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ASYMPTOTIC STABILIZATION FOR BRESSE TRANSMISSION SYSTEMS WITH FRACTIONAL DAMPING

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ABSTRACT. In this article, we study the asymptotic stability of Bresse transmission systems with two fractional dampings. The dissipation mechanism of control is given by the fractional damping term and acts on two equations. The relationship between the stability of the system, the fractional damping index $\theta \in [0,1]$ and the different wave velocities is obtained. By using the semigroup method, we obtain the well-posedness of the system. We also prove that when the wave velocities are unequal or equal with $\theta \neq 0$, the system is not exponential stable, and it is polynomial stable. In addition, the precise decay rate is obtained by the multiplier method and the frequency domain method. When the wave velocities are equal with $\theta = 0$, the system is exponential stable.

1. INTRODUCTION

In the previous decades, various types of equations models have been used to describe chemical, biological, physical, and engineering systems. In recent years, the mathematical model of arc-shaped elastic structures has been greatly promoted by more and more practical problems, and arc-shaped elastic structures are also widely studied in the fields of ocean, engineering, aviation, architecture and so on. Following the main idea of the deformation of elastic structures, we consider the circular arch problem given by the equations of motion, also known as the Bresse system (see [28] for details),

$$\rho_1\varphi_{tt} = Q_x + lN, \tag{1.1}$$
$$\rho_2\psi_{tt} = M_x - Q, \tag{1.2}$$
$$\rho_1 w_{tt} = N_x - lQ, \tag{1.3}$$

where

$$N = \kappa_0 l(w_x - l\varphi), \tag{1.4}$$
$$Q = \kappa(\varphi_x + \psi + lw), \tag{1.5}$$
$$M = b\psi_x, \tag{1.6}$$

are the stress-strain relations for elastic behavior. Here $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\kappa = k'GA$, $\kappa_0 = EA$, $b = EI$, $l = R^{-1}$. Here ρ is the material density, E is the elastic

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modulus, G is the shear modulus, and k' is the shear coefficient, A is the cross-sectional area, I is the area of the cross-sectional second moment, and R is the radius of curvature. These coefficients are normal numbers related to the physical properties of the beam. Functions φ, ψ and w denote vertical, shear angular and longitudinal displacement.

In this article, we are interested in the asymptotic stability of Bresse systems (from coupled equations (1.1)-(1.6) whose dampings are given by fractional damping terms, and act on two equations respectively. The system is written as

$$\begin{gathered}\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi + lw)_x - \kappa_0 l(w_x - l\varphi) + \gamma_1(-\partial_{xx})^\theta \varphi_t = 0 \quad \text{in } (0,L)\times\mathbb{R}^+,\\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + lw) + \gamma_2(-\partial_{xx})^\theta \psi_t = 0 \quad \text{in } (0,L)\times\mathbb{R}^+,\\ \rho_1 w_{tt} - \kappa_0(w_x - l\varphi)_x + \kappa l(\varphi_x + \psi + lw) + \gamma_3(-\partial_{xx})^\theta w_t = 0 \quad \text{in } (0,L)\times\mathbb{R}^+,\end{gathered} \tag{1.7}$$

where θ is a parameter in the interval $[0,1]$ and damping coefficient $\gamma_i \ge 0$, $i = 1,2,3$. We consider the Dirichlet-Neumann-Neumann boundary conditions

$$\varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = w_x(0,t) = w_x(L,t) = 0 \quad \text{in } \mathbb{R}^+, \tag{1.8}$$

and the initial conditions

$$\begin{aligned} \varphi(x,0) &= \varphi_0, \quad \varphi_t(x,0) = \varphi_1 \quad \text{in } (0,L),\\ \psi(x,0) &= \psi_0, \quad \psi_t(x,0) = \psi_1 \quad \text{in } (0,L),\\ w(x,0) &= w_0, \quad w_t(x,0) = w_1 \quad \text{in } (0,L).\end{aligned} \tag{1.9}$$

This fractional damping is an intermediate dissipation mechanism not previously considered in Bresse systems. In special cases, the mechanism includes friction damping ($\theta = 0$) and Kelvin-Voigt damping ($\theta = 1$). In books [13, 25] we find the following definition of fractional order operators: For $\alpha > 0$, the bounded linear operator $A^{-\alpha}$ is defined by

$$A^{-\alpha} := \frac{1}{2\pi i}\int_\gamma \lambda^{-\alpha}(\lambda I - A)^{-1} d\lambda,$$

where γ is a piecewise smooth path in $\Sigma\ \mathbb{R}^+$ going from $\infty e^{-i\delta}$ to $\infty e^{i\delta}$ for some $\delta > 0$. We refer to [13, 25] for other relevant results on fractional powers.

First, we introduce some relevant results that motivated this work. To stabilize the Bresse system, various kinds of damping are used and some decay results are established. From a large number of literature, three basic damping mechanisms can be distinguished, namely friction damping, Kelvin-Voigt damping and damping with memory. By comparison, the friction damping term is relatively simple, and the study of local Kelvin-Voigt damping is too much, while the damping with memory is more complex, because the damping term is represented by various forms of convolution products of the kernel. In the following content, we will briefly introduce the asymptotic stability of Bresse system under these three damping mechanisms.

Guesmia [22] studied that when the friction damping only acts on a vertical displacement, under the Dirichlet-Neumann-Neumann boundary conditions, if

$$l \ne m\pi, \quad \forall m \in \mathbb{Z}, \tag{1.10}$$

the system is not exponentially stable. If (1.10) holds and

$$l^2 \ne \frac{\kappa_0\rho_2 - b\rho_1}{\kappa_0\rho_2}(m\pi^2) - \frac{\kappa\rho_1}{\rho_2(\kappa+\kappa_0)}, \quad \forall m \in \mathbb{Z}, \tag{1.11}$$

the system is polynomial attenuated (the meaning of coefficient is consistent with that in this paper). Alabau Boussouira et al. studied in [5] that when the friction damping only acts on the shear angular displacement, under the complete Dirichlet boundary conditions, its stability is related to wave velocity $\frac{\kappa}{\rho_1}$, $\frac{b}{\rho_2}$ and $\frac{\kappa_0}{\rho_1}$. Denote the difference of wave velocity

$$\chi_0 = \frac{\kappa}{\rho_1} - \frac{b}{\rho_2} \quad \text{and} \quad \chi_1 = \kappa - \kappa_0. \tag{1.12}$$

when the wave velocity is equal, i.e. $\chi_0, \chi_1 = 0$, the system is decay exponentially. When the wave velocity is not equal, i.e. $\chi_0 \neq 0, \chi_1 \neq 0$ or $\chi_0 \neq 0, \chi_1 = 0$, the system attenuates in polynomial form of $t^{-1/6}$ or $t^{-1/3}$. The optimality of polynomial decay is proved in [16]. Under the Dirichlet-Neumann-Neumann boundary conditions, if $\chi_0 \neq 0$ or $\chi_1 = 0$, the system is not exponentially stable. Finally, numerical analysis is given to verify their conclusions. Afilal et al. [3] obtained that when the friction damping only acts on the longitudinal displacement, under the mixed boundary conditions

$$\varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \varphi_x(L,t) = \psi(L,t) = w(L,t) = 0 \quad \text{in } \mathbb{R}^+,$$

if $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2} = \frac{\kappa_0}{\rho_1}$,

$$l \neq \frac{\pi}{2} + m\pi, \quad \forall m \in \mathbb{Z}, \tag{1.13}$$

and

$$l^2 \neq \frac{\kappa_0\rho_2 + b\rho_1}{\kappa_0\rho_2}\big(\frac{\pi}{2} + m\pi\big) - \frac{\kappa\rho_1}{\rho_2(\kappa+\kappa_0)}, \quad \forall m \in \mathbb{Z}, \tag{1.14}$$

the system is exponentially stable. If only (1.13) and (1.14) hold, the system is polynomial decay. The numerical analysis is also given.

When there are two friction damping in the system, Alves et al. [2] proved that if there is no friction damping on the longitudinal displacement, the system is exponentially stable under the Dirichlet-Neumann-Neumann boundary conditions when $\chi_1 = 0$, and the system is non exponentially stable when $\chi_1 \neq 0$. At the same time, they also proved that the decay is polynomial at the optimal rate $t^{-1/2}$. Wehbe et al. [40] got that when there are two locally distributed feedbacks on the shear angular displacement and longitudinal displacement,

$$\begin{gathered}
\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi + lw)_x - \kappa_0 l(w_x - l\varphi) = 0 \quad \text{in } (0,L)\times\mathbb{R}^+,\\
\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + lw) + a_1(x)\psi_t = 0 \quad \text{in } (0,L)\times\mathbb{R}^+,\\
\rho_1 w_{tt} - \kappa_0(w_x - l\varphi)_x + \kappa l(\varphi_x + \psi + lw) + a_1(x)w_t = 0 \quad \text{in } (0,L)\times\mathbb{R}^+,
\end{gathered}$$

where the positive continuous functions $a_j(x)$, $j = 1,2$ satisfy the conditions

$$a_j(x) \geq a_- > 0 \quad \text{for every } x \in \Theta := (0,c)\cup(d,L), \quad 0 < c < d < L.$$

It turned out that under the Dirichlet-Neumann-Neumann boundary conditions, the system is exponentially stable when $\chi_0 = 0$. When $\chi_0 \neq 0$, then for any positive integer $m \geq 1$, there exists a constant $C_m > 0$ independent of initial value $U_0 \in D(\mathcal{A}_j^m)$, $j = 1,2$ such that

$$\|S_j(t)U_0\|^2_{\mathcal{H}_j} \leq C_m\big(\frac{\ln t}{t}\big)^m \ln^2 t\|U_0\|^2_{D(\mathcal{A}_j^m)}, \quad \forall t > 0.$$

For the Kelvin-Voigt damping system, Akil [1] studied the stability of Bresse system with only one discontinuous local Kelvin-Voigt damping on the axial force:

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi + lw)_x - \kappa_0 l(w_x - l\varphi) - ld(x)(w_{tx} - l\varphi_t) = 0 \quad \text{in } (0,L)\times\mathbb{R}^+,$$
$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + lw) = 0 \quad \text{in } (0,L)\times\mathbb{R}^+,$$
$$\rho_1 w_{tt} - [\kappa_0(w_x - l\varphi)_x + d(x)(w_{tx} - l\varphi_t)]_x + \kappa l(\varphi_x + \psi + lw) = 0 \quad \text{in } (0,L)\times\mathbb{R}^+.$$

Suppose that there exists $0 < \alpha < \beta < L$ and a positive constant d_0 such that

$$d(x) = \begin{cases} d_0 & \text{if } x \in (\alpha,\beta), \\ 0, & \text{if } x \in (0,\alpha)\cup(\beta,L), \end{cases}$$

and under the complete Dirchlet boundary conditions, they proved that whether the wave velocities are equal or not, the system exhibits polynomial decay. When $\chi_0 = 0$, the decay rate is t^{-1}. When $\chi_0 \neq 0$, the decay rate is $t^{-1/2}$. For other results on friction damping and Kelvin-Voigt damping, see [4, 14, 37] and their references.

Recently some scholars have also studied the stability of Bresse systems whose damping term is dissipated through memory. When the memory terms of the three equations exist simultaneously, it has the following form

$$\int_0^\infty g(s)\varphi_{xx}(x,t-s)ds, \quad \int_0^\infty g(s)\psi_{xx}(x,t-s)ds, \quad \int_0^\infty g(s)w_{xx}(x,t-s)ds,$$

where $g:\mathbb{R}_+ \to \mathbb{R}_+$ is differentiable, non-increasing and integrable function on $\mathbb{R}_+$. Guesmia and Kafini [23] (three infinite memories), Guesmia and Kirane [24] (two infinite memories), Guesmia [19] (one infinite memory only acts on the longitudinal displacement) and De Lima Santos et al. [36] (one infinite memory only acts on the shear angular displacement) obtained the asymptotic stability of the one-dimensional linear Bresse system under infinite memory, respectively. When the kernel function decays exponential at infinity, if the wave propagation velocity is the same, the exponential stability of the corresponding systems is obtained in these papers, otherwise it will lead to polynomial stability with decay rate $t^{-1/2}$. Guesmia in [20] studied that an infinite memory only acts on the vertical displacement, they proved that even if the wave propagation velocity is the same and the kernel function has exponential decay at infinity, the exponential stability is not tenable, but it decays as a polynomial with $t^{-1/4}$. In addition, the authors in [7] considered the stability of Bresse system with memory term acting on shear angular displacement under arbitrary growth of the relaxation function at infinity. They not only proved that the system is well-posedness, but also presented two general decay estimates: a uniform stability estimate under (1.12) and another general weak stability result. Some other authors have also considered the others dissipation mechanisms in Bresse systems, such as thermoelastic Bresse systems, (see [15, 21, 26, 30]).

It is well known that the Bresse system evolved from the Timoshenko beam equation. If $R \to \infty$ and $g = 0$, then $l \to 0$ (see above for specific physical meanings), the model is simplified to Timoshenko beam equation (see [18]). If $R \to \infty$ and $g \neq 0$, then $l \to 0$, the model is simplified to Timoshenko beam equation with past history (see [31]). It is noteworthy that Higidio Portillo Oquendo et al. [32] dealt with the asymptotic behavior of the solution for a Timoshenko system with a fractional damping,

$$\begin{gathered}\rho_1\phi_{tt} - \kappa(\phi_{xx} + \psi_x) = 0 \quad \text{in } (0, L) \times \mathbb{R}^+, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\phi_x + \psi) + (-\partial_{xx})^\theta \psi_t = 0 \quad \text{in } (0, L) \times \mathbb{R}^+,\end{gathered}$$

satisfying the boundary conditions

$$\phi(0, t) = \phi(L, t) = \psi_x(0, t) = \psi_x(L, t) = 0 \quad \text{on } \mathbb{R}^+.$$

Here, the parameter $\theta \in [0, 1]$, the damping only acts on one equation of the system, and the exponentially decreasing kernel is considered. The authors obtained the exact decay rate, which depends on the difference of the propagation speeds of the two waves. To be precise, when the equations have different propagation speeds, if $\theta \le 1/2$ then the system decays polynomially with rate $t^{-1/(2-2\theta)}$, if $\theta \ge 1/2$ then the system decays polynomially with rate $t^{-1/(2\theta)}$; when the equations have the same propagation speed and $\theta \in (0, 1]$, the system decays polynomially with rate $t^{-1/(2\theta)}$, and these decay rates are optimal; when $\theta = 0$ and the equations have the same propagation speed, the exponential decay of the system is obtained. Furthermore, Astudillo and Oquendo [6] studied the stability of the Timoshenko beam equation with fractional memory term under the exponentially decreasing kernels. The relationship between stability, the wave velocity and the fractional damping exponent is studied by using semi group method, and obtained the corresponding exponential stability and precise polynomial decay rates.

The stability of some other Bresse systems with fractional derivatives have also been studied. In [9], Oquendo and Suárez introduced two internal damping terms expressed by the generalized Caputo fractional derivative and studied the asymptotic stability of the following viscoelastic Bresse systems,

$$\begin{gathered}\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi + lw)_x - \kappa_0 l(w_x - l\varphi) + a_1(x)\partial_t^{\alpha,\eta}\varphi = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^\infty g(s)\psi_{xx}(t - s)ds + \kappa(\varphi_x + \psi + lw) + a_2(x)\partial_t^{\beta,\eta}\psi = 0, \\ \rho_1 w_{tt} - \kappa_0(w_x - l\varphi)_x + \kappa l(\varphi_x + \psi + lw) = 0,\end{gathered}$$

in $(0, L) \times \mathbb{R}^+$, where the symbol $\partial_t^{\alpha,\eta}$ (or $\partial_t^{\beta,\eta}$) refers to the generalized Caputo fractional derivative corresponding to the time variable t of order α (or β) and it is expressed for the order α by

$$\partial_t^{\alpha,\eta} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t - s)^{-\alpha} e^{-\eta(t-s)} \frac{df}{ds}(s)ds.$$

The authors not only proved the strong stability, lack of exponential stability and polynomial stability of the system, but also gave an accurate decay rate (see Theorem 3.8 of [9] for details), and also used numerical simulation to verify their results.

Earlier, Benaissa and Kasmi [8] considered the Bresse system with three control boundary conditions of fractional derivative type, and they obtained the polynomial decay result. There are many studies on fractional damping. For other types of references, the readers can see [11, 12, 27, 33, 38, 39] and the references therein.

Inspired by these works, we study the asymptotic behavior of the Bresse system (1.7)-(1.9). Firstly we introduce some notation. For $1 \le p \le \infty$, $L^p := L^p(0, L)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. For the convenience of notation, we will use $\|\cdot\|$ instead of $\|\cdot\|_{L^2}$ and $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{L^2}$. Let s be a nonnegative number, $H^s := H^s(0,L)$ denotes the usual Sobolev space, equipped with the norm $\|\cdot\|_{H^s}$. In the following, C denotes a generic positive constant.

The set

$$L^2_*(0,L) := \big\{h \in L^2(0,L) : \int_0^L h(x)dx = 0\big\}$$

is a closed subspace with the L^2-norm, therefore it is a Hilbert space. As we know, the operators

$$E := -\partial_{xx} : D(E) \subset L^2(0,L) \to L^2(0,L), \tag{1.15}$$

$$E_* := -\partial_{xx} : D(E_*) \subset L^2_*(0,L) \to L^2_*(0,L), \tag{1.16}$$

with respective domains

$$D(E) =: H^2(0,L) \cap H^1_0(0,L),$$

$$D(E_*) := \big\{\psi, w \in H^2(0,L) \cap L^2_*(0,L) : \psi_x(0) = \psi_x(L) = 0, w_x(0) = w_x(L) = 0\big\},$$

are positive, self-adjoint and have compact inverse. Therefore, the operators E^σ , E^σ_* are bounded for $\sigma \le 0$, and positive self-adjoint for $\sigma \in \mathbb{R}$. Furthermore, the embeddings

$$D(E^{\sigma_1}) \hookrightarrow D(E^{\sigma_2}), \quad D(E^{\sigma_1}_*) \hookrightarrow D(E^{\sigma_2}_*)$$

are continuous for $\sigma_1 > \sigma_2$. The norms in $D(E^\sigma)$ and $D(E^\sigma_*)$ for $\sigma \ge 0$ are given by $\|\varphi\|_{D(E^\sigma)} := \|E^\sigma\varphi\|$, $\|\psi\|_{D(E^\sigma_*)} := \|E^\sigma_*\psi\|$, and $\|w\|_{D(E^\sigma_*)} := \|E^\sigma_* w\|$ respectively.

Because the operators E and E_* are positive, self-adjoint and they have compact inverse, the spectrum of these operators is constituted only by positive eigenvalues. The eigenvalues for both operators are given by ξ_n^2, where

$$\xi_n = \frac{n\pi}{L}, \quad n \in \mathbb{N},$$

and the corresponding unitary eigenfunctions associated to these eigenvalues are

$$e_n(x) = \sqrt{\frac{2}{L}}\sin(\xi_n x), \quad e^*_n(x) = \sqrt{\frac{2}{L}}\cos(\xi_n x). \tag{1.17}$$

The sequences $\{e_n\}$ and $\{e^*_n\}$ form the bases of the spaces $L^2(0,L)$ and $L^2_*(0,L)$ respectively, then for $\varphi \in L^2(0,L)$ and $\psi, w \in L^2_*(0,L)$ we have

$$\varphi = \sum_{n=1}^\infty \langle\varphi, e_n\rangle e_n, \quad \psi = \sum_{n=1}^\infty \langle\psi, e^*_n\rangle e^*_n, \quad w = \sum_{n=1}^\infty \langle w, e^*_n\rangle e^*_n.$$

Note that, for $\varphi \in D(E^{\sigma+1/2})$, we have the following identities

$$E^{\sigma+1/2}\varphi = \sum_{n=1}^\infty \xi_n^{2\sigma+1}\langle\varphi, e_n\rangle e_n, \quad E^\sigma_*\partial_x\varphi = \sum_{n=1}^\infty \xi_n^{2\sigma+1}\langle\varphi, e_n\rangle e^*_n,$$

by Parseval's identity, we obtain

$$\|E^{\sigma+1/2}\varphi\| = \|E^\sigma_*\partial_x\varphi\|. \tag{1.18}$$

In particular, for $\sigma = 0$ we have $\|E^{1/2}\varphi\| = \|\partial_x\varphi\|$. In a similar way, for $\psi, w \in D(E^{\sigma+1/2}_*)$ it follows that

$$\|E^{\sigma+1/2}_*\psi\| = \|E^\sigma\partial_x\psi\|, \quad \|E^{\sigma+1/2}_* w\| = \|E^\sigma\partial_x w\|. \tag{1.19}$$

At the same time, for $\varphi \in D(E^{\sigma_0})$ and $\psi, w \in D(E^{\sigma_0}_*)$, with $\sigma_0 = \max\{\sigma, 1/2\}$, we easily verify that

$$\langle E^\sigma_*\psi, \varphi_x\rangle = -\langle\psi_x, E^\sigma\varphi\rangle, \quad \langle E^\sigma_* w, \varphi_x\rangle = -\langle w_x, E^\sigma\varphi\rangle. \tag{1.20}$$

Our main results deal with the asymptotic behavior of the solution of this system. The innovation of this paper is to extend the dissipation mechanism of control in some literatures to the case of fractional damping, and is to study the asymptotic stability of the system (1.7)-(1.9) when there are only two fractional damping respectively. It is found that the stability is related to the wave speed (1.12) and the value of $\theta \in [0,1]$. These results are clarified in Theorems 4.11, which are mainly stated as follows:

(i) When fractional damping acts on vertical displacement and shear angular displacement, that is, $\gamma_1 > 0, \gamma_2 > 0, \gamma_3 = 0$, if $\chi_1 = 0$ and $\theta \in (0,1]$, then the semigroup $e^{t\mathcal{A}}$ decays polynomially with rate $t^{-1/(2\theta)}$, when $\theta = 0$, then the semigroup $e^{t\mathcal{A}}$ decay exponentially; if $\chi_1 \neq 0$ and $\theta \leq 1/2$, then the semigroup $e^{t\mathcal{A}}$ decays polynomially with rate $t^{-1/(2-2\theta)}$, when $\theta \geq 1/2$, then the semigroup $e^{t\mathcal{A}}$ decays polynomially with rate $t^{-1/(2\theta)}$.

