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# GLOBAL EXISTENCE AND ASYMPTOTIC PROFILE FOR A DAMPED WAVE EQUATION WITH VARIABLE-COEFFICIENT DIFFUSION

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Abstract. We considered a Cauchy problem of a one-dimensional semilinear wave equation with variable-coefficient diffusion, time-dependent damping, and perturbations. The global well-posedness and the asymptotic profile are given by employing scaling variables and the energy method. The lower bound estimate of the lifespan to the solution is obtained as a byproduct.

## 1. Introduction

We investigate the asymptotic profile and lifespan estimate of solutions to a one-dimensional semilinear wave equation with variable-coefficient diffusion, time-dependent damping, and perturbations

$$\begin{gathered} \partial_t^2 u - \partial_x(a(x)\partial_x u) + b(t)\partial_t u = c(t)\partial_x u + d(t)u + N(u, \partial_x u, \partial_t u), \\ \text{for } t > 0,\ x \in \mathbb{R}, \\ u(0,x) = \varepsilon u_0(x), \quad \partial_t u(0,x) = \varepsilon u_1(x), \quad x \in \mathbb{R}, \end{gathered} \tag{1.1}$$

where $\varepsilon$ is a small parameter describing the smallness of the initial data, the diffusion coefficient $a(x)$ is Lipshitz continuous and possesses a positive lower bound, the coefficients $b(t)$, $c(t)$, $d(t)$ are smooth, $b(t) \sim (1+t)^{-\beta}$, $\beta \in [-1,1)$, $c(t)\partial_x u$, $d(t)u$ can be regarded as small perturbations and the nonlinear term

$$|N(u, \partial_x u, \partial_t u)| \le C|u|^{p_1}|\partial_x u|^{p_2}|\partial_t u|^{p_3},$$

and the precise assumptions regarding these terms and the initial data will be provided in Section 2.

The primary objective is to establish the global well-posedness and asymptotic profile of the solutions to (1.1) under the following conditions

$$p_1 + 2p_2 + \big(3 - \frac{2\beta}{1+\beta}\big)p_3 > 3, \quad p_2 + p_3 \le 1. \tag{1.2}$$

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The second objective is to determine the lower bound estimate of the lifespan of the solutions when

$$p_1 + 2p_2 + \big(3 - \frac{2\beta}{1+\beta}\big)p_3 \le 3, \quad p_2 + p_3 \le 1. \tag{1.3}$$

Equation (1.1) typically arises from gene and population dynamics models in Biology, where $\partial_x(a(x)\partial_x)$ is called the diffusion operator, c.f. [4]. The spatial distribution of individuals is described by Brownian motion, so the population densities are solutions to the corresponding reaction diffusion equations. However, when $n = 1$, the process is often substituted with damped wave equations, and the disturbance terms in (1.1) stem from the non-uniformity of the medium. The derivation of (1.1) and its physical background can be found in [2],[6],[10],[14],[20].

Todorova and Yordanov [15] demonstrated the existence of a critical exponent, denoted as $p_F(n) = 1 + \frac{2}{n}$, which plays a pivotal role in distinguishing between global existence and non-existence of solutions to the equation

$$\begin{gathered} \partial_t^2 u - \Delta u + \partial_t u = |u|^p, \quad t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = \varepsilon u_0(x), \quad \partial_t u(0,x) = \varepsilon u_1(x), \quad x \in \mathbb{R}^n. \end{gathered} \tag{1.4}$$

More precisely, when $p_F(n) < p \le \frac{n}{n-2}, n \ge 3$ or $p_F(n) < p < \infty,\ n = 1, 2$, (1.4) has a global solution, however when $1 < p < p_F(n)$, for all $n \in \mathbb{N}$, the solution blows up in finite time even for small initial values. Subsequently, Zhang [21] proved that the case of $p = p_F(n)$ belongs to the blow-up scenario. It should be noted that $p_F(n)$ is known as the Fujita exponent, and it serves as the critical index for the corresponding Cauchy problems of the heat equation (see [3]). Furthermore, the lifespan of the solution to (1.4) can be estimated as follows

$$\text{Lifespan}(u) \sim \begin{cases} C\varepsilon^{(-\frac{1}{p-1} - \frac{n}{2})^{-1}}, & 1 < p < p_F(n), \\ e^{C\varepsilon^{-(p-1)}}, & p = p_F(n), \\ \infty, & p > p_F(n). \end{cases} \tag{1.5}$$

More details can be found in Ikeda-Ogawa [7], Ikeda-Wakasugi [9], Lai-Zhou [11], Li [12], Li-Zhou [13].

Wirth [17, 18, 19] studied systematically the influence of the index $\gamma$ on the behavior of the solutions to the linear wave equations with time-dependent damping

$$\partial_t^2 u - \Delta u + \Gamma(t)\partial_t u = 0,$$

where $\Gamma(t) = (1+t)^{-\gamma}$, $\gamma \in \mathbb{R}$. He classified the behavior as follows: when $\gamma < -1$, the damping term is referred to as "over-damping", in which case the solution does not decay to zero as $t \to \infty$; for $-1 \le \gamma < 1$, the damping term is called "effective" because the solution behaves similarly to the corresponding heat equation, and the asymptotic profile of the solution is described by the scaled Gaussian; when $\gamma > 1$, the damping term is labeled "non-effective", indicating that the solution behaves similarly to the corresponding wave equation. From this perspective, the assumption regarding the index $\beta$ in (1.1) is reasonable.

