (3.13) for $p = 2$ as $\varepsilon \to 0$, we obtain

$$\big\|\mathcal{T}_i^{\delta,\varepsilon}\big[u_i^{\delta,\varepsilon} - \mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_i}(u_i^{\delta,\varepsilon})\big]\big\|_{L^2(\Omega;L^{2^*}(\mathbb{R}^N))} \leq C\frac{\varepsilon|Y|^{1/2}}{\delta^{\frac{N}{2}-1}}\|\nabla u_i^{\delta,\varepsilon}\|_{L^2(\Omega_i^{\delta,\varepsilon})^N}, \to 0, \quad (4.87)$$

for $i = 1, 2$.

Meanwhile, by (3.6) and (3.9), we have the convergence

$$u_i^{\delta,\varepsilon} \to u_i \quad \text{strongly in } L^2(\Omega).$$

This along with a similar proof of (4.25), we have

$$\mathcal{M}^{\delta,\varepsilon}_{\frac{1}{\delta}Y_i}(u_i^{\delta,\varepsilon})1_{\frac{1}{\delta}Y_i} \to u_i \quad \text{strongly in } L^2(\Omega; L^2_{\rm loc}(\mathbb{R}^N)). \quad (4.88)$$

Now for $\gamma < 0$, as shown in Theorems 4.9, 4.11, and 4.13, we have $u_1 = u_2$. Therefore, by linearity of $\mathcal{T}_i^{\delta,\varepsilon}$, Theorem 3.10 (i), and (4.88), in view of (4.87), we obtain

$$\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \to u_1 \quad \text{strongly in } L^2(\Omega; L^2_{\rm loc}(\mathbb{R}^N)). \quad (4.89)$$

However, as seen in the proof of (4.28), $\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) = 0$ in $\Omega \times B$. Finally, this and (4.89) imply that $u_1 = 0$.

Remark 4.16. When $k_1 = +\infty$ and $\gamma < 0$, then u_1 vanishes in Ω. This means that the size of the very small inclusions is too big that it forces the limit function u_1 to be zero.

4.5. **Case $k_1 = 0$.** When k_1 in (A3) is zero, then $k_1^2 = 0$ so that the limit problems in Theorems 4.10, 4.12, and (4.14) become

$$\begin{gathered} -\operatorname{div}(A^{\rm hom}\nabla u_1) = f \quad \text{in } \Omega, \\ u_1 = 0 \quad \text{in } \partial\Omega, \end{gathered}$$

with their corresponding homogenized matrices given in (4.42), (4.72) and (4.86), respectively.

Remark 4.17. When $k_1 = 0$ and $\gamma < 0$, then the very small inclusions do not contribute in the limit problem. This means that the size of the very small inclusions are too small to influence the limit problem and so we do not have the appearance of a zero order strange term.

Furthermore, let us mention that the proof of the homogenization results for $k_1 = 0$ requires slight modifications. For instance, let us consider the case when

$\gamma < -1$. Let us focus on Step 2 on the proof of Theorem (4.9). To resolve this, we will use Theorem 3.9 (viii). In particular, in place of (4.23), we will use

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \rightharpoonup U_i \quad \text{weakly in } L^2(\Omega; L^2_{\rm loc}(\mathbb{R}^N)).$$

The rest of the arguments are similar.

5. Corrector results

In this section, we prove some convergence for the associated energies to problem (1.1). As a consequence, corrector results will be obtained using the periodic unfolding method for problem (1.1) where $\gamma < 0$. We first recall a classical result which is essential in this part (see e.g. [11]).

Theorem 5.1. *Let $\{D_\varepsilon\}_\varepsilon$ be a sequence of $N \times N$ matrix fields in $\mathcal{M}(\alpha,\beta,\mathcal{O})$ for some open set $\mathcal{O}$ such that $D_\varepsilon \to D$ almost everywhere on $\mathcal{O}$ (or more generally, in measure in $\mathcal{O}$). If the sequence of vector fields $\{\zeta_\varepsilon\}_\varepsilon$ converges weakly to ζ in $L^2(\mathcal{O})^N$, then*

$$\liminf_{\varepsilon\to 0}\int_{\mathcal{O}} D_\varepsilon\zeta_\varepsilon\zeta_\varepsilon\,dx \geq \int_{\mathcal{O}} D\zeta\zeta\,dx.$$

