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STABILITY AND RATE OF DECAY FOR SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS WITH MARKOV SWITCHING

SHUAISHUAI LU, XUE YANG

Abstract. In this article, we present the almost sure asymptotic stability and a general rate of decay for solutions to stochastic differential equations (SDEs) with Markov switching. By establishing a suitable Lyapunov function and using an exponential Martingale inequality and the Borel-Cantelli theorem, we give sufficient conditions for the asymptotic stability. Also, we obtain sufficient conditions for the construction of two kinds of Lyapunov functions. Finally give two examples to illustrate the validity of our results.

1. Introduction

As an important property of stochastic differential equations (SDEs), the stability research has been a popular direction. Many results have been achieved on various stability problems of SDEs, see [2, 15, 16, 24, 26]. For instance, Mao introduced the concept of polynomial decay rate stability into stochastic differential systems. In the subsequent research, the concept of stability with general decay rate was extended in [4, 5, 6]. Recently, the concept of the partial practical stability of SDEs with general decay rate was introduced by [2].

Stochastic differential equation with Markov switching is an important type of stochastic equations [7, 22]. It plays an important role in prediction model, physics, ecological engineering, financial stock market, network control, etc., and can be used to explain the physical process of sudden change of environment or transformation models under different conditions [19, 21, 25, 28, 34, 38]. Therefore, it is very worth studying the stability of this kind of equation. The p-moment and exponential stability of SDEs with Markov switching has been studied in [17, 30, 31, 37]. In these references, Lyapunov methods are used to study stability. It is interesting to note that in [17], the theory of M-matrices is used to establish some sufficient criteria for the exponential stability and these criteria are much easier to determine than the results obtained using the Lyapunov methods. In the stability theory of SDEs with Markov switching, the almost sure stability is also very significant, see [18, 35, 36, 27]. A sufficient condition that the equation is almost sure stability is given in [35]. The important thing is that the sufficient condition given in [35] is

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independent of the moment stability of the system. In addition, in [11, 14, 32, 33], some problems of discrete Markov switching systems are studied, and in [9, 10] the stability is studied for nonlinear stochastic delay systems with asynchronous Markov switching.

Lyapunov functions are often used to prove the stability of differential systems. A function is called a Lyapunov candidate function if it has the possibility to prove the stability of the differential system at an equilibrium. With the development of Lyapunov's first and second methods, more and more work is based on Lyapunov methods to study the stability of differential systems, see [1, 8, 12, 23].

In the analysis of asymptotic stability of stochastic differential systems with Markov switching, many article study the exponential asymptotic stability of solutions, i.e., if the deterministic or stochastic system is not stable, our goal is to add a noise term to make the solution path of the stochastic system exponentially stable. However, this type of results fails to be applied, for instance, when the deterministic model is non autonomous. In this case, it may occur that the stability cannot always be exponential, or even sub exponential or super exponential. This fact has inspired this article. The main purpose of this article is to extend exponential stability to general decay stability, such as polynomial decay, logarithmic decay, sub-exponential decay, super exponential decay and so on. Using Lyapunov method, we establish sufficient conditions for the almost sure asymptotic stability of equations with the general decay rate. At the same time, this paper discusses two kinds of Lyapunov functions to prove the rationality of our theorems. Therefore, this paper generalizes the results in [17].

The structure of this article is as follows: In Section 2 we give the basic concepts of SDEs with Markov switching and related definitions. In Section 3, the sufficient conditions ensuring the almost sure asymptotic stability on a general decay rate of SDEs with Markov switching are given, and the relevant proofs are given by using Markov inequality, Borel-Cantelli theorem and exponential Martingale inequality. Then, in Section 4 we present two examples to illustrate the theoretical findings.

2. SDEs with Markov switching

We assume that $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete probability space, $\{\mathscr{F}_t\}_{t\geq 0}$ is a filtration in the probability space. Then $\mathscr{F}_0$ is right continuous and contains all the $\mathbb{P}$-null test sets. We're going to use $\|\cdot\|$ for the Euclidean norm in $\mathbb{R}^n$. If A is a matrix or a vector, A^T is its transpose. If A is a matrix, the norm is expressed as $\|A\| = \sqrt{\operatorname{trace}(AA^T)}$. If A is a symmetric matrix, its maximum and minimum eigenvalues are denoted by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ respectively. We use $m \vee n$ to represent $\max\{m, n\}$ and $m \wedge n$ to represent $\min\{m, n\}$. Now consider the stochastic differential equation with Markov switching

$$\mathrm{d}x(t) = f(x(t), t, r(t))\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}W(t), \tag{2.1}$$

where

$$f : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n, \quad g : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^{n\times w},$$

and $\{W(t)\}_{t\geq 0}$ is the w-dimensional Brownian motion and $r(t)(t \geq 0)$ is the right continuous Markov chain in finite state space $S = \{1, 2, 3, \dots, N\}$, whose composition $\Gamma = (\gamma_{ij})_{N\times N}$ is generated as follows:

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & i \neq j, \\ 1 + \gamma_{ij}\Delta + o(\Delta), & i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is transition rate from i to state j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j\neq i}^{N} (\gamma_{ij}).$$

Therefore, (2.1) can be rewritten as the result of the following N equations:

$$\mathrm{d}x(t) = f(x(t), t, i)\mathrm{d}t + g(x(t), t, i)\mathrm{d}W(t), \quad t \geq 0,\ 1 \leq i \leq N. \tag{2.2}$$

In this article we assume that Markov chains $r(t)$ and Brownian motion $W(t)$ are independent each other. To ensure the existence and uniqueness solutions of (2.1), the following assumptions are made:

(H1) f and g satisfy the following linear growth conditions and local Lipschitz conditions:
  (1) There is $h > 0$ such that for all $(x, t, i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S$,
  $$\|f(x, t, i)\| \vee \|g(x, t, i)\| \leq h(1 + \|x\|);$$
  (2) For each $k = 1, 2, 3, \dots$, there is an $h_k > 0$ such that
  $$\|f(x, t, i) - f(y, t, i)\| \vee \|g(x, t, i) - g(y, t, i)\| \leq h_k\|x - y\|,$$
  for all $t \geq 0$, $i \in S$ and $x, y \in \mathbb{R}^n$ with $\|x\| \vee \|y\| \leq k$.

It is known [20, Theorem 3.16 and Lemma 4.1] and [17, (H)]) that if system (2.1) satisfies (H1), then for any initial value $x_0 \in \mathbb{R}^n$, there exists a unique continuous solution $x(t, t_0, x_0)$, denoted by $x(t)$, and for any $p > 0$,

$$\mathbb{E}[\sup\{\|x(s)\|^p : t_0 \leq s \leq t\}] < \infty, \quad t \geq t_0.$$

Definition 2.1. Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ represent the family of all non-negative functions on $\mathbb{R}^n \times \mathbb{R}_+ \times S$ that are twice differentiable in x and continuously differentiable in t. Suppose $V(x, t, i) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ has the following:

$$V_t = \frac{\partial V(x, t, i)}{\partial t}, \quad V_x = \frac{\partial V(x, t, i)}{\partial x}, \quad V_{xx} = (\frac{\partial^2 V(x, t, i)}{\partial x_i \partial x_j})_{n\times n}.$$

Then we define an operator L acting on $V(x, t, i)$ and $LV : \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}$, where

$$\begin{aligned} LV(x, t, i) &= V_t(x, t, i) + V_x(x, t, i)f(x, t, i) \\ &\quad + \frac{1}{2}\operatorname{trace}[g^T(x, t, i)V_{xx}(x, t, i)g(x, t, i)] + \sum_{j=1}^{N} \gamma_{ij} V(x, t, j). \end{aligned}$$

Definition 2.2. Let $\alpha(t) > 0$ be such that $\alpha(t) \to \infty$ as $t \to \infty$. A non-trivial solution $x(t)$ of system (2.1) is almost sure asymptotic stable with decay function $\alpha(t)$ and order at least $\gamma > 0$, if its generalized Lyapunov exponent is less than or equal to $-\gamma$ with probability one, i.e.,

$$\limsup t \to +\infty \frac{\ln(\|x(t)\|)}{\ln \alpha(t)} \leq -\gamma, \quad \text{a.s.} \tag{2.3}$$

The following lemmas will play important roles in our derivation.