Gallay and Raugel [4] conducted a comprehensive study on the asymptotic expansions for the damped wave equation

$$\partial_t^2 u - \partial_x(a(x)\partial_x u) + u_t = N(u, u_x, u_t), \quad x \in \mathbb{R},\ t \ge 0. \tag{1.6}$$

Among their notable findings, they successfully obtained the first-order asymptotic profile by employing scaling variables

$$y = \frac{x}{\sqrt{t+t_0}}, \quad s = \log(t+t_0), \quad t_0 > 0 \text{ is fixed}, \tag{1.7}$$

and the methodology primarily relied on energy-based approaches. Subsequently, Wakasugi [16] explored a similar problem of the equation

$$\partial_t^2 u - \Delta u + b(t)\partial_t u = c(t)\cdot\nabla u + d(t)u + N(u,\nabla u,\partial_t u), \quad t>0,\ x\in\mathbb{R}^n, \tag{1.8}$$

where the scaling variables (1.7) were replaced with the following new variables

$$y = \frac{x}{B(t)+1}, \quad s = \log\big(B(t)+1\big), \quad B(t) = \int_0^t \frac{d\tau}{b(\tau)}. \tag{1.9}$$

When the motion occurs within an inhomogeneous medium, the diffusion coefficients in (1.8) depend on the space variable $x$, as highlighted in [4]. This naturally prompts the question: how will the solution behave when considering variable diffusion coefficients in (1.8)? In this article, we focus our attention on this intriguing problem. By employing the scaling variables given in (1.9), the equation (1.1) can be transformed into a first-order differential system, and the abstract theory of operator semigroup can be used to obtain the local well-posedness. Building upon the foundational work of [4] and [16], we employ spectral decomposition and the energy method to investigate global existence and asymptotic behavior. However, as the diffusion coefficient of (1.1) is no longer constant, the energy functional utilized by Wakasugi in [16] becomes inapplicable. To overcome this issue, some modifications on the energy functionals have to be made such that an a prior estimate on the blowup quantity can be obtained, and as a result, the global existence and asymptotic behavior can be achieved.

This article is organized as follows. In Section 2, we present a set of assumptions on the coefficients and the nonlinear term, and outline our main results. Section 3 is dedicated to establishing local well-posedness using the semigroup method. In Section 4, we prove the global well-poesdness and the asymptotic profile. The lower bound estimate of the lifespan for (1.1) is derived in Section 5.

**Notation.** $f \lesssim g$ ($f \gtrsim g$) means there exists a constant $C>0$ such that $f \le Cg$ ($f \ge Cg$), and $f \sim g$ when $g \lesssim f \lesssim g$. $H^{k,m}(\mathbb{R})$ is the weighted Sobolev space equipped with the norm $\|f\|_{H^{k,m}(\mathbb{R})} = \sum_{|\alpha|\le k} \|\langle x\rangle^m D_x^\alpha f\|_{L^2(\mathbb{R})}$, where $k\in\mathbb{Z}$, $m \ge 0$, $\langle x\rangle = (1+|x|^2)^{1/2}$, and in situations where no ambiguity arises, we sometimes omit $R$ in the norm $H^{k,m}(\mathbb{R})$. $C^k\big(I;X\big)$ denotes the space of $k$-times continuously differentiable mapping from $I$ to $X$ with respect to the topology in $X$. Moreover, the positive constant $C$ varies from line to line in this paper.

## 2. Main results

We use the following assumptions:

(A1) The initial data $(u_0,u_1) \in H^{1,1}(\mathbb{R}) \times H^{0,1}(\mathbb{R})$.

(A2) The coefficient of the damping term $b(t)$ satisfies

$$C^{-1}(1+t)^{-\beta} \le b(t) \le C(1+t)^{-\beta}, \quad \Big|\frac{db(t)}{dt}\Big| \le C(1+t)^{-1}b(t), \tag{2.1}$$

where $\beta \in [-1,1)$.

(A3) The coefficient functions $c(t)$, $d(t)$ satisfy

$$|c(t)| \le C(1+t)^{-\gamma}, \quad |d(t)| \le C(1+t)^{-\nu} \tag{2.2}$$

for some $\gamma > 1+\beta/2$, $\nu > 1+\beta$.

(A4) The nonlinear term $N$ satisfies

$$\begin{gathered} |N(z)| \le C|z_1|^{p_1}|z_2|^{p_2}|z_3|^{p_3}, \quad p_i \ge 1 \text{ or } p_i = 0,\ p_1 > 1,\ p_2+p_3 \le 1, \\ p_1 + 2p_2 + (3 - \frac{2\beta}{1+\beta})p_3 > 3. \end{gathered} \tag{2.3}$$

In addition, to ensure the existence of local-in-time solutions, we assume

$$|N(z) - N(w)| \le C|z_1 - w_1|(|z_1| + |w_1|)^{p_1-1}, \tag{2.4}$$

where $z = (z_1, z_2, z_3)$, $w = (w_1, w_2, w_3)$, $z_i, w_i \in \mathbb{R}$ $(i = 1, 2, 3)$, and $-\frac{2\beta}{1+\beta}$ is regarded as a sufficiently large number when $\beta = -1$.

(A5) $a : \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz continuous and $a(x) \ge \underline{a} > 0$. Set $a(x) = \tilde{a}(x) + a_0(x)$, where $\tilde{a}(x) = \lim_{x\to\pm\infty} a(x)$ satifies

$$\tilde{a}(x) = \begin{cases} a_+, & \text{if } x > 0, \\ a_-, & \text{if } x < 0. \end{cases} \tag{2.5}$$

Moreover, for each $\mu > -1/2$, we assume that

$$(1+|x|)^\mu a_0(x) \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}). \tag{2.6}$$

**Remark 2.1.** Assumption (2.6) is reasonable. Indeed, if $a(x) = \frac{1}{(1+|x|)^{1+\mu+\varepsilon}} + 1$, $\mu > -\frac{1}{2}$, and $varepsilon > 0$, then $a_+ = a_- = 1$ and it is easy to verify that $a_0(x)(1+|x|)^\mu \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Let

$$B(t) = \int_0^t \frac{1}{b(\tau)} d\tau, \quad \mathcal{G}(t,x) = \frac{2}{\sqrt{4\pi t}\big(\sqrt{a_+} + \sqrt{a_-}\big)} e^{-\frac{x^2}{4t\tilde{a}(x)}}, \tag{2.7}$$

Equation (2.1) shows that $B(t)$ is strictly increasing and $\lim_{t\to+\infty} B(t) = +\infty$.