Furthermore if

$$\limsup_{\varepsilon\to 0}\int_{\mathcal{O}} D_\varepsilon\zeta_\varepsilon\zeta_\varepsilon\,dx \leq \int_{\mathcal{O}} D\zeta\zeta\,dx,$$

then

$$\lim_{\varepsilon\to 0}\int_{\mathcal{O}} D_\varepsilon\zeta_\varepsilon\zeta_\varepsilon\,dx = \int_{\mathcal{O}} D\zeta\zeta\,dx \quad \text{and} \quad \zeta_\varepsilon \to \zeta \quad \text{strongly in } L^2(\mathcal{O})^N.$$

5.1. **Case $\boldsymbol{\gamma < -1}$.** Let us now have the convergence of the energy for this case.

Theorem 5.2. *Let $\gamma < -1$. Under the assumptions of Theorems 4.9 and 4.10, one has*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{i=1}^{2}\int_{\Omega_i^{\delta,\varepsilon}} A^\varepsilon\nabla u_i^{\delta,\varepsilon}\nabla u_i^{\delta,\varepsilon}\,dx\\
&= \int_\Omega A^{\rm hom}\nabla u_1\nabla u_1\,dx + k_1^2\int_\Omega \Theta u_1^2\,dx\\
&= \frac{1}{|Y|}\int_{\Omega\times Y} A(x,y)(\nabla G_1(\cdot,\infty)+\nabla_y\widehat{u})(\nabla G_1(\cdot,\infty)+\nabla_y\widehat{u})\,dx\,dy\\
&\quad + \frac{k_1^2}{|Y|}\int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} F(x,z)\nabla_z G_1(x,z)\nabla_z G_1(x,z)\,dx\,dz,
\end{aligned} \tag{5.1}$$

and

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{2}\int_{\Lambda_i^{\delta,\varepsilon}} |\nabla u_i^{\delta,\varepsilon}|^2 dx = 0. \tag{5.2}$$

Moreover, we have the strong convergences

$$\mathcal{T}_1^\varepsilon(\nabla u_1^{\delta,\varepsilon})\mathbf{1}_{\Omega\times Y_1^\delta} \to \nabla u_1 + \nabla_y\widehat{u}_1 \quad \text{strongly in } L^2(\Omega\times Y_1)^N, \tag{5.3}$$

$$\mathcal{T}_2^\varepsilon(\nabla u_2^{\delta,\varepsilon})\mathbf{1}_{\Omega\times Y_2^\delta} \to \nabla_y\widehat{u}_2 \quad \text{strongly in } L^2(\Omega\times Y_2)^N, \tag{5.4}$$

and for $i = 1, 2$,

$$\nabla_z\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}))1_{\frac{1}{\delta}Y_i^\delta} \to \nabla_z G_1 \quad \text{strongly in } L^2(\Omega\times\mathbb{R}^N)^N. \tag{5.5}$$

Proof. Let $v_i = u_i^{\delta,\varepsilon}$ be test functions in (4.2) for $i = 1, 2$. Then

$$\begin{aligned}&\sum_{i=1}^{2}\int_{\Omega_i^{\delta,\varepsilon}} A^\varepsilon \nabla u_i^{\delta,\varepsilon}\nabla u_i^{\delta,\varepsilon}\,dx\\&= \sum_{i=1}^{2}\Big[\int_{\Omega_i^{\delta,\varepsilon}} f u_i^{\delta,\varepsilon}\,dx\Big] - \varepsilon^\gamma \int_{\Gamma^{\delta,\varepsilon}} h^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}-u_2^{\delta,\varepsilon})(u_1^{\delta,\varepsilon}-u_2^{\delta,\varepsilon})\,d\sigma_x.\end{aligned} \tag{5.6}$$