Lemma 2.3. *Assume* (H1) *and that* $f(0,t,i)=0$ *and* $g(0,t,i)=0$. *Then for all* $x_0 \in \mathbb{R}^n$, *such that* $x_0 \neq 0$, *we have*

$$\mathbb{P}(x(t,t_0,x_0) \neq 0, \forall t \geq t_0) = 1.$$

The proof of the above lemma can be found in [17, Lemma 2.1]. We require that the assumptions of Lemma 2.3 hold for the rest of this article, i.e., assume that for all $t \in \mathbb{R}_+$ and $i \in S$,

$$f(0,t,i) \equiv 0, \quad g(0,t,i) \equiv 0.$$

Lemma 2.4 (Exponential Martingale inequality). *Let* $F(t) \in L^2(\mathbb{R}_+,\mathbb{R}^n)$ *and* T, ϵ, η *be any positive numbers. Then*

$$\mathbb{P}[\sup\{Y(t) : 0 \leq t \leq T\} > \eta] \leq e^{-\epsilon\eta},$$

where

$$Y(t) = \int_0^t F(s)\mathrm{d}W(s) - \frac{\epsilon}{2}\int_0^t \|F(s)\|^2 \mathrm{d}s.$$

The proof of the above lemma can be found in [18, Theorem 1.7.4]. With the above preparations, the main result of this paper is to seek sufficient conditions about the almost sure asymptotic stability on a general decay rate of SDEs with Markov switching.

3. Almost sure asymptotic stability of SDEs with Markov switching

In this section, based on the research in [17] on SDEs with Markov switching, we discuss the almost sure asymptotic stability on a general decay rate of stochastic differential systems with Markov switching. We are in a position to state the first result.

Theorem 3.1. *Assume* (H1) *and that there exist a function* $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ *a continuous function* $G(t) > 0$, *constants* $p \in \mathbb{N}^+$, $\beta > 0$, $\sigma > 0$, $m \geq 0$, $M \geq 0$, *such that for all* $t \geq 0$, $x \in \mathbb{R}^n$ *and* $i \in S$, *the following conditions hold:*

(1) $LV(x,t,i) \leq -\beta V(x,t,i)$;
(2) $\alpha(t)^m\|x\|^p \leq G(t)V(x,t,i)$ *and* $\lim_{t\to+\infty} \frac{\ln G(t)}{\ln \alpha(t)} = a$, $a \in \mathbb{R}$;
(3) $\limsup_{t\to+\infty} \frac{t}{\ln \alpha(t)} = M$;
(4) $\|V_x(x,t,i)g(x,t,i)\|^2 \leq \sigma\|V(x,t,i)\|^2$.

Let $x_0 \in \mathbb{R}^n (x_0 \neq 0)$. *Then*

$$\limsup_{t\to+\infty} \frac{\ln(\|x(t)\|)}{\ln \alpha(t)} < -\gamma^*, \quad a.s.,$$

where $\gamma^* = \frac{m+\beta M-a}{p}$.

Furthermore, if $\gamma^* > 0$, *the solution* $x(t)$ *of* (2.1) *is deemed to converge to zero with decay function* $\alpha(t)$ *and order at least* γ^* *with probability one.*

Proof. Since system (2.1) satisfies condition (H1), without loss of generality, we assume that the initial moment is $t_0 = 0$, so for any initial value $x_0 \in \mathbb{R}^n (x_0 \neq 0)$, there exists a unique continuous solution $x(t)$ of system (2.1). And Lemma 2.3 states that almost all sample paths of $x(t)$ will never arrive at the origin, i.e., $x(t) \neq 0$ almost surely for any $t \geq 0$.

For each $n \geq 1$, we define a stopping time

$$\tau_n = \inf\{t \geq 0 : \|x(t)\| > n\}.$$

The condition (H1) is satisfied, which means that the solution process of system (2.1) is non-explosive, and we can get $\tau_n \to \infty$ as $n \to \infty$. Next, we use the generalized Itô formula to study $e^{\beta t} V(x(t), t, r(t))$, then we obtain

$$\begin{aligned}
&\mathbb{E}[e^{\beta(t\wedge\tau_n)} V(x(t \wedge \tau_n), t \wedge \tau_n, r(t \wedge \tau_n))] \\
&= \mathbb{E}[V(x(0), 0, r(0))] + \mathbb{E}\int_0^{t\wedge\tau_n} e^{\beta s}[\beta V(x(s), s, r(s)) + LV(x(s), s, r(s))]\mathrm{d}s \\
&\leq D,
\end{aligned}$$

where $D = \mathbb{E}[V(x(0), 0, r(0))]$. Letting $n \to \infty$, we have

$$\mathbb{E}[V(x(t), t, r(t))] \leq De^{-\beta t}. \tag{3.1}$$

For all continuous functions $V(x, t, i) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, by the generalized Itô formula we have

$$\begin{aligned}
V(x(t), t, r(t)) &= V(x(0), 0, r(0)) + \int_0^t LV(x(s), s, r(s))\mathrm{d}s \\
&\quad + \int_0^t V_x(x(s), s, r(s))g(x(s), s, r(s))\mathrm{d}W(s) \\
&\leq V(x(0), 0, r(0)) + \int_0^t V_x(x(s), s, r(s))g(x(s), s, r(s))\mathrm{d}W(s).
\end{aligned}$$

Hence we obtain that for any $k \geq 1$, $\delta \in (0, \frac{1}{32\sigma})$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{\delta(k-1)\leq t\leq\delta k} V(x(t), t, r(t))\Big) \\
&\leq \mathbb{E}(V(x(\delta(k-1)), \delta(k-1), r(\delta(k-1)))) \\
&\quad + \mathbb{E}(\sup_{\delta(k-1)\leq t\leq\delta k} \int_{\delta(k-1)}^t V_x(x(s), s, r(s))g(x(s), s, r(s))\mathrm{d}W(s)).
\end{aligned}$$