**Definition 2.2.** $u$ is a mild solution to (1.1) on the interval $[0, T)$ if (1.1) holds in the sense of distributions and $u \in C([0,T); H^{1,1}(\mathbb{R})) \cap C^1([0,T); H^{0,1}(\mathbb{R}))$. Moreover, if $u \in C([0,T); H^{2,1}(\mathbb{R})) \cap C^1([0,T); H^{1,1}(\mathbb{R})) \cap C^2([0,T); H^{0,1}(\mathbb{R}))$, then $u$ is called a strong solution.

**Definition 2.3.** For a fixed $\varepsilon > 0$, the lifespan of the mild solution to (1.1) is defined as

$$T(\varepsilon) := \sup\{T \in (0,\infty) : \text{there exists a unique mild solution to (1.1) on } [0,T)\}.$$

Our main results are the following three theorems.

**Theorem 2.4** (Local well-posedness)**.** *Under assumptions* (A1)–(A5)*, there exists a* $T = T(\varepsilon\|(u_0,u_1)\|_{H^{1,1}\times H^{0,1}}) > 0$*, such that* (1.1) *has a unique mild solution* $u$ *on* $[0,T)$*. Furthermore, if* $(u_0,u_1) \in H^{2,1}(\mathbb{R}) \times H^{1,1}(\mathbb{R})$*,* $u$ *becomes a strong solution. In addition, if* $T = T(\varepsilon) < \infty$*, then*

$$\lim_{t\to T(\varepsilon)} \|(u,u_t)\|_{H^{1,1}(\mathbb{R})\times H^{0,1}(\mathbb{R})} = \infty. \tag{2.8}$$

*In particular, for each* $T_0 > 0$*, there exists a* $\varepsilon_0 > 0$*, such that for each* $\varepsilon \in (0,\varepsilon_0]$*, we have* $T_0 < T(\varepsilon)$ *(which is the lifespan of* (1.1)*), i.e., the solution* $u$ *will exist on* $[0,T_0]$*.*

By introducing scaling variables, (1.1) can be transformed into an abstract evolution equation, and the abstract semigroup theory on semilinear evolution equations can be applied to prove Theorem 2.4. Utilizing a standard a priori energy estimate, we can demonstrate the existence of a solution at any given time for sufficiently small initial data. This is achieved through the application of Banach's fixed point theorem.

**Theorem 2.5** (Global well-posedness and asymptotic profile). *Under assumptions* (A1)-(A5)*, there exists a* $\varepsilon_1 > 0$*, such that for each* $\varepsilon \in (0, \varepsilon_1]$, (1.1) *has a unique mild solution* $u \in C([0,\infty); H^{1,1}(\mathbb{R})) \cap C^1([0,\infty); H^{0,1}(\mathbb{R}))$. *In addition, the limit* $\alpha^* = \lim_{t\to\infty} \int_{\mathbb{R}} u(t,x)dx$ *exists and*

$$\|u(t,\cdot)-\alpha^*\mathcal{G}(B(t)+1,\cdot)\|_{L^2(\mathbb{R})} \lesssim \varepsilon^2(B(t)+1)^{-\frac{1}{4}-\lambda}\|(u_0,u_1)\|_{H^{1,1}(\mathbb{R})\times H^{0,1}(\mathbb{R})}, \tag{2.9}$$

*where* $\lambda = \min\{\frac{1}{4} + \frac{\mu_0}{2}, \lambda_0, \lambda_1\}$, $\mu_0 = \min\{0, \mu\}$,

$$\lambda_0 = \min\{\frac{1-\beta}{1+\beta}, \frac{\gamma}{1+\beta} - \frac{1}{2}, \frac{\nu}{1+\beta} - 1\}, \quad \lambda_1 = \frac{1}{2}\{p_1 + 2p_2 + (3 - \frac{2\beta}{1+\beta})p_3 - 3\},$$

*in which if* $\beta = -1$, $p_3 \neq 0$, *then* $\frac{1}{1+\beta}$ *and* $\frac{-2\beta p_3}{1+\beta}$ *are regarded as sufficiently large numbers.*

We will make the spectral decomposition on the unknown functions, namely decompose these functions into the leading terms and the remainder terms, respectively. By employing an a priori estimate (Proposition 4.3) based on the energy argument, we can establish the proof of Theorem 2.5. Instead of (2.3), we require that

$$\begin{gathered} p_1 + 2p_2 + \big(3 - \frac{2\beta}{1+\beta}\big)p_3 < 3, \quad \beta \in (-1,1), \\ p_1 > 1,\ p_i \geq 1 \text{ or } p_i = 0\ (i = 2,3),\ p_2 + p_3 \leq 1, \end{gathered}$$

which is equivalent to

$$\beta \in (-1,1),\ 1 < p_1 < 3,\ p_2 = p_3 = 0. \tag{2.10}$$

**Theorem 2.6** (Lower bound estimate of lifespan). *Under the assumptions* (A1)–(A5)*, there exist* $\varepsilon_2 > 0$, $C > 0$*, such that for each* $\varepsilon \in (0, \varepsilon_2]$,

$$B(T(\varepsilon)) + 1 \geq C\varepsilon^{-\frac{2(p_1-1)}{3-p_1}}, \tag{2.11}$$

*where* $B(t)$ *is given by* (2.7).

This theorem will be proved by employing a similar argument as those in Theorem 2.5. Here we remark that the lower bound on the lifespan is sharp in some situations. However, as indicated in Remark 5.2, we are currently unable to established an upper bound of the lifespan.