Unfolding the left-hand side of (5.6) using $\mathcal{T}_\varepsilon$, in view of Remark 2.2 and Theorem 2.3 (ii), we obtain

$$\begin{aligned}&\sum_{i=1}^{2}\Big[\frac{1}{|Y|}\int_{\Omega\times Y_i} \mathcal{T}_i^\varepsilon(A^\varepsilon)\mathcal{T}_i^\varepsilon(\nabla u_i^{\delta,\varepsilon})\mathcal{T}_i^\varepsilon(\nabla u_i^{\delta,\varepsilon})\,dx + \int_{\Lambda_i^{\delta,\varepsilon}} A^\varepsilon\nabla u_i^{\delta,\varepsilon}\nabla u_i^{\delta,\varepsilon}\,dx\Big]\\&= \sum_{i=1}^{2}\Big[\int_{\Omega_i^{\delta,\varepsilon}} f u_i^{\delta,\varepsilon}\,dx\Big] - \varepsilon^\gamma \int_{\Gamma^{\delta,\varepsilon}} h^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}-u_2^{\delta,\varepsilon})(u_1^{\delta,\varepsilon}-u_2^{\delta,\varepsilon})\,d\sigma_x.\end{aligned} \tag{5.7}$$

Now, to investigate the convergence of the energy, we set $Y_2^\delta = \sqrt{\delta}\,\overline{B}$ and $Y_1^\delta = Y \setminus Y_2^\delta$. From here, we note that $\frac{1}{\delta}Y_2^\delta = \frac{1}{\sqrt{\delta}}\overline{B}$. With a change of variable $y = \delta z$ in Y_2^δ, transforming the first and third term in the left-hand side of (5.7), in view of Remark 3.5 (2), and (3.11), gives

$$\begin{aligned}&\sum_{i=1}^{2}\Big[\frac{1}{|Y|}\int_{\Omega\times Y_i^\delta} \mathcal{T}_i^\varepsilon(A^\varepsilon)\mathcal{T}_i^\varepsilon(\nabla u_i^{\delta,\varepsilon})\mathcal{T}_i^\varepsilon(\nabla u_i^{\delta,\varepsilon})\,dx\,dy\\&+ \Big(\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\Big)^2\frac{1}{|Y|}\int_{\Omega\times\frac{1}{\delta}Y_i^\delta} \mathcal{T}_i^{\delta,\varepsilon}(A^\varepsilon)\nabla_z\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})\nabla_z\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})\,dx\,dz\\&+ \int_{\Lambda_i^{\delta,\varepsilon}} A^\varepsilon\nabla u_i^{\delta,\varepsilon}\nabla u_i^{\delta,\varepsilon}\,dx\Big]\\&= \sum_{i=1}^{2}\Big[\int_{\Omega_i^{\delta,\varepsilon}} f u_i^{\delta,\varepsilon}\,dx\Big] - \varepsilon^\gamma \int_{\Gamma^{\delta,\varepsilon}} h^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}-u_2^{\delta,\varepsilon})(u_1^{\delta,\varepsilon}-u_2^{\delta,\varepsilon})\,d\sigma_x.\end{aligned} \tag{5.8}$$

For conciseness in (5.8), we set

$$\begin{gathered}\mathcal{A}^{\delta,\varepsilon} = \sum_{i=1}^{2}\frac{1}{|Y|}\int_{\Omega\times Y_i^\delta} \mathcal{T}_i^\varepsilon(A^\varepsilon)\mathcal{T}_i^\varepsilon(\nabla u_i^{\delta,\varepsilon})\mathcal{T}_i^\varepsilon(\nabla u_i^{\delta,\varepsilon})\,dx\,dy,\\ \mathcal{B}^{\delta,\varepsilon} = \sum_{i=1}^{2}\Big(\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\Big)^2\frac{1}{|Y|}\int_{\Omega\times\frac{1}{\delta}Y_i^\delta} \mathcal{T}_i^{\delta,\varepsilon}(A^\varepsilon)\nabla_z\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})\nabla_z\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})\,dx\,dz,\\ \mathcal{C}^{\delta,\varepsilon} = \sum_{i=1}^{2}\int_{\Lambda_i^{\delta,\varepsilon}} A^\varepsilon\nabla u_i^{\delta,\varepsilon}\nabla u_i^{\delta,\varepsilon}\,dx.\end{gathered} \tag{5.9}$$

From (4.8)–(4.10), we have

$$\mathcal{T}_1^\varepsilon(\nabla u_1^{\delta,\varepsilon})\mathbf{1}_{\Omega\times Y_1^\delta} \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \quad \text{weakly in } L^2(\Omega\times Y_1)^N, \tag{5.10}$$

$$\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon)\mathbf{1}_{\Omega\times Y_2^\delta} \rightharpoonup \nabla_y \widehat{u}_2 \quad \text{weakly in } L^2(\Omega\times Y_2)^N, \tag{5.11}$$

$$\nabla_z\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})\mathbf{1}_{\frac{1}{\delta}Y_i^\delta} \rightharpoonup \nabla_z G_1 \quad \text{weakly in } L^2(\Omega\times\mathbb{R}^N)^N. \tag{5.12}$$