An application of the Burkholder-Davis-Gundy inequality leads to

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{\delta(k-1)\leq t\leq\delta k} \int_{\delta(k-1)}^t V_x(x(s), s, r(s))g(x(s), s, r(s))\mathrm{d}W(s)\Big) \\
&\leq \sqrt{32}\mathbb{E}\Big(\int_{\delta(k-1)}^{\delta k} \|V_x(x(s), s, r(s))g(x(s), s, r(s))\|^2\mathrm{d}s\Big)^{1/2} \\
&\leq \sqrt{32\sigma}\mathbb{E}\Big(\int_{\delta(k-1)}^{\delta k} \|V(x(s), s, r(s))\|^2\mathrm{d}s\Big)^{1/2} \\
&\leq \sqrt{32\sigma\delta}\mathbb{E}\big(\sup_{\delta(k-1)\leq t\leq\delta k} V(x(t), t, r(t))\big),
\end{aligned} \tag{3.2}$$

which yields

$$\mathbb{E}\Big(\sup_{\delta(k-1)\leq t\leq\delta k} V(x(t), t, r(t))\Big) \leq \frac{1}{1-\sqrt{32\sigma\delta}}\mathbb{E}(V(x(\delta(k-1)), \delta(k-1), r(\delta(k-1)))).$$

Then by (3.1) we have

$$\mathbb{E}\big[\sup_{\delta(k-1)\le t\le \delta k} V(x(t),t,r(t))\big] \le \frac{D}{1-\sqrt{32\sigma\delta}} e^{-\beta\delta(k-1)}.$$

By Markov's inequality, for any given time $t \ge 0$ and any $\varrho \in (0, \beta\delta)$, we obtain

$$\begin{aligned}
&\mathbb{P}\big\{\omega : \sup_{\delta(k-1)\le t\le \delta k}\{V(x(t),t,r(t))\} > e^{(\varrho-\beta\delta)k}\big\}\\
&\le \frac{De^{-\beta\delta(k-1)}}{(1-\sqrt{32\sigma\delta})e^{(\varrho-\beta\delta)k}} = \frac{De^{\beta\delta}}{(1-\sqrt{32\sigma\delta})} e^{-\varrho k}.
\end{aligned} \tag{3.3}$$

Since $\sum_{k=1}^{\infty} e^{-\varrho k} < \infty$, we have

$$\sum_{k=1}^{\infty} \mathbb{P}\{\omega : \sup_{\delta(k-1)\le t\le \delta k}\{V(x(t),t,r(t))\} > e^{(\varrho-\beta\delta)k}\} < +\infty.$$

Then we apply the Borel-Cantelli lemma to obtain that, for almost all $\omega \in \Omega$, $\exists \bar{k} > 0$, where $\bar{k}$ only related to $\omega \in \Omega$, with

$$\mathbb{P}\big\{\omega : \sup_{\delta(k-1)\le t\le \delta k}\{V(x(t),t,r(t))\} > e^{(\varrho-\beta\delta)k}\big\} = 0, \quad \forall \delta(k-1) \le t \le \delta k, k \ge \bar{k},$$

i.e.

$$V(x(t),t,r(t)) \le e^{(\varrho-\beta\delta)k}, \quad \text{a.s.}$$

Further

$$V(x(t),t,r(t)) \le e^{(\varrho-\beta\delta)k} \le e^{\frac{(\varrho-\beta\delta)}{\delta}t}, \quad \forall \delta(k-1) \le t \le \delta k, k \ge \bar{k}. \tag{3.4}$$

By (3.4), when $\forall \delta(k-1) \le t \le \delta k, k \ge \bar{k}$, we obtain

$$\alpha(t)^m(\|x(t)\|^p) \le G(t)V(x(t),t,r(t)) \le G(t)e^{\frac{(\varrho-\beta\delta)}{\delta}t},$$

which implies

$$m\ln\alpha(t) + \ln(\|x(t)\|^p) \le \ln G(t) + \frac{(\varrho-\beta\delta)}{\delta}t.$$

Further

$$\frac{\ln(\|x(t)\|^p)}{\ln\alpha(t)} \le -m + \frac{(\varrho-\beta\delta)}{\delta}\frac{t}{\ln\alpha(t)} + \frac{\ln G(t)}{\ln\alpha(t)}.$$

Now letting $k \to +\infty$ yields

$$\limsup_{t\to+\infty} \frac{\ln(\|x(t)\|^p)}{\ln\alpha(t)} \le \lim_{t\to+\infty}\Big(-m + \frac{(\varrho-\beta\delta)}{\delta}\frac{t}{\ln\alpha(t)} + \frac{\ln G(t)}{\ln\alpha(t)}\Big).$$

By conditions (2), (3) and the property of $\alpha(t)$, we have

$$\limsup_{t\to+\infty} \frac{\ln(\|x(t)\|)}{\ln\alpha(t)} \le -\frac{m + (\beta - \frac{\varrho}{\delta})M - a}{p}, \quad \text{a.s.}$$

Let $\varrho \to 0$ and $\gamma^* := \frac{m+\beta M-a}{p}$, we have

$$\limsup_{t\to+\infty} \frac{\ln(\|x(t)\|)}{\ln\alpha(t)} \le -\gamma^*, \quad \text{a.s.}$$

$\square$

Remark 3.2. (1) Assuming that all the conditions of [17, Theorem 3.2] hold, with $m = 0$, $G(t) = \frac{1}{c_1}$, $a = 0$, $M = 1$, and $\beta = \frac{\lambda}{c_2}$, we have that Theorem 3.1 holds and the Lyapunov exponent is less than or equal to $\gamma^* = \frac{m+\beta M-a}{p} = \frac{\lambda}{pc_2}$, which is the same as the conclusion of [17]. In other words, we can obtain the same result and decay rate. Hence Theorem 3.1 is a generalization of the results in [17].

(2) The two important questions of the above Theorem are: one is the selection of the decay function, and the other is whether the Lyapunov function used in the theorem exists. The first question is generally easy to solve. For example, we can choose $\alpha(t) = O(e^t)$ or $\alpha(t) = O(\ln(t + 1))$, so we can get the almost sure exponential stability or the almost sure logarithmic stability. The key question is the selection of Lyapunov functions. In the following, we will construct suitable Lyapunov functions.

It is rare to find the Lyapunov functions satisfying Theorem 3.1 when judging the stability of the system. However, in this article it is not complex to determine the stability of (2.1) with $V(x, t, i) = (x^T Q_i x)^{\frac{p}{2}} (1 \le i \le N)$ and $p > 0$. For instance, let $p = 1$, $x \in \mathbb{R}^n$ and Q_i are n-dimensional symmetric positive-definite matrices.