## 3. Proof of Theorem 2.4

3.1. **Preliminaries.** Let

$$s = \log(B(t)+1), \quad y = (B(t)+1)^{-1/2}x \tag{3.1}$$

and

$$v(s,y) = e^{s/2}u(t(s), e^{s/2}y), \quad w(s,y) = b(t(s))e^{3s/2}u_t(t(s), e^{s/2}y).$$

Then

$$\begin{aligned} u(t,x) &= \big(B(t)+1\big)^{-1/2} v\big(\log(B(t)+1), (B(t)+1)^{-1/2}x\big),\\ u_t(t,x) &= b^{-1}(t)\big(B(t)+1\big)^{-3/2} w\big(\log(B(t)+1), (B(t)+1)^{-1/2}x\big), \end{aligned} \tag{3.2}$$

where $t(s) = B^{-1}(e^s - 1)$ ($B^{-1}$ denotes the inverse function of $B$). Equation (1.1) is transformed into

$$\begin{gathered} v_s - \frac{y}{2}v_y - \frac{1}{2}v = w, \quad s>0,\ y\in\mathbb{R},\\ \frac{e^{-s}}{b(t(s))}(w_s - \frac{y}{2}w_y - \frac{3}{2}w) + w = (a(e^{s/2}y)v_y)_y + r(s,y), \quad s>0,\ y\in\mathbb{R},\\ v(0,y) = \varepsilon v_0(y) = \varepsilon u_0(x), \quad y\in\mathbb{R},\\ w(0,y) = \varepsilon w_0(y) = \varepsilon b(0)u_1(x), \quad y\in\mathbb{R}, \end{gathered} \tag{3.3}$$

where

$$\begin{aligned} r(s,y) = \frac{1}{b(t(s))^2}\frac{db(t(s))}{dt}w + e^{s/2}c(t(s))v_y + e^s d(t(s))v\\ + e^{3s/2}N(e^{-s/2}v, e^{-s}v_y, b^{-1}(t(s))e^{-3s/2}w). \end{aligned} \tag{3.4}$$

**Lemma 3.1** ([16]). *We have*

$$\frac{db(t(s))}{ds} = \frac{db(t(s))}{dt}b(t(s))e^s, \quad \frac{d}{ds}(\frac{1}{b^2(t(s))}) = -\frac{2}{b^2(t(s))}\frac{db(t(s))}{dt}e^s. \tag{3.5}$$

**Lemma 3.2** ([16]). *Under assumption* (A2)*, the following estimates hold*

(i) *when* $\beta\in(-1,1)$,

$$b(t(s)) \sim e^{-\frac{\beta s}{1+\beta}}, \quad \frac{e^{-s}}{b^2(t(s))} \sim e^{-\frac{(1-\beta)s}{1+\beta}}, \quad \frac{1}{b^2(t(s))}\Big|\frac{db(t(s))}{dt}\Big| \le Ce^{-\frac{(1-\beta)s}{1+\beta}}.$$

(ii) *when* $\beta = -1$ ,

$$b(t(s)) \sim \exp(e^s), \quad \frac{e^{-s}}{b^2(t(s))} \sim \exp(-2e^s - s), \quad \frac{1}{b^2(t(s))}\Big|\frac{db(t(s))}{dt}\Big| \le C\exp(-2e^s).$$

**Lemma 3.3** (Gagliardo-Nirenberg inequality, [16]). *Let* $1 < p < \infty$ $(n = 1,2)$, *then for each* $f \in H^{1,0}(\mathbb{R}^n)$,

$$\|f\|_{L^{2p}} \le C\|\nabla f\|_{L^2}^{\sigma}\|f\|_{L^2}^{1-\sigma},$$

*where* $\sigma = n(p-1)/(2p)$.

For completeness, we recall the following results on the existence of solutions to semilinear evolution equations in abstract Banach spaces, see Proposition 4.3.3, Theorem 4.3.4 and Proposition 4.3.9 in [1] for details.

**Lemma 3.4** ([1]). *Let* $T>0$, $X$ *be a Banach space,* $A$ *be a* $m$*-dissipative operator in* $X$ *with dense domain* $D(A)$. *For any* $x\in X$ *and a local Lipschitz mapping* $f : X \to X$, *consider the semilinear problem*

$$\begin{gathered} u'(t) = Au(t) + f(u(t)),\ t\in[0,T],\\ u(0) = x,\\ u\in C([0,T]; D(A)) \cap C^1([0,T]; X) \end{gathered} \tag{3.6}$$

*and the associated integral equation*

$$u(t) = T(t)x + \int_0^t T(t-s)f(u(s))ds, \tag{3.7}$$

*where $(T(t))_{t\geq 0}$ is the contraction semigroup generated by $A$, then the following results hold:*

(i) *Let $M > 0$ and let $x \in X$ be such that $\|x\| \leq M$, then there exits a unique solution $u \in C([0, T_M]; X)$ to* (3.7)*.*

(ii) *Assume that $X$ is reflexive. Let $T > 0$, $x \in X$, and let $u \in C([0,T]; X)$ be a solution to* (3.7)*. Then, if $x \in D(A)$, $u$ is the solution to problem* (3.6)*.*

(iii) *There exits a function $T : X \to (0, \infty]$ with the following properties: for all $x \in X$, there exists $u \in C([0, T(x)); X)$ such that for all $0 < T < T(x)$, $u$ is the unique solution to (3.7). In addition, we have the following alternatives:*
  (a) *$T(x) = \infty$;*
  (b) *$T(x) < \infty$ and $\lim_{t\uparrow T(x)} \|u(t)\| = \infty$.*

3.2. **Proof of Theorem 2.4.** For using Lemma 3.4, we introduce

$$U(t,x) = \langle x\rangle u, \quad U_0(x) = \langle x\rangle u_0, \quad U_1(x) = \langle x\rangle u_1. \tag{3.8}$$

Then

$$\begin{aligned} U_x &= \langle x\rangle^{-1} xu + \langle x\rangle u_x, \\ (a(x)U_x)_x &= \langle x\rangle (a(x)u_x)_x + 2a(x)\langle x\rangle^{-1} xu_x \\ &\quad + \dot{a}(x)\langle x\rangle^{-1} xu + a(x)\langle x\rangle^{-3}(\langle x\rangle^2 - |x|^2)u, \end{aligned}$$

so

$$\begin{gathered} U_{tt} + b(t)U_t = (a(x)U_x)_x + \tilde{c}(t,x)U_x + \tilde{d}(t,x)U + \tilde{N}(U, U_x, U_t), \\ \text{for } t > 0,\ x \in \mathbb{R}, \\ U(0,x) = \varepsilon U_0(x), \quad U_t(0,x) = \varepsilon U_1(x), \quad x \in \mathbb{R}, \end{gathered} \tag{3.9}$$

where

$$\begin{gathered} \tilde{c}(t,x) = c(t) - 2a(x)\langle x\rangle^{-2} x, \\ \tilde{d}(t,x) = d(t) - c(t)\langle x\rangle^{-2} x - \dot{a}(x)\langle x\rangle^{-2} x - a(x)\langle x\rangle^{-4}(\langle x\rangle^2 - 3|x|^2), \\ \tilde{N}(U, U_x, U_t) = \langle x\rangle N(\langle x\rangle^{-1} U, \langle x\rangle^{-1} U_x - \langle x\rangle^{-3} xU, \langle x\rangle^{-1} U_t). \end{gathered}$$