Therefore, with (4.12) and in view of (4.11) with $V = G_1$ and $\Psi = \widehat{u}$, same arguments as in the proof of Theorem 4.9, Theorem 5.1 together with (5.6), (5.7), (5.9) - (5.12), and Theorem 3.9 (ii) yield

$$
\begin{aligned}
&\int_\Omega fG_1(\cdot,\infty)\,dx \\
&= \frac{1}{|Y|}\int_{\Omega\times Y} A(x,y)(\nabla G_1(\cdot,\infty)+\nabla_y\widehat{u})(\nabla G_1(\cdot,\infty)+\nabla_y\widehat{u})\,dx\,dy \\
&\quad + \frac{k_1^2}{|Y|}\int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} F(x,z)\nabla_z G_1(x,z)\nabla_z G_1(x,z)\,dx\,dz \\
&\le \liminf_{\varepsilon\to 0}\mathcal{A}^{\delta,\varepsilon} + \liminf_{\varepsilon\to 0}\mathcal{B}^{\delta,\varepsilon} \le \liminf_{\varepsilon\to 0}(\mathcal{A}^{\delta,\varepsilon}+\mathcal{B}^{\delta,\varepsilon}) \\
&= \liminf_{\varepsilon\to 0}\Big[\sum_{i=1}^2\Big(\int_{\Omega_i^{\delta,\varepsilon}} A^\varepsilon\nabla u_i^{\delta,\varepsilon}\nabla u_i^{\delta,\varepsilon}\,dx\Big) - \mathcal{C}^{\delta,\varepsilon}\Big] \\
&\le \limsup_{\varepsilon\to 0}\Big[\sum_{i=1}^2\Big(\int_{\Omega_i^{\delta,\varepsilon}} A^\varepsilon\nabla u_i^{\delta,\varepsilon}\nabla u_i^{\delta,\varepsilon}\,dx\Big) - \mathcal{C}^{\delta,\varepsilon}\Big] \\
&\le \limsup_{\varepsilon\to 0}\sum_{i=1}^2\int_{\Omega_i^{\delta,\varepsilon}} A^\varepsilon\nabla u_i^{\delta,\varepsilon}\nabla u_i^{\delta,\varepsilon}\,dx \\
&= \lim_{\varepsilon\to 0}\Big[\sum_{i=1}^2\Big(\int_{\Omega_i^{\delta,\varepsilon}} fu_i^{\delta,\varepsilon}\,dx\Big) - \varepsilon^\gamma\int_{\Gamma^{\delta,\varepsilon}} h^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}-u_2^{\delta,\varepsilon})(u_1^{\delta,\varepsilon}-u_2^{\delta,\varepsilon})\,d\sigma_x\Big] \\
&= \int_\Omega fG_1(\cdot,\infty)\,dx,
\end{aligned}
\tag{5.13}
$$

which implies that these inequalities are actually equalities. Hence, equations (5.1) and (5.2) hold true. Finally, the convergences in (5.3)–(5.5) follows from (5.13) and with the application of some properties of limits as well as lim sup and lim inf. □

Corollary 5.3. *Under the assumptions of Theorem 5.2, we have the corrector results for $i = 1, 2$,*

$$
\|\nabla u_i^{\delta,\varepsilon}\mathbf{1}_{\Omega_i^{\sqrt{\delta},\varepsilon}} - \nabla u_1 - \sum_{j=1}^N \mathcal{U}_i^\varepsilon\Big(\frac{\partial G_1}{\partial x_j}(x,\infty)\Big)\mathcal{U}_i^\varepsilon\Big(\nabla_y\widehat{\chi}_j(x,y)\big|_{Y_i}\Big)\|_{L^2(\Omega_i^{\delta,\varepsilon})} \to 0, \tag{5.14}
$$

and

$$
\|\nabla u_i^{\delta,\varepsilon}\|_{L^2(\Omega_i^{\delta,\varepsilon}\setminus\Omega_i^{\sqrt{\delta},\varepsilon})} \to \frac{k_1}{|Y|^{1/2}}\|\nabla_z G_1\|_{L^2(\Omega\times\mathbb{R}^N)}. \tag{5.15}
$$