(ii) When fractional damping acts on vertical displacement and longitudinal displacements, that is, $\gamma_1 > 0, \gamma_3 > 0, \gamma_2 = 0$, if $\chi_0 = 0$ and $\theta \in (0,1]$, then the semigroup $e^{t\mathcal{A}}$ decays polynomially with rate $t^{-1/(2\theta)}$, when $\theta = 0$, then the semigroup $e^{t\mathcal{A}}$ decay exponentially; if $\chi_0 \neq 0$ and $\theta \leq 1/2$, then the semigroup $e^{t\mathcal{A}}$ decays polynomially with rate $t^{-1/(2-2\theta)}$, when $\theta \geq 1/2$, then the semigroup $e^{t\mathcal{A}}$ decays polynomially with rate $t^{-1/(2\theta)}$.

(iii) When fractional damping acts on longitudinal displacements and shear angular displacement, that is, $\gamma_2 > 0, \gamma_3 > 0, \gamma_1 = 0$, if $\chi_0 = 0$ and $\theta \in (0,1]$, then the semigroup $e^{t\mathcal{A}}$ decays polynomially with rate $t^{-1/(2\theta)}$, when $\theta = 0$, then the semigroup $e^{t\mathcal{A}}$ decay exponentially; if $\chi_0 \neq 0$ and $\theta \leq 1/2$, then the semigroup $e^{t\mathcal{A}}$ decays polynomially with rate $t^{-1/(2-2\theta)}$, when $\theta \geq 1/2$, then the semigroup $e^{t\mathcal{A}}$ decays polynomially with rate $t^{-1/(2\theta)}$.

The outline of this article is the followings. In section 2 we study the well-posedness result of solution to system (1.7)-(1.9). In section 3, we prove the case of lack of exponential stability. In section 4, we give the asymptotic behavior of the corresponding semigroups, including exponential stability and polynomial stability, and precise decay rates are obtained.

2. Well-posedness of solution

In this section, we use the semigroup theory to obtain the existence and uniqueness of solution for system (1.7)-(1.9). We denote the state space by

$$\mathcal{H} := H_0^1(0,L) \times L^2(0,L) \times H_*^1(0,L) \times L_*^2(0,L) \times H_*^1(0,L) \times L_*^2(0,L), \tag{2.1}$$

where $H_*^1(0,L) := H^1(0,L) \cap L_*^2(0,L)$. Note that $\mathcal{H}$ is an Hilbert space with the inner product

$$\begin{aligned}\langle U_1, U_2\rangle_{\mathcal{H}} &= \rho_1\langle \tilde{\varphi}_1, \tilde{\varphi}_2\rangle + \rho_2\langle \tilde{\psi}_1, \tilde{\psi}_2\rangle + \rho_1\langle \tilde{w}_1, \tilde{w}_2\rangle + \kappa_0\langle \partial_x w_1 - l\varphi_1, \partial_x w_2 - l\varphi_2\rangle \\ &\quad + \kappa\langle \partial_x\varphi_1 + \psi_1 + lw_1, \partial_x\varphi_2 + \psi_2 + lw_2\rangle + b\langle \partial_x\psi_1, \partial_x\psi_2\rangle,\end{aligned} \tag{2.2}$$

and the norm

$$\begin{aligned}\|U\|_{\mathcal{H}}^2 &= \rho_1\|\tilde{\varphi}\|^2 + \rho_2\|\tilde{\psi}\|^2 + \rho_1\|\tilde{w}\|^2 + \kappa\|\partial_x\varphi + \psi + lw\|^2 \\ &\quad + \kappa_0\|\partial_x w - l\varphi\|^2 + b\|\partial_x\psi\|^2,\end{aligned} \tag{2.3}$$

where $U_i = (\varphi_i, \tilde{\varphi}_i, \psi_i, \tilde{\psi}_i, w_i, \tilde{w}_i)^T$, $i = 1, 2$.

If we consider the vector $U(t) = (\varphi(t), \tilde{\varphi}(t), \psi(t), \tilde{\psi}(t), w(t), \tilde{w}(t))^T$, then system (1.7)-(1.9) can be written as the Cauchy problem

$$\begin{gathered} \frac{d}{dt}U(t) = \mathcal{A}U(t), \\ U(0) = U_0, \end{gathered} \tag{2.4}$$

where $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1)^T$ is the vector of initial data and the operator $\mathcal{A}$ is given by

$$\mathcal{A}U = \begin{pmatrix} \tilde{\varphi} \\ \frac{\kappa}{\rho_1}(\varphi_{xx} + \psi_x + lw_x) + \frac{\kappa_0 l}{\rho_1}(w_x - l\varphi) - \frac{\gamma_1}{\rho_1}E^\theta\tilde{\varphi} \\ \tilde{\psi} \\ \frac{b}{\rho_2}\psi_{xx} - \frac{\kappa}{\rho_2}(\varphi_x + \psi + lw) - \frac{\gamma_2}{\rho_2}E_*^\theta\tilde{\psi} \\ \tilde{w} \\ \frac{\kappa_0}{\rho_1}(w_{xx} - l\varphi_x) - \frac{kl}{\rho_1}(\varphi_x + \psi + lw) - \frac{\gamma_3}{\rho_1}E_*^\theta\tilde{w} \end{pmatrix}, \tag{2.5}$$

with

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = \Big\{ &U \in \mathcal{H} : \tilde{\varphi} \in H_0^1(0,L),\ \tilde{\psi} \in H_*^1(0,L),\ \tilde{w} \in H_*^1(0,L), \\ &\varphi \in H_0^1(0,L) \cap H^2(0,L),\ \psi, w \in H_*^1(0,L) \cap H^2(0,L), \\ &\kappa E\varphi + \gamma_1 E^\theta\tilde{\varphi} \in L^2(0,L),\ bE_*\psi + \gamma_2 E^\theta\tilde{\psi} \in L_*^2(0,L), \\ &\kappa_0 E_* w + \gamma_3 E_*^\theta \tilde{w} \in L_*^2(0,L) \Big\}. \end{aligned} \tag{2.6}$$

We use the following Lumer-Phillips theorem [34] to prove the existence of solution of Cauchy problem (2.4).

Theorem 2.1 ([34]). *Let $\mathcal{A}$ a linear operator with dense domain $D(\mathcal{A})$ in a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$, then the operator $\mathcal{A}$ is the generator of a C_0-semigroup of contractions on $\mathcal{H}$.*

We give the well-posedness result of solution as the following theorem.

Theorem 2.2. *For $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1)^T \in \mathcal{H}$, there exists a unique solution of Cauchy problem* (2.4)

$$U = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w})^T \in C([0,\infty); \mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, then the solution is more regular, i.e.

$$U = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w})^T \in C([0,\infty); D(\mathcal{A})) \cap C^1([0,\infty); \mathcal{H}).$$

Proof. We prove that the operator $\mathcal{A}$ in (2.5) satisfies the conditions of Theorem 2.1. Firstly, from (2.6) we can obtain that the domain of the operator $\mathcal{A}$ is dense in $\mathcal{H}$. In addition, for any $U = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w})^T \in \mathcal{D}(\mathcal{A})$ we obtain

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle = -\gamma_1\|E^{\theta/2}\tilde{\varphi}\|^2 - \gamma_2\|E_*^{\theta/2}\tilde{\psi}\|^2 - \gamma_3\|E_*^{\theta/2}\tilde{w}\|^2 \le 0. \tag{2.7}$$

Therefore, the operator $\mathcal{A}$ is dissipative. Secondly, we need to check that $0 \in \rho(\mathcal{A})$. To do this, for any $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$, let us prove that the problem $\mathcal{A}U = F$ has a unique solution $U = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w})^T$ in $D(\mathcal{A})$. According to the definition of the operator $\mathcal{A}$, the system can be written as

$$\tilde{\varphi} = f_1, \tag{2.8a}$$

$$\kappa(\varphi_{xx} + \psi_x + lw_x) + \kappa_0 l(w_x - l\varphi) - \gamma_1 E^\theta\tilde{\varphi} = \rho_1 f_2, \tag{2.8b}$$

$$\tilde{\psi} = f_3, \tag{2.8c}$$
$$b\psi_{xx} - \kappa(\varphi_x + \psi + lw) - \gamma_2 E_*^\theta \tilde{\psi} = \rho_2 f_4, \tag{2.8d}$$
$$\tilde{w} = f_5, \tag{2.8e}$$
$$\kappa_0(w_{xx} - l\varphi_x) - kl(\varphi_x + \psi + lw) - \gamma_3 E_*^\theta \tilde{w} = \rho_1 f_6, \tag{2.8f}$$

then from (2.8b), (2.8d) and (2.8e), we obtain

$$\kappa(\varphi_{xx} + \psi_x + lw_x) + \kappa_0 l(w_x - l\varphi) = h_1, \tag{2.9a}$$
$$b\psi_{xx} - \kappa(\varphi_x + \psi + lw) = h_2, \tag{2.9b}$$
$$\kappa_0(w_{xx} - l\varphi_x) - kl(\varphi_x + \psi + lw) = h_3, \tag{2.9c}$$

where $h_1 = \rho_1 f_2 + \gamma_1 E^\theta f_1$, $h_2 = \rho_2 f_4 + \gamma_2 E_*^\theta f_3$ and $h_3 = \rho_1 f_6 + \gamma_3 E_*^\theta f_5$. Multiplying (2.9a) by $\Phi \in H_0^1(0,L)$, (2.9b) by $\Psi \in H_*^1(0,L)$, and (2.9c) by $W \in H_*^1(0,L)$, summing them, then system (2.9a)-(2.9c) can be studied as a variational problem

$$\mathcal{B}((\varphi,\psi,w),(\Phi,\Psi,W)) = \mathcal{L}(\Phi,\Psi,W), \tag{2.10}$$

where

$$\begin{gathered}\mathcal{B}((\varphi,\psi,w),(\Phi,\Psi,W)) \\ = \kappa\langle\varphi_x + \psi + lw, \Phi_x + \Psi + lW\rangle + b\langle\psi_x, \Psi_x\rangle + \kappa_0\langle w_x - l\varphi, W_x - l\Phi\rangle, \\ \mathcal{L}(\Phi,\Psi,W) = -\langle h_1, \Phi\rangle - \langle h_2, \Psi\rangle - \langle h_3, W\rangle.\end{gathered}$$

We can verify that $\mathcal{B}$ is a continuous sesquilinear form on $(H_0^1(0,L) \times H_*^1(0,L) \times H_*^1(0,L))^2$ and $\mathcal{L}$ is a continuous linear form on $H^{-1}(0,L) \times H_*^{-1}(0,L) \times H_*^{-1}(0,L)$. At the same time, taking $(\Phi,\Psi,W) = (\varphi,\psi,w)$, we have

$$\mathcal{B}((\varphi,\psi,w),(\varphi,\psi,w)) = \kappa\|\varphi_x + \psi + lw\|^2 + b\|\psi_x\|^2 + \kappa_0\|w_x - l\varphi\|^2. \tag{2.11}$$

Thus, we obtain the coercivity of this sesquilinear form. Now, applying Lax-Milgram theorem and considering (2.8a), (2.8c) and (2.8e), we have a unique solution $U \in \mathcal{H}$. Since the solution satisfies system (2.8a)-(2.8f) in a weak sense, by these equations, we can obtain that $U \in D(\mathcal{A})$. Finally, from (2.10) and (2.11), we deduce

$$\begin{aligned}&\kappa\|\varphi_x + \psi + lw\|^2 + b\|\psi_x\|^2 + \kappa_0\|w_x - l\varphi\|^2 \\ &= -\rho_1\langle f_2, \varphi\rangle - \rho_2\langle f_4, \psi\rangle - \rho_1\langle f_6, w\rangle - \gamma_1\langle E^\theta f_1, \varphi\rangle \\ &\quad - \gamma_2\langle E_*^\theta f_3, \psi\rangle - \gamma_3\langle E_*^\theta f_5, w\rangle.\end{aligned} \tag{2.12}$$

From (2.7), we obtain

$$-\gamma_1\|E^{\theta/2}\tilde{\varphi}\|^2 - \gamma_2\|E_*^{\theta/2}\tilde{\psi}\|^2 - \gamma_3\|E_*^{\theta/2}\tilde{w}\|^2 \le -CRe\langle\mathcal{A}U, U\rangle \le C\|F\|\|U\|. \tag{2.13}$$

Substituting (2.13) to (2.12), and using the Cauchy-Schwarz, Young's and poincaré inequalities, we obtain

$$\begin{aligned}&\kappa\|\varphi_x + \psi + lw\|^2 + b\|\psi_x\|^2 + \kappa_0\|w_x - l\varphi\|^2 \\ &\le \varepsilon(\|\varphi_x\|^2 + \|\psi_x\|^2 + \|w_x\|^2 + C(\|F\|\|U\| + \|F\|^2)\end{aligned}$$

for any constant $\varepsilon > 0$. Using this inequality gives

$$\begin{aligned}\|\varphi_x\|^2 &\le C(\|\varphi_x + \psi + lw\|^2 + \|\psi\|^2 + \|w\|^2) \\ &\le C(\|\varphi_x + \psi + lw\|^2 + \|\psi_x\|^2 + \|w_x\|^2),\end{aligned}$$

and

$$\|w_x\|^2 \le C(\|w_x - l\varphi\|^2 + \|\varphi\|^2)$$

$$\begin{aligned} &\leq C(\|w_x - l\varphi\|^2 + \|\varphi_x\|^2) \\ &\leq C(\|w_x - l\varphi\|^2 + \|\varphi_x + \psi + lw\|^2 + \|\psi_x\|^2 + \varepsilon\|w_x\|^2), \end{aligned}$$

for fixing constant ε enough small. By using poincaré inequality we obtain

$$\kappa\|\varphi_x + \psi + lw\|^2 + \|\psi_x\|^2 + \kappa_0\|w_x - l\varphi\|^2 \leq C(\|F\|\|U\| + \|F\|^2).$$

Furthermore, from (2.8a), (2.8c), (2.8e), it follows that

$$\rho_1\|\varphi\|^2 + \rho_2\|\psi\|^2 + \rho_1\|w\|^2 \leq C\|F\|^2.$$

Therefore, from the above inequalities we conclude

$$\|U\|^2 \leq C\|F\|^2,$$

that is, $0 \in \rho(\mathcal{A})$. From Theorem 2.1, we obtain that $\mathcal{A}$ is the generator of a C_0-semigroup of contractions in $\mathcal{H}$, and the well-posedness of the Cauchy problem (2.4) is a result of the semigroup theory. The proof of Theorem 2.2 is complete. □

3. Lack of exponential stability

In this section, we show that the semigroup associated with the Bresse system is not exponentially stable. We will use Pruss's theorem [35] to prove the lack of exponential stability. That is, we will show that there exists a sequence of values λ_n such that

$$\|(\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \to \infty.$$

It is equivalent to prove the existence of a sequence $\{F_n\} \subset \mathcal{H}$ and a sequence of complex numbers $\{\lambda_n\} \subset i\mathbb{R}$, with F_n is bounded in $\mathcal{H}$ such that

$$\|(\lambda_n I - \mathcal{A})^{-1}F_n\|_{\mathcal{H}} \to \infty,$$

where

$$(\lambda_n - \mathcal{A})U_n = F_n$$

with U_n not bounded. Taking $F_n = (f_1, f_2, f_3, f_4, f_5, f_6)^T$, we write firstly the spectral equation in terms of its components as follows,

$$\lambda_n\varphi - \tilde{\varphi} = f_1, \tag{3.1a}$$

$$\lambda_n\tilde{\varphi} - \frac{\kappa}{\rho_1}(\varphi_x + \psi + lw)_x - \frac{\kappa_0 l}{\rho_1}(w_x - l\varphi) + \frac{\gamma_1}{\rho_1}E^\theta\tilde{\varphi} = f_2, \tag{3.1b}$$

$$\lambda_n\psi - \tilde{\psi} = f_3, \tag{3.1c}$$

$$\lambda_n\tilde{\psi} - \frac{b}{\rho_2}\psi_{xx} + \frac{\kappa}{\rho_2}(\varphi_x + \psi + lw) + \frac{\gamma_2}{\rho_2}E_*^\theta\tilde{\psi} = f_4, \tag{3.1d}$$

$$\lambda_n w - \tilde{w} = f_5, \tag{3.1e}$$

$$\lambda_n\tilde{w} - \frac{\kappa_0}{\rho_1}(w_x - l\varphi)_x + \frac{kl}{\rho_1}(\varphi_x + \psi + lw) + \frac{\gamma_3}{\rho_1}E_*^\theta\tilde{w} = f_6. \tag{3.1f}$$

The main result of this section is stated as follows.

Theorem 3.1. (i) *When $\gamma_1, \gamma_2 > 0, \gamma_3 = 0$, if $\chi_1 \neq 0$, or $\chi_1 = 0$ and $\theta \in (0,1]$, then the semigroup associated to system* (1.7)-(1.9) *is not exponentially stable.*

(ii) *When $\gamma_1, \gamma_3 > 0, \gamma_2 = 0$, if $\chi_0 \neq 0$, or $\chi_0 = 0$ and $\theta \in (0,1]$, then the semigroup associated to system* (1.7)-(1.9) *is not exponentially stable.*

(iii) *When $\gamma_2, \gamma_3 > 0, \gamma_1 = 0$, if $\chi_0 \neq 0$, or $\chi_0 = 0$ and $\theta \in (0,1]$, then the semigroup associated to system* (1.7)-(1.9) *is not exponentially stable.*

Proof. Using Pruss's theorem [35], and taking $f_1 = f_3 = f_5 = 0$ in (3.1a)-(3.1f) we obtain

$$\lambda_n\varphi = \tilde{\varphi}, \tag{3.2a}$$

$$\lambda_n\tilde{\varphi} - \frac{\kappa}{\rho_1}(\varphi_x + \psi + lw)_x - \frac{\kappa_0 l}{\rho_1}(w_x - l\varphi) + \frac{\gamma_1}{\rho_1}E^\theta\tilde{\varphi} = f_2, \tag{3.2b}$$

$$\lambda_n\psi = \tilde{\psi}, \tag{3.2c}$$

$$\lambda_n\tilde{\psi} - \frac{b}{\rho_2}\psi_{xx} + \frac{\gamma_2}{\rho_2}E_*^\theta\tilde{\psi} + \frac{\kappa}{\rho_2}(\varphi_x + \psi + lw) = f_4, \tag{3.2d}$$

$$\lambda_n w = \tilde{w}, \tag{3.2e}$$

$$\lambda_n\tilde{w} - \frac{\kappa_0}{\rho_1}(w_x - l\varphi)_x + \frac{kl}{\rho_1}(\varphi_x + \psi + lw) + \frac{\gamma_3}{\rho_1}E_*^\theta\tilde{w} = f_6. \tag{3.2f}$$

Substituting (3.2a), (3.2c), and (3.2e) into (3.2b), (3.2d) and (3.2f) respectively, we obtain

$$\begin{gathered}
\lambda_n^2\varphi - \frac{\kappa}{\rho_1}(\varphi_x + \psi + lw)_x - \frac{\kappa_0 l}{\rho_1}(w_x - l\varphi) + \frac{\gamma_1}{\rho_1}\lambda_n E^\theta\varphi = f_2, \\
\lambda_n^2\psi - \frac{b}{\rho_2}\psi_{xx} + \frac{\kappa}{\rho_2}(\varphi_x + \psi + lw) + \frac{\gamma_2}{\rho_2}\lambda_n E_*^\theta\psi = f_4, \\
\lambda_n^2 w - \frac{\kappa_0}{\rho_1}(w_x - l\varphi)_x + \frac{kl}{\rho_1}(\varphi_x + \psi + lw) + \frac{\gamma_3}{\rho_1}\lambda_n E_*^\theta w = f_6.
\end{gathered} \tag{3.3}$$

Because of the Dirichlet-Neumann-Neumann boundary conditions (1.8), we take φ, ψ, w are of the form $\varphi = A_n\sin(\frac{n\pi}{L}x)$, $\psi = B_n\cos(\frac{n\pi}{L}x)$ and $w = C_n\cos(\frac{n\pi}{L}x)$ with $n \in \mathbb{N}$, where A_n, B_n and C_n depend on λ_n and will be explicitly determined below. After performing some simplifications, we will obtain a system of the form

$$\Lambda\bar{U} = \Xi,$$

with

$$\bar{U} := (A_n, B_n, C_n)^T, \quad \Xi := \Big(\frac{f_2}{\sin(\frac{n\pi}{L}x)}, \frac{f_4}{\cos(\frac{n\pi}{L}x)}, \frac{f_6}{\cos(\frac{n\pi}{L}x)}\Big)^T,$$

$$\Lambda := \begin{pmatrix} P_1(\lambda_n) & \frac{\kappa}{\rho_1}(\frac{n\pi}{L}) & \frac{l(\kappa+\kappa_0)}{\rho_1}(\frac{n\pi}{L}) \\ \frac{\kappa}{\rho_2}(\frac{n\pi}{L}) & P_2(\lambda_n) & \frac{l\kappa}{\rho_2} \\ \frac{l(\kappa+\kappa_0)}{\rho_1}(\frac{n\pi}{L}) & \frac{l\kappa}{\rho_1} & P_3(\lambda_n) \end{pmatrix},$$

where

$$\begin{gathered}
P_1(\lambda_n) := \lambda_n^2 + \frac{\kappa}{\rho_1}(\frac{n\pi}{L})^2 + \frac{\kappa_0 l^2}{\rho_1} + \frac{\gamma_1}{\rho_1}\lambda_n(\frac{n\pi}{L})^{2\theta}, \\
P_2(\lambda_n) := \lambda_n^2 + \frac{b}{\rho_2}(\frac{n\pi}{L})^2 + \frac{\kappa}{\rho_2} + \frac{\gamma_2}{\rho_2}\lambda_n(\frac{n\pi}{L})^{2\theta}, \\
P_3(\lambda_n) := \lambda_n^2 + \frac{\kappa_0}{\rho_1}(\frac{n\pi}{L})^2 + \frac{\kappa l^2}{\rho_1} + \frac{\gamma_3}{\rho_1}\lambda_n(\frac{n\pi}{L})^{2\theta}.
\end{gathered}$$

By solving this system, the expressions of A_n, B_n, C_n are obtained. And to do that, we set

$$\det(\Lambda) := \begin{vmatrix} P_1(\lambda_n) & \frac{\kappa}{\rho_1}(\frac{n\pi}{L}) & \frac{l(\kappa+\kappa_0)}{\rho_1}(\frac{n\pi}{L}) \\ \frac{\kappa}{\rho_2}(\frac{n\pi}{L}) & P_2(\lambda_n) & \frac{l\kappa}{\rho_2} \\ \frac{l(\kappa+\kappa_0)}{\rho_1}(\frac{n\pi}{L}) & \frac{l\kappa}{\rho_1} & P_3(\lambda_n) \end{vmatrix}.$$

Next, we will discuss the non-exponential stability of the system in the following three cases.