Let $\mathcal{U} = (U, U_t)^{\mathrm{T}}$, $\mathcal{U}_0 = (U_0, U_1)^{\mathrm{T}}$. Then (3.9) is equivalent to

$$\begin{gathered} \mathcal{U}_t = A\mathcal{U} + N(\mathcal{U}), \\ \mathcal{U}(0) = \varepsilon \mathcal{U}_0, \end{gathered} \tag{3.10}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ \partial_x(a(x)\partial_x) & 0 \end{pmatrix}, \quad N(\mathcal{U}) = \begin{pmatrix} 0 \\ -bU_t + \tilde{c}U_x + \tilde{d}U + \tilde{N}(U, U_x, U_t) \end{pmatrix}.$$

Set $X = H^{1,0}(\mathbb{R}) \times L^2(\mathbb{R})$, $D(A) = H^{2,0}(\mathbb{R}) \times H^{1,0}(\mathbb{R})$. Note that

$$A^* = \begin{pmatrix} 0 & -1 \\ -\partial_x(a(x)\partial_x) & 0 \end{pmatrix} = -A,$$

i.e., $A$ is skew-adjoint, so $A$ is an $m$-dissipative operator and $D(A)$ is dense in $X$ (c.f. [1, Corollary 2.4.9]). Therefore, $A$ generates a contraction semigroup $e^{tA}$. Consider the integral form of (3.10),

$$\mathcal{U}(t)=\varepsilon e^{tA}\mathcal{U}_0+\int_0^t e^{A(t-s)}N(\mathcal{U}(s))ds. \tag{3.11}$$

By Lemma 3.4, it suffices to verify that $N(\mathcal{U})$ is local Lipschitz. Indeed,

$$\begin{aligned}&N(\mathcal{U})-N(\mathcal{V})\\&=\begin{pmatrix}0\\ -b(t)(U_t-V_t)+\tilde{c}(U_x-V_x)+\tilde{d}(U-V)+\tilde{N}(U,U_x,U_t)-\tilde{N}(V,V_x,V_t)\end{pmatrix}\end{aligned}$$

for $M>0$ and $\mathcal{U}=(U,U_t)^{\mathrm{T}}$, $\mathcal{V}=(V,V_t)^{\mathrm{T}}$ in $B(0,M)\subset X$, we have

$$\begin{aligned}&\|N(\mathcal{U})-N(\mathcal{V})\|_{H^{1,0}\times L^2}\\&=\|-b(t)(U_t-V_t)+\tilde{c}(U_x-V_x)+\tilde{d}(U-V)+\tilde{N}(U,U_x,U_t)-\tilde{N}(V,V_x,V_t)\|_{L^2}.\end{aligned}$$

From (2.1)-(2.4), it follows that

$$\begin{aligned}&\|-b(t)(U_t-V_t)+\tilde{c}(U_x-V_x)+\tilde{d}(U-V)\|_{L^2}\\&\le C\big(\|U-V\|_{H^{1,0}}+\|U_t-V_t\|_{L^2}\big)\\&=C\|\mathcal{U}-\mathcal{V}\|_{H^{1,0}\times L^2}\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\|\tilde{N}(U,U_x,U_t)-\tilde{N}(V,V_x,V_t)\|_{L^2}&\le C\||U-V|(|U|+|V|)^{p_1-1}\|_{L^2}\\&\le C(M)\|\mathcal{U}-\mathcal{V}\|_{H^{1,0}\times L^2},\end{aligned} \tag{3.13}$$

so

$$\|\mathcal{N}(\mathcal{U})-\mathcal{N}(\mathcal{V})\|_{H^{1,0}\times L^2}\le (C(M)+C)\|\mathcal{U}-\mathcal{V}\|_{H^{1,0}\times L^2}.$$

By (i) of Lemma 3.4, (3.11) has a unique solution $\mathcal{U}\in H^{1,0}(\mathbb{R})\times L^2(\mathbb{R})$, which means (1.1) has a unique mild solution $u$. Furthermore, if $\mathcal{U}_0\in H^{2,0}(\mathbb{R})\times H^{1,0}(\mathbb{R})$, then $\mathcal{U}\in C([0,T];D(A))\cap C^1([0,T];X)$ is a strong solution to (3.10) by Lemma 3.4 (ii), i.e., $U\in C([0,T];H^{2,0}(\mathbb{R}))\cap C^1([0,T];H^{1,0}(\mathbb{R}))\cap C^2([0,T];L^2(\mathbb{R}))$ is a solution to (3.9), and $u$ is a strong solution to (1.1). In addition, if the lifespan $T(\varepsilon)<\infty$, by Lemma 3.4 (iii),

$$\lim_{t\to T(\varepsilon)}\|(u,u_t)(t)\|_{H^{1,1}(\mathbb{R})\times H^{0,1}(\mathbb{R})}=\infty.$$

We now prove that the solution $u$ exists at any given time $T_0>0$. Indeed, consider the non-homogeneous linear equation of (3.9)

$$\begin{gathered}U_{tt}+b(t)U_t=(a(x)U_x)_x+\tilde{c}(t,x)U_x+\tilde{d}(t,x)U+\tilde{N}(t,x),\quad t>0,\ x\in\mathbb{R},\\ U(0,x)=\varepsilon U_0(x),\quad U_t(0,x)=\varepsilon U_1(x),\quad x\in\mathbb{R}.\end{gathered} \tag{3.14}$$