Furthermore, for $i = 1, 2$,

$$
\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \to G_1 \quad \text{strongly in } L^2(\Omega; L^2_{\mathrm{loc}}(\mathbb{R}^N)). \tag{5.16}
$$

Proof. First, we note that from (4.12) and (4.46), we have

$$
\widehat{u}_1 = \widehat{u}\big|_{\Omega\times Y_1} = \sum_{j=1}^N \frac{\partial G_1}{\partial x_j}(x,\infty)\widehat{\chi}_j(x,y)\big|_{Y_1},
$$

which implies

$$
\nabla_y\widehat{u}_1 = \nabla_y\sum_{j=1}^N \frac{\partial G_1}{\partial x_j}(x,\infty)\widehat{\chi}_j(x,y)\big|_{Y_1} = \sum_{j=1}^N \frac{\partial G_1}{\partial x_j}(x,\infty)\nabla_y\widehat{\chi}_j(x,y)\big|_{Y_1}.
$$

This along with (5.2), (5.3), Theorem 2.5 (ii), linearity of $\mathcal{U}_1^\varepsilon$, triangle inequality, and Theorem 2.5 (i), yield

$$\begin{aligned}
&\|\nabla u_1^{\delta,\varepsilon}\mathbf{1}_{\Omega_1^{\sqrt{\delta},\varepsilon}} - \nabla u_1 - \sum_{j=1}^N \mathcal{U}_1^\varepsilon\Big(\frac{\partial G_1}{\partial x_j}(x,\infty)\Big)\mathcal{U}_1^\varepsilon\big(\nabla_y \widehat{\chi}_j(x,y)\big|_{Y_1}\big)\|_{L^2(\Omega_1^{\delta,\varepsilon})}\\
&= \|\nabla u_1^{\delta,\varepsilon}\mathbf{1}_{\Omega_1^{\sqrt{\delta},\varepsilon}} - \nabla u_1 - \mathcal{U}_1^\varepsilon(\nabla_y \widehat{u}_1)\|_{L^2(\Omega_1^{\delta,\varepsilon})}\\
&\le \|\nabla u_1^{\delta,\varepsilon}\mathbf{1}_{\Omega_1^{\sqrt{\delta},\varepsilon}} - \mathcal{U}_1^\varepsilon(\nabla u_1) - \mathcal{U}_1^\varepsilon(\nabla_y \widehat{u}_1)\|_{L^2(\Omega_1^{\delta,\varepsilon})} + \|\mathcal{U}_1^\varepsilon(\nabla u_1) - \nabla u_1\|_{L^2(\Omega_1^{\delta,\varepsilon})} \to 0,
\end{aligned}$$

which implies (5.14) for $i=1$. Meanwhile, from (4.12) and (4.46), we have

$$\widehat{u}_2 = \widehat{u}\big|_{\Omega\times Y_2} + y_\gamma \nabla u_1 = \sum_{j=1}^N \frac{\partial G_1}{\partial x_j}(x,\infty)\widehat{\chi}_j(x,y)\big|_{Y_2} + y_\Gamma \nabla u_1,$$

where y_Γ is given in (4.13). Moreover, a number of computations yield $\nabla_y(y_\gamma \nabla u_1) = \nabla u_1$. This implies

$$\begin{aligned}
\nabla_y \widehat{u}_2 &= \nabla_y\Big(\sum_{j=1}^N \frac{\partial G_1}{\partial x_j}(x,\infty)\widehat{\chi}_j(x,y)\big|_{Y_2} + y_\Gamma \nabla u_1\Big)\\
&= \sum_{j=1}^N \frac{\partial G_1}{\partial x_j}(x,\infty)\nabla_y \widehat{\chi}_j(x,y)\big|_{Y_2} + \nabla u_1.
\end{aligned}$$