Case (i) When γ_1, $\gamma_2 > 0$, $\gamma_3 = 0$. We consider the expression

$$C_n = \frac{\begin{vmatrix} P_1(\lambda_n) & \frac{\kappa}{\rho_1}(\frac{n\pi}{L}) & \frac{f_2}{\sin(\frac{n\pi}{L}x)} \\ \frac{\kappa}{\rho_2}(\frac{n\pi}{L}) & P_2(\lambda_n) & \frac{f_4}{\cos(\frac{n\pi}{L}x)} \\ \frac{l(\kappa+\kappa_0)}{\rho_1}(\frac{n\pi}{L}) & \frac{l\kappa}{\rho_1} & \frac{f_6}{\cos(\frac{n\pi}{L}x)} \end{vmatrix}}{\begin{vmatrix} P_1(\lambda_n) & \frac{\kappa}{\rho_1}(\frac{n\pi}{L}) & \frac{l(\kappa+\kappa_0)}{\rho_1}(\frac{n\pi}{L}) \\ \frac{\kappa}{\rho_2}(\frac{n\pi}{L}) & P_2(\lambda_n) & \frac{l\kappa}{\rho_2} \\ \frac{l(\kappa+\kappa_0)}{\rho_1}(\frac{n\pi}{L}) & \frac{l\kappa}{\rho_1} & P_3(\lambda_n) \end{vmatrix}}.$$

Taking $f_2 = f_4 = 0$, $f_6 = \cos(\frac{n\pi}{L}x)$, and $P_3(\lambda_n) = \lambda_n^2 + \frac{\kappa_0}{\rho_1}(\frac{n\pi}{L})^2 + \frac{\kappa l^2}{\rho_1} = c_0 \in \mathbb{R}$. Here, if $\chi_1 \neq 0$, then $c_0 := \frac{l^2(\kappa+\kappa_0)^2}{\rho_1(\kappa-\kappa_0)}$, while if $\chi_1 = 0$, then c_0 is a given constant. Thus we have

$$\lambda_n^2 = c_0 - \frac{\kappa_0}{\rho_1}(\frac{n\pi}{L})^2 - \frac{\kappa l^2}{\rho_1}.$$

For large values of n, we have $\lambda_n \in i\mathbb{R}$ and $|\lambda_n| \sim O(n)$. So we obtain

$$C_n = \frac{P_1(\lambda_n)P_2(\lambda_n) - \frac{\kappa^2}{\rho_1\rho_2}(\frac{n\pi}{L})^2}{\det_1(\lambda_n)},$$

where

$$\begin{aligned} \det_1(\lambda_n) &= P_1(\lambda_n)P_2(\lambda_n)c_0 + \frac{2\kappa^2 l^2(\kappa+\kappa_0)}{\rho_1^2\rho_2}(\frac{n\pi}{L})^2 - P_2(\lambda_n)\frac{l^2(\kappa+\kappa_0)^2}{\rho_1^2}(\frac{n\pi}{L})^2 \\ &\quad - P_1(\lambda_n)\frac{\kappa^2 l^2}{\rho_1\rho_2} - c_0\frac{\kappa^2}{\rho_1\rho_2}(\frac{n\pi}{L})^2, \end{aligned}$$

with the polynomials reduced to

$$\begin{aligned} P_1(\lambda_n) &= c_0 - \frac{\kappa_0}{\rho_1}(\frac{n\pi}{L})^2 - \frac{\kappa l^2}{\rho_1} + \frac{\kappa}{\rho_1}(\frac{n\pi}{L})^2 + \frac{\kappa_0 l^2}{\rho_1} + \frac{\gamma_1}{\rho_1}\lambda_n(\frac{n\pi}{L})^{2\theta} \\ &= c_0 + \frac{|\chi_1|}{\rho_1}(\frac{n\pi}{L})^2 + \frac{|\chi_1| l^2}{\rho_1} + \frac{\gamma_1}{\rho_1}\lambda_n(\frac{n\pi}{L})^{2\theta}, \end{aligned}$$

and

$$\begin{aligned} P_2(\lambda_n) &= c_0 - \frac{\kappa_0}{\rho_1}(\frac{n\pi}{L})^2 - \frac{\kappa l^2}{\rho_1} + \frac{b}{\rho_2}(\frac{n\pi}{L})^2 + \frac{\kappa}{\rho_2} + \frac{\gamma_2}{\rho_2}\lambda_n(\frac{n\pi}{L})^{2\theta} \\ &= c_0 + (\frac{b}{\rho_2} - \frac{\kappa_0}{\rho_1})(\frac{n\pi}{L})^2 - \frac{\kappa l^2}{\rho_1} + \frac{\kappa}{\rho_2} + \frac{\gamma_2}{\rho_2}\lambda_n(\frac{n\pi}{L})^{2\theta}. \end{aligned}$$

Next, we discuss the classifications.

For the subcase $\chi_1 = 0$, we obtain

$$|P_1(\lambda_n)| = c_0 + \frac{\gamma_1}{\rho_1}\lambda_n(\frac{n\pi}{L})^{2\theta} \sim O(n^{1+2\theta}),$$

$$
\begin{aligned}
|P_2(\lambda_n)| &= c_0 + (\frac{b}{\rho_2} - \frac{\kappa}{\rho_1})(\frac{n\pi}{L})^2 - \frac{\kappa l^2}{\rho_1} + \frac{\kappa}{\rho_2} + \frac{\gamma_2}{\rho_2}\lambda_n(\frac{n\pi}{L})^{2\theta} \\
&\sim \begin{cases} O(n^{1+2\theta}), & \text{if } \chi_0 = 0, \\ O(n^{1+2\theta}), & \text{if } \chi_0 \neq 0 \text{ and } \theta \geq 1/2, \\ O(n^2), & \text{if } \chi_0 \neq 0 \text{ and } \theta \leq 1/2. \end{cases}
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
|P_1(\lambda_n)P_2(\lambda_n) - \frac{\kappa^2}{\rho_1\rho_2}(\frac{n\pi}{L})^2| &\sim P_1(\lambda_n)P_2(\lambda_n) \\
&\sim \begin{cases} O(n^{2+4\theta}), & \text{if } \chi_0 = 0, \\ O(n^{2+4\theta}), & \text{if } \chi_0 \neq 0,\ \theta \geq 1/2, \\ O(n^{3+2\theta}), & \text{if } \chi_0 \neq 0,\ \theta \leq 1/2, \end{cases}
\end{aligned}
$$

$$
|\det_1(\lambda_n)| \sim \begin{cases} O(n^{2+4\theta}), & \text{if } \ \theta \geq 1/2,\ \chi_0 = 0, \\ O(n^{3+2\theta}), & \text{if } \theta \leq 1/2,\ \chi_0 = 0, \end{cases}
$$

$$
|\det_1(\lambda_n)| \sim \begin{cases} O(n^{2+4\theta}), & \text{if } \theta \geq 1/2,\ \chi_0 \neq 0, \\ O(n^4), & \text{if } \theta \leq 1/2,\ \chi_0 \neq 0, \end{cases}
$$

According to the ratio, it is found that whether χ_0 is 0 or not, the asymptotic behavior of C_n can be estimated as

$$
|C_n| \sim \begin{cases} O(1), & \text{if } \theta \geq 1/2, \\ O(n^{2\theta-1}), & \text{if } \theta \leq 1/2. \end{cases}
$$

For the subcase $\chi_1 \neq 0$, we have

$$
|P_1(\lambda_n)|,\ |P_2(\lambda_n)| \sim \begin{cases} O(n^{1+2\theta}), & \text{if } \theta \geq 1/2, \\ O(n^2), & \text{if } \theta \leq 1/2, \end{cases}
$$

then

$$
|P_1(\lambda_n)P_2(\lambda_n) - \frac{\kappa^2}{\rho_1\rho_2}(\frac{n\pi}{L})^2| \sim \begin{cases} O(n^{2+4\theta}), & \text{if } \theta \geq 1/2, \\ O(n^4), & \text{if } \theta \leq 1/2, \end{cases}
$$

Substituting $c_0 = \frac{l^2(\kappa+\kappa_0)^2}{\rho_1(\kappa-\kappa_0)}$ and $P_1(\lambda_n)$ into $\det_1(\lambda_n)$, we obtain

$$
\begin{aligned}
|\det_1(\lambda_n)| &= c_0 P_2(\lambda_n)\big[P_1(\lambda_n) - \frac{l^2(\kappa+\kappa_0)^2}{c_0\rho_1^2}(\frac{n\pi}{L})^2\big] + \frac{2\kappa^2 l^2(\kappa+\kappa_0)}{\rho_1^2\rho_2}(\frac{n\pi}{L})^2 \\
&\quad - P_1(\lambda_n)\frac{\kappa^2 l^2}{\rho_1\rho_2} - c_0\frac{\kappa^2}{\rho_1\rho_2}(\frac{n\pi}{L})^2 \\
&= c_0 P_2(\lambda_n)\big[c_0 + \frac{|\chi_1|}{\rho_1}(\frac{n\pi}{L})^2 + \frac{|\chi_1| l^2}{\rho_1} + \frac{\gamma_1}{\rho_1}\lambda_n(\frac{n\pi}{L})^{2\theta} - \frac{|\chi_1|}{\rho_1}(\frac{n\pi}{L})^2\big] \\
&\quad + \frac{2\kappa^2 l^2(\kappa+\kappa_0)}{\rho_1^2\rho_2}(\frac{n\pi}{L})^2 - P_1(\lambda_n)\frac{\kappa^2 l^2}{\rho_1\rho_2} - c_0\frac{\kappa^2}{\rho_1\rho_2}(\frac{n\pi}{L})^2 \\
&= c_0 P_2(\lambda_n)\big[c_0 + \frac{|\chi_1| l^2}{\rho_1} + \frac{\gamma_1}{\rho_1}\lambda_n(\frac{n\pi}{L})^{2\theta}\big] + \frac{2\kappa^2 l^2(\kappa+\kappa_0)}{\rho_1^2\rho_2}(\frac{n\pi}{L})^2 \\
&\quad - P_1(\lambda_n)\frac{\kappa^2 l^2}{\rho_1\rho_2} - c_0\frac{\kappa^2}{\rho_1\rho_2}(\frac{n\pi}{L})^2
\end{aligned}
$$

$$\sim \begin{cases} O(n^{2+4\theta}), & \text{if } \theta \geq 1/2, \\ O(n^{3+2\theta}), & \text{if } \theta \leq 1/2. \end{cases}$$

So, the asymptotic behavior of C_n can be estimated as

$$|C_n| \sim \begin{cases} O(1), & \text{if } \theta \geq 1/2, \\ O(n^{1-2\theta}), & \text{if } \theta \leq 1/2. \end{cases}$$

Hence,

$$\|U_n\|_{\mathcal{H}}^2 \geq \rho_1 \|\tilde{w}\|_{L^2(0,L)}^2 = \rho_1 |\lambda_n C_n|^2 \int_0^L \cos^2(\frac{n\pi}{L}x)dx = \frac{\rho_1 L}{2} |\lambda_n C_n|^2.$$

So, when $\chi_1 \neq 0$, we have

$$\|U_n\|_{\mathcal{H}} \geq \sqrt{\frac{\rho_1 L}{2}} |\lambda_n|\, |C_n| \sim \begin{cases} O(n), & \text{if } \theta \geq 1/2, \\ O(n^{2-2\theta}), & \text{if } \theta \leq 1/2, \end{cases}$$

$$\lim_{n\to\infty} \|U_n\|_{\mathcal{H}} = +\infty,$$

while when $\chi_1 = 0$, we have

$$\|U_n\|_{\mathcal{H}} \geq \sqrt{\frac{\rho_1 L}{2}} |\lambda_n|\, |C_n| \sim \begin{cases} O(n), & \text{if } \theta \geq 1/2, \\ O(n^{2\theta}), & \text{if } \theta \leq 1/2. \end{cases}$$

If $\theta \in (0,1]$, then $\lim_{n\to\infty} \|U_n\|_{\mathcal{H}} = +\infty$. This means that the corresponding semigroup is not exponentially stable when $\chi_1 \neq 0$ or $\chi_1 = 0$ and $\theta \in (0,1]$. The first result of this theorem is proved.

Case (ii) When γ_1, $\gamma_3 > 0$, and $\gamma_2 = 0$, we consider the expression

$$B_n = \frac{\begin{vmatrix} P_1(\lambda_n) & \frac{f_2}{\sin(\frac{n\pi}{L}x)} & \frac{l(\kappa+\kappa_0)}{\rho_1}(\frac{n\pi}{L}) \\ \frac{\kappa}{\rho_2}(\frac{n\pi}{L}) & \frac{f_4}{\cos(\frac{n\pi}{L}x)} & \frac{l\kappa}{\rho_2} \\ \frac{l(\kappa+\kappa_0)}{\rho_1}(\frac{n\pi}{L}) & \frac{f_6}{\cos(\frac{n\pi}{L}x)} & P_3(\lambda_n) \end{vmatrix}}{\begin{vmatrix} P_1(\lambda_n) & \frac{\kappa}{\rho_1}(\frac{n\pi}{L}) & \frac{l(\kappa+\kappa_0)}{\rho_1}(\frac{n\pi}{L}) \\ \frac{\kappa}{\rho_2}(\frac{n\pi}{L}) & P_2(\lambda_n) & \frac{l\kappa}{\rho_2} \\ \frac{l(\kappa+\kappa_0)}{\rho_1}(\frac{n\pi}{L}) & \frac{l\kappa}{\rho_1} & P_3(\lambda_n) \end{vmatrix}}.$$

We take $f_2 = f_6 = 0$, $f_4 = \cos(\frac{n\pi}{L}x)$, and $P_2(\lambda_n) = \lambda_n^2 + \frac{b}{\rho_2}(\frac{n\pi}{L})^2 + \frac{\kappa}{\rho_2} = b_0 \in \mathbb{R}$. Here, if $\chi_0 \neq 0$, then $b_0 := \frac{\kappa^2}{\kappa\rho_2 - b\rho_1}$, while if $\chi_0 = 0$, then b_0 is a given constant. Then

$$\lambda_n^2 = b_0 - \frac{b}{\rho_2}(\frac{n\pi}{L})^2 - \frac{\kappa}{\rho_2}.$$

For large values of n, we have $\lambda_n \in i\mathbb{R}$ and $|\lambda_n| \sim O(n)$. So, we obtain

$$B_n = \frac{P_1(\lambda_n)P_3(\lambda_n) - \frac{l^2(\kappa+\kappa_0)^2}{\rho_1^2}(\frac{n\pi}{L})^2}{\det_2(\lambda_n)},$$

where

$$\begin{aligned} \det_2(\lambda_n) &= P_1(\lambda_n)P_2(\lambda_n)b_0 + \frac{2\kappa^2 l^2(\kappa+\kappa_0)}{\rho_1^2\rho_2}(\frac{n\pi}{L})^2 - b_0\frac{l^2(\kappa+\kappa_0)^2}{\rho_1^2}(\frac{n\pi}{L})^2 \\ &\quad - P_1(\lambda_n)\frac{\kappa^2 l^2}{\rho_1\rho_2} - P_3(\lambda_n)\frac{\kappa^2}{\rho_1\rho_2}(\frac{n\pi}{L})^2, \end{aligned}$$

with the polynomials reduced to

$$\begin{aligned} P_1(\lambda_n) &= b_0 - \frac{b}{\rho_2}(\frac{n\pi}{L})^2 - \frac{\kappa}{\rho_2} + \frac{\kappa}{\rho_1}(\frac{n\pi}{L})^2 + \frac{\kappa_0 l^2}{\rho_1} + \frac{\gamma_1}{\rho_1}\lambda_n(\frac{n\pi}{L})^{2\theta}, \\ &= b_0 + |\chi_0|(\frac{n\pi}{L})^2 - \frac{\kappa}{\rho_2} + \frac{\kappa_0 l^2}{\rho_1} + \frac{\gamma_1}{\rho_1}\lambda_n(\frac{n\pi}{L})^{2\theta}, \end{aligned}$$

and

$$\begin{aligned} P_3(\lambda_n) &= b_0 - \frac{b}{\rho_2}(\frac{n\pi}{L})^2 - \frac{\kappa}{\rho_2} + \frac{\kappa_0}{\rho_1}(\frac{n\pi}{L})^2 + \frac{\kappa l^2}{\rho_1} + \frac{\gamma_3}{\rho_1}\lambda_n(\frac{n\pi}{L})^{2\theta}, \\ &= b_0 + (\frac{\kappa_0}{\rho_1} - \frac{b}{\rho_2})(\frac{n\pi}{L})^2 - \frac{\kappa}{\rho_2} + \frac{\kappa l^2}{\rho_1} + \frac{\gamma_3}{\rho_1}\lambda_n(\frac{n\pi}{L})^{2\theta}. \end{aligned}$$

Through calculation, we find that the estimate of the asymptotic behavior of B_n is consistent with the estimate of C_n in the item (i) of this Theorem, and only replace χ_1 and χ_0 in C_n with χ_0 and χ_1 in B_n respectively. So we will not go into details here.

Case (iii) When γ_2, $\gamma_3 > 0$, and $\gamma_1 = 0$, we consider the expression

$$A_n = \frac{\begin{vmatrix} \frac{f_2}{\sin(\frac{n\pi}{L}x)} & \frac{\kappa}{\rho_1}(\frac{n\pi}{L}) & \frac{l(\kappa+\kappa_0)}{\rho_1}(\frac{n\pi}{L}) \\ \frac{f_4}{\cos(\frac{n\pi}{L}x)} & P_2(\lambda_n) & \frac{l\kappa}{\rho_2} \\ \frac{f_6}{\cos(\frac{n\pi}{L}x)} & \frac{l\kappa}{\rho_1} & P_3(\lambda_n) \end{vmatrix}}{\begin{vmatrix} P_1(\lambda_n) & \frac{\kappa}{\rho_1}(\frac{n\pi}{L}) & \frac{l(\kappa+\kappa_0)}{\rho_1}(\frac{n\pi}{L}) \\ \frac{\kappa}{\rho_2}(\frac{n\pi}{L}) & P_2(\lambda_n) & \frac{l\kappa}{\rho_2} \\ \frac{l(\kappa+\kappa_0)}{\rho_1}(\frac{n\pi}{L}) & \frac{l\kappa}{\rho_1} & P_3(\lambda_n) \end{vmatrix}}.$$

We Take $f_2 = \sin(\frac{n\pi}{L}x)$, $f_4 = \cos(\frac{n\pi}{L}x)$, $f_6 = 0$, and $P_1(\lambda_n) = \lambda_n^2 + \frac{\kappa}{\rho_1}(\frac{n\pi}{L})^2 + \frac{\kappa_0 l^2}{\rho_1} = a_0$, where $c_0 \in \mathbb{R}$ is a given constant. Then

$$\lambda_n^2 = a_0 - \frac{\kappa}{\rho_1}(\frac{n\pi}{L})^2 - \frac{\kappa_0 l^2}{\rho_1}.$$

For large values of n, we have $\lambda_n \in i\mathbb{R}$ and $|\lambda_n| \sim O(n)$. So, we obtain

$$A_n = \frac{P_2(\lambda_n)P_3(\lambda_n) - \frac{\kappa l^2(\kappa+\kappa_0)}{\rho_1^2}(\frac{n\pi}{L})^2 - \frac{\kappa^2 l^2}{\rho_1\rho_2} - P_3(\lambda_n)\frac{\kappa}{\rho_1}(\frac{n\pi}{L})^2}{\det_3(\lambda_n)},$$

where

$$\begin{aligned} \det_3(\lambda_n) &= P_2(\lambda_n)P_3(\lambda_n)a_0 + \frac{2\kappa^2 l^2(\kappa+\kappa_0)}{\rho_1^2\rho_2}(\frac{n\pi}{L})^2 - P_2(\lambda_n)\frac{l^2(\kappa+\kappa_0)^2}{\rho_1^2}(\frac{n\pi}{L})^2 \\ &\quad - a_0(\lambda_n)\frac{\kappa^2 l^2}{\rho_1\rho_2} - P_3(\lambda_n)\frac{\kappa^2}{\rho_1\rho_2}(\frac{n\pi}{L})^2, \end{aligned}$$

with the polynomials reduced to

$$\begin{aligned} P_2(\lambda_n) &= a_0 - \frac{\kappa}{\rho_1}(\frac{n\pi}{L})^2 - \frac{\kappa_0 l^2}{\rho_1} + \frac{b}{\rho_2}(\frac{n\pi}{L})^2 + \frac{\kappa}{\rho_2} + \frac{\gamma_2}{\rho_2}\lambda_n(\frac{n\pi}{L})^{2\theta}, \\ &= a_0 + |\chi_0|(\frac{n\pi}{L})^2 + \frac{\kappa}{\rho_2} - \frac{\kappa_0 l^2}{\rho_1} + \frac{\gamma_2}{\rho_2}\lambda_n(\frac{n\pi}{L})^{2\theta}, \end{aligned}$$

and

$$\begin{aligned} P_3(\lambda_n) &= a_0 - \frac{\kappa}{\rho_1}(\frac{n\pi}{L})^2 - \frac{\kappa_0 l^2}{\rho_1} + \frac{\kappa_0}{\rho_1}(\frac{n\pi}{L})^2 + \frac{\kappa l^2}{\rho_1} + \frac{\gamma_3}{\rho_1}\lambda_n(\frac{n\pi}{L})^{2\theta}, \\ &= a_0 + \frac{|\chi_1|}{\rho_1}(\frac{n\pi}{L})^2 - \frac{|\chi_1| l^2}{\rho_1} + \frac{\gamma_3}{\rho_1}\lambda_n(\frac{n\pi}{L})^{2\theta}. \end{aligned}$$

Next, we discuss the classification. For the subcase $\chi_0 = 0$, we have

$$|P_2(\lambda_n)| = a_0 + \frac{\kappa}{\rho_2} - \frac{\kappa_0 l^2}{\rho_1} + \frac{\gamma_2}{\rho_2}\lambda_n(\frac{n\pi}{L})^{2\theta} \sim O(n^{1+2\theta}),$$

$$|P_3(\lambda_n)| \sim \begin{cases} O(n^{1+2\theta}), & \text{if } \chi_1 = 0, \\ O(n^{1+2\theta}), & \text{if } \chi_1 \neq 0,\ \theta \geq 1/2, \\ O(n^2), & \text{if } \chi_1 \neq 0.\ \theta \leq 1/2. \end{cases}$$