From definition 2.1, we have

$$
\begin{gathered}
V_t(x,t,i) = 0, \quad V_x(x,t,i) = (x^T Q_i x)^{(-1/2)} x^T Q_i, \\
V_{xx}(x,t,i) = (x^T Q_i x)^{(-1/2)} Q_i - \frac{1}{2}(x^T Q_i x)^{(-3/2)} Q_i x x^T Q_i,
\end{gathered}
$$

$$
\begin{aligned}
\frac{1}{2}\operatorname{trace}[g^T(x,t,i) V_{xx}(x,t,i) g(x,t,i)] &= \frac{1}{2}(x^T Q_i x)^{(-3/2)} \operatorname{trace}[g^T(x,t,i) Q_i g(x,t,i)] \\
&\quad - \frac{1}{4}(x^T Q_i x)^{(-3/2)} \|x^T Q_i g(x,t,i)\|^2;
\end{aligned}
$$

therefore,

$$
\begin{aligned}
LV(x,t,i) &= V_t(x,t,i) + V_x(x,t,i) f(x,t,i) \\
&\quad + \frac{1}{2}\operatorname{trace}[g^T(x,t,i) V_{xx}(x,t,i) g(x,t,i)] + \sum_{j=1}^N \gamma_{ij} V(x,t,j) \\
&= (x^T Q_i x)^{(-1/2)} x^T Q_i f(x,t,i) \\
&\quad + \frac{1}{2}(x^T Q_i x)^{(-1/2)} \operatorname{trace}[g^T(x,t,i) Q_i g(x,t,i)] \\
&\quad - \frac{1}{2}(x^T Q_i x)^{(-3/2)} \|x^T Q_i g(x,t,i)\|^2 + \sum_{j=1}^N \gamma_{ij} (x^T Q_j x)^{1/2}.
\end{aligned}
\tag{3.5}
$$

Corollary 3.3. *Consider* (2.1). *Assume* (H1) *and let $p = 1$ in Theorem 3.1. In addition, for all $t \ge 0$, $x \in \mathbb{R}^n$ and $i \in S$, assume the following conditions hold:*

(1) $Z(x,t,i) \le -\lambda \|x\|^2$, *where*

$$
\begin{aligned}
Z(x,t,i) = x^T Q_i f(x,t,i) &+ \frac{1}{2}\operatorname{trace}[g^T(x,t,i) Q_i g(x,t,i)] \\
&- \frac{1}{2}(x^T Q_i x)^{(-1)} \|x^T Q_i g(x,t,i)\|^2 \\
&+ (x^T Q_i x)^{(\frac{1}{2})} \sum_{j=1}^N \gamma_{ij} (x^T Q_j x)^{1/2};
\end{aligned}
$$

(2) $\limsup_{t\to+\infty} \frac{t}{\ln \alpha(t)} = M$;
(3) $\|x^T Q_i g(x,t,i)\|^2 \le \sigma \|x^T Q_i x\|^2$.

Then the conclusion in Theorem 3.1 holds.

Proof. Let $V(x,t,i) = (x^T Q_i x)^{1/2}$. We know that the specified Lyapunov functions V are not differentiable when the spatial variable (denoted x) is zero. However, by Lemma 2.3, we just need the Lyapunov function to be differentiable in $\mathbb{R}^n \setminus \{0\}$ with respect to the variable x. Let $\mathbb{R}^n_0 := \mathbb{R}^n \setminus \{0\}$, so it is obvious that $(x^T Q_i x)^{1/2} \in C^{2,1}(\mathbb{R}^n_0 \times \mathbb{R}_+ \times S; \mathbb{R}_+)$. Because matrix Q_i is symmetric positive-definite matrix, we obtain

$$\lambda_{\max}(Q_i) \ge \lambda_{\min}(Q_i) > 0.$$

Therefore, for all $(x,t,i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S$,

$$\begin{aligned}[\min\{\lambda_{\min}(Q_i) : 1 \le i \le N\}]^{1/2}\|x\| \le (x^T Q_i x)^{1/2} \le [\max\{\lambda_{\max}(Q_i) : 1 \\ \le i \le N\}]^{1/2}\|x\|,\end{aligned}$$

and we have the following conclusions:

(1) $V(x,t,i) \le [\max\{\lambda_{\max}(Q_i) : 1 \le i \le N\}]^{1/2}\|x\|$.
(2) $(x^T Q_i x)^{1/2} \ge [\min\{\lambda_{\min}(Q_i) : 1 \le i \le N\}]^{1/2}\|x\|$.

For Theorem 3.1, we can make $G(t) = [\min\{\lambda_{\min}(Q_i) : 1 \le i \le N\}]^{(-\frac{1}{2})}\alpha(t)^m$, so

$$G(t)(x^T Q_i x)^{1/2} \ge \alpha(t)^m \|x\|.$$

Further

$$\lim_{t\to+\infty} \frac{\ln G(t)}{\ln \alpha(t)} = m.$$

Corresponding to Theorem 3.1, we obtain $a = m$.

(3) From condition (1) in this theorem, we obtain

$$\begin{aligned} LV(x,t,i) &= (x^T Q_i x)^{(-1/2)} x^T Q_i f(x,t,i) \\ &\quad + \frac{1}{2}(x^T Q_i x)^{(-1/2)} \operatorname{trace}[g^T(x,t,i) Q_i g(x,t,i)] \\ &\quad - \frac{1}{2}(x^T Q_i x)^{(-3/2)} \|x^T Q_i g(x,t,i)\|^2 + \sum_{j=1}^N \gamma_{ij}(x^T Q_j x)^{1/2} \\ &\le -\frac{\lambda}{\max\{\lambda_{\max}(Q_i) : 1 \le i \le N\}}(x^T Q_i x)^{1/2}, \end{aligned} \tag{3.6}$$

which implies $\beta = \frac{\lambda}{\max\{\lambda_{\max}(Q_i):1\le i\le N\}}$.

Summing up, from the analysis of (1), (2) and (3) we see that $V(x(t),t,i) = (x^T(t) Q_i x(t))^{1/2}$ satisfies all the conditions of Theorem 3.1, so

$$\limsup_{t\to+\infty} \frac{\ln(\|x(t)\|)}{\ln \alpha(t)} \le -\gamma^*, \quad \text{a.s.}$$

By Theorem 3.1, we can take $\delta \in (0, \frac{1}{32\sigma})$, thus $\gamma^* = \beta M$. If $M > 0$, the solution $x(t)$ of (2.1) converges to zero with decay function $\alpha(t)$ and order at least βM with probability one. $\square$

Remark 3.4. (1) If (H1) and Lemma 2.3 hold, we know that almost all sample paths of any solution of system (2.1) beginning from a nonzero state will never arrive at the origin, so we only need Lyapunov functions $V \in C^{2,1}(\mathbb{R}_0^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ in Theorem 3.1 and Corollary 3.3. The following Theorem 3.5 and Corollary 3.7 are equally applicable.

(2) The above proof process shows that when analyzing the almost sure asymptotic stability of a stochastic differential system with Markov switching, the qualified matrices $Q_i (1 \le i \le N)$ can be selected to construct the Lyapunov function according to Theorem 3.1. If it is a one-dimensional system, we need to choose Q_i as N positive numbers, and the proof process is the same as Corollary 3.3.

In the following section, we will further extend the application of Theorem 3.1. In Theorem 3.1, the condition (1) we want to ensure is $LV(x,t,i) \le -\beta V(x,t,i)$, and then our main task is to generalize this condition to $LV(x,t,i) \le h_1(t)V(x,t,i)$ where $h_1(t) \in \mathbb{R}$ for any $t \in \mathbb{R}_+$. To ensure the well-posedness of the above system, the following assumptions are made:

(H2) f and g satisfy
  (1) There is a nonnegative function $\phi_1(t)$ such that for all $(x,t,i) \in \mathbb{R}^n \times \mathbb{R}_+ \times S$,
$$\|f(x,t,i)\|^2 \vee \|g(x,t,i)\|^2 \le \phi_1(t)(1 + \|x\|^2);$$
  (2) There is a nonnegative function $\phi_2(t)$ such that for all $t \ge 0$, $i \in S$ and $x, y \in \mathbb{R}^n$,
$$\|f(x,t,i) - f(y,t,i)\| \vee \|g(x,t,i) - g(y,t,i)\| \le \phi_2(t)\|x - y\|.$$

Then, according to theabove conditions, there exists a unique global solution with initial values $x_0 \in \mathbb{R}^n$ defined in an interval $[t_0, T)$. Since we study the asymptotic behavior of solutions, we assume $T = +\infty$.