This along with (5.2), (5.4), Theorem 2.5 (ii), linearity of $\mathcal{U}_2^\varepsilon$, triangle inequality, and Theorem 2.5 (i) yield

$$\begin{aligned}
&\|\nabla u_2^{\delta,\varepsilon}\mathbf{1}_{\Omega_2^{\sqrt{\delta},\varepsilon}} - \nabla u_1 - \sum_{j=1}^N \mathcal{U}_2^\varepsilon\Big(\frac{\partial G_1}{\partial x_j}(x,\infty)\Big)\mathcal{U}_2^\varepsilon\big(\nabla_y \widehat{\chi}_j(x,y)\big|_{Y_2}\big)\|_{L^2(\Omega_2^{\delta,\varepsilon})}\\
&= \|\nabla u_2^{\delta,\varepsilon}\mathbf{1}_{\Omega_2^{\sqrt{\delta},\varepsilon}} - \nabla u_1 - \mathcal{U}_2^\varepsilon(\nabla_y \widehat{u}_2 - \nabla u_1)\|_{L^2(\Omega_2^{\delta,\varepsilon})}\\
&\le \|\nabla u_2^{\delta,\varepsilon}\mathbf{1}_{\Omega_2^{\sqrt{\delta},\varepsilon}} - \mathcal{U}_2^\varepsilon(\nabla_y \widehat{u}_2)\|_{L^2(\Omega_2^{\delta,\varepsilon})} + \|\mathcal{U}_2^\varepsilon(\nabla u_1) - \nabla u_1\|_{L^2(\Omega_2^{\delta,\varepsilon})} \to 0,
\end{aligned}$$

which shows (5.14) for $i=2$.

Let us prove (5.15). Indeed, from (3.11), (A3), and (5.5) we have by unfolding

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\|\nabla u_i^{\delta,\varepsilon}\|^2_{L^2(\Omega_i^{\delta,\varepsilon}\setminus\Omega_i^{\sqrt{\delta},\varepsilon})}\\
&= \lim_{\varepsilon\to 0}\Big(\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\Big)^2 \frac{1}{|Y|}\int_{\Omega\times\frac{1}{\delta}Y_i^\delta} \nabla_z \mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})\nabla_z \mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})\,dx\,dz\\
&= \frac{k_1^2}{|Y|}\int_{\Omega\times\mathbb{R}^N} |\nabla_z G_1|^2\,dx\,dz\\
&= \frac{k_1^2}{|Y|}\|\nabla_z G_1\|^2_{L^2(\Omega\times\mathbb{R}^N)}.
\end{aligned}$$

Finally, we prove (5.16). Let ω be an open and bounded set and choose $R>0$ such that $\omega\cup B\subset \mathbf{B}(O,R)$, the ball in $\mathbb{R}^N$ with center at O of radius R. In view of (4.28) and (4.29), a Poincaré inequality holds on the space $\mathbf{B}(O,R)$. By Definition 3.4 and since $\widehat{\Omega}_i^{\delta,\varepsilon} = \Omega_i^{\delta,\varepsilon}\setminus\Lambda_i^{\delta,\varepsilon}$, we obtain

$$\|\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) - G_1\|^2_{L^2(\widehat{\Omega}_i^{\delta,\varepsilon}\times\mathbf{B}(O,R))}$$

$$
\begin{aligned}
&= \|\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) - G_1\|^2_{L^2(\Omega_i^{\delta,\varepsilon}\times \mathbf{B}(O,R))} - \|\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) - G_1\|^2_{L^2(\Lambda_i^{\delta,\varepsilon}\times \mathbf{B}(O,R))}\\
&\leq C\|\nabla_z \mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) - \nabla_z G_1\|^2_{L^2(\Omega_i^{\delta,\varepsilon}\times \mathbf{B}(O,R))} - C\|\nabla_z \mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon})\\
&\quad - \nabla_z G_1\|^2_{L^2(\Lambda_i^{\delta,\varepsilon}\times \mathbf{B}(O,R))} + \|\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) - G_1\|^2_{L^2(\Lambda_i^{\delta,\varepsilon}\times \mathbf{B}(O,R))}\\
&\leq C\Big(\|\nabla_z \mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) - \nabla_z G_1\|^2_{L^2(\Omega_i^{\delta,\varepsilon}\times \mathbf{B}(O,R))} + \|\nabla_z G_1\|^2_{L^2(\Lambda_i^{\delta,\varepsilon}\times \mathbf{B}(O,R))}\\
&\quad + \|G_1\|^2_{L^2(\Lambda_i^{\delta,\varepsilon}\times \mathbf{B}(O,R))}\Big),
\end{aligned}
$$

where C is a generic constant.