Thus, we have

$$\begin{aligned} &|P_2(\lambda_n)P_3(\lambda_n) - \frac{\kappa l^2(\kappa+\kappa_0)}{\rho_1^2}(\frac{n\pi}{L})^2 - \frac{\kappa^2 l^2}{\rho_1\rho_2} - P_3(\lambda_n)\frac{\kappa}{\rho_1}(\frac{n\pi}{L})^2| \\ &\sim \begin{cases} O(n^{2+4\theta}), & \text{if } \chi_1 = 0, \\ O(n^{2+4\theta}), & \text{if } \chi_1 \neq 0,\ \theta \geq 1/2, \\ O(n^{3+2\theta}), & \text{if } \chi_1 \neq 0,\ \theta \leq 1/2, \end{cases} \end{aligned}$$

$$|\det_1(\lambda_n)| \sim \begin{cases} O(n^{2+4\theta}), & \text{if } \theta \geq 1/2,\ \chi_1 = 0, \\ O(n^{3+2\theta}), & \text{if } \theta \leq 1/2,\ \chi_1 = 0. \end{cases}$$

$$|\det_1(\lambda_n)| \sim \begin{cases} O(n^{2+4\theta}), & \text{if } \theta \geq 1/2,\ \chi_1 \neq 0, \\ O(n^4), & \text{if } \theta \leq 1/2,\ \chi_1 \neq 0. \end{cases}$$

According to the ratio, it is found that whether χ_1 is 0 or not, the asymptotic behavior of A_n can be estimated as

$$|A_n| \sim \begin{cases} O(1), & \text{if } \theta \geq 1/2, \\ O(n^{2\theta-1}), & \text{if } \theta \leq 1/2. \end{cases}$$

For the subcase $\chi_0 \neq 0$, we have $P_3(\lambda_n)$ is unchanged and

$$|P_2(\lambda_n)| \sim \begin{cases} O(n^{1+2\theta}), & \text{if } \theta \geq 1/2, \\ O(n^2), & \text{if } \theta \leq 1/2. \end{cases}$$

Then

$$|P_2(\lambda_n)P_3(\lambda_n)| \sim \begin{cases} O(n^{2+4\theta}), \text{if } \theta \geq 1/2,\ \chi_1 = 0, \\ O(n^{3+2\theta}), \text{if } \theta \leq 1/2,\ \chi_1 = 0, O(n^{2+4\theta}), \text{if } \theta \geq 1/2,\ \chi_1 \neq 0, \\ O(n^4), \text{if } \theta \leq 1/2,\ \chi_1 \neq 0, \end{cases}$$

$$|P_3(\lambda_n)\frac{\kappa}{\rho_1}(\frac{n\pi}{L})^2| \sim \begin{cases} O(n^{3+2\theta}), & \text{if } \chi_1 = 0, \\ O(n^{3+2\theta}), & \text{if } \theta \geq 1/2,\ \chi_1 \neq 0, \\ O(n^4), & \text{if } \theta \leq 1/2,\ \chi_1 \neq 0. \end{cases}$$

Thus we obtain

$$|P_2(\lambda_n)P_3(\lambda_n) - \frac{\kappa l^2(\kappa+\kappa_0)}{\rho_1^2}(\frac{n\pi}{L})^2 - \frac{\kappa^2 l^2}{\rho_1\rho_2} - P_3(\lambda_n)\frac{\kappa}{\rho_1}(\frac{n\pi}{L})^2|$$

$$\sim \begin{cases} O(n^{2+4\theta}), & \text{if } \theta \geq 1/2, \\ O(n^4),, & \text{if } \theta \leq 1/2, \end{cases}$$

when $\chi_1 = 0$ or $\chi_1 \neq 0$; and

$$|\det_3(\lambda_n)| \sim \begin{cases} O(n^{2+4\theta}), & \text{if } \theta \geq 1/2, \\ O(n^4), & \text{if } \theta \leq 1/2, \end{cases}$$

when $\chi_1 = 0$ or $\chi_1 \neq 0$. Therefore regardless of the value of χ_1 and θ, the asymptotic behavior of A_n can be estimated as $|A_n| \sim O(1)$. It follows that

$$\|U_n\|_{\mathcal{H}}^2 \geq \rho_1 \|\tilde{\varphi}\|_{L^2(0,L)}^2 = \rho_1 |\lambda_n A_n|^2 \int_0^L \sin^2(\frac{n\pi}{L}x)dx = \frac{\rho_1 L}{2} |\lambda_n A_n|^2.$$

So when $\chi_0 \neq 0$, we have

$$\|U_n\|_{\mathcal{H}} \geq \sqrt{\frac{\rho_1 L}{2}} |\lambda_n|\, |A_n| \sim O(n) \quad \text{and} \quad \lim_{n\to\infty} \|U_n\|_{\mathcal{H}} = +\infty,$$

while when $\chi_0 = 0$, we have

$$\|U_n\|_{\mathcal{H}} \geq \sqrt{\frac{\rho_1 L}{2}} |\lambda_n|\, |A_n| \sim \begin{cases} O(n), & \text{if } \theta \geq 1/2, \\ O(n^{2\theta}), & \text{if } \theta \leq 1/2, \end{cases}$$

and if $\theta \in (0, 1]$, then $\lim_{n\to\infty} \|U_n\|_{\mathcal{H}} = +\infty$. This means that the corresponding semigroup is not exponentially stable when $\chi_1 \neq 0$ or $\chi_1 = 0$ and $\theta \in (0, 1]$. This shows the third result of Theorem 3.1, which completes the proof. □

4. Stability results

In this section we study the asymptotic behavior of the semigroup $e^{t\mathcal{A}}$ associated to the system (1.7)-(1.9) when fractional damping is applied to two equations separately. The following spectral characteristics of exponential and polynomial stability of semigroups will be used to obtain the stability results. Firstly we give the following useful theorems.

Theorem 4.1 ([17]). *Let $\mathcal{A}$ be the generator of a C_0-semigroup of contractions on a Hilbert space. Then, the semigroup $e^{t\mathcal{A}}$ is exponentially stable if and only if $i\mathbb{R} \subset \rho(\mathcal{A})$ and*

$$\limsup_{|\lambda|\to\infty} \|(i\lambda I - \mathcal{A})^{-1}\| < \infty.$$

Theorem 4.2 ([10, 29]). *Let $\mathcal{A}$ the generator of a C_0-semigroup of bounded operators on a Hilbert space with $i\mathbb{R} \subset \rho(\mathcal{A})$. Then we have*

$$\|e^{t\mathcal{A}} U_0\| \leq C t^{-1/\theta} \|U_0\|_{D(\mathcal{A})}, \quad \forall t > 0,\ U_0 \in D(\mathcal{A}),$$

if and only if

$$\limsup_{|\lambda|\to\infty} |\lambda|^{-\theta} \|(i\lambda I - \mathcal{A})^{-1}\| < \infty.$$

In the remainder of this article, C and C_δ represent positive constants that assume different values at different locations. In most cases, it may be $C_\delta \to \infty$ when $\delta \to 0^+$.

Using the above theorems, we obtain some estimates for the solution

$$U = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w})^T$$

of the equation $(i\lambda I-\mathcal{A})U=F$, where $\lambda\in\mathbb{R}$ and $F=(f_1,f_2,f_3,f_4,f_5,f_6)^T\in\mathcal{H}$. Then the system can be decomposed into the following forms

$$i\lambda\varphi-\tilde{\varphi}=f_1, \tag{4.1a}$$

$$i\lambda\rho_1\tilde{\varphi}-\kappa(\varphi_x+\psi+lw)_x-\kappa_0 l(w_x-l\varphi)+\gamma_1E^\theta\tilde{\varphi}=\rho_1f_2, \tag{4.1b}$$

$$i\lambda\psi-\tilde{\psi}=f_3, \tag{4.1c}$$

$$i\lambda\rho_2\tilde{\psi}-b\psi_{xx}+\kappa(\varphi_x+\psi+lw)+\gamma_2E_*^\theta\tilde{\psi}=\rho_2f_4, \tag{4.1d}$$

$$i\lambda w-\tilde{w}=f_5, \tag{4.1e}$$

$$i\lambda\rho_1\tilde{w}-\kappa_0(w_x-l\varphi)_x+kl(\varphi_x+\psi+lw)+\gamma_3E_*^\theta\tilde{w}=\rho_1f_6. \tag{4.1f}$$

Before stating the main stability results we need to introduce some lemmas. In all the lemmas below, we assume that $\theta\in[0,1]$, $U\in D(\mathcal{A})$ is the solution of the equation $(i\lambda I-\mathcal{A})U=F$ for real number $\lambda>0$. Note that similar to (2.13) and using (2.7), we obtain the first estimate

$$\gamma_1\|E^{\theta/2}\tilde{\varphi}\|^2+\gamma_2\|E_*^{\theta/2}\tilde{\psi}\|^2+\gamma_3\|E^{\theta/2}\tilde{w}\|^2\le CRe\langle(i\lambda I-\mathcal{A})U,U\rangle\le C\|F\|\|U\|; \tag{4.2}$$

that is,

$$\|E^{\theta/2}\tilde{\varphi}\|^2\le C\|F\|\|U\|, \tag{4.3}$$

$$\|E_*^{\theta/2}\tilde{\psi}\|^2\le C\|F\|\|U\|, \tag{4.4}$$

$$\|E^{\theta/2}\tilde{w}\|^2\le C\|F\|\|U\|. \tag{4.5}$$

Using (4.1a) and taking into account estimates (4.3), we have

$$\lambda^2\|E^{\theta/2}\varphi\|^2\le\|E^{\theta/2}\tilde{\varphi}\|^2+\|E^{\theta/2}f_1\|^2\le C\big(\|F\|\|U\|+\|F\|^2\big), \tag{4.6}$$

in the same way, we obtain

$$\lambda^2\|E^{\theta/2}\psi\|^2\le C\big(\|F\|\|U\|+\|F\|^2\big), \tag{4.7}$$

$$\lambda^2\|E^{\theta/2}w\|^2\le C\big(\|F\|\|U\|+\|F\|^2\big). \tag{4.8}$$

Lemma 4.3. *Let $\delta>0$ and $\alpha\le 0$. There exists positive constant C_δ, such that for $|\lambda|\ge\delta$ the solution $U=(\varphi,\tilde{\varphi},\psi,\tilde{\psi},w,\tilde{w})^T$ of system* (4.1a)-(4.1f) *satisfies the following estimates: (i)*

$$\begin{aligned}\lambda^2\|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2&\le C_\delta\big(\|E_*^{\frac{\alpha}{2}+\frac12}\tilde{\varphi}\|^2+\|E_*^{\frac{\alpha}{2}}\tilde{\psi}\|^2+\|E_*^{\frac{\alpha}{2}}\tilde{w}\|^2\big)\\&\quad+\varepsilon_1\lambda^2\|E^{\alpha+\frac{\theta}{2}}\tilde{\varphi}\|^2+C_{\varepsilon_1}\big(\|F\|\|U\|+\|F\|^2\big),\end{aligned}$$

and (ii)

$$\begin{aligned}\|E^{\frac{\alpha}{2}+\frac12}\tilde{\varphi}\|^2&\le C_\delta\big(\lambda^2\|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2+\|E_*^{\frac{\alpha}{2}}\tilde{\psi}\|^2+\|E_*^{\frac{\alpha}{2}}\tilde{w}\|^2\big)\\&\quad+\varepsilon_1\lambda^2\|E^{\alpha+\frac{\theta}{2}}\tilde{\varphi}\|^2+C_{\varepsilon_1}\big(\|F\|\|U\|+\|F\|^2\big),\end{aligned}$$

where ε_1 is positive or zero if the the damping coefficient γ_1 is present or not present respectively.

Proof. Multiplying (4.1b) by $i\lambda$, taking the inner product with $E^\alpha\tilde{\varphi}$ and using the definition of the operator E, applying the self-adjointness of E^σ for $\sigma\in\mathbb{R}$, we obtain

$$\begin{aligned}\rho_1\lambda^2\|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2&=i\lambda\kappa\langle E\varphi,E^\alpha\tilde{\varphi}\rangle-i\lambda\kappa\langle\psi_x,E^\alpha\tilde{\varphi}\rangle-i\lambda\kappa l\langle w_x,E^\alpha\tilde{\varphi}\rangle-i\lambda\kappa_0 l\langle w_x,E^\alpha\tilde{\varphi}\rangle\\&\quad+i\lambda\kappa_0l^2\langle\varphi,E^\alpha\tilde{\varphi}\rangle+i\lambda\gamma_1\langle E^\theta\tilde{\varphi},E^\alpha\tilde{\varphi}\rangle-i\lambda\rho_1\langle f_2,E^\alpha\tilde{\varphi}\rangle.\end{aligned}$$

Substituting (4.1a) in the previous equality, we obtain

$$\begin{aligned}\rho_1\lambda^2\|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2 &= \kappa\|E^{\frac{\alpha}{2}+\frac{1}{2}}\tilde{\varphi}\|^2 + \kappa\langle E^{\frac{\alpha}{2}+\frac{1}{2}}f_1, E^{\frac{\alpha}{2}+\frac{1}{2}}\tilde{\varphi}\rangle - i\lambda\kappa\langle\psi_x, E^{\alpha}\tilde{\varphi}\rangle\\ &\quad - i\lambda(\kappa+\kappa_0)l\langle w_x, E^{\alpha}\tilde{\varphi}\rangle + \kappa_0 l^2\|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2 + \kappa_0 l^2\langle E^{\frac{\alpha}{2}}f_1, E^{\frac{\alpha}{2}}\tilde{\varphi}\rangle\\ &\quad + i\lambda\gamma_1\langle E^{\theta/2}\tilde{\varphi}, E^{\alpha+\frac{\theta}{2}}\tilde{\varphi}\rangle - i\lambda\rho_1\langle f_2, E^{\alpha}\tilde{\varphi}\rangle.\end{aligned} \tag{4.9}$$

Next, let's estimate some items in (4.9). First, we estimate the items with f in (4.9). Applying Cauchy-Schwarz and Young's inequalities, for ε small enough to be chosen later, we have

$$\begin{gathered}|\kappa\langle E^{\frac{\alpha}{2}+\frac{1}{2}}f_1, E^{\frac{\alpha}{2}+\frac{1}{2}}\tilde{\varphi}\rangle| \le C\|E^{\frac{\alpha}{2}+\frac{1}{2}}f_1\|^2 + C\|E^{\frac{\alpha}{2}+\frac{1}{2}}\tilde{\varphi}\|^2 \le C\|F\|^2 + C\|E^{\frac{\alpha}{2}+\frac{1}{2}}\tilde{\varphi}\|^2,\\ |\kappa_0 l^2\langle E^{\frac{\alpha}{2}}f_1, E^{\frac{\alpha}{2}}\tilde{\varphi}\rangle| \le C\|E^{\frac{\alpha}{2}}f_1\|^2 + C\|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2 \le C\|F\|^2 + C\|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2,\\ |i\lambda\rho_1\langle f_2, E^{\alpha}\tilde{\varphi}\rangle| \le C\|E^{\frac{\alpha}{2}}f_2\|^2 + \varepsilon\lambda^2\|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2 \le C\|F\|^2 + \varepsilon\lambda^2\|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2.\end{gathered}$$

Then, using integration by parts, the self-adjointness of E^{σ} and (1.19), applying Cauchy-Schwarz, Young's inequalities and (4.3), for ε small enough to be chosen later, we have

$$\begin{aligned}|i\lambda\kappa\langle\psi_x, E^{\alpha}\tilde{\varphi}\rangle| &= |i\lambda\kappa\langle E_*^{\frac{\alpha}{2}}\psi, E_*^{\frac{\alpha}{2}}\tilde{\varphi}_x\rangle|\\ &= |i\lambda\kappa\langle E_*^{\frac{\alpha}{2}}\psi, E^{\frac{\alpha}{2}+\frac{1}{2}}\tilde{\varphi}\rangle|\\ &= |\kappa\langle E_*^{\frac{\alpha}{2}}(\tilde{\psi}+f_3), E^{\frac{\alpha}{2}+\frac{1}{2}}\tilde{\varphi}\rangle|\\ &\le C\|E_*^{\frac{\alpha}{2}}\tilde{\psi}\|^2 + C\|E^{\frac{\alpha}{2}+\frac{1}{2}}\tilde{\varphi}\|^2 + C\|F\|^2,\\ |i\lambda(\kappa+\kappa_0)l\langle w_x, E^{\alpha}\tilde{\varphi}\rangle| &= |i\lambda(\kappa+\kappa_0)l\langle E_*^{\frac{\alpha}{2}}w, E_*^{\frac{\alpha}{2}}\tilde{\varphi}_x\rangle|\\ &= |(\kappa+\kappa_0)l\langle E_*^{\frac{\alpha}{2}}(\tilde{w}+f_5), E^{\frac{\alpha}{2}+\frac{1}{2}}\tilde{\varphi}\rangle|\\ &\le C\|E_*^{\frac{\alpha}{2}}\tilde{w}\|^2 + C\|E^{\frac{\alpha}{2}+\frac{1}{2}}\tilde{\varphi}\|^2 + \|F\|^2,\\ |i\lambda\gamma_1\langle E^{\theta/2}\tilde{\varphi}, E^{\alpha+\frac{\theta}{2}}\tilde{\varphi}\rangle| &\le C\|E^{\theta/2}\tilde{\varphi}\|^2 + \varepsilon_1\|E^{\alpha+\frac{\theta}{2}}\tilde{\varphi}\|^2\\ &\le \varepsilon_1\|E^{\alpha+\frac{\theta}{2}}\tilde{\varphi}\|^2 + C_{\varepsilon_1}\|F\|\|U\|.\end{aligned}$$

Substituting the above estimates into (4.9) and considering $\alpha \le 0$, we obtain the first result of Lemma 4.3. Similarly, rewriting (4.9) introduces the item (ii) of this Lemma. The proof is complete. □

Lemma 4.4. *Let $\delta > 0$ and $\alpha \le 0$. There exists positive constant C_δ, such that for $|\lambda| \ge \delta$ the solution $U = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w})^T$ of system* (4.1a)-(4.1f) *satisfies the following estimates: (i)*

$$\begin{aligned}\lambda^2\|E^{\frac{\alpha}{2}}\tilde{\psi}\|^2 &\le C_\delta\big(\|E_*^{\frac{\alpha}{2}+\frac{1}{2}}\tilde{\psi}\|^2 + \|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2 + \|E_*^{\frac{\alpha}{2}}\tilde{w}\|^2\big) + \varepsilon_2\lambda^2\|E^{\alpha+\frac{\theta}{2}}\tilde{\psi}\|^2\\ &\quad + C_{\varepsilon_2}\big(\|F\|\|U\| + \|F\|^2\big),\end{aligned}$$

(ii)

$$\begin{aligned}\|E^{\frac{\alpha}{2}+\frac{1}{2}}\tilde{\psi}\|^2 &\le C_\delta\big(\lambda^2\|E^{\frac{\alpha}{2}}\tilde{\psi}\|^2 + \|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2 + \|E_*^{\frac{\alpha}{2}}\tilde{w}\|^2\big) + \varepsilon_2\lambda^2\|E^{\alpha+\frac{\theta}{2}}\tilde{\varphi}\|^2\\ &\quad + C_{\varepsilon_2}\big(\|F\|\|U\| + \|F\|^2\big),\end{aligned}$$

(iii)

$$\lambda^2\|E^{\frac{\alpha}{2}}\tilde{w}\|^2 \le C_\delta\big(\|E_*^{\frac{\alpha}{2}+\frac{1}{2}}\tilde{w}\|^2 + \|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2 + \|E_*^{\frac{\alpha}{2}}\tilde{\psi}\|^2\big) + \varepsilon_3\lambda^2\|E^{\alpha+\frac{\theta}{2}}\tilde{w}\|^2$$

$$+ C_{\varepsilon_3}\big(\|F\|\|U\| + \|F\|^2\big),$$

and (iv)

$$\|E^{\frac{\alpha}{2}+\frac{1}{2}}\tilde{w}\|^2 \le C_\delta\big(\lambda^2\|E^{\frac{\alpha}{2}}\tilde{w}\|^2 + \|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2 + \|E_*^{\frac{\alpha}{2}}\tilde{\psi}\|^2\big) + \varepsilon_3\lambda^2\|E^{\alpha+\frac{\theta}{2}}\tilde{w}\|^2 \\ + C_{\varepsilon_3}\big(\|F\|\|U\| + \|F\|^2\big),$$

where ε_2 and ε_3 are positive or zero if the the damping coefficient γ_2 and γ_3 are present or not present, respectively.