For $\tilde{N}\in L^1((0,T_0];L^2(\mathbb{R}))$, there exists a unique distribution solution which has the standard energy estimate [5]

$$\sup_{0<t<T_0}\|(U,U_t)(t)\|_{H^{1,0}\times L^2}\le C(T_0)\Big(\varepsilon\|(U_0,U_1)\|_{H^{1,0}\times L^2}+\int_0^{T_0}\|\tilde{N}(t)\|_{L^2}dt\Big). \tag{3.15}$$

Let

$$K=\big\{U\in C([0,T_0];H^{1,0}(\mathbb{R}))\cap C^1([0,T_0];L^2(\mathbb{R})):$$

$$\sup_{0<t\le T_0} \|(U,U_t)(t)\|_{H^{1,0}\times L^2} \le 2C(T_0)I_0\varepsilon\}$$

and define $\mathcal{L}: V \to U$, where $I_0 = \|(U_0,U_1)\|_{H^{1,0}\times L^2}$, $V\in K$ and $U$ is the solution to (3.14) with $\tilde{N} = \tilde{N}(V,V_x,V_t)(t,x) = \langle x\rangle N(\langle x\rangle^{-1}V, \langle x\rangle^{-1}V_x - \langle x\rangle^{-3}xV, \langle x\rangle^{-1}V_t)$.

(i) $\mathcal{L}(K)\subset K$. There are three cases according to (2.3).

**Case 1:** If $p_1>1,\ p_2=0,\ p_3=1$, then

$$|\tilde{N}(t)| \le C|\langle x\rangle||\langle x\rangle^{-1}V|^{p_1}|\langle x\rangle^{-1}V_t| \le C|V|^{p_1}|V_t|,$$

and by Sobolev's embedding theorem,

$$\|\tilde{N}(t)\|_{L^2} \le C\|V\|_{L^\infty}^{p_1}\|V_t\|_{L^2} \le C\|V\|_{H^{1,0}}^{p_1}\|V_t\|_{L^2} \le C\big(2C(T_0)I_0\varepsilon\big)^{p_1+1}. \tag{3.16}$$

**Case 2:** If $p_1>1$, $p_2=1$, and $p_3=0$, then

$$\begin{gathered}
|\tilde{N}(t)| \le C|\langle x\rangle|\,|\langle x\rangle^{-1}V|^{p_1}|\langle x\rangle^{-1}V_x - \langle x\rangle^{-3}xV| \le C(|V|^{p_1+1}+|V|^{p_1}|V_x|)\\
\|\tilde{N}(t)\|_{L^2} \le C(\|V\|_{L^\infty}^{p_1}\|V\|_{L^2} + \|V\|_{L^\infty}^{p_1}\|V_x\|_{L^2}) \le C\big(2C(T_0)I_0\varepsilon\big)^{p_1+1}.
\end{gathered}$$

**Case 3:** If $p_1>1, p_2=p_3=0$ and $|\tilde{N}(t)| \le C|\langle x\rangle|\,|\langle x\rangle^{-1}V|^{p_1} \le C|V|^{p_1}$, then the Gagliardo-Nirenberg inequality yields

$$\begin{aligned}
\|\tilde{N}(t)\|_{L^2} &\le C\||V|^{p_1}\|_{L^2} = C\||V|\|_{L^{2p_1}}^{p_1}\\
&\le C\|V_x\|_{L^2}^{\frac{p_1-1}{2}}\|V\|_{L^2}^{\frac{p_1+1}{2}}\\
&\le C\|V_x\|_{H^{1,0}}^{\frac{p_1-1}{2}}\|V\|_{H^{1,0}}^{\frac{p_1+1}{2}}\\
&\le C(2C(T_0)I_0\varepsilon)^{p_1}.
\end{aligned} \tag{3.17}$$

It follows from (3.16)-(3.17) that

$$\|\tilde{N}(t)\|_{L^2} \le C(2C(T_0)I_0\varepsilon)^{p_1}$$

if $\varepsilon<1$ is sufficiently small. Substituting this estimate in (3.15), we have

$$\begin{aligned}
\sup_{0<t<T_0}\|(V,V_t)(t)\|_{H^{1,0}\times L^2} &\le C(T_0)I_0\varepsilon + C(T_0)(2C(T_0)I_0\varepsilon)^{p_1}T_0\\
&\le 2C(T_0)\varepsilon I_0
\end{aligned} \tag{3.18}$$

when $\varepsilon$ is sufficiently small.

(ii) $\mathcal{L}$ is contractive. For any $V^1,V^2\in K$, $U^1-U^2=\mathcal{L}(V^1)-\mathcal{L}(V^2)$ is a solution to

$$\begin{gathered}
U_{tt}+b(t)U_t = (a(x)U_x)_x + \tilde{c}(t,x)U_x + \tilde{d}(t,x)U + F(V^1,V^2), \quad t>0,\ x\in\mathbb{R},\\
U(0,x)=0, \quad \partial_t U(0,x)=0, \quad x\in\mathbb{R},
\end{gathered}$$

where $F(V^1,V^2)=\tilde{N}(V^1,V^1_x,V^1_t)-\tilde{N}(V^2,V^2_x,V^2_t)$. Obviously, $F$ is an element of $L^1\big((0,T_0];L^2(\mathbb{R})\big)$, so

$$\sup_{0<t<T_0}\|(U^1,U^1_t)-(U^2,U^2_t)\|_{H^{1,0}\times L^2} \le C(T_0)\Big(\int_0^{T_0}\|F(t)\|_{L^2}dt\Big). \tag{3.19}$$

Then by (2.4), $F(V^1,V^2)\le C|V^1-V^2|(|V^1|+|V^2|)^{p_1-1}$.