For δ small enough, $\omega \subset \mathbf{B}(O,R) \subset \mathbb{R}^N$. This and when (5.5) is applied to the first term in the right-hand side, and (4.29) to the remaining two terms, then the left-hand side above approaches zero and so we obtain

$$
\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \to G_1 \quad \text{strongly in } L^2(\Omega\times\omega),
$$

which yields (5.16). □

5.2. **Case $\boldsymbol{\gamma \in (-1,0)}$.** For this case, we start by giving the energy convergence.

Theorem 5.4. *Let $\gamma \in (-1,0)$. Under the assumptions of Theorems 4.11 and 4.12, one has*

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{i=1}^{2}\int_{\Omega_i^{\delta,\varepsilon}} A^\varepsilon \nabla u_i^{\delta,\varepsilon}\nabla u_i^{\delta,\varepsilon}\,dx\\
&= \int_\Omega A^{\rm hom}\nabla u_1\nabla u_1\,dx + k_1^2\int_\Omega \Theta u_1^2\,dx\\
&= \frac{1}{|Y|}\int_{\Omega\times Y} A(x,y)(\nabla G_1(\cdot,\infty)+\nabla_y\widehat{u})(\nabla G_1(\cdot,\infty)+\nabla_y\widehat{u})\,dx\,dy\\
&\quad + \frac{k_1^2}{|Y|}\int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} F(x,z)\nabla_z G_1(x,z)\nabla_z G_1(x,z)\,dx\,dz,
\end{aligned}
\tag{5.17}
$$

$$
\lim_{\varepsilon\to 0}\sum_{i=1}^{2}\int_{\Lambda_i^{\delta,\varepsilon}} |\nabla u_i^{\delta,\varepsilon}|^2 dx = 0. \tag{5.18}
$$

Moreover, we have the following strong convergences hold

$$
\mathcal{T}_1^\varepsilon(\nabla u_1^{\delta,\varepsilon})\mathbf{1}_{\Omega\times Y_1^\delta} \to \nabla u_1 + \nabla_y\widehat{u}_1 \quad \textit{strongly in } L^2(\Omega\times Y_1)^N, \tag{5.19}
$$

$$
\mathcal{T}_2^\varepsilon(\nabla u_2^{\delta,\varepsilon})\mathbf{1}_{\Omega\times Y_2^\delta} \to 0 \quad \textit{strongly in } L^2(\Omega\times Y_2)^N, \tag{5.20}
$$

$$
\nabla_z \mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}))1_{\frac{1}{\delta}Y_1^\delta} \to \nabla_z G_1 \quad \textit{strongly in } L^2(\Omega\times\mathbb{R}^N)^N, \tag{5.21}
$$

$$
\nabla_z \mathcal{T}_2^{\delta,\varepsilon}(u_2^{\delta,\varepsilon}))1_{\frac{1}{\delta}Y_2^\delta} \to 0 \quad \textit{strongly in } L^2(\Omega\times\mathbb{R}^N)^N. \tag{5.22}
$$

Proof. For this case, the proof of (5.19)–(5.22) is similar to that of Theorem 5.2, the difference being convergences (5.20) and (5.22) which are immediate from (4.55) and (4.57). □

The following result is proved similarly to the one in Corollary 5.3.

Corollary 5.5. *Under the assumptions of Theorem 5.4, we have the following corrector results:*

$$\begin{gathered}
\|\nabla u_1^{\delta,\varepsilon}\mathbf{1}_{\Omega_i^{\sqrt{\delta},\varepsilon}} - \nabla u_1 \\
- \sum_{j=1}^N \mathcal{U}_1^\varepsilon\Big(\frac{\partial G_1}{\partial x_j}(x,\infty)\Big)\mathcal{U}_1^\varepsilon\big(\nabla_y \widehat{\chi}_j(x,y)\big|_{Y_i}\big)\|_{L^2(\Omega_1^{\delta,\varepsilon})} \to 0, \\
\|\nabla u_1^{\delta,\varepsilon}\|_{L^2(\Omega_i^{\delta,\varepsilon}\setminus\Omega_i^{\sqrt{\delta},\varepsilon})} \to \frac{k_1}{|Y|^{1/2}}\|\nabla_z G_1\|_{L^2(\Omega\times\mathbb{R}^N)}, \\
\|\nabla u_2^{\delta,\varepsilon}\|_{L^2(\Omega_2^{\delta,\varepsilon})} \to 0.
\end{gathered}$$