Proof. Multiplying (4.1d) and (4.1f) by $i\lambda$, taking the inner product with $E_*^\alpha\tilde{\psi}$ and $E_*^\alpha\tilde{w}$, respectively. Using the definition of the operator E_*, applying the self-adjointness of E^σ for $\sigma \in \mathbb{R}$, by integration by parts, we obtain the results of this Lemma. Since the corresponding steps and estimates are similar to Lemma 4.3, we will not repeat them here. □

Lemma 4.5. *Let $\delta > 0$ and $\alpha \le 0$. There exists positive constant C_δ, such that for $|\lambda| \ge \delta$ the solution $U = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w})^T$ of system* (4.1a)-(4.1f) *satisfies the following estimates,*

$$\|E_*^{\frac{\alpha}{2}}(\varphi_x + \psi + lw)\|^2 \le \frac{\rho_1\rho_2}{\kappa}|\chi_0|\lambda^2\langle\psi, E_*^\alpha\varphi_x\rangle + C\|E_*^{\frac{\alpha}{2}}\tilde{\psi}\|^2 + C\|E_*^{\frac{\alpha}{2}}\tilde{w}\|^2 \\ + C\|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2 - \gamma_2\langle E_*^\theta\tilde{\psi}, E_*^\alpha(\varphi_x + \psi + lw)\rangle \\ - \frac{b}{\kappa}\gamma_1\langle\psi_x, E_*^{\alpha+\theta}\tilde{\varphi}\rangle - \frac{bl}{\kappa}\gamma_3\langle\psi, E_*^{\alpha+\theta}\tilde{w}\rangle + C\|F\|^2.$$

Proof. Taking the inner product of (4.1d) and $E_*^\alpha(\varphi_x + \psi + lw)$, using the fact that E^σ is self-adjoint for any $\sigma \in \mathbb{R}$, and applying (4.1c), we deduce that

$$\begin{aligned} &\kappa\|E_*^{\frac{\alpha}{2}}(\varphi_x + \psi + lw)\|^2 \\ &= \rho_2\lambda^2\langle\psi, E_*^\alpha(\varphi_x + \psi + lw)\rangle + \rho_2\langle(i\lambda f_3 + f_4), E_*^\alpha(\varphi_x + \psi + lw)\rangle \\ &\quad + b\langle\psi_{xx}, E_*^\alpha(\varphi_x + \psi + lw)\rangle - \gamma_2\langle E_*^\theta\tilde{\psi}, E_*^\alpha(\varphi_x + \psi + lw)\rangle. \end{aligned} \tag{4.10}$$

Next, we estimate the third term on the right of the equality. Using the definition of f_* and E, integration by parts, from (4.1b), (4.1a), (4.1e), and (4.1f) we have

$$\begin{aligned} &b\langle\psi_{xx}, E_*^\alpha(\varphi_x + \psi + lw)\rangle \\ &= -b\langle\psi_x, E_*^\alpha(\varphi_x + \psi + lw)_x\rangle \\ &= -\frac{b}{\kappa}\langle\psi_x, E_*^\alpha(i\lambda\rho_1\tilde{\varphi} - \kappa_0 l(w_x - l\varphi) + \gamma_1 E^\theta\tilde{\varphi} - \rho_1 f_2)\rangle \\ &= \frac{\rho_1 b}{\kappa}\langle\psi_x, E_*^\alpha(i\lambda f_1 + f_2)\rangle - \frac{\rho_1 b}{\kappa}\lambda^2\langle\psi, E_*^\alpha\varphi_x\rangle - \frac{\kappa_0 bl}{\kappa}\langle\psi, E_*^\alpha(w_x - l\varphi)_x\rangle \\ &\quad - \frac{b\gamma_1}{\kappa}\langle\psi_x, E_*^{\alpha+\theta}\tilde{\varphi}\rangle \\ &= \frac{\rho_1 b}{\kappa}\langle\psi_x, E_*^\alpha(i\lambda f_1 + f_2)\rangle - \frac{\rho_1 b}{\kappa}\lambda^2\langle\psi, E_*^\alpha\varphi_x\rangle - \frac{b\gamma_1}{\kappa}\langle\psi_x, E_*^{\alpha+\theta}\tilde{\varphi}\rangle \\ &\quad - \frac{bl}{\kappa}\langle\psi, E_*^\alpha(i\lambda\rho_1\tilde{w} + \kappa l(\varphi_x + \psi + lw) + \gamma_3 E_*^\theta\tilde{w} - \rho_1 f_6)\rangle \\ &= \frac{\rho_1 b}{\kappa}\langle\psi_x, E_*^\alpha(i\lambda f_1 + f_2)\rangle - \frac{\rho_1 b}{\kappa}\lambda^2\langle\psi, E_*^\alpha\varphi_x\rangle - \frac{b\gamma_1}{\kappa}\langle\psi_x, E_*^{\alpha+\theta}\tilde{\varphi}\rangle \\ &\quad + \frac{bl\rho_1}{\kappa}\lambda^2\langle\psi, E_*^\alpha w\rangle + \frac{bl\rho_1}{\kappa}\langle\psi, E_*^\alpha(i\lambda f_5 + f_6)\rangle - \frac{bl\gamma_3}{\kappa}\langle\psi, E_*^{\alpha+\theta}\tilde{w}\rangle \end{aligned}$$

$$- bl^2\langle\psi, E_*^{\alpha}(\varphi_x+\psi+lw)\rangle.$$

Substituting the above expression in (4.10), from the definition of (1.12) we obtain

$$\begin{aligned}
&\kappa\|E_*^{\frac{\alpha}{2}}(\varphi_x+\psi+lw)\|^2\\
&= |\chi_0|\frac{\rho_1\rho_2}{\kappa}\lambda^2\langle\psi, E_*^{\alpha}\varphi_x\rangle + \rho_2\lambda^2\|E_*^{\frac{\alpha}{2}}\psi\|^2 + (l\rho_2+\frac{\rho_1 bl}{\kappa})\lambda^2\langle\psi, E_*^{\alpha}w\rangle\\
&\quad + \frac{\rho_1 b}{\kappa}\langle\psi_x, E_*^{\alpha}(i\lambda f_1+f_2)\rangle + \rho_2\langle(i\lambda f_3+f_4), E_*^{\alpha}(\varphi_x+\psi+lw)\rangle\\
&\quad + \frac{\rho_1 bl}{\kappa}\langle\psi, E_*^{\alpha}(i\lambda f_5+f_6)\rangle - bl^2\langle\psi, E_*^{\alpha}(\varphi_x+\psi+lw)\rangle\\
&\quad - \frac{b\gamma_1}{\kappa}\langle\psi_x, E_*^{\alpha+\theta}\tilde{\varphi}\rangle - \gamma_2\langle E_*^{\theta}\tilde{\psi}, E_*^{\alpha}(\varphi_x+\psi+lw)\rangle - \frac{bl\gamma_3}{\kappa}\langle\psi, E_*^{\alpha+\theta}\tilde{w}\rangle.
\end{aligned} \tag{4.11}$$

Now, we estimate some items on the right side of formula (4.11). Using the self-adjointness of E_*^{σ} and (1.19), Cauchy-Schwarz and Young's inequalities, we have

$$\begin{aligned}
\frac{\rho_1 b}{\kappa}\langle\psi_x, E_*^{\alpha}(i\lambda f_1+f_2)\rangle &= \frac{\rho_1 b}{\kappa}\langle E_*^{\alpha}\psi_x, f_2\rangle - i\lambda\frac{\rho_1 b}{\kappa}\langle E_*^{\alpha}\psi_x, f_1\rangle\\
&= \frac{\rho_1 b}{\kappa}\langle E_*^{\alpha}\psi_x, f_2\rangle + i\lambda\frac{\rho_1 b}{\kappa}\langle E_*^{\alpha}\psi, (f_1)_x\rangle\\
&\le C\|E_*^{\alpha+\frac12}\psi\|\|F\| + C\lambda^2\|E_*^{\alpha}\psi\|^2 + C\|F\|^2\\
&\le C\lambda^2\|E_*^{\frac{\alpha}{2}}\psi\|^2 + C\|F\|\|U\| + C\|F\|^2,
\end{aligned}$$

since $\alpha<0$, then $\alpha+\frac12\le\frac12$ and $\alpha\le\frac{\alpha}{2}$. Considering the continuous embedding $D(E_*^{\sigma_1})\hookrightarrow D(E_*^{\sigma_2})$, for $\sigma_1\ge\sigma_2$, using equivalent norm and $\|E_*^{1/2}\psi\|\le\|U\|$ to obtain the above estimate. Similarly,

$$\frac{\rho_1 bl}{\kappa}\langle\psi, E_*^{\alpha}(i\lambda f_5+f_6)\rangle \le C\lambda^2\|E_*^{\frac{\alpha}{2}}\psi\|^2 + C\|F\|^2.$$

Using Cauchy-Schwarz and Young's inequalities, by (1.18)-(1.19) and the continuous embedding $\alpha-\frac12\le\frac{\alpha}{2}$, we have

$$\begin{aligned}
&\rho_2\langle(i\lambda f_3+f_4), E_*^{\alpha}(\varphi_x+\psi+lw)\rangle\\
&= \rho_2\langle(f_4), E_*^{\alpha}(\varphi_x+\psi+lw)\rangle + i\lambda\rho_2\langle f_3, E_*^{\alpha}(\varphi_x+\psi+lw)\rangle\\
&\le C\|E_*^{\alpha}(\varphi_x+\psi+lw)\|\|F\| + C|\lambda|\|E_*^{1/2}f_3\|\|E_*^{\alpha-\frac12}(\varphi_x+\psi+lw)\|\\
&\le C\|E_*^{\alpha}(\varphi_x+\psi+lw)\|^2 + C\lambda^2\|E^{\alpha}\varphi\|^2 + C\lambda^2\|E_*^{\alpha-\frac12}\psi\|^2\\
&\quad + C\lambda^2\|E_*^{\alpha-\frac12}w\|^2 + C\|F\|^2\\
&\le C\|E_*^{\frac{\alpha}{2}}(\varphi_x+\psi+lw)\|^2 + C\lambda^2\|E^{\frac{\alpha}{2}}\varphi\|^2 + C\lambda^2\|E_*^{\frac{\alpha}{2}}\psi\|^2\\
&\quad + C\lambda^2\|E_*^{\frac{\alpha}{2}}w\|^2 + C\|F\|^2.
\end{aligned}$$

We also have from Cauchy-Schwarz and Young's inequalities

$$\begin{aligned}
(l\rho_2+\frac{\rho_1 bl}{\kappa})\lambda^2\langle\psi, E_*^{\alpha}w\rangle &= (l\rho_2+\frac{\rho_1 bl}{\kappa})\lambda^2\langle E_*^{\frac{\alpha}{2}}\psi, E_*^{\frac{\alpha}{2}}w\rangle\\
&\le C\lambda^2\|E_*^{\frac{\alpha}{2}}\psi\|^2 + C\lambda^2\|E_*^{\frac{\alpha}{2}}w\|^2,
\end{aligned}$$

$$
\begin{aligned}
|bl^2\langle\psi, E_*^{\alpha}(\varphi_x+\psi+lw)\rangle| &= |bl^2\langle E_*^{\frac{\alpha}{2}}\psi, E_*^{\frac{\alpha}{2}}(\varphi_x+\psi+lw)\rangle| \\
&\le C\|E_*^{\frac{\alpha}{2}}\psi\|^2 + C\|E_*^{\frac{\alpha}{2}}(\varphi_x+\psi+lw)\|^2.
\end{aligned}
$$

Substituting the above estimates into (4.11), using the continuous embedding and (4.1a), (4.1c) and (4.1e), we conclude that

$$
\begin{aligned}
&\|E_*^{\frac{\alpha}{2}}(\varphi_x+\psi+lw)\|^2 \\
&\le \frac{\rho_1\rho_2}{\kappa}|\chi_0|\lambda^2\langle\psi, E_*^{\alpha}\varphi_x\rangle + C\lambda^2\|E_*^{\frac{\alpha}{2}}\psi\|^2 + C\lambda^2\|E_*^{\frac{\alpha}{2}}w\|^2 + C\lambda^2\|E^{\frac{\alpha}{2}}\varphi\|^2 \\
&\quad - \frac{b}{\kappa}\gamma_1\langle\psi_x, E_*^{\alpha+\theta}\tilde\varphi\rangle - \gamma_2\langle E_*^{\theta}\tilde\psi, E_*^{\alpha}(\varphi_x+\psi+lw)\rangle - \frac{bl}{\kappa}\gamma_3\langle\psi, E_*^{\alpha+\theta}\tilde w\rangle + C\|F\|^2 \\
&\le \frac{\rho_1\rho_2}{\kappa}|\chi_0|\lambda^2\langle\psi, E_*^{\alpha}\varphi_x\rangle + C\|E_*^{\frac{\alpha}{2}}\tilde\psi\|^2 + C\|E_*^{\frac{\alpha}{2}}\tilde w\|^2 + C\|E^{\frac{\alpha}{2}}\tilde\varphi\|^2 \\
&\quad - \frac{b}{\kappa}\gamma_1\langle\psi_x, E_*^{\alpha+\theta}\tilde\varphi\rangle - \gamma_2\langle E_*^{\theta}\tilde\psi, E_*^{\alpha}(\varphi_x+\psi+lw)\rangle - \frac{bl}{\kappa}\gamma_3\langle\psi, E_*^{\alpha+\theta}\tilde w\rangle + C\|F\|^2.
\end{aligned}
$$

The proof is complete. □

Lemma 4.6. *Let $\delta > 0$ and $\alpha \le 0$. There exists positive constant C_δ, such that for $|\lambda| \ge \delta$ the solution $U = (\varphi, \tilde\varphi, \psi, \tilde\psi, w, \tilde w)^T$ of system* (4.1a)-(4.1f) *satisfies the following estimates:*

$$
\begin{aligned}
&\|E_*^{\frac{\alpha}{2}}\tilde w\|^2 + \|E_*^{\frac{\alpha}{2}}(w_x - l\varphi)\|^2 \\
&\le |\chi_1|\rho_1\lambda^2\langle\varphi, E_*^{\alpha}w_x\rangle + C\|E_*^{\frac{\alpha}{2}}\tilde\psi\|^2 + C\|E^{\frac{\alpha}{2}}\tilde\varphi\|^2 + C\|E_*^{\frac{\alpha}{2}}(\varphi_x+\psi+lw)\|^2 \\
&\quad + \gamma_1\langle E^{\theta}\tilde\varphi, E_*^{\alpha}(w_x - l\varphi)\rangle + \frac{\kappa\gamma_3}{\kappa_0}\langle\varphi_x+\psi+lw, E_*^{\alpha+\theta}\tilde w\rangle + C\|F\|^2.
\end{aligned}
$$

Proof. Performing the duality product between (4.1b) and $E_*^{\alpha}(w_x - l\varphi)$, using (4.1a), and integrating by parts, we obtain

$$
\begin{aligned}
&\kappa_0 l\|E_*^{\frac{\alpha}{2}}(w_x - l\varphi)\|^2 \\
&= \langle i\lambda\rho_1\tilde\varphi - \kappa(\varphi_x+\psi+lw)_x + \gamma_1E^{\theta}\tilde\varphi - \rho_1f_2, E_*^{\alpha}(w_x - l\varphi)\rangle \\
&= \rho_1\langle -\lambda^2\varphi - i\lambda f_1, E_*^{\alpha}(w_x - l\varphi)\rangle - \rho_1\langle f_2, E_*^{\alpha}(w_x - l\varphi)\rangle \\
&\quad + \kappa\langle(\varphi_x+\psi+lw), E_*^{\alpha}(w_x - l\varphi)_x\rangle + \gamma_1\langle E^{\theta}\tilde\varphi, E_*^{\alpha}(w_x - l\varphi)\rangle \\
&= -\rho_1\lambda^2\langle\varphi, E_*^{\alpha}w_x\rangle - \rho_1\langle i\lambda f_1 + f_2, E_*^{\alpha}(w_x - l\varphi)\rangle + \rho_1 l\lambda^2\|E^{\frac{\alpha}{2}}\varphi\|^2 \\
&\quad + \gamma_1\langle E^{\theta}\tilde\varphi, E_*^{\alpha}(w_x - l\varphi)\rangle + \kappa\langle(\varphi_x+\psi+lw), E_*^{\alpha}(w_x - l\varphi)_x\rangle.
\end{aligned} \tag{4.12}
$$

Substituting (4.1f) into the last item of the above estimate, and using (4.1e) yield

$$\begin{aligned}
&\kappa\langle(\varphi_x+\psi+lw),E_*^\alpha(w_x-l\varphi)_x\rangle\\
&=\kappa\langle(\varphi_x+\psi+lw),\frac{1}{\kappa_0}E_*^\alpha(i\lambda\rho_1\tilde{w}+\kappa l(\varphi_x+\psi+lw)+\gamma_3E_*^\theta\tilde{w}-\rho_1f_6)\rangle\\
&=\frac{\rho_1\kappa}{\kappa_0}\langle(\varphi_x+\psi+lw),E_*^\alpha(-\lambda^2w-i\lambda f_5)\rangle+\frac{\kappa^2l}{\kappa_0}\|E_*^{\frac{\alpha}{2}}(\varphi_x+\psi+lw)\|^2\\
&\quad+\frac{\kappa\gamma_3}{\kappa_0}\langle(\varphi_x+\psi+lw),E_*^{\alpha+\theta}\tilde{w}\rangle-\frac{\kappa\rho_1}{\kappa_0}\langle(\varphi_x+\psi+lw),E_*^\alpha f_6\rangle\\
&=\frac{\rho_1\kappa}{\kappa_0}\lambda^2\langle\varphi,E_*^\alpha w_x\rangle-\frac{\rho_1\kappa}{\kappa_0}\lambda^2\langle\psi,E_*^\alpha w\rangle+\frac{\kappa\gamma_3}{\kappa_0}\langle(\varphi_x+\psi+lw),E_*^{\alpha+\theta}\tilde{w}\rangle\\
&\quad+\frac{\kappa^2l}{\kappa_0}\|E_*^{\frac{\alpha}{2}}(\varphi_x+\psi+lw)\|^2-\frac{\kappa\rho_1}{\kappa_0}\langle(\varphi_x+\psi+lw),E_*^\alpha(i\lambda f_5+f_6)\rangle\\
&\quad-\frac{\rho_1\kappa l}{\kappa_0}\lambda^2\langle w,E_*^\alpha w\rangle.
\end{aligned}\tag{4.13}$$

Substituting (4.13) into (4.12), from the definition of (1.12) we have

$$\begin{aligned}
&\kappa_0l\|E_*^{\frac{\alpha}{2}}(w_x-l\varphi)\|^2\\
&=|\chi_1|\rho_1\lambda^2\langle\varphi,E_*^\alpha w_x\rangle+\rho_1l\lambda^2\|E^{\frac{\alpha}{2}}\varphi\|^2+\frac{\kappa^2l}{\kappa_0}\|E_*^{\frac{\alpha}{2}}(\varphi_x+\psi+lw)\|^2\\
&\quad-\frac{\rho_1\kappa l}{\kappa_0}\lambda^2\|E_*^{\frac{\alpha}{2}}w\|^2-\frac{\rho_1\kappa}{\kappa_0}\lambda^2\langle\psi,E_*^\alpha w\rangle-\rho_1\langle i\lambda f_1+f_2,E_*^\alpha(w_x-l\varphi)\rangle\\
&\quad-\frac{\kappa\rho_1}{\kappa_0}\langle(\varphi_x+\psi+lw),E_*^\alpha(i\lambda f_5+f_6)\rangle+\frac{\kappa\gamma_3}{\kappa_0}\langle(\varphi_x+\psi+lw),E_*^{\alpha+\theta}\tilde{w}\rangle\\
&\quad+\gamma_1\langle E^\theta\tilde{\varphi},E_*^\alpha(w_x-l\varphi)\rangle.
\end{aligned}\tag{4.14}$$

Now, we estimate some terms on the right-hand side of (4.14). Using the self-adjointness of E_*^σ and (1.19), Cauchy-Schwarz and Young's inequalities, we have

$$\begin{aligned}
&|\rho_1\langle i\lambda f_1+f_2,E_*^\alpha(w_x-l\varphi)\rangle|\\
&=|\rho_1\langle f_2,E_*^\alpha(w_x-l\varphi)\rangle|+|\rho_1\langle i\lambda f_1,E_*^\alpha(w_x-l\varphi)\rangle|\\
&=|\rho_1\langle f_2,E_*^\alpha(w_x-l\varphi)\rangle|+|\rho_1\langle i\lambda E_*^{1/2}f_1,E_*^{\alpha-\frac{1}{2}}w_x\rangle-|l\rho_1\langle i\lambda f_1,E^\alpha\varphi\rangle\|\\
&\le C\|F\|\|E_*^\alpha(w_x-l\varphi)\|+C|\lambda|\|F\|\|E_*^\alpha w\|+C|\lambda|\|F\|\|E^\alpha\varphi\|\\
&\le C\|F\|^2+C\|E_*^\alpha(w_x-l\varphi)\|^2+C\lambda^2\|E_*^\alpha w\|^2+C\lambda^2\|E^\alpha\varphi\|^2,
\end{aligned}$$

and

$$|\frac{\rho_1\kappa}{\kappa_0}\lambda^2\langle(\psi,E_*^\alpha w)|=|\frac{\rho_1\kappa}{\kappa_0}\langle\lambda E_*^{\frac{\alpha}{2}}\psi,\lambda E_*^{\frac{\alpha}{2}}w\rangle|\le C\lambda^2\|E_*^{\frac{\alpha}{2}}\psi\|^2+C\lambda^2\|E_*^{\frac{\alpha}{2}}w\|^2.$$

Considering the continuous embedding $D(E_*^{\sigma_1})\hookrightarrow D(E_*^{\sigma_2})$ for $\sigma_1\ge\sigma_2$, using (1.19), Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned}
&|\frac{\kappa\rho_1}{\kappa_0}\langle(\varphi_x+\psi+lw),E_*^\alpha(i\lambda f_5+f_6)\rangle|\\
&=|\frac{\kappa\rho_1}{\kappa_0}\langle(\varphi_x+\psi+lw),E_*^\alpha f_6\rangle|+|\frac{\kappa\rho_1}{\kappa_0}\langle i\lambda(\varphi_x+\psi+lw),E_*^\alpha f_5\rangle|\\
&=|\frac{\kappa\rho_1}{\kappa_0}\langle(\varphi_x+\psi+lw),E_*^\alpha f_6\rangle|-\frac{\kappa\rho_1}{\kappa_0}i\lambda\langle E^\alpha\varphi,(f_5)_x\rangle+\frac{\kappa\rho_1}{\kappa_0}i\lambda\langle E_*^\alpha\psi,f_5\rangle\\
&\quad+\frac{\kappa\rho_1}{\kappa_0}i\lambda l\langle E_*^\alpha w,f_5\rangle
\end{aligned}$$

$$\le C\|F\|^2 + C\|E_*^\alpha(\varphi_x+\psi+lw)\|^2 + C\lambda^2\|E_*^\alpha w\|^2 + C\lambda^2\|E^\alpha\varphi\|^2 + C\lambda^2\|E_*^\alpha\psi\|^2.$$

Substituting the above estimates in (4.14), using the continuous embedding, $\alpha \le \alpha/2$, and (4.1a), (4.1c), (4.1e), we conclude that

$$\begin{aligned}
&\|E_*^{\frac{\alpha}{2}}\tilde{w}\|^2 + \|E_*^{\frac{\alpha}{2}}(w_x - l\varphi)\|^2 \\
&\le |\chi_1|\rho_1\lambda^2\langle\varphi, E_*^\alpha w_x\rangle + C\lambda^2\|E_*^{\frac{\alpha}{2}}\psi\|^2 + C\lambda^2\|E^{\frac{\alpha}{2}}\varphi\|^2 + C\|E_*^{\frac{\alpha}{2}}(\varphi_x+\psi+lw)\|^2 \\
&\quad + \gamma_1\langle E^\theta\tilde{\varphi}, E_*^\alpha(w_x - l\varphi)\rangle + \frac{\kappa\gamma_3}{\kappa_0}\langle\varphi_x+\psi+lw, E_*^{\alpha+\theta}\tilde{w}\rangle + \|F\|^2 \\
&\le |\chi_1|\rho_1\lambda^2\langle\varphi, E_*^\alpha w_x\rangle + C\|E_*^{\frac{\alpha}{2}}\tilde{\psi}\|^2 + C\|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2 + C\|E_*^{\frac{\alpha}{2}}(\varphi_x+\psi+lw)\|^2 \\
&\quad + \gamma_1\langle E^\theta\tilde{\varphi}, E_*^\alpha(w_x - l\varphi)\rangle + \frac{\kappa\gamma_3}{\kappa_0}\langle\varphi_x+\psi+lw, E_*^{\alpha+\theta}\tilde{w}\rangle + \|F\|^2.
\end{aligned}$$

The proof is complete. □

Lemma 4.7. *The solution $U = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w})^T$ of system* (4.1a)-(4.1f) *satisfies*

$$\|E_*^{1/2}\psi\|^2 = \|\psi_x\|^2 \le C_\delta\big(\|\tilde{\psi}\|^2 + \|\varphi\|^2 + \|w\|^2 + \|F\|\|U\| + \|F\|^2\big).$$

Proof. Taking the inner product of (4.1d) with ψ, using (1.19) and integrating by parts, we obtain

$$\begin{aligned}
b\|E_*^{1/2}\psi\|^2 &= b\|\psi_x\|^2 \\
&= -i\lambda\rho_2\langle\tilde{\psi},\psi\rangle - \kappa\langle(\varphi_x+\psi+lw),\psi\rangle - \gamma_2\langle E_*^\theta\tilde{\psi},\psi\rangle + \rho_2\langle f_4,\psi\rangle \\
&= \rho_2\|\tilde{\psi}\|^2 + \rho_2\langle\tilde{\psi}, f_3\rangle - \kappa\langle\varphi,\psi_x\rangle + \kappa\|\psi\|^2 + l\kappa\langle w,\psi\rangle \\
&\quad - \gamma_2\langle E_*^{\theta/2}\tilde{\psi}, E_*^{\theta/2}\psi\rangle + \rho_2\langle f_4,\psi\rangle.
\end{aligned}$$

Applying Cauchy-Schwarz and Young's inequalities, by (4.4) and (4.1c) we have

$$\begin{aligned}
b\|E_*^{1/2}\psi\|^2 &= b\|\psi_x\|^2 \\
&\le C\big(\|\tilde{\psi}\|^2 + \|\varphi\|^2 + \|w\|^2 + \|\psi\|^2 + \|E_*^{\theta/2}\tilde{\psi}\|^2 + \|E_*^{\theta/2}\psi\|^2\big) \\
&\le C\big(\|\tilde{\psi}\|^2 + \|\varphi\|^2 + \|w\|^2 + \|F\|\|U\| + \|F\|^2\big),
\end{aligned}$$

which completes the proof. □

Next, we prove the asymptotic stability of system (1.7)-(1.9) when the fractional damping acts on two of the equations. Firstly we consider the case $\gamma_1, \gamma_2 > 0$ and $\gamma_3 = 0$.