Theorem 3.5. *Assume* (H2)*, there exist continuous functions $V \in C^{2,1}(\mathbb{R}_0^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, $G(t) \ge 0$, $h_1(t) \in \mathbb{R}$ and $h_2(t) \ge 0$ for all $t \in \mathbb{R}_+$, constants $p \in \mathbb{N}^+$, $m \ge 0, M \ge 0$, $\vartheta_1 \in \mathbb{R}$, $\vartheta_2 \ge 0$, such that for all $t \ge t_0$, $x \in \mathbb{R}_0^n$ and $i \in S$, the following conditions hold:*

(1) $\alpha(t)^m \|x\|^p \le G(t)V(x,t,i)$ *and* $\lim_{t\to+\infty} \frac{\ln G(t)}{\ln \alpha(t)} = a$, $a \in \mathbb{R}$;
(2) $LV(x,t,i) \le h_1(t)V(x,t,i)$ *and* $\limsup_{t\to+\infty} \frac{\int_0^t h_1(s)\mathrm{d}s}{\ln \alpha(t)} \le \vartheta_1$;
(3) $\|V_x(x,t,i)g(x,t,i)\|^2 \ge h_2(t)V^2(x,t,i)$ *and* $\liminf_{t\to+\infty} \frac{\int_0^t h_2(s)\mathrm{d}s}{\ln \alpha(t)} \ge \vartheta_2$;
(4) $\limsup_{t\to+\infty} \frac{t}{\ln \alpha(t)} = M$.

Let $x_0 \in \mathbb{R}^n (x_0 \ne 0)$. Then
$$\limsup_{t\to+\infty} \frac{\ln(\|x(t)\|)}{\ln \alpha(t)} \le -\gamma^*, \quad a.s.,$$
where
$$\gamma^* = \begin{cases} \frac{1}{p}(m - \vartheta_1 - M - a), & M > \frac{1}{2}\vartheta_2, \\ \frac{1}{p}(m - \vartheta_1 - a + \frac{1}{2}\vartheta_2 - \frac{3}{2}\sqrt{M\vartheta_2}), & M \le \frac{1}{2}\vartheta_2. \end{cases}.$$
Further, if $\gamma^* > 0$, the solution $x(t)$ of equation (2.1) *is almost sure asymptotic stable with decay function $\alpha(t)$ and order at least γ^* with probability one.*

Proof. Since system (2.1) satisfies condition (H2), for any initial value $x(t_0) = x_0 \in \mathbb{R}^n (x_0 \neq 0)$ there exists a unique continuous solution $x(t, t_0, x_0)$.

Applying the Itô formula to $\ln V(\cdot)$ along the trajectory $x(\cdot)$ for (2.1), for any $t \geq t_0$,

$$\begin{aligned} \ln\left(V(x(t), t, r(t))\right) = \ln\left(V(x(t_0), t_0, r(t_0))\right) + M_t + \int_{t_0}^{t} \frac{LV(x(s), s, r(s))}{V(x(s), s, r(s))} \mathrm{d}s \\ - \frac{1}{2} \int_{t_0}^{t} \frac{\|V_x(x(s), s, r(s)) g(x(s), s, r(s))\|^2}{V^2(x(s), s, r(s))} \mathrm{d}s, \end{aligned} \tag{3.7}$$

where

$$M_t = \int_{t_0}^{t} \frac{V_x(x(s), s, r(s)) g(x(s), s, r(s))}{V(x(s), s, r(s))} \mathrm{d}W(s)$$

is a continuous martingale.

The next main task is to study M_t. We choose the standard initial value $x(t_0) = x_0$ and guarantee that $\mathbb{E}\,|x_0|^2 < \infty$. According to exponential Martingale inequality, letting $T = k$, $\epsilon = \varepsilon$, $\eta = \frac{k-1}{\varepsilon}$, and $\varepsilon \in (0,1)$, $k \in \mathbb{N}^+ (k > 1)$. Then by Lemma 2.4 we have

$$\mathbb{P}\big[\sup_{0 \leq t \leq k} \{M_t - Y(t, \varepsilon)\} > \frac{k-1}{\varepsilon}\big] \leq e^{-(k-1)},$$

where

$$Y(t, \varepsilon) = \frac{\varepsilon}{2} \int_{t_0}^{t} \frac{\|V_x(x(s), s, r(s)) g(x(s), s, r(s))\|^2}{V^2(x(s), s, r(s))} \mathrm{d}s.$$

Since $\sum_{k=2}^{\infty} e^{-(k-1)} < \infty$, by Borel-Cantelli Lemma, we obtain that for almost all $\omega \in \Omega$,

$$\mathbb{P}\big[\liminf_{t \to +\infty} (\sup_{0 \leq t \leq k} \{M_t - Y(t, \varepsilon)\}) \leq \frac{k-1}{\varepsilon}\big] = 1.$$

In other words, there exists $\tilde{k} > 0$ where $\tilde{k}$ only related to $\omega \in \Omega$ such that for all $k - 1 \leq t \leq k (k \geq \tilde{k})$, we have

$$M_t \leq \frac{\varepsilon}{2} \int_{t_0}^{t} \frac{\|V_x(x(s), s, r(s)) g(x(s), s, r(s))\|^2}{V^2(x(s), s, r(s))} \mathrm{d}s + \frac{k-1}{\varepsilon}, \quad \text{a.s.}$$

Using (2)–(4), it follows that by (3.7),

$$\ln\left(V(x(t), t, r(t))\right) \leq \ln\left(V(x(t_0), t_0, r(t_0))\right) + \frac{t}{\varepsilon} + \int_{t_0}^{t} h_1(s) \mathrm{d}s - \frac{(1-\varepsilon)}{2} \int_{t_0}^{t} h_2(s) \mathrm{d}s.$$

Further, for all $k - 1 \leq t \leq k (k \geq \tilde{k})$, when $k \to +\infty$ we obtain

$$\limsup_{t \to +\infty} \frac{\ln V(x(t), t, r(t))}{\ln \alpha(t)} \leq \vartheta_1 - \frac{(1-\varepsilon)}{2} \vartheta_2 + \frac{1}{\varepsilon} M. \tag{3.8}$$

By condition (1), we have

$$m \ln \alpha(t) + p \ln(\|x(t)\|) \leq \ln G(t) + \ln\left(V(x(t), t, r(t))\right);$$

further

$$\frac{\ln \|x(t)\|}{\ln \alpha(t)} \leq \frac{1}{p} \big[-m + \frac{\ln\left(V(x(t), t)\right)}{\ln \alpha(t)} + \frac{\ln G(t)}{\ln \alpha(t)} \big].$$

By (3.8) and the property of $\alpha(t)$, letting $k \to \infty$ yields

$$\limsup_{t \to +\infty} \frac{\ln(\|x(t)\|)}{\ln \alpha(t)} \leq -\frac{1}{p} (m - \vartheta_1 + \frac{(1-\varepsilon)}{2} \vartheta_2 - \frac{1}{\varepsilon} M - a). \tag{3.9}$$

Let $\gamma(\varepsilon)=\frac{1}{p}(m-\vartheta_1+\frac{(1-\varepsilon)}{2}\vartheta_2-\frac{1}{\varepsilon}M-a)$, which shows that the order of decay depends on the parameter ε. The next main task is to find the optimal valued $\gamma^*=\sup_{\varepsilon\in(0,1)}\gamma(\varepsilon)$.