Moreover,

$$\begin{gathered}
\mathcal{T}_1^{\delta,\varepsilon}(u_1^{\delta,\varepsilon}) \to G_1 \quad \textit{strongly in } L^2(\Omega; L^2_{\rm loc}(\mathbb{R}^N)), \\
\mathcal{T}_2^{\varepsilon}(u_2^{\delta,\varepsilon}) \to u_1 \quad \textit{strongly in } L^2(\Omega, H^1(Y_2)).
\end{gathered}$$

5.3. **Case $\boldsymbol{\gamma = -1}$.** The proofs for the next results are similar to the previous cases with appropriate modifications.

Theorem 5.6. *Let $\gamma=-1$. Under the assumptions of Theorems 4.13 and 4.14, one has*

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_{i=1}^2 \int_{\Omega_i^{\delta,\varepsilon}} A^\varepsilon \nabla u_i^{\delta,\varepsilon}\nabla u_i^{\delta,\varepsilon}\,dx \\
&= \int_\Omega A^{\rm hom}\nabla u_1\nabla u_1\,dx + k_1^2\int_\Omega \Theta u_1^2\,dx \\
&= \frac{1}{|Y|}\int_{\Omega\times Y} A(x,y)(\nabla G_1(\cdot,\infty)+\nabla_y\widehat{u})(\nabla G_1(\cdot,\infty)+\nabla_y\widehat{u})\,dx\,dy \\
&\quad + \frac{k_1^2}{|Y|}\int_{\Omega\times(\mathbb{R}^N\setminus\overline{B})} F(x,z)\nabla_z G_1(x,z)\nabla_z G_1(x,z)\,dx\,dz,
\end{aligned}$$

$$\lim_{\varepsilon\to 0}\sum_{i=1}^2\int_{\Lambda_i^{\delta,\varepsilon}} |\nabla u_i^{\delta,\varepsilon}|^2 dx = 0.$$

Moreover, we have the strong convergences for $i=1,2$,

$$\begin{gathered}
\mathcal{T}_1^\varepsilon(\nabla u_1^{\delta,\varepsilon})\mathbf{1}_{\Omega\times Y_1^\delta} \to \nabla u_1 + \nabla_y\widehat{u}_1 \quad \textit{strongly in } L^2(\Omega\times Y_1)^N, \\
\mathcal{T}_2^\varepsilon(\nabla u_2^{\delta,\varepsilon})\mathbf{1}_{\Omega\times Y_2^\delta} \to \nabla_y\widehat{u}_2 \quad \textit{strongly in } L^2(\Omega\times Y_2)^N, \\
\nabla_z\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}))1_{\frac{1}{\delta}Y_i^\delta} \to \nabla_z G_1 \quad \textit{strongly in } L^2(\Omega\times\mathbb{R}^N)^N.
\end{gathered}$$

Corollary 5.7. *Under the assumptions of Theorem 5.6, we have the corrector results for $i=1,2$,*

$$\begin{gathered}
\|\nabla u_i^{\delta,\varepsilon}\mathbf{1}_{\Omega_i^{\sqrt{\delta},\varepsilon}} - \nabla u_1 \\
- \sum_{j=1}^N \mathcal{U}_i^\varepsilon\Big(\frac{\partial G_1}{\partial x_j}(x,\infty)\Big)\mathcal{U}_i^\varepsilon\big(\nabla_y \widehat{\chi}_j(x,y)\big|_{Y_i}\big)\|_{L^2(\Omega_i^{\delta,\varepsilon})} \to 0, \\
\|\nabla u_i^{\delta,\varepsilon}\|_{L^2(\Omega_i^{\delta,\varepsilon}\setminus\Omega_i^{\sqrt{\delta},\varepsilon})} \to \frac{k_1}{|Y|^{1/2}}\|\nabla_z G_1\|_{L^2(\Omega\times\mathbb{R}^N)}.
\end{gathered}$$

Furthermore, for $i = 1, 2$,

$$\mathcal{T}_i^{\delta,\varepsilon}(u_i^{\delta,\varepsilon}) \to G_1 \quad \textit{strongly in } L^2(\Omega; L^2_{\text{loc}}(\mathbb{R}^N)).$$

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