Lemma 4.8. *Let $\delta > 0$ and $\gamma_1, \gamma_2 > 0$, $\gamma_3 = 0$. There exists positive constant C_δ, such that for $|\lambda| \ge \delta$ the solution $U = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w})^T$ of system* (4.1a)-(4.1f) *satisfies the following:*

(i) $\|\tilde{w}\|^2 \le C_\delta\lambda^{2\theta}\big(\|F\|\|U\| + \|F\|^2\big)$ *when* $\chi_1 = 0$,

(ii) $\|\tilde{w}\|^2 \le C_\delta\lambda^{2-2\theta}\big(\|F\|\|U\| + \|F\|^2\big)$ *when* $\chi_1 \ne 0$ *and* $\theta \le 1/2$,

(iii) $\|\tilde{w}\|^2 \le C_\delta\lambda^{2\theta}\big(\|F\|\|U\| + \|F\|^2\big)$ *when* $\chi_1 \ne 0$ *and* $\theta \ge 1/2$.

Proof. Adding (4.1a), (4.1c) and (4.1e), we obtain

$$i\lambda(\varphi_x+\psi+lw) = (\tilde{\varphi}_x + \tilde{\psi} + l\tilde{w}) + (f_1)_x + f_3 + f_5.$$

The inner product of the above formula and $\kappa E_*^{\alpha}(\varphi_x+\psi+lw)$, by partial integral and (4.1b) to deduce

$$\begin{aligned}
&\kappa\|E_*^{\frac{\alpha}{2}}(\varphi_x+\psi+lw)\|^2\\
&=\frac{\kappa}{i\lambda}\langle(\tilde\varphi_x+\tilde\psi+l\tilde w)+(f_1)_x+f_3+f_5,E_*^{\alpha}(\varphi_x+\psi+lw)\rangle\\
&=\frac{1}{i\lambda}\langle\tilde\varphi,E_*^{\alpha}(-i\lambda\rho_1\tilde\varphi+\kappa_0 l(w_x-l\varphi)-\gamma_1E^{\theta}\tilde\varphi+\rho_1f_2)\rangle\\
&\quad+\frac{\kappa}{i\lambda}\langle(\tilde\psi+l\tilde w)+(f_1)_x+f_3+f_5,E_*^{\alpha}(\varphi_x+\psi+lw)\rangle\\
&=\rho_1\|E_*^{\frac{\alpha}{2}}\tilde\varphi\|^2+\frac{\gamma_1}{i\lambda}\|E_*^{\frac{\alpha}{2}+\frac{\theta}{2}}\tilde\varphi\|^2+\frac{\kappa_0 l}{i\lambda}\langle\tilde\varphi,E_*^{\alpha}(w_x-l\varphi)\rangle+\frac{\rho_1}{i\lambda}\langle\tilde\varphi,E_*^{\alpha}f_2\rangle\\
&\quad+\frac{\kappa}{i\lambda}\langle(\tilde\psi+l\tilde w)+(f_1)_x+f_3+f_5,E_*^{\alpha}(\varphi_x+\psi+lw)\rangle\\
&\le C\|E_*^{\frac{\alpha}{2}}\tilde\varphi\|^2+\frac{C}{\lambda}\|E_*^{\theta/2}\tilde\varphi\|^2+C\|E_*^{\frac{\alpha}{2}}\psi\|^2+C\|E_*^{\frac{\alpha}{2}}w\|^2\\
&\quad+\frac{C}{\lambda^2}\|E_*^{\frac{\alpha}{2}}(w_x-l\varphi)\|^2+\|F\|^2.
\end{aligned}\tag{4.15}$$

The last step follows from the self-adjointness of E_*^{σ}, Cauchy-Schwarz and Young's inequalities, the continuous embedding and (4.1c), (4.1e). It can be seen from (4.15) that the estimate here is different from Lemma 4.5, which is independent of χ_0.

Substituting (4.15) into the result of Lemma 4.6, note that $\gamma_3=0$, using continuous embedding and (4.1c), (4.1e), (4.3), we obtain

$$\begin{aligned}
&\|E_*^{\frac{\alpha}{2}}\tilde w\|^2+\|E_*^{\frac{\alpha}{2}}(w_x-l\varphi)\|^2\\
&\le|\chi_1|\rho_1\lambda^2\langle\varphi,E_*^{\alpha}w_x\rangle+C\|E_*^{\frac{\alpha}{2}}\tilde\psi\|^2+C\|E^{\frac{\alpha}{2}}\tilde\varphi\|^2+C\|E_*^{\frac{\alpha}{2}}(\varphi_x+\psi+lw)\|^2\\
&\quad+\gamma_1\langle E^{\theta}\tilde\varphi,E_*^{\alpha}(w_x-l\varphi)\rangle+C\|F\|^2\\
&\le|\chi_1|\rho_1\lambda^2\langle\varphi,E_*^{\alpha}w_x\rangle+C\|E_*^{\frac{\alpha}{2}}\tilde\psi\|^2+C\|E^{\frac{\alpha}{2}}\tilde\varphi\|^2+\frac{C}{\lambda}\|E_*^{\theta/2}\tilde\varphi\|^2+C\|E_*^{\frac{\alpha}{2}}\psi\|^2\\
&\quad+C\|E_*^{\frac{\alpha}{2}}w\|^2+C\|E^{\theta/2}\tilde\varphi\|^2+C\|E^{\alpha+\frac{\theta}{2}}(w_x-l\varphi)\|^2+\|F\|^2\\
&\le|\chi_1|\rho_1\lambda^2\langle\varphi,E_*^{\alpha}w_x\rangle+C\|E_*^{\frac{\alpha}{2}}\tilde\psi\|^2+C\|E^{\frac{\alpha}{2}}\tilde\varphi\|^2+C\|E^{\alpha+\frac{\theta}{2}}(w_x-l\varphi)\|^2\\
&\quad+\|F\|\,\|U\|+\|F\|^2.
\end{aligned}\tag{4.16}$$

When $\chi_1=0$, taking $\alpha=-\theta$ in (4.16), then from the continuous embedding we have

$$\begin{aligned}
\|E_*^{-\theta/2}\tilde w\|^2+\|E_*^{-\theta/2}(w_x-l\varphi)\|^2&\le C\|E_*^{-\theta/2}\tilde\psi\|^2+C\|E^{-\theta/2}\tilde\varphi\|^2+\|F\|\|U\|+\|F\|^2\\
&\le C\big(\|F\|\|U\|+\|F\|^2\big),
\end{aligned}$$

since $-\theta/2<\theta/2$, we have (4.3) and (4.4). So, we obtain

$$\|E_*^{-\theta/2}\tilde w\|^2\le C\big(\|F\|\|U\|+\|F\|^2\big),\tag{4.17}$$

$$\|E_*^{-\theta/2}(w_x-l\varphi)\|^2\le C\big(\|F\|\|U\|+\|F\|^2\big).\tag{4.18}$$

On the other hand, we have

$$\|E_*^{-\theta/2}w_x\|^2\le C\big(\|E_*^{-\theta/2}(w_x-l\varphi)\|^2+\|E_*^{-\theta/2}\varphi\|^2\big).$$

Then using (1.19) and (4.18) we have

$$\|E_*^{\frac{1}{2}-\frac{\theta}{2}} w\|^2 = \|E_*^{-\theta/2} w_x\|^2 \le C\big(\|F\|\|U\| + \|F\|^2\big). \tag{4.19}$$

From (4.1e) we have $\tilde{w} = i\lambda w - f_5$, then by (4.19) we obtain

$$\begin{aligned} \|E_*^{\frac{1}{2}-\frac{\theta}{2}} \tilde{w}\|^2 &= \|E_*^{\frac{1}{2}-\frac{\theta}{2}} (i\lambda w - f_5)\|^2 \\ &\le C\lambda^2 \|E_*^{\frac{1}{2}-\frac{\theta}{2}} w\|^2 + C\|F\|^2 \\ &\le C_\delta \lambda^2 \big(\|F\|\|U\| + \|F\|^2\big). \end{aligned} \tag{4.20}$$

Then, by applying the interpolation inequalities, from (4.18) and (4.20) we conclude that

$$\begin{aligned} \|\tilde{w}\| &\le \|E_*^{-\theta/2} \tilde{w}\|^{1-\theta} \|E_*^{\frac{1}{2}-\frac{\theta}{2}} \tilde{w}\|^{\theta} \\ &\le C_\delta \big(\sqrt{\|F\|\|U\| + \|F\|^2}\big)^{1-\theta} \big(|\lambda| \sqrt{\|F\|\|U\| + \|F\|^2}\big)^{\theta} \\ &\le C_\delta |\lambda|^{\theta} \sqrt{\|F\|\|U\| + \|F\|^2}. \end{aligned} \tag{4.21}$$

This yields

$$\|\tilde{w}\| \le C_\delta \lambda^{2\theta} \big(\|F\|\|U\| + \|F\|^2\big).$$

This leads to the first result of this Lemma.

When $\chi_1 \neq 0$, we have

$$\begin{aligned} &\|E_*^{\frac{\alpha}{2}} \tilde{w}\|^2 + \|E_*^{\frac{\alpha}{2}} (w_x - l\varphi)\|^2 \\ &\le C\lambda^2 \langle \varphi, E_*^{\alpha} w_x \rangle + C\|E_*^{\frac{\alpha}{2}} \tilde{\psi}\|^2 + C\|E^{\frac{\alpha}{2}} \tilde{\varphi}\|^2 + C\|E^{\alpha+\frac{\theta}{2}} (w_x - l\varphi)\|^2 \\ &\quad + \|F\|\|U\| + \|F\|^2 \\ &\le C\lambda^2 \langle \varphi, E_*^{\alpha} (w_x - l\varphi) \rangle + C\lambda^2 \langle \varphi, E_*^{\alpha} \varphi \rangle + C\|E_*^{\frac{\alpha}{2}} \tilde{\psi}\|^2 + C\|E^{\frac{\alpha}{2}} \tilde{\varphi}\|^2 \\ &\quad + C\|E^{\alpha+\frac{\theta}{2}} (w_x - l\varphi)\|^2 + \|F\|\|U\| + \|F\|^2 \\ &\le C\lambda^2 \|E^{\frac{\alpha}{2}} \tilde{\varphi}\|^2 + C\|E_*^{\frac{\alpha}{2}} \tilde{\psi}\|^2 + C\|E^{\alpha+\frac{\theta}{2}} (w_x - l\varphi)\|^2 + \|F\|\|U\| + \|F\|^2. \end{aligned} \tag{4.22}$$

Considering $\alpha = \theta - 1$ and $\theta \le 1/2$ in (4.22), then $\frac{\theta}{2} + \alpha \le \frac{\alpha}{2}$, from the continuous embedding we obtain

$$\begin{aligned} &\|E_*^{\frac{\theta}{2}-\frac{1}{2}} \tilde{w}\|^2 + \|E_*^{\frac{\theta}{2}-\frac{1}{2}} (w_x - l\varphi)\|^2 \\ &\le C\lambda^2 \|E_*^{\frac{\theta}{2}-\frac{1}{2}} \tilde{\varphi}\|^2 + C\|E_*^{\frac{\theta}{2}-\frac{1}{2}} \tilde{\psi}\|^2 + \|F\|\|U\| + \|F\|^2. \end{aligned}$$

Taking into account $\alpha = \theta - 1$ in the item (i) of Lemma 4.3, we obtain

$$\begin{aligned} \lambda^2 \|E^{\frac{\theta}{2}-\frac{1}{2}} \tilde{\varphi}\|^2 &\le C\big(\|E^{\theta/2} \tilde{\varphi}\|^2 + \|E_*^{\frac{\theta}{2}-\frac{1}{2}} \tilde{\psi}\|^2 + \|E_*^{\frac{\theta}{2}-\frac{1}{2}} \tilde{w}\|^2\big) + \|F\|\|U\| + \|F\|^2 \\ &\le C\|E_*^{\frac{\theta}{2}-\frac{1}{2}} \tilde{w}\|^2 + C\big(\|F\|\|U\| + \|F\|^2\big), \end{aligned}$$

since $\theta \le 1/2$, then $\alpha + \frac{\theta}{2} \le \frac{\alpha}{2}$, $\frac{\theta}{2} - \frac{1}{2} < \frac{\theta}{2}$ and (4.4). Substituting the last formula to obtain

$$\|E_*^{\frac{\theta}{2}-\frac{1}{2}} \tilde{w}\|^2 + \|E_*^{\frac{\theta}{2}-\frac{1}{2}} (w_x - l\varphi)\|^2 \le C\big(\|F\|\|U\| + \|F\|^2\big);$$

that is,

$$\|E_*^{\frac{\theta}{2}-\frac{1}{2}} \tilde{w}\|^2 \le C\big(\|F\|\|U\| + \|F\|^2\big), \tag{4.23}$$

$$\|E_*^{\frac{\theta}{2}-\frac{1}{2}} (w_x - l\varphi)\|^2 \le C\big(\|F\|\|U\| + \|F\|^2\big), \tag{4.24}$$

Similar to the estimates of (4.19) and (4.6), we deduce that

$$\|E_*^{\theta/2}\tilde{w}\|^2 \le C\lambda^2\big(\|F\|\|U\| + \|F\|^2\big), \tag{4.25}$$

Then, by applying the interpolation inequalities, from (4.23) and (4.25) we conclude that

$$\begin{aligned}\|\tilde{w}\| &\le \|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{w}\|^\theta \|E_*^{\theta/2}\tilde{w}\|^{1-\theta}\\ &\le C_\delta\big(\sqrt{\|F\|\|U\| + \|F\|^2}\big)^\theta \big(|\lambda|\sqrt{\|F\|\|U\| + \|F\|^2}\big)^{1-\theta}\\ &\le C_\delta|\lambda|^{1-\theta}\sqrt{\|F\|\|U\| + \|F\|^2}.\end{aligned} \tag{4.26}$$

Thus it yields

$$\|\tilde{w}\| \le C_\delta\lambda^{2-2\theta}\big(\|F\|\|U\| + \|F\|^2\big).$$

This leads to the second result of this Lemma.

For the third result of this lemma, that is, $\theta \ge 1/2$, considering $\alpha = -\theta$ in (4.22), then $\frac{\theta}{2} + \alpha \le \frac{\alpha}{2}$, because of the continuous embedding, we deduce that

$$\begin{aligned}&\|E_*^{-\theta/2}\tilde{w}\|^2 + \|E_*^{-\theta/2}(w_x - l\varphi)\|^2\\ &\le C\lambda^2\|E_*^{-\theta/2}\tilde{\varphi}\|^2 + C\|E_*^{-\theta/2}\tilde{\psi}\|^2 + \|F\|\|U\| + \|F\|^2.\end{aligned}$$

Taking into account $\alpha = -\theta$ in the item (i) of Lemma 4.3, we obtain

$$\begin{aligned}\lambda^2\|E_*^{-\theta/2}\tilde{\varphi}\|^2 &\le C\big(\|E_*^{\frac{1}{2}-\frac{\theta}{2}}\tilde{\varphi}\|^2 + \|E_*^{-\theta/2}\tilde{\psi}\|^2 + \|E_*^{-\theta/2}\tilde{w}\|^2\big) + \|F\|\|U\| + \|F\|^2\\ &\le C\|E_*^{-\theta/2}\tilde{w}\|^2\big) + C\big(\|F\|\|U\| + \|F\|^2\big),\end{aligned}$$

since $\theta \ge 1/2$, then $\alpha + \frac{\theta}{2} \le \frac{\alpha}{2}$, $\frac{1}{2} - \frac{\theta}{2} \le \frac{\theta}{2}$ and (4.4). furthermore, $-\frac{\theta}{2} < \frac{\theta}{2}$ is an identity. Substituting the last formula to obtain

$$\|E_*^{-\theta/2}\tilde{w}\|^2 + \|E_*^{-\theta/2}(w_x - l\varphi)\|^2 \le C\big(\|F\|\|U\| + \|F\|^2\big);$$

that is,

$$\|E_*^{-\theta/2}\tilde{w}\|^2 \le C\big(\|F\|\|U\| + \|F\|^2\big), \tag{4.27}$$

$$\|E_*^{-\theta/2}(w_x - l\varphi)\|^2 \le C\big(\|F\|\|U\| + \|F\|^2\big), \tag{4.28}$$

Similar to the estimates in the previous two parts, we have

$$\|E_*^{\frac{1}{2}-\frac{\theta}{2}}\tilde{w}\|^2 \le C\lambda^2\big(\|F\|\|U\| + \|F\|^2\big), \tag{4.29}$$

Using the same interpolation inequalities as in (4.21), and applying (4.27), (4.29) we conclude that

$$\|\tilde{w}\| \le C_\delta\lambda^{2\theta}\big(\|F\|\|U\| + \|F\|^2\big).$$

The third conclusion of this Lemma has also been proved. So the proof is complete. □

Secondly we consider the case $\gamma_1, \gamma_3 > 0$, and $\gamma_2 = 0$.

Lemma 4.9. *Let $\delta > 0$ and $\gamma_1, \gamma_3 > 0$, $\gamma_2 = 0$. then there exists positive constant C_δ, such that for $|\lambda| \ge \delta$ the solution $U = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w})^T$ of system* (4.1a)-(4.1f) *satisfies*

(i) $\|\tilde{\psi}\|^2 \le C_\delta\lambda^{2\theta}\big(\|F\|\|U\| + \|F\|^2\big)$ *when* $\chi_0 = 0$,

(ii) $\|\tilde{\psi}\|^2 \le C_\delta\lambda^{2-2\theta}\big(\|F\|\|U\| + \|F\|^2\big)$ *when* $\chi_0 \ne 0$ *and* $\theta \le 1/2$,

(iii) $\|\tilde{\psi}\|^2 \le C_\delta\lambda^{2\theta}\big(\|F\|\|U\| + \|F\|^2\big)$ *when* $\chi_0 \ne 0$ *and* $\theta \ge 1/2$.