Applying Sobolev's embedding theorem, we have

$$\|F(V^1,V^2)\|_{L^2} \le C\|V^1-V^2\|_{L^2}\|(|V^1|+|V^2|)^{p_1-1}\|_{L^\infty}$$

$$
\begin{aligned}
&\le C\|V^1-V^2\|_{H^{1,0}}\Big(\|V^1\|_{H^{1,0}}^{p_1-1}+\|V^2\|_{H^{1,0}}^{p_1-1}\Big)\\
&\le C(2C(T_0)I_0\varepsilon)^{p_1-1}\|V^1-V^2\|_{H^{1,0}\times L^2},
\end{aligned}
$$

so

$$
\begin{aligned}
&\sup_{0<t<T_0}\|(U^1,U^1_t)-(U^2,U^2_t)\|_{H^{1,0}\times L^2}\\
&\le C(T_0)(2C(T_0)I_0\varepsilon)^{p_1-1}T_0\|V^1-V^2\|_{H^{1,0}\times L^2}\le\frac12\|V^1-V^2\|_{H^{1,0}\times L^2}
\end{aligned}
\tag{3.20}
$$

when $\varepsilon$ is sufficiently small.

It is easy to see that there exits a small $\varepsilon_0>0$, such that for each $0<\varepsilon\le\varepsilon_0$, (3.18) and (3.20) are valid. Therefore by Banach's fixed point theorem, there exists a $U\in K$ such that $\mathcal{L}U=U$, i.e., $U$ is the unique solution to (3.9). Moreover, if $T_0>T(\varepsilon)$, (iii) of Lemma 3.4 shows that $\sup_{0<t<T_0}\|(U^1,U^1_t)-(U^2,U^2_t)\|_{H^{1,0}\times L^2}=\infty$, which contradicts (3.18), so $T_0\le T(\varepsilon)$.

## 4. Proof of Theorem 2.5

We begin with the definition of solutions to system (3.3).

**Definition 4.1.** (i) $(v,w)\in C\big([0,S);H^{1,1}(\mathbb{R})\times H^{0,1}(\mathbb{R})\big)$ is a mild solution to (3.3) if $u\in C\big([0,T(S));H^{1,1}(\mathbb{R})\big)\cap C^1\big([0,T(S));H^{0,1}(\mathbb{R})\big)$ is a mild solution to (1.1), where $T(S)=B^{-1}(e^S-1)$, see (2.7) and (3.1).

(ii) $(v,w)\in C\big([0,S);H^{2,1}(\mathbb{R})\times H^{1,1}(\mathbb{R})\big)\cap C^1\big([0,S);H^{1,1}(\mathbb{R})\times H^{0,1}(\mathbb{R})\big)$ is a strong solution to (3.3) if $u\in C([0,T(S));H^{2,1}(\mathbb{R}))\cap C^1\big([0,T(S));H^{1,1}(\mathbb{R})\big)\cap C^2\big([0,T(S));H^{0,1}(\mathbb{R})\big)$ is a strong solution to (1.1).

(iii) The lifespan of the mild solution to (3.3) is defined as

$$
S(\varepsilon):=\sup\{S\in(0,\infty):\text{there is a unique mild solution }(v,w)\text{ to (3.3) on [0,S)}\}.
$$

From Definition 4.1 and Theorem 2.4, we have the following result.

**Corollary 4.2.** *Under assumptions* (A1)–(A5)*, there exists a $S=S(\varepsilon\|(v_0,w_0)\|)>0$, such that* (3.3) *has a unique mild solution $(v,w)$ on $[0,S)$. Furthermore, if $(u_0,u_1)\in H^{2,1}(\mathbb{R})\times H^{1,1}(\mathbb{R})$, then $(v,w)$ becomes a strong solution. Moreover, if the lifespan $T=T(\varepsilon)$ of the mild solution to* (1.1) *is finite, then*

$$
\lim_{s\to S(\varepsilon)}\|(v,w)\|_{H^{1,1}(\mathbb{R})\times H^{0,1}(\mathbb{R})}=\infty.
$$

*In particular, for each given $S_0>0$, there exists a $\varepsilon_0^*>0$, such that for each $\varepsilon\in[0,\varepsilon_0^*)$, the solution to* (3.3) *will exist on $[0,S_0]$, i.e., $S_0<S(\varepsilon)$.*

In the sequel, to simplify calculations, applying a simple density argument and using the continuous dependence on initial data guaranteed by Theorem 2.4, we can assume that the initial data $(v_0,\ w_0)\in H^{2,1}\times H^{1,1}$, and $(v,w)$ is the strong solution to (3.3). We can establish the following crucial priori estimate for the unique mild solution to (3.3), which plays a fundamental role in the proof of Theorem 2.5.

**Proposition 4.3.** *Under assumptions* (A1)–(A5)*, there exist $s_0>0$, $\tilde\varepsilon_0>0$, and $C_*>0$ such that for each $\varepsilon\in[0,\tilde\varepsilon_0)$, if $(v,w)$ is the unique mild solution to* (3.3) *on $[0,S]$, where $S>s_0$, then*

$$
\|v(s)\|_{H^{1,1}}^2+\frac{e^{-s}}{b^2(t(s))}\|w(s)\|_{H^{0,1}}^2\le C_*\varepsilon^2\|(v_0,w_0)\|_{H^{1,1}\times H^{0,1}}^2,\quad\forall s\in[s_0,S].\tag{4.1}
$$

To prove Proposition 4.3, we will decompose $v$ and $w$ into the leading terms and the remainder terms. Subsequently, we will employ the energy argument to derive the decay estimates for the remainder terms.

4.1. **Proof of Proposition 4.3.** Let

$$\alpha(s)=\int_{\mathbb{R}} v(s,y)dy, \quad \dot{\alpha}(s)=\frac{d\alpha(s)}{ds}, \tag{4.2}$$

since $v(s)\in H^{1,1}(\mathbb{R})$, for all $s\in[0,S)$, $\alpha(s)$ is well defined by Sobolev's embedding theorem.