Obviously we can obtain that

$$\frac{\mathrm{d}\gamma(\varepsilon)}{\mathrm{d}\varepsilon}=\frac{1}{p}(\frac{1}{\varepsilon^2}M-\frac{1}{2}\vartheta_2),$$

which implies that

$$\gamma^*=\frac{1}{p}(m-\vartheta_1-M-a),\quad M>\frac{1}{2}\vartheta_2,$$
$$\gamma^*=\frac{1}{p}(m-\vartheta_1-a+\frac{1}{2}\vartheta_2-\frac{3}{2}\sqrt{M\vartheta_2}),\quad M\le\frac{1}{2}\vartheta_2.$$

Thus if $\gamma^*>0$, we obtain

$$\limsup_{t\to+\infty}\frac{\ln(\|x(t)\|)}{\ln\alpha(t)}\le-\gamma^*,\quad \text{a.s.},$$

where

$$\gamma^*=\begin{cases}\frac{1}{p}(m-\vartheta_1-M-a), & M>\frac{1}{2}\vartheta_2,\\ \frac{1}{p}(m-\vartheta_1-a+\frac{1}{2}\vartheta_2-\frac{3}{2}\sqrt{M\vartheta_2}), & M\le\frac{1}{2}\vartheta_2.\end{cases}$$

□

Remark 3.6. It should be noted that the Lyapunov function constructed according to Theorem 3.1 is independent of time t. Let $Q:\mathbb{R}_+\to\mathbb{R}^{n\times n}$ be a C^1 positive-definite function with $Q(t)=Q(t)^T$[29]. We take the Lyapunov function as $V(x,t,i)=x^TQ_i(t)x$, where $x\in\mathbb{R}^n$.

Corollary 3.7. *Consider* (2.1) *and let f be $C^1(\mathbb{R}_0^n\times\mathbb{R}_+\times S;\mathbb{R}_+)$ in x. There exist continuous functions $V\in C^{2,1}(\mathbb{R}_0^n\times\mathbb{R}_+\times S;\mathbb{R}_+)$, $G(t)\ge0$, $\varphi_1(t)$, $\varphi_2(t)\in\mathbb{R}$ and $h_2(t)\ge0$, constants $p=2$, $m\ge0$, $\widetilde{a}\in\mathbb{R}$, $M\ge0$, $\vartheta_1\in\mathbb{R}$, $\vartheta_2\ge0$. For all $t\ge t_0$, $x\in\mathbb{R}_0^n$ and $i\in S$, the following conditions hold:*

1. $\dot{Q}_i(t)+\frac{\partial f^T(x,t,i)}{\partial x}Q_i(t)+Q_i(t)\frac{\partial f(x,t,i)}{\partial x}\le\varphi_1(t)Q_i(t)$;
2. $\operatorname{trace}[g^T(x,t,i)Q_i(t)g(x,t,i)]+\sum_{j=1}^N\gamma_{ij}V(x,t,j)\le\varphi_2(t)x^TQ_i(t)x$, $h_1(t):=\varphi_1(t)+\varphi_2(t)$ *and* $\limsup_{t\to+\infty}\frac{\int_0^t h_1(s)\mathrm{d}s}{\ln\alpha(t)}\le\vartheta_1$;
3. $\|x^TQ_i(t)g(x,t,i)\|^2\ge h_2(t)\|x^TQ_i(t)x\|^2$ *and* $\liminf_{t\to+\infty}\frac{\int_0^t h_2(s)\mathrm{d}s}{\ln\alpha(t)}\ge\vartheta_2$;
4. $\limsup_{t\to+\infty}\frac{t}{\ln\alpha(t)}=M$ *and* $\liminf_{t\to+\infty}\frac{\ln Q_i(t)}{\ln\alpha(t)}\ge\widetilde{a}$.

where $\|Q_i(t)\|$ is the determinant of the matrix $Q_i(t)$ at time t. Then the conclusion in Theorem 3.5 holds.

Proof. Let $V(x,t,i)=x^TQ_i(t)x$. It is obvious that $V(x,t,i)\in C^{2,1}(\mathbb{R}_0^n\times\mathbb{R}_+\times S;\mathbb{R}_+)$. And by Definition 2.1, we obtain

$$V_t(x,t,i)=x^T\dot{Q}_i(t)x,$$
$$V_x(x,t,i)f(x,t,i)=x^TQ_i(t)f(x,t,i)+f^T(x,t,i)Q_i(t)x,$$
$$V_{xx}(x,t,i)=Q_i(t),$$

and

$$\begin{aligned} LV(x,t,i) &= V_t(x,t,i) + V_x(x,t,i)f(x,t,i) \\ &\quad + \frac{1}{2}\operatorname{trace}[g^T(x,t,i)V_{xx}(x,t,i)g(x,t,i)] + \sum_{j=1}^N \gamma_{ij}V(x,t,j) \\ &= x^T\dot{Q}_i(t)x + x^TQ_i(t)f(x,t,i) + f^T(x,t,i)Q_i(t)x \\ &\quad + \operatorname{trace}[g^T(x,t,i)Q_i(t)g(x,t,i)] + \sum_{j=1}^N \gamma_{ij}x^TQ_j(t)x. \end{aligned} \tag{3.10}$$

Analyzing the above equality, we have

$$\begin{aligned} LV(x(t),t,i) &= x^T(t)\dot{Q}_i(t)x(t) + x^T(t)Q_i(t)[f(x(t),t,i) - f(0,t,i)] \\ &\quad + [f(x(t),t,i) - f(0,t,i)]^TQ_i(t)x(t) \\ &\quad + \operatorname{trace}[g^T(x(t),t,i)Q_i(t)g(x(t),t,i)] \\ &\quad + \sum_{j=1}^N \gamma_{ij}x^T(t)Q_j(t)x(t). \end{aligned}$$

Since f be $C^1(\mathbb{R}_0^n \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ in x, by Lagrange Mean Value Theorem, there exists $\varepsilon_t \in (0, x(t))$ such that

$$f(x(t),t,i) - f(0,t,i) = f_x(\varepsilon_t x(t),t,i)x(t),$$

which implies that

$$\begin{aligned} LV(x(t),t,i) &= x^T(t)\dot{Q}_i(t)x(t) + x^T(t)Q_i(t)f_x(\varepsilon_t x(t),t,i)x(t) \\ &\quad + x^T(t)Q_i(t)f_x^T(\varepsilon_t x(t),t,i)x(t) \\ &\quad + \operatorname{trace}[g^T(x(t),t,i)Q_i(t)g(x(t),t,i)] \\ &\quad + \sum_{j=1}^N \gamma_{ij}x^T(t)Q_j(t)x(t). \end{aligned}$$

By conditions (1) and (2), we have

$$\begin{aligned} LV(x(t),t,i) &\le (\varphi_1(t) + \varphi_2(t))x(t)^TQ_i(t)x(t) \\ &= h_1(t)V(x(t),t,i), \quad \forall 1 \le i \le N. \end{aligned}$$

We can make $G(t) = \frac{\alpha(t)^m}{Q_i(t)}$. To sum up, Corollary 3.7 satisfies all conditions of Theorem 3.5, which allow us to conclude that

$$\limsup_{t\to+\infty} \frac{\ln(\|x(t)\|)}{\ln \alpha(t)} \le -\gamma^*,$$

where

$$\gamma^* = \begin{cases} \frac{1}{2}(\widetilde{a} - \vartheta_1 - M), & M > \frac{1}{2}\vartheta_2, \\ \frac{1}{2}(\widetilde{a} - \vartheta_1 + \frac{1}{2}\vartheta_2 - \frac{3}{2}\sqrt{M\vartheta_2}), & M \le \frac{1}{2}\vartheta_2. \end{cases}$$

□

4. Examples

To illustrate the validity of our main results, we provide two examples.