Proof. Taking the inner product of (4.1b) with $E_*^\alpha\psi_x$ to deduce that

$$\begin{aligned}
&\kappa\|E_*^{\frac{\alpha}{2}}\psi_x\|^2\\
&=\langle i\lambda\rho_1\tilde\varphi-\kappa(\varphi_x+lw)_x-\kappa_0 l(w_x-l\varphi)+\gamma_1E^\theta\tilde\varphi-\rho_1f_2,E_*^\alpha\psi_x\rangle\\
&=-\rho_1\lambda^2\langle\varphi,E_*^\alpha\psi_x\rangle-i\lambda\rho_1\langle f_1,E_*^\alpha\psi_x\rangle-\kappa\langle\varphi_{xx},E_*^\alpha\psi_x\rangle\\
&\quad-l(\kappa+\kappa_0)\langle w_x,E_*^\alpha\psi_x\rangle+\kappa_0l^2\langle\varphi,E_*^\alpha\psi_x\rangle+\gamma_1\langle E_*^{\theta/2}\tilde\varphi,E_*^{\alpha+\frac{\theta}{2}}\psi_x\rangle\\
&\quad-\rho_1\langle f_2,E_*^\alpha\psi_x\rangle.
\end{aligned}\tag{4.30}$$

Next, we estimate the item $-\kappa\langle\varphi_{xx},E_*^\alpha\psi_x\rangle$ in (4.30). For this end, taking the inner product of (4.1d) with $\kappa E_*^\alpha\varphi_x$, noting that $\gamma_2=0$, we have

$$\begin{aligned}
-\kappa\langle\varphi_{xx},E_*^\alpha\psi_x\rangle&=\frac{\kappa}{b}\langle i\lambda\rho_2\tilde\psi+\kappa(\varphi_x+\psi+lw)-\rho_2f_4,E_*^\alpha\varphi_x\rangle\\
&=-\frac{\kappa\rho_2}{b}\lambda^2\langle E_*^\alpha\psi,\varphi_x\rangle-i\lambda\frac{\kappa\rho_2}{b}\langle f_3,E_*^\alpha\varphi_x\rangle+\frac{\kappa^2}{b}\|E_*^\alpha\varphi_x\|^2\\
&\quad+\frac{\kappa^2}{b}\langle\psi,E_*^\alpha\varphi_x\rangle+\frac{\kappa^2l}{b}\langle w,E_*^\alpha\varphi_x\rangle-\frac{\kappa\rho_2}{b}\langle f_4,E_*^\alpha\varphi_x\rangle.
\end{aligned}$$

Substitute the last formula and use integration by parts to obtain

$$\begin{aligned}
\kappa\|E_*^{\frac{\alpha}{2}}\psi_x\|^2&=-\rho_1\lambda^2\langle\varphi,E_*^\alpha\psi_x\rangle-i\lambda\rho_1\langle f_1,E_*^\alpha\psi_x\rangle-l(\kappa+\kappa_0)\langle w_x,E_*^\alpha\psi_x\rangle\\
&\quad+\kappa_0l^2\langle\varphi,E_*^\alpha\psi_x\rangle+\gamma_1\langle E_*^{\theta/2}\tilde\varphi,E_*^{\alpha+\frac{\theta}{2}}\psi_x\rangle-\rho_1\langle f_2,E_*^\alpha\psi_x\rangle\\
&\quad+\frac{\kappa\rho_2}{b}\lambda^2\langle\psi_x,E_*^\alpha\varphi\rangle-i\lambda\frac{\kappa\rho_2}{b}\langle f_3,E_*^\alpha\varphi_x\rangle+\frac{\kappa^2}{b}\|E_*^\alpha\varphi_x\|^2\\
&\quad+\frac{\kappa^2}{b}\langle\psi,E_*^\alpha\varphi_x\rangle+\frac{\kappa^2l}{b}\langle w,E_*^\alpha\varphi_x\rangle-\frac{\kappa\rho_2}{b}\langle f_4,E_*^\alpha\varphi_x\rangle,
\end{aligned}\tag{4.31}$$

where

$$\begin{gathered}
|i\lambda\rho_1\langle f_1,E_*^\alpha\psi_x\rangle|=|i\lambda\rho_1\langle(f_1)_x,E_*^\alpha\psi\rangle|\le C\lambda^2\|E_*^{\frac{\alpha}{2}}\psi\|^2+C\|F\|^2,\\
|l(\kappa+\kappa_0)\langle w_x,E_*^\alpha\psi_x\rangle|\le C\lambda^2\|E_*^{\frac{\alpha}{2}}w_x\|^2+C\lambda^2\|E_*^{\frac{\alpha}{2}}\psi_x\|^2,\\
|\kappa_0l^2\langle\varphi,E_*^\alpha\psi_x\rangle|\le C\lambda^2\|E_*^{\frac{\alpha}{2}}\varphi\|^2+C\lambda^2\|E_*^{\frac{\alpha}{2}}\psi_x\|^2.
\end{gathered}$$

The estimates for the remaining items are similar to the those above. Substituting these estimates into (4.31) and using Cauchy-Schwarz, Young's inequalities, (1.12) and (4.3) yield

$$\begin{aligned}
&\kappa\|E_*^{\frac{\alpha}{2}}\psi_x\|^2\\
&\le|\chi_0|\lambda^2\langle\varphi,E_*^\alpha\psi_x\rangle+C\lambda^2\|E^{\frac{\alpha}{2}}\psi\|^2+C\lambda^2\|E_*^{\frac{\alpha}{2}}\varphi\|^2+C\|E_*^{\frac{\alpha}{2}}\varphi_x\|^2\\
&\quad+C\|E_*^{\frac{\alpha}{2}}w_x\|^2+C\|E_*^{\alpha+\frac{\theta}{2}}\psi_x\|^2+\|F\|\|U\|+\|F\|^2.
\end{aligned}\tag{4.32}$$

For the case $\chi_0=0$, taking $\alpha=-\theta$ in (4.32), because of the continuous embedding we have

$$\begin{aligned}
\|E_*^{-\theta/2}\psi_x\|^2&\le C\|E_*^{-\theta/2}\tilde\psi\|^2+C\|E_*^{-\theta/2}\varphi_x\|^2+C\|E_*^{-\theta/2}\tilde\varphi\|^2\\
&\quad+C\|E_*^{-\theta/2}w_x\|^2+\|F\|\|U\|+\|F\|^2.
\end{aligned}\tag{4.33}$$

Taking into account $\alpha=-\theta$ in the item (ii) of Lemma 4.3, using (1.18) we obtain

$$\|E_*^{-\theta/2}\varphi_x\|^2=\|E_*^{\frac{1}{2}-\frac{\theta}{2}}\varphi\|^2$$

$$\leq C\big(\|E_*^{-\theta/2}\tilde{\varphi}\|^2 + \|E_*^{-\theta/2}\psi\|^2 + \|E_*^{-\theta/2}w\|^2\big) + \|F\|\|U\| + \|F\|^2.$$

The estimate of $\|E^{-\theta/2}w_x\|^2$ can be obtained in the same way. Substituting into (4.33) and using (1.19), we obtain

$$\begin{aligned}\|E_*^{\frac{1}{2}-\frac{\theta}{2}}\psi\|^2 &= \|E_*^{-\theta/2}\psi_x\|^2\\ &\leq C\|E_*^{-\theta/2}\tilde{\psi}\|^2 + C\|E_*^{-\theta/2}\tilde{\varphi}\|^2 + C\|E_*^{-\theta/2}\tilde{w}\|^2 + \|F\|\|U\| + \|F\|^2\\ &\leq C\|E_*^{-\theta/2}\tilde{\psi}\|^2 + \|F\|\|U\| + \|F\|^2,\end{aligned} \tag{4.34}$$

since (4.1a) and (4.1c). Consider $\alpha = -\theta$ in the item (i) of Lemma 4.4 and divide by λ^2, by (4.1a), (4.1c) and (4.1e) to obtain

$$\|E_*^{-\theta/2}\tilde{\psi}\|^2 \leq C\big(\|E_*^{\frac{1}{2}-\frac{\theta}{2}}\psi\|^2 + \|E_*^{-\theta/2}\varphi\|^2 + \|E_*^{-\theta/2}w\|^2\big) + \|F\|^2. \tag{4.35}$$

From the $\gamma_2 = 0$, we have $\varepsilon_2 = 0$. Substituting (4.34) into (4.35) we have

$$\begin{aligned}\|E_*^{-\theta/2}\tilde{\psi}\|^2 &\leq C\big(\|E_*^{-\theta/2}\tilde{\psi}\|^2 + \|E_*^{-\theta/2}\varphi\|^2 + \|E_*^{-\theta/2}w\|^2\big) + \|F\|\|U\| + \|F\|^2\\ &\leq C\big(\|E_*^{-\theta/2}\tilde{\psi}\|^2 + \|F\|\|U\| + \|F\|^2\big).\end{aligned} \tag{4.36}$$

Thus,

$$\|E_*^{-\theta/2}\tilde{\psi}\|^2 \leq C\big(\|F\|\|U\| + \|F\|^2\big). \tag{4.37}$$

Taking into account $\alpha = -\theta$ in the item (ii) of Lemma 4.4 and $\varepsilon_2 = 0$, using $-\frac{\theta}{2} < \frac{\theta}{2}$ and (4.37), we deduce that

$$\|E_*^{\frac{1}{2}-\frac{\theta}{2}}\tilde{\psi}\|^2 \leq C\lambda^2\big(\|F\|\|U\| + \|F\|^2\big). \tag{4.38}$$

Similar to the interpolation inequality as (4.21), use (4.37) and (4.38) to conclude

$$\|\tilde{\psi}\| \leq C_\delta\lambda^{2\theta}\big(\|F\|\|U\| + \|F\|^2\big).$$

This leads to the first result of Lemma 4.9.

For the case $\chi_0 \neq 0$, we have

$$\begin{aligned}\|E_*^{\frac{\alpha}{2}}\psi_x\|^2 &\leq C\|E_*^{\frac{\alpha}{2}}\psi_x\|^2 + C\lambda^2\|E_*^{\frac{\alpha}{2}}\tilde{\varphi}\|^2 + C\lambda^2\|E_*^{\frac{\alpha}{2}}\psi\|^2 + C\|E_*^{\frac{\alpha}{2}}\varphi_x\|^2\\ &\quad + C\|E_*^{\frac{\alpha}{2}}w_x\|^2 + C\|E_*^{\alpha+\frac{\theta}{2}}\psi_x\|^2 + \|F\|\|U\| + \|F\|^2\\ &\leq C\lambda^2\|E_*^{\frac{\alpha}{2}}\tilde{\varphi}\|^2 + C\lambda^2\|E_*^{\frac{\alpha}{2}}\psi\|^2 + C\|E_*^{\frac{\alpha}{2}}\varphi_x\|^2 + C\|E_*^{\frac{\alpha}{2}}w_x\|^2\\ &\quad + C\|E_*^{\alpha+\frac{\theta}{2}}\psi_x\|^2 + \|F\|\|U\| + \|F\|^2.\end{aligned} \tag{4.39}$$

Considering $\alpha = \theta - 1$ and $\theta \leq 1/2$ in (4.39), we have $\frac{\theta}{2} + \alpha \leq \frac{\alpha}{2}$, by continuous embedding, (1.18), (1.19), (4.3), and (4.5), we obtain

$$\begin{aligned}\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\psi_x\|^2 &\leq C\lambda^2\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\varphi}\|^2 + \|E_*^{\frac{\theta}{2}-\frac{1}{2}}\varphi_x\|^2 + \|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\psi}\|^2 + C\|E_*^{\frac{\theta}{2}-\frac{1}{2}}w_x\|^2\\ &\quad + \|F\|\|U\| + \|F\|^2\\ &\leq C\lambda^2\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\varphi}\|^2 + \|E_*^{\theta/2}\varphi\|^2 + \|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\psi}\|^2 + C\|E_*^{\theta/2}w\|^2\\ &\quad + \|F\|\|U\| + \|F\|^2\\ &\leq C\lambda^2\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\varphi}\|^2 + \|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\psi}\|^2 + \|F\|\|U\| + \|F\|^2.\end{aligned}$$

Taking into account $\alpha=\theta-1$ in the item (i) of Lemma 4.3, note that $\theta\le 1/2$ and (4.3), substituting the last formula to obtain

$$\begin{aligned}\|E_*^{\frac{\theta}{2}-\frac12}\psi_x\|^2 &\le C\|E_*^{\theta/2}\tilde\varphi\|^2+C\|E_*^{\frac{\theta}{2}-\frac12}\tilde\psi\|^2+C\|E_*^{\frac{\theta}{2}-\frac12}\tilde w\|^2+\|F\|\|U\|+\|F\|^2\\ &\le C\|E_*^{\frac{\theta}{2}-\frac12}\tilde\psi\|^2+\big(\|F\|\|U\|+\|F\|^2\big).\end{aligned}$$

Similar to the process of (4.35)-(4.36), $-\frac{\theta}{2}$ is replaced by $\frac{\theta}{2}-\frac12$, $\frac12-\frac{\theta}{2}$ is replaced by $\frac{\theta}{2}$, we have

$$\|E_*^{\frac{\theta}{2}-\frac12}\tilde\psi\|^2\le C\big(\|F\|\|U\|+\|F\|^2\big). \tag{4.40}$$

Taking $\alpha=\theta-1$ in the item (ii) of Lemma 4.4 again, we obtain

$$\|E_*^{\theta/2}\tilde\psi\|^2\le C\lambda^2\big(\|F\|\|U\|+\|F\|^2\big). \tag{4.41}$$

Using the same interpolation inequalities with (4.26), and applying (4.40)-(4.41), we conclude that

$$\|\tilde\psi\|\le C_\delta\lambda^{2-2\theta}\big(\|F\|\|U\|+\|F\|^2\big).$$

The second conclusion of Lemma 4.9 follows.

For the third result of this Lemma, that is $\theta\ge 1/2$, then $\frac12-\frac{\theta}{2}\le\frac{\theta}{2}$. Considering $\alpha=-\theta$ in (4.39), then $\frac{\theta}{2}+\alpha\le\frac{\alpha}{2}$, by the continuous embedding we deduce

$$\begin{aligned}\|E_*^{-\theta/2}\psi_x\|^2 &\le C\lambda^2\|E_*^{-\theta/2}\tilde\varphi\|^2+\|E_*^{-\theta/2}\varphi_x\|^2+\|E_*^{-\theta/2}\tilde\psi\|^2+C\|E_*^{-\theta/2}w_x\|^2\\ &\quad+\|F\|\|U\|+\|F\|^2\\ &\le C\|E_*^{\frac12-\frac{\theta}{2}}\tilde\varphi\|^2+\|E_*^{-\theta/2}\tilde\psi\|^2+\|E_*^{-\theta/2}\tilde w\|^2+\|E_*^{\frac12-\frac{\theta}{2}}\varphi\|^2\\ &\quad+C\|E_*^{\frac12-\frac{\theta}{2}}w\|^2+\|F\|\|U\|+\|F\|^2\\ &\le\|E_*^{-\theta/2}\tilde\psi\|^2+\|F\|\|U\|+\|F\|^2.\end{aligned}$$

Here, we use (1.18) and (1.19), and consider $\alpha=-\theta$ in item (i) of lemma 4.4 and $\theta\ge 1/2$ to obtain the above inequality. Then similar to the process of (4.35)-(4.38), we obtain

$$\|E_*^{-\theta/2}\tilde\psi\|^2\le C\big(\|F\|\|U\|+\|F\|^2\big), \tag{4.42}$$

$$\|E_*^{\frac12-\frac{\theta}{2}}\tilde\psi\|^2\le C\lambda^2\big(\|F\|\|U\|+\|F\|^2\big). \tag{4.43}$$

Similar to the interpolation inequality as (4.21), apply (4.42) and (4.43) to conclude that

$$\|\tilde\psi\|\le C_\delta\lambda^{2\theta}\big(\|F\|\|U\|+\|F\|^2\big).$$

The third conclusion of Lemma 4.9 can be obtained. The proof is complete. □

Thirdly we consider the case $\gamma_2,\gamma_3>0$ and $\gamma_1=0$.

Lemma 4.10. *Let $\delta>0$ and $\gamma_2,\gamma_3>0$, $\gamma_1=0$. Then there exists positive constant C_δ, such that for $|\lambda|\ge\delta$ the solution $U=(\varphi,\tilde\varphi,\psi,\tilde\psi,w,\tilde w)^T$ of system* (4.1a)-(4.1f) *satisfies*

(i) $\|\tilde\varphi\|^2\le C_\delta\lambda^{2\theta}\big(\|F\|\|U\|+\|F\|^2\big)$ *when* $\chi_0=0$,

(ii) $\|\tilde\varphi\|^2\le C_\delta\lambda^{2-2\theta}\big(\|F\|\|U\|+\|F\|^2\big)$ *when* $\chi_0\ne 0$ *and* $\theta\le 1/2$,

(iii) $\|\tilde\varphi\|^2\le C_\delta\lambda^{2\theta}\big(\|F\|\|U\|+\|F\|^2\big)$ *when* $\chi_0\ne 0$ *and* $\theta\ge 1/2$.

Proof. From (4.1a) and (4.1e) we obtain

$$i\lambda(w_x - l\varphi) = (\tilde{w}_x - l\tilde{\varphi}) + (f_5)_x - lf_1.$$

Taking the inner product with $\kappa_0 E_*^{\alpha}(w_x - l\varphi)$ we deduce that

$$\begin{aligned}
&\kappa_0\|E_*^{\frac{\alpha}{2}}(w_x - l\varphi)\|^2\\
&= \frac{\kappa_0}{i\lambda}\langle(\tilde{w}_x - l\tilde{\varphi}) + (f_5)_x - lf_1, E_*^{\alpha}(w_x - l\varphi)\rangle\\
&= \frac{1}{i\lambda}\langle\tilde{w}, E_*^{\alpha}(-i\lambda\rho_1\tilde{w} - \kappa l(\varphi_x + \psi + lw) - \gamma_3 E_*^{\theta}\tilde{w} + \rho_1 f_6)\rangle\\
&\quad + \frac{\kappa_0}{i\lambda}\langle -l\tilde{\varphi} + (f_5)_x - lf_1, E_*^{\alpha}(w_x - l\varphi)\rangle\\
&= \rho_1\|E_*^{\frac{\alpha}{2}}\tilde{w}\|^2 + \frac{\gamma_3}{i\lambda}\|E_*^{\frac{\alpha}{2}+\frac{\theta}{2}}\tilde{w}\|^2 - \frac{\kappa l}{i\lambda}\langle\tilde{w}, E_*^{\alpha}(\varphi_x + \psi + lw)\rangle\\
&\quad + \frac{\kappa_0}{i\lambda}\langle -l\tilde{\varphi} + (f_5)_x - lf_1, E_*^{\alpha}(w_x - l\varphi)\rangle + \frac{\rho_1}{i\lambda}\langle\tilde{w}, E_*^{\alpha} f_6\rangle\\
&\le C\|E_*^{\frac{\alpha}{2}}\tilde{w}\|^2 + \frac{C}{\lambda}\|E_*^{\theta/2}\tilde{w}\|^2\\
&\quad + C\|E_*^{\frac{\alpha}{2}}(\varphi_x + \psi + lw)\|^2 + C\|E^{\frac{\alpha}{2}}\varphi\|^2 + \|F\|^2,
\end{aligned} \tag{4.44}$$

here we used the facts of $\alpha \le 0$, (4.1a), the self-adjointness of E_*^{σ}, Cauchy-Schwarz and Young's inequalities. It can be seen from (4.44) that the estimate here is different from Lemma 4.6, which is independent of χ_1.

From the result of Lemma 4.5, the fact $\gamma_1 = 0$, and using (4.4), the self-adjointness of E_*^{σ}, Cauchy-Schwarz and Young's inequalities we deduce that

$$\begin{aligned}
&\|E_*^{\frac{\alpha}{2}}(\varphi_x + \psi + lw)\|^2\\
&\le \frac{\rho_1\rho_2}{\kappa}|\chi_0|\lambda^2\langle\psi, E_*^{\alpha}\varphi_x\rangle + C\|E_*^{\frac{\alpha}{2}}\tilde{\psi}\|^2 + C\|E_*^{\frac{\alpha}{2}}\tilde{w}\|^2 + C\|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2\\
&\quad + C\|E_*^{\theta/2}\tilde{\psi}\|^2 + C\|E_*^{\frac{\alpha}{2}+\theta}\psi\|^2 + C\|E_*^{\frac{\theta}{2}+\alpha}(\varphi_x + \psi + lw)\|^2 + C\|F\|^2\\
&\le C|\chi_0|\big(\lambda^2\|E_*^{\frac{\alpha}{2}}\tilde{\psi}\|^2 + \|E_*^{\frac{\alpha}{2}}\varphi_x\|^2\big) + C\|E_*^{\frac{\alpha}{2}}\tilde{\psi}\|^2 + C\|E_*^{\frac{\alpha}{2}}\tilde{w}\|^2 + C\|E^{\frac{\alpha}{2}}\tilde{\varphi}\|^2\\
&\quad + C\|E_*^{\frac{\alpha}{2}+\theta}\psi\|^2 + C\|E_*^{\frac{\theta}{2}+\alpha}(\varphi_x + \psi + lw)\|^2 + \|F\|\|U\| + \|F\|^2.
\end{aligned} \tag{4.45}$$

Case 1: $\chi_0 = 0$. Taking $\alpha = -\theta$ in (4.45), we obtain $\frac{\alpha}{2} + \theta = \frac{\theta}{2}$ and $\frac{\theta}{2} + \alpha = \frac{\alpha}{2}$, because of (4.4), (4.5), (4.1c), and the continuous embedding, we have

$$\begin{aligned}
&\|E_*^{-\theta/2}(\varphi_x + \psi + lw)\|^2\\
&\le C\|E^{-\theta/2}\tilde{\psi}\|^2 + C\|E_*^{-\theta/2}\tilde{\varphi}\|^2 + C\|E_*^{-\theta/2}\tilde{w}\|^2 + \|F\|\|U\| + \|F\|^2\\
&\le C\|E_*^{-\theta/2}\tilde{\varphi}\|^2 + \|F\|\|U\| + \|F\|^2.
\end{aligned} \tag{4.46}$$

On the other hand,

$$\|E_*^{-\theta/2}\varphi_x\|^2 \le C\big(\|E_*^{-\theta/2}(\varphi_x + \psi + lw)\|^2 + \|E_*^{-\theta/2}\psi\|^2 + \|E_*^{-\theta/2}w\|^2\big). \tag{4.47}$$

Thus by $-\frac{\theta}{2} < \frac{\theta}{2}$, (4.46), (4.1c), (4.1e), (4.4) and (4.5), we have

$$\begin{aligned}
\|E_*^{-\theta/2}\varphi_x\|^2 &\le C\|E_*^{-\theta/2}\tilde{\varphi}\|^2 + C\|E_*^{-\theta/2}\psi\|^2 + C\|E_*^{-\theta/2}w\|^2 + \|F\|\|U\| + \|F\|^2\\
&\le C\|E_*^{-\theta/2}\tilde{\varphi}\|^2 + \|F\|\|U\| + \|F\|^2.
\end{aligned} \tag{4.48}$$

Taking into account $\alpha = -\theta$ in the item (i) of Lemma 4.3, dividing λ^2, and from $\gamma_1 = 0$ and (1.18) we obtain

$$\begin{aligned}\|E^{-\theta/2}\tilde{\varphi}\|^2 &\le C\big(\|E^{\frac{1}{2}-\frac{\theta}{2}}\varphi\|^2 + \|E_*^{-\theta/2}\psi\|^2 + \|E_*^{-\theta/2}w\|^2\big) + \|F\|^2 \\ &\le C\|E^{-\theta/2}\varphi_x\|^2 + C\big(\|F\|\|U\| + \|F\|^2\big).\end{aligned}$$

The above estimate and (4.48) yield

$$\|E_*^{-\theta/2}\tilde{\varphi}\|^2 \le C\big(\|F\|\|U\| + \|F\|^2\big). \tag{4.49}$$

Taking $\alpha = -\theta$ in the item (ii) of Lemma 4.3, from (4.49) we obtain

$$\|E_*^{\frac{1}{2}-\frac{\theta}{2}}\tilde{\varphi}\|^2 \le C\lambda^2\big(\|F\|\|U\| + \|F\|^2\big). \tag{4.50}$$

Using the interpolation inequality similar to (4.21), by (4.49)-(4.50) we obtain

$$\|\tilde{\varphi}\| \le C_\delta \lambda^{2\theta}\big(\|F\|\|U\| + \|F\|^2\big).$$

This leads to the first result of Lemma 4.10.