**Lemma 4.4.** *We have*

$$\frac{d\alpha(s)}{ds}=\int_{\mathbb{R}} w(s,y)dy, \tag{4.3}$$

$$\frac{e^{-s}}{b^2(t(s))}\frac{d^2\alpha(s)}{ds^2}=\frac{e^{-s}}{b^2(t(s))}\dot{\alpha}(s)-\dot{\alpha}(s)+\int_{\mathbb{R}} r(s,y)dy, \tag{4.4}$$

*where $r(s,y)$ is given by* (3.4)*.*

*Proof.* Note $v(s)\in C^1\big([0,S);H^{1,1}(\mathbb{R})\big)$ and $w(s)\in C\big([0,S);H^{0,1}(\mathbb{R})\big)$, (4.3) follows from (3.3) and integration by parts

$$\frac{d\alpha(s)}{ds}=\int_{\mathbb{R}}\big(\frac{y}{2}v_y+\frac{v}{2}+w\big)dy=\int_{\mathbb{R}}\big((\frac{y}{2}v)_y+w\big)dy=\int_{\mathbb{R}} w(s,y)dy.$$

Differentiating (4.3) and using (3.3), we have

$$\begin{aligned}
&\frac{e^{-s}}{b^2(t(s))}\frac{d^2\alpha(s)}{ds^2}\\
&=\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} w_s(s,y)dy\\
&=\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}(\frac{3}{2}w+\frac{y}{2}w_y)dy-\int_{\mathbb{R}}(w-(a(ye^{s/2})v_y)_y)dy+\int_{\mathbb{R}} r(s,y)dy\\
&=\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} w(s,y)dy-\int_{\mathbb{R}} wdy+\int_{\mathbb{R}}(a(ye^{s/2})v_y)_ydy+\int_{\mathbb{R}} r(s,y)dy\\
&=\frac{e^{-s}}{b^2(t(s))}\frac{d\alpha}{ds}(s)-\frac{d\alpha}{ds}(s)+\int_{\mathbb{R}} r(s,y)dy.
\end{aligned}$$

□

We set

$$\begin{gathered}
\varphi_0(y)=\frac{1}{\sqrt{4\pi}}\frac{2}{\sqrt{a_+}+\sqrt{a_-}}\exp(-\frac{|y|^2}{4\tilde{a}(y)}),\\
\psi_0(y)=(\tilde{a}(y)\varphi_0'(y))_y=-\frac{y}{2}\varphi_0'(y)-\frac{1}{2}\varphi_0,
\end{gathered} \tag{4.5}$$

where $a_\pm$ and $\tilde{a}(x)$ are given by (2.5) and it is easy to verify that

$$\int_{\mathbb{R}}\varphi_0(y)dy=1 \quad \text{and} \quad \int_{\mathbb{R}}\psi_0(y)dy=0. \tag{4.6}$$

We decompose $(v,w)$ as

$$\begin{gathered}
v(s,y)=\alpha(s)\varphi_0(y)+f(s,y),\\
w(s,y)=\dot{\alpha}(s)\varphi_0(y)+\alpha(s)\psi_0(y)+g(s,y).
\end{gathered} \tag{4.7}$$

By Definition 4.1,

$$(f,g)\in C\big([0,S);H^{2,1}(\mathbb{R})\times H^{1,1}(\mathbb{R})\big)\cap C^1\big([0,S);H^{1,1}(\mathbb{R})\times H^{0,1}(\mathbb{R})\big), \tag{4.8}$$

and from (4.5) and (4.7) it follows that

$$\begin{gathered}\int_{\mathbb{R}} f(s,y)dy=\int_{\mathbb{R}}\big(v(s,y)-\alpha(s)\varphi_0(y)\big)dy=\alpha(s)-\alpha(s)=0,\\ \int_{\mathbb{R}} g(s,y)dy=\int_{\mathbb{R}}\big(w(s,y)-\dot\alpha(s)\varphi_0(y)-\alpha(s)\psi_0(y)\big)=\dot\alpha(s)-\dot\alpha(s)=0.\end{gathered}\tag{4.9}$$

Substituting (4.7) into (3.3) gives

$$\begin{gathered}f_s-\frac{y}{2}f_y-\frac12 f=g,\quad s>0,\ y\in\mathbb{R},\\ \frac{e^{-s}}{b^2(t(s))}(g_s-\frac32 g-\frac{y}{2}g_y)+g\\ =(a(ye^{s/2})f_y)_y+\alpha(s)(a_0(ye^{s/2})\varphi_0'(y))_y+h(s,y),\quad s>0,\ y\in\mathbb{R},\\ f(0,y)=v(0,y)-\alpha(0)\varphi_0(y),\quad y\in\mathbb{R},\\ g(0,y)=w(0,y)-\dot\alpha(0)\varphi_0(y)-\alpha(0)\psi_0(y),\quad y\in\mathbb{R},\end{gathered}\tag{4.10}$$

where

$$\begin{aligned}h(s,y)=&\frac{e^{-s}}{b^2(t(s))}\Big(-2\dot\alpha(s)\psi_0(y)+\alpha(s)(\frac{y}{2}\psi_0'(y)+\frac32\psi_0(y))\Big)\\ &+r(s,y)-\varphi_0(y)\int_{\mathbb{R}} r(s,y)dy,\end{aligned}\tag{4.11}$$

here we have used (4.5) and $a(x)=\tilde a(x)+a_0(x)$. From (4.6) and (4.9) we deduce that $\int_{\mathbb{R}} h(s,y)dy=0$.

To obtain decay estimates for $f$ and $g$, we define

$$F(s,y)=\int_{-\infty}^y f(s,z)dz,\quad G(s,y)=\int_{-\infty}^y g(s,z)dz.$$

**Lemma 4.5** (Hardy-type inequality, [16]). *If $f\in H^{0,1}(\mathbb{R})$ satisfies $\int_{\mathbb{R}} f(y)dy=0$, then for $F(y)=\int_{-\infty}^y f(z)dz$, it holds*

$$\int_{\mathbb{R}} F^2(y)dy\le 4\int_{\mathbb{R}} y^2f^2(y)dy. \tag{4.12}$$

From (4.8)-(4.10) and Lemma 4.5,

$$(F,G)\in C\big([0,S);H^{3,0}(\mathbb{