Example 4.1. Let $W(t)$ be a standard Brownian motion on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. We consider the following one-dimensional stochastic differential equation with Markovian switching,

$$\mathrm{d}x(t) = f(x(t), t, r(t))\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}W(t), \tag{4.1}$$

with initial value $x_0 \in \mathbb{R}_0^n$ and $r(t)$ be a right-continuous Markov chain takingvalues in $S = \{1, 2\}$ with its generator $\Gamma = \begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}$. Let

$$f(x,t,i) = \begin{cases} -3x, & i = 1, \\ \frac{1}{2}x, & i = 2, \end{cases} \quad g(x,t,i) = \begin{cases} 3x, & i = 1, \\ 5x, & i = 2, \end{cases} \quad V(x,t,i) = \begin{cases} x^2, & i = 1, \\ 4x^2, & i = 2. \end{cases}$$

By Corollary 3.3, we can define a Lyapunov function $(V(x,t,i))^{1/2}$, which implies $Q_1 = 1$ nd $Q_2 = 4$. Let $\alpha(t) = 2te^t$, $m = 0$ and $p = 1$, which shows that $M = 1$, $G(t) = 1$ and $a = 0$. By computing the Itô operator, we have

$$\begin{aligned} Z(x,t,1) &= xf(x,t,1) + \frac{1}{2}\|g(x,t,1)\|^2 - \frac{1}{2}\|x\|^{(-2)}\|xg(x,t,1)\|^2 \\ &\quad + \|x\|(\gamma_{11} + 2\gamma_{11})\|x\| \\ &= -\|x\|^2, \\ Z(x,t,2) &= 4xf(x,t,2) + 2\|x\|(\gamma_{21} + 2\gamma_{21})\|x\| \leq -2\|x\|^2; \end{aligned}$$

then, for each $i \in S$, we have

$$Z(x,t,i) \leq -\|x\|^2,$$

which implies $\lambda = 1$.

In addition, we have

$$\begin{aligned} \|x^T Q_1 g(x,t,1)\|^2 &= 9x^4, \\ \|x^T Q_2 g(x,t,2)\|^2 &= 400x^4. \end{aligned}$$

Let $\sigma = 100$, so we obtain $\|x^T Q_i g(x,t,i)\|^2 \leq 100\|x^T Q_i x\|^2$. Then it is not hard to show that $\beta = \frac{\lambda}{max\{\lambda_{max}(Q_i):1\leq i\leq N\}} = \frac{1}{4}$ by Corollary 3.3, i.e., all conditions of Corollary 3.3 and (H1) are satisfied. Therefore, the solution of system (4.1) converges to zero with decay function $2te^t$ and order at least 1/4 with probability one.

Example 4.2. To show the validity of Theorem 3.5, let us consider the equation

$$\mathrm{d}x(t) = f(x(t), t, r(t))\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}W(t), \tag{4.2}$$

where $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with its generator $\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$. Let

$$f(x,t,i) = \begin{cases} k_1 tx, & i = 1, \\ \frac{1}{2}e^{-t}x, & i = 2, \end{cases} \quad g(x,t,i) = \begin{cases} k_2 t^{1/2}x, & i = 1, \\ k_3 t^{1/2}x, & i = 2, \end{cases}$$

with initial value $x_0 \in \mathbb{R}_0^n$ and define a Lyapunov function

$$V(x,t,i) = \begin{cases} tx^2, & i = 1, \\ x^2, & i = 2. \end{cases}$$

From the above definition, we can obtain that system (4.2) satisfies the condition (H2) and the assumptions of Lemma 2.3. We will be interested in analyzing the asymptotic behavior of solutions. Assuming initial time $t_0 \geq 2$. It is clear that $Q_1(t) = t$ when $i = 1$, hence we have

$$\dot{Q}_1(t) + \frac{\partial f^T(x,y,1)}{\partial x} Q_1(t) + Q_1(t) \frac{\partial f(x,y,1)}{\partial x} = (\frac{1}{t} + 2k_1 t) Q_1,$$
$$\operatorname{trace}[g^T(x,t,1) Q_1(t) g(x,t,1)] + \sum_{j=1}^{N} \gamma_{1j} V(x,t,j) = (\frac{1}{t} + k_2^2 t - 1) tx^2.$$

By Corollary 3.7, for t sufficiently large, we obtain

$$LV(x,t,1) \leq (\frac{2}{t} - 1 + 2k_1 t + k_2^2 t) tx^2 \leq (2k_1 t + k_2^2 t) V(x,t,1).$$

When $i = 2$, we know $Q_2 = 1$ and this shows that

$$\begin{aligned} LV(x,t,2) &= 2xf(x,t,2) + g^2(x,t,2) + 2tx^2 - 2x^2 \\ &= (e^{-t} - 2 + 2t + k_3^2 t) x^2 \leq (2t + k_3^2 t) V(x,t,2). \end{aligned}$$

Therefore, $h_1(t) = (2k_1 t + k_2^2 t) \vee (2t + k_3^2 t)$. We obtain $LV(x,t,i) \leq h_1(t) V(x,t,i)$ for each $i \in \{1,2\}$.

On the other hand, we have

$$\|V_x(x,t,1) g(x,t,1)\|^2 = 4k_2^2 t^3 x^4,$$
$$\|V_x(x,t,2) g(x,t,2)\|^2 = 4k_3^2 t x^4.$$

Letting $h_2(t) = 4k_2^2 t \wedge 4k_3^2 t$, we obtain $\|V_x(x,t,i) g(x,t,i)\|^2 \geq h_2(t) V^2(x,t,i)$ for all $i \in \{1,2\}$.

Taking $\alpha(t) = e^{t^2}$, $m = 1$, $p = 2$, $k_1 = \frac{1}{4}$, $k_2 = 2$, $k_3 = 2$ and

$$G(t) = \begin{cases} \frac{1}{t} e^{t^2}, & i = 1, \\ e^{t^2}, & i = 2, \end{cases}$$

and $M = 0$, $h_1(t) = 6t$, and $h_2(t) = 16t$. We can check that assumptions in Theorem 3.5 hold with $\widetilde{a} = 0$, $\vartheta_1 = 3$, $\vartheta_2 = 8$, and $M \leq \frac{1}{2}\vartheta_2$, which implies $\gamma^* = \frac{1}{2}(\widetilde{a} - \vartheta_1 + \frac{1}{2}\vartheta_2 - \frac{3}{2}\sqrt{M\vartheta_2}) = \frac{1}{2}$. Hence we deduce that the solution of system (4.2) is almost sure asymptotic stable with decay function e^{t^2} and order at least $\frac{1}{2}$ with probability one.