Case 2: $\chi_0 \neq 0$. Taking $\alpha = \theta - 1$ in (4.45) and noting $\theta \le 1/2$, from (4.4), (4.1c), (1.19) and continuous embedding we obtain

$$\begin{aligned}&\|E_*^{\frac{\theta}{2}-\frac{1}{2}}(\varphi_x + \psi + lw)\|^2 \\ &\le C|\chi_0|\big(\lambda^2\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\psi}\|^2 + \|E_*^{\frac{\theta}{2}-\frac{1}{2}}\varphi_x\|^2\big) + C\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{w}\|^2 + C\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\varphi}\|^2 \\ &\quad + C\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\psi}\|^2 + \|F\|\|U\| + \|F\|^2 \\ &\le C\lambda^2\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\psi}\|^2 + C\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\varphi_x\|^2 + C\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\varphi}\|^2 + \|F\|\|U\| + \|F\|^2.\end{aligned} \tag{4.51}$$

Taking into account $\alpha = \theta - 1$ in the item (i) of Lemma 4.4, from (4.4) we obtain

$$\begin{aligned}&\lambda^2\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\psi}\|^2 \\ &\le C\big(\|E^{\theta/2}\tilde{\psi}\|^2 + \|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\varphi}\|^2 + \|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{w}\|^2\big) + \|F\|\|U\| + \|F\|^2 \\ &\le C\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\varphi}\|^2 + \|F\|\|U\| + \|F\|^2.\end{aligned} \tag{4.52}$$

Substituting (4.52) into (4.51) gives

$$\|E_*^{\frac{\theta}{2}-\frac{1}{2}}(\varphi_x + \psi + lw)\|^2 \le C\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\varphi_x\|^2 + C\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\varphi}\|^2 + \|F\|\|U\| + \|F\|^2.$$

In addition, similar to (4.47)-(4.50), $-\frac{\theta}{2}$ is replaced by $\frac{\theta}{2} - \frac{1}{2}$, $\frac{1}{2} - \frac{\theta}{2}$ is replaced by $\frac{\theta}{2}$, we have

$$\|E_*^{\theta/2}\varphi\|^2 = \|E_*^{\frac{\theta}{2}-\frac{1}{2}}\varphi_x\|^2 \le C\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\varphi}\|^2 + \|F\|\|U\| + \|F\|^2, \tag{4.53}$$

$$\|E_*^{\frac{\theta}{2}-\frac{1}{2}}\tilde{\varphi}\|^2 \le C\big(\|F\|\|U\| + \|F\|^2\big), \tag{4.54}$$

$$\|E_*^{\theta/2}\tilde{\varphi}\|^2 \le C\lambda^2\big(\|F\|\|U\| + \|F\|^2\big). \tag{4.55}$$

Using the same interpolation inequalities with similar as (4.26), and applying (4.54)-(4.55), we conclude that

$$\|\tilde{\varphi}\| \le C_\delta \lambda^{2-2\theta}\big(\|F\|\|U\| + \|F\|^2\big).$$

The second conclusion of Lemma 4.10 follows.

Case 3: For the third result of this Lemma, that is, $\theta \geq 1/2$. Taking $\alpha = -\theta$ in (4.45), and from the continuous embedding we obtain

$$\begin{aligned} &\|E_*^{-\theta/2}(\varphi_x + \psi + lw)\|^2 \\ &\leq C\lambda^2\|E_*^{-\theta/2}\tilde{\psi}\|^2 + C\|E_*^{-\theta/2}\varphi_x\|^2 + C\|E_*^{-\theta/2}\tilde{w}\|^2 \\ &\quad + C\|E_*^{-\theta/2}\tilde{\varphi}\|^2 + \|F\|\|U\| + \|F\|^2 \\ &\leq C\lambda^2\|E_*^{-\theta/2}\tilde{\psi}\|^2 + C\|E_*^{-\theta/2}\varphi_x\|^2 + C\|E_*^{-\theta/2}\tilde{\varphi}\|^2 + \|F\|\|U\| + \|F\|^2. \end{aligned} \tag{4.56}$$

Next, we use Lemma 4.4 to estimate the first item in (4.54). Consider $\alpha = -\theta$ and $\theta \geq 1/2$ in the item (i) of Lemma 4.3, by (4.4) we deduce that

$$\begin{aligned} \lambda^2\|E_*^{-\theta/2}\tilde{\psi}\|^2 &\leq C\|E_*^{\frac{1}{2}-\frac{\theta}{2}}\tilde{\psi}\|^2 + C\|E_*^{-\theta/2}\tilde{\varphi}\|^2 + C\|E_*^{-\theta/2}\tilde{w}\|^2 + \|F\|\|U\| + \|F\|^2 \\ &\leq C\|E_*^{-\theta/2}\tilde{\varphi}\|^2 + \|F\|\|U\| + \|F\|^2. \end{aligned}$$

Substituting this into (4.56) gives

$$\|E_*^{-\theta/2}(\varphi_x + \psi + lw)\|^2 \leq C\|E_*^{-\theta/2}\tilde{\varphi}\|^2 + C\|E_*^{-\theta/2}\varphi_x\|^2 + \|F\|\|U\| + \|F\|^2. \tag{4.57}$$

In addition, similar to (4.47)-(4.50), we have

$$\|E_*^{\frac{1}{2}-\frac{\theta}{2}}\varphi\|^2 = \|E_*^{-\theta/2}\varphi_x\|^2 \leq C\|E_*^{-\theta/2}\tilde{\varphi}\|^2 + \|F\|\|U\| + \|F\|^2, \tag{4.58}$$

$$\|E_*^{-\theta/2}\tilde{\varphi}\|^2 \leq C\big(\|F\|\|U\| + \|F\|^2\big), \tag{4.59}$$

$$\|E_*^{\frac{1}{2}-\frac{\theta}{2}}\tilde{\varphi}\|^2 \leq C\lambda^2\big(\|F\|\|U\| + \|F\|^2\big). \tag{4.60}$$

Using the same interpolation inequalities as for (4.21), and applying (4.59)-(4.60) we conclude that

$$\|\tilde{\varphi}\| \leq C_\delta\lambda^{2\theta}\big(\|F\|\|U\| + \|F\|^2\big).$$

The third conclusion of this Lemma can be obtained. The proof is complete. □

Finally, we present the stability results of system (1.7)-(1.9) (or problem (2.4)).

Theorem 4.11. *Assume that χ_0 and χ_1 defined by (1.12) and $\theta \in [0, 1]$. Then the semigroup $e^{t\mathcal{A}}$ corresponding to problem (2.4) is stable as follows:*

(i) *Assume $\gamma_1, \gamma_2 > 0, \gamma_3 = 0$.*

- *If $\chi_1 = 0$ and the exponents $\theta = 0$, then the semigroup is stable exponentially, i.e., there exist positive constants C and δ_0 such that*
$$\|e^{t\mathcal{A}}U_0\| \leq Ce^{-\delta_0 t}, \quad \forall t > 0.$$
- *If $\chi_1 = 0$ and the exponents $\theta_1 \in (0, 1]$, then the semigroup is stable polynomially with the estimate*
$$\|e^{t\mathcal{A}}U_0\| \leq Ct^{-1/(2\theta)}\|U_0\|_{D(\mathcal{A})}, \quad \forall t > 0,\ U_0 \in D(\mathcal{A}).$$
- *If $\chi_1 \neq 0$, then the semigroup is stable polynomially with the estimates*
$$\|e^{t\mathcal{A}}U_0\| \leq \begin{cases} Ct^{-1/(2-2\theta)}\|U_0\|_{D(\mathcal{A})}, & \text{if } \theta \in [0, 1/2], \\ Ct^{-1/(2\theta)}\|U_0\|_{D(\mathcal{A})}, & \text{if } \theta \in [1/2, 1], \end{cases}$$
for $\forall t > 0$, $U_0 \in D(\mathcal{A})$.

(ii) *Assume $\gamma_1, \gamma_3 > 0, \gamma_2 = 0$.*

- *If $\chi_0 = 0$ and the exponents $\theta = 0$, then the semigroup is stable exponentially, i.e., there exist positive constants C and δ_0 such that*
$$\|e^{t\mathcal{A}}U_0\| \le Ce^{-\delta_0 t}, \quad \forall t > 0.$$
- *If $\chi_0 = 0$ and the exponents $\theta_1 \in (0,1]$, then the semigroup is stable polynomially with the estimate as following,*
$$\|e^{t\mathcal{A}}U_0\| \le Ct^{-1/(2\theta)}\|U_0\|_{D(\mathcal{A})}, \quad \forall t > 0,\ U_0 \in D(\mathcal{A}).$$
- *If $\chi_0 \neq 0$, then the semigroup is stable polynomially with the estimate*
$$\|e^{t\mathcal{A}}U_0\| \le \begin{cases} Ct^{-1/(2-2\theta)}\|U_0\|_{D(\mathcal{A})}, & \text{if } \theta \in [0,1/2],\\ Ct^{-1/(2\theta)}\|U_0\|_{D(\mathcal{A})}, & \text{if } \theta \in [1/2,1], \end{cases}$$
for $\forall t > 0$, $U_0 \in D(\mathcal{A})$.

(iii) *Assume $\gamma_2, \gamma_3 > 0, \gamma_1 = 0$.*

- *If $\chi_1 = 0$ and the exponents $\theta = 0$, then the semigroup is stable exponentially, i.e., there exist positive constants C and δ_0 such that*
$$\|e^{t\mathcal{A}}U_0\| \le Ce^{-\delta_0 t}, \quad \forall t > 0.$$
- *If $\chi_1 = 0$ and the exponents $\theta_1 \in (0,1]$, then the semigroup is stable polynomially with the estimate*
$$\|e^{t\mathcal{A}}U_0\| \le Ct^{-1/(2\theta)}\|U_0\|_{D(\mathcal{A})}, \quad \forall t > 0,\ U_0 \in D(\mathcal{A}).$$
- *If $\chi_1 \neq 0$, then the semigroup is stable polynomially with the estimate*
$$\|e^{t\mathcal{A}}U_0\| \le \begin{cases} Ct^{-1/(2-2\theta)}\|U_0\|_{D(\mathcal{A})}, & \text{if } \theta \in [0,1/2],\\ Ct^{-1/(2\theta)}\|U_0\|_{D(\mathcal{A})}, & \text{if } \theta \in [1/2,1], \end{cases}$$
for $\forall t > 0$, $U_0 \in D(\mathcal{A})$.

Proof. We use Theorems 4.1 and 4.2 to prove these stability results. So, we check the conditions of these two Theorems.

Step 1. We prove $i\mathbb{R} \subset \rho(\mathcal{A})$ through a contradictory argument. Suppose that $i\mathbb{R} \not\subset \rho(\mathcal{A})$. From Theorem 1, we know that $0 \in \rho(\mathcal{A})$, then we denote λ_0 the maximum positive number such that $(-i\lambda_0, i\lambda_0) \subset \rho(\mathcal{A})$, therefore $-i\lambda_0$ or $i\lambda_0$ is an element of the spectrum $\sigma(\mathcal{A})$. Suppose that $i\lambda_0 \in \sigma(\mathcal{A})$ (if $-i\lambda_0 \in \sigma(\mathcal{A})$ the process is similar). Then, for $\delta \in (0, \lambda_0)$, there exist a sequence of real numbers $\{\lambda_n\}$, such that $\lambda_n \in [\delta, \lambda_0)$, $\lambda_n \to \lambda_0$, and a sequence of unit vectors $\{U_n = (\varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n, w_n, \tilde{w}_n)^T\} \subset D(\mathcal{A})$ satisfying
$$\|(i\lambda_n - \mathcal{A})U_n\| = \|F_n\| \to 0$$
as $n \to \infty$. That is, if $F_n = (f_{1n}, f_{2n}, f_{3n}, f_{4n}, f_{5n}, f_{6n})^T$, then we have
$$i\lambda\varphi_n - \tilde{\varphi}_n = f_{1n} \to 0 \quad \text{in } H_0^1(0,L), \tag{4.61a}$$
$$i\lambda\rho_1\tilde{\varphi}_n - \kappa(\varphi_{nx} + \psi_n + lw_n)_x - \kappa_0 l(w_{nx} - l\varphi_n) + \gamma_1 E^\theta\tilde{\varphi}_n = \rho_1 f_{2n} \to 0, \quad \text{in } L^2(0,L) \tag{4.61b}$$
$$i\lambda\psi_n - \tilde{\psi}_n = f_{3n} \to 0 \quad \text{in } H_*^1(0,L), \tag{4.61c}$$
$$i\lambda\rho_2\tilde{\psi}_n - b\psi_{xx} + \kappa(\varphi_{nx} + \psi_n + lw_n) + \gamma_2 E_*^\theta\tilde{\psi}_n = \rho_2 f_{4n} \to 0 \quad \text{in } L_*^2(0,L), \tag{4.61d}$$
$$i\lambda w_n - \tilde{w}_n = f_{5n} \to 0 \quad \text{in } H_*^1(0,L), \tag{4.61e}$$

$$i\lambda\rho_1\tilde{w}_n - \kappa_0(w_{nx} - l\varphi_n)_x + kl(\varphi_{nx} + \psi_n + lw_n) + \gamma_3 E_*^\theta \tilde{w}_n = \rho_1 f_{6n} \to 0 \quad \text{in } L_*^2(0,L). \tag{4.61f}$$

In the same way as we obtained in (4.2), we have

$$\gamma_1\|E^{\theta/2}\tilde{\varphi}_n\|^2 + \gamma_2\|E_*^{\theta/2}\tilde{\psi}_n\|^2 + \gamma_3\|E^{\theta/2}\tilde{w}_n\|^2 \le C\|F\|_n\|U\|_n \to 0. \tag{4.62}$$

Then the following estimates hold

$$\|\tilde{\varphi}_n\|^2 \le \|E^{\theta/2}\tilde{\varphi}_n\|^2 \le C\|F\|_n\|U\|_n \to 0, \tag{4.63}$$

$$\|\tilde{\psi}_n\|^2 \le \|E^{\theta/2}\tilde{\psi}_n\|^2 \le C\|F\|_n\|U\|_n \to 0, \tag{4.64}$$

$$\|\tilde{w}_n\|^2 \le \|E^{\theta/2}\tilde{w}_n\|^2 \le C\|F\|_n\|U\|_n \to 0. \tag{4.65}$$

From Lemma 4.7, (4.63), (4.65), (4.1c) and (4.1e) we find that

$$\begin{aligned} \|E_*^{1/2}\psi_n\|^2 &= \|\psi_{nx}\|^2 \\ &\le C_\delta\big(\|\tilde{\psi}_n\|^2 + \|\varphi_n\|^2 + \|w_n\|^2 + \|F\|_n\|U\|_n + \|F\|_n^2\big) \\ &\le C\big(\|F\|_n\|U\|_n + \|F\|_n^2\big) \to 0. \end{aligned} \tag{4.66}$$

On the other hand,

$$\begin{aligned} \|\varphi_{nx} + \psi_n + lw_n\|^2 &\le C(\|\varphi_{nx}\|^2 + \|\psi_n\|^2 + \|w_n\|^2) \\ &= C(\|E_*^{1/2}\varphi_n\|^2 + \|\psi_n\|^2 + \|w_n\|^2) \\ &\le C\big(\frac{1}{\lambda^2}\|E^{1/2}\tilde{\varphi}_n\|^2 + \frac{1}{\lambda^2}\|\tilde{\psi}_n\|^2 + \frac{1}{\lambda^2}\|\tilde{w}_n\|^2 + \|F\|^2\big). \end{aligned}$$

Taking $\alpha = 0$ in the item (ii) of Lemma 4.3 and dividing λ^2, we obtain

$$\begin{aligned} \frac{1}{\lambda^2}\|E^{1/2}\tilde{\varphi}_n\|^2 &\le C\big(\|\tilde{\varphi}_n\|^2 + \|\psi_n\|^2 + \|w_n\|^2\big) \\ &\quad + \varepsilon_1\|E^{\theta/2}\tilde{\varphi}_n\|^2 + C_{\varepsilon_1}\big(\|F\|_n\|U\|_n + \|F\|_n^2\big). \end{aligned}$$

Whether ε_1 is zero or not, we can get from Lemmas 4.8, 4.9, and 4.10,

$$\frac{1}{\lambda^2}\|E^{1/2}\tilde{\varphi}_n\|^2 \to 0.$$

thus, from (4.63) and (4.65) we deduce that

$$\|\varphi_{nx} + \psi_n + lw_n\|^2 \to 0. \tag{4.67}$$

Similarly, it follows that

$$\begin{aligned} \|w_{nx} - l\psi_n\|^2 &\le C(\|w_{nx}\|^2 + \|\varphi_n\|^2) = C(\|E_*^{1/2}w_n\|^2 + \|\varphi_n\|^2) \\ &\le C\big(\frac{1}{\lambda^2}\|E^{1/2}\tilde{w}_n\|^2 + \frac{1}{\lambda^2}\|\tilde{\varphi}_n\|^2\big) \to 0. \end{aligned} \tag{4.68}$$

The estimates (4.62)-(4.68) imply that $\|U_n\| \to 0$ which is absurd with $\|U_n\| = 1$, for all $n \in \mathbb{N}$. Consequently $i\mathbb{R} \subset \rho(\mathcal{A})$.

Step 2. Let $U = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w})^T$ be the solution of the system $(i\lambda - \mathcal{A})U = F$. Once we have proven $i\mathbb{R} \subset \rho(\mathcal{A})$, and then according to Theorems 4.1 and 4.2, we need to prove the decay rate of the desired semigroup $e^{t\mathcal{A}}$. Because the proving process is very similar, we only prove the item (i) of Theorem 4.11, and the other two results have similar proof.

Assume $\gamma_1, \gamma_2 > 0, \gamma_3 = 0$. From (4.2) we have

$$\|E^{\theta/2}\tilde{\varphi}_n\|^2 + \gamma_2\|E_*^{\theta/2}\tilde{\psi}_n\|^2 \le C\|F\|_n\|U\|_n;$$

that is,

$$\|\tilde{\varphi}_n\|^2 \le \|E^{\theta/2}\tilde{\varphi}_n\|^2 \le C\|F\|_n\|U\|_n, \tag{4.69}$$

$$\|\tilde{\psi}_n\|^2 \le \|E^{\theta/2}\tilde{\psi}_n\|^2 \le C\|F\|_n\|U\|_n. \tag{4.70}$$

From (4.66)-(4.68) we have

$$\begin{aligned}\|\psi_{nx}\|^2 &\le C_\delta\big(\|\tilde{\psi}_n\|^2 + \|\varphi_n\|^2 + \|w_n\|^2 + \|F\|_n\|U\|_n + \|F\|_n^2\big)\\ &\le C\big(\|w_n\|^2 + \|F\|_n\|U\|_n + \|F\|_n^2\big),\end{aligned} \tag{4.71}$$

$$\begin{aligned}&\|\varphi_{nx} + \psi_n + lw_n\|^2\\ &\le C(\|E_*^{1/2}\varphi_n\|^2 + \|\psi_n\|^2 + \|w_n\|^2)\\ &\le C\big(\|\tilde{\varphi}_n\|^2 + \|E^{\theta/2}\tilde{\varphi}_n\|^2 + \frac{1}{\lambda^2}\|\tilde{\psi}_n\|^2 + \frac{1}{\lambda^2}\|\tilde{w}_n\|^2\big) + (\|F\|_n\|U\|_n + \|F\|_n^2)\\ &\le C\frac{1}{\lambda^2}\|\tilde{w}_n\|^2 + \|F\|_n\|U\|_n + \|F\|_n^2.\end{aligned} \tag{4.72}$$

$$\begin{aligned}\|w_{nx} - l\psi_n\|^2 &\le C(\|E_*^{1/2}w_n\|^2 + \|\varphi_n\|^2)\\ &\le C\big(\|\tilde{w}_n\|^2 + \|\varphi_n\|^2 + \|\psi_n\|^2 + \frac{1}{\lambda^2}\|\tilde{\varphi}_n\|^2 + \|F\|_n^2\big)\\ &\le C\|\tilde{w}_n\|^2 + C\|F\|_n^2.\end{aligned} \tag{4.73}$$

The above estimates imply that

$$\|U_n\|^2 \le \|\tilde{w}_n\|^2 + \|F\|_n\|U\|_n + \|F\|_n^2.$$

It can be seen from Lemma 4.8 that, when $\chi_1 = 0$ and $\theta = 0$, we obtain

$$\|U_n\|^2 \le C_\delta\big(\|F\|_n\|U\|_n + \|F\|_n^2\big);$$

when $\chi_1 = 0$ or $\chi_1 \ne 0$ and $\theta \ge 1/2$, we obtain

$$\|U_n\|^2 \le C_\delta\lambda^{2\theta}\big(\|F\|_n\|U\|_n + \|F\|_n^2\big);$$

when $\chi_1 \ne 0$ and $\theta \le 1/2$, we obtain

$$\|U_n\|^2 \le C_\delta\lambda^{2-2\theta}\big(\|F\|_n\|U\|_n + \|F\|_n^2\big).$$

Hence, applying Young's inequality, it follows that

$$\|U_n\|^2 \le C_\delta\|F\|_n^2, \quad \text{or} \quad \|U_n\|^2 \le C_\delta\lambda^{2\theta}\|F\|_n^2, \quad \text{or} \quad \|U_n\|^2 \le C_\delta\lambda^{2-2\theta}\|F\|_n^2,$$

which is the desired result. Consequently, when $\chi_1 = 0$ and $\theta = 0$, by Theorem 4.1, we obtain that the semigroup is exponentially stable. When $\chi_1 = 0$ and $\theta \in (0, 1]$, by Theorem 4.2 the semigroup decays polynomially with the rate $t^{-1/(2\theta)}$. When $\chi_1 \ne 0$ and $\theta \in [0, 1/2]$, by Theorem 4.2 the semigroup decays polynomially with the rate $t^{-1/(2-2\theta)}$. When $\chi_1 \ne 0$ and $\theta \in [1/2, 1]$, by Theorem 4.2 the semigroup decays polynomially with the rate $t^{-1/(2\theta)}$. Then the item (i) of Theorem 4.11 is be established. Similarly, we can get the other two results. Thus the proof is complete. □

Remark 4.12. Alves et al. [2] studied the asymptotic behavior of the system when friction damping acts on the vertical displacement and the angle displacement at the same time (that is, $\theta = 0$ and $\gamma_3 = 0$ in system (1.7)-(1.9)). They obtained the exponential decay and polynomial decay of the system respectively. These results are consistent with our conclusions obtained in the item (i) of Theorem 4.11. Furthermore, Alves et al. showed that the polynomial decay rate $t^{-\frac{1}{2}}$ is optimal. Thus we believe that our polynomial stability results are optimal, which will be our next research topic.

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