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References

[1] S. P. Bhat, D. S. Bernstein; Finite-time stability of continuous autonomous systems, *SIAM J. Control Optim.*, **38** (2000), 751-766.

[2] T. Caraballo, F. Ezzine, M. A. Hammami; Partial stability analysis of stochastic differential equations with a general decay rate, *J. Engrg. Math.*, **130** (2021), 17.

[3] T. Caraballo, F. Ezzine, M. A. Hammami, L. Mchiri; Practical stability with respect to a part of variables of stochastic differential equations, *Stochastics*, **93** (2021), 647-664.
[4] T. Caraballo, M. A. Hammami, L. Mchiri; Practical asymptotic stability of nonlinear stochastic evolution equations, *Stoch. Anal. Appl.*, **32** (2014), 77-87.
[5] T. Caraballo, M. J. Garrido-Atienza, J. Real; Asymptotic stability of nonlinear stochastic evolution equations, *Stochastic Anal. Appl.*, **21** (2003), 301-327.
[6] T. Caraballo, M. J. Garrido-Atienza, J. Real; Stochastic stabilization of differential systems with general decay rate, *Systems Control Lett.*, **48** (2003), 397-406.
[7] E. J. C. Dela Vega, R. J. Elliott; Backward stochastic differential equations with regime-switching and sublinear expectations, *Stochastic Process. Appl.*, **148** (2022), 278-298.
[8] O. Ignatyev; Partial asymptotic stability in probability of stochastic differential equations, *Statist. Probab. Lett.*, **79** (2009), 597-601.
[9] Y. Kang, D. Zhai, G. Liu, Y. B. Zhao; On input-to-state stability of switched stochastic nonlinear systems under extended asynchronous switching, *IEEE Transactions on Cybernetics*, **46** (2016), 1092-1105.
[10] Y. Kang, D. Zhai, G. Liu, Y. B. Zhao, P. Zhao; Stability analysis of a class of hybrid stochastic retarded systems under asynchronous switching, *IEEE Trans. Automat. Control*, **59** (2014), 1511-1523.
[11] F. Li, C. Du, C. Yang, W. Gui; Passivity-based asynchronous sliding mode control for delayed singular Markovian jump systems, *IEEE Trans. Automat. Control*, **63** (2018), 2715-2721.
[12] H. Lin, P. J. Antsaklis; Stability and stabilizability of switched linear systems: a survey of recent results, *IEEE Trans. Automat. Control*, **54** (2009), 308-322.
[13] J. Liu, B. L. S. Prakasa Rao; On conditional Borel-Cantelli lemmas for sequences of random variables, *J. Math. Anal. Appl.*, **399** (2013), 156-165.
[14] M. Liu, Danial W. C. Ho, Y. Niu; Stabilization of Markovian jump linear system over networks with random communication delay, *Automatica J. IFAC*, **45** (2009), 416-421.
[15] S. Lu, X. Yang; Partial practical stability and asymptotic stability of stochastic differential equations driven by LšŠvy noise with a general decay rate, *J. Appl. Anal. Comput.*, **13** (2023), 553-574.
[16] X. Mao; Almost sure polynomial stability for a class of stochastic differential equations, *Quart. J. Math. Oxford Ser.*, **43** (1992), 339-348.
[17] X. Mao; Stability of stochastic differential equations with Markovian switching, *Stochastic Process. Appl.*, **79** (1999), 45-67.
[18] X. Mao; stochastic differential equations and applications, Second edition. *Horwood Publishing Limited, Chichester*, (2008).
[19] X. Mao, A. Truman, C. Yuan; Euler-Maruyama approximations in mean-reverting stochastic volatility model under regime-switching, *J. Appl. Math. Stoch. Anal.*, (2006), Article ID 80967.
[20] X. Mao, C. Yuan; stochastic differential equations with Markovian switching, *Imperial College Press, London*, (2006).
[21] M. Mariton; Jump Linear Systems in Automatic Control, *Marcel Dekker, New York*, (1990).
[22] H. Mei, G. Yin; Convergence and convergence rates for approximating ergodic means of functions of solutions to stochastic differential equations with Markov switching, *Stochastic Process. Appl.*, **125** (2015), 3104-3125.
[23] K. Peiffer, N. Rouche; Liapunov's second method applied to partial stability, *J. Mécanique*, **8** (1969), 323-334.
[24] Y. Saito, T. Mitsui; Stability analysis of numerical schemes for stochastic differential equations, *SIAM J. Numer. Anal.*, **33** (1996), 2254-2267.
[25] S. P. Sethi, Q. Zhang; Hierarichical Decision Making in Stochastic Manufacturing Systems, *Birkhäuser, Boston*, (1994).
[26] L. Shen, J. Sun; p-th moment exponential stability of stochastic differential equations with impulse effect, *Sci. China Inf. Sci.*, **54** (2011), 1702-1711.
[27] A. V. Skorokhod; Asymptotic methods in the theory of stochastic differential equations, *American Mathematical Society, Providence, RI.*, (1989).
[28] D. D. Sworder, J. E. Boyd; Estimation Problems in Hybrid Systems, *Cambridge University Press, Cambridge, UK*, (1999).
[29] H. Wang, X. Yang, Y. Li, X. Li; LaSalle type stationary oscillation theorems for affine-periodic systems, *Discrete Contin. Dyn. Syst. Ser. B*, **22** (2017), 2907-2921.

[30] H. Wu, J. Sun; p-moment stability of stochastic differential equations with impulsive jump and Markovian switching, *Automatica J. IFAC*, **42** (2006), 1753-1759.
[31] S. Wu, D. Han, X. Meng; p-moment stability of stochastic differential equations with jumps, *Appl. Math. Comput.*, **152** (2004), 505-519.
[32] Z. Wu, P. Shi, Z. Shu, H. Su, R. Lu; Passivity-based asynchronous control for Markov jump systems, *IEEE Trans. Automat. Control*, **62** (2017), 2020-2025.
[33] J. Xiong, J. Lam; Stabilization of discrete-time Markovian jump linear systems via time-delayed controllers, *Automatica J. IFAC*, **42** (2006), 747-753.
[34] G. Yin, R. H. Liu, Q. Zhang; Recursive algorithms for stock liquidation: a stochastic optimization approach, *SIAM J. Optim.*, **13** (2002) 240-263.
[35] C. Yuan, J. Lygeros; On the exponential stability of switching diffusion processes, *IEEE Trans. Automat. Control*, **50** (2005), 1422-1426.
[36] C. Yuan, X. Mao; Robust stability and controllability of stochastic differential delay equations with Markovian switching, *Automatica J. IFAC*, **40** (2004), 343-354.
[37] Q. Zhu; Razumikhin-type theorem for stochastic functional differential equations with LšŠvy noise and Markov switching, *Internat. J. Control*, **90** (2017), 1703-1712.
[38] C. Zhu, G. Yin; On competitive Lotka-Volterra model in random environments, *J. Math. Anal. Appl*, **357** (2009) 154-170.

Shuaishuai Lu
College of Mathematics, Jilin University, Changchun 130012, China
Email address: stluss@outlook.com

Xue Yang
College of Mathematics, Jilin University, Changchun, 130012, China
Email address: xueyang@jlu.edu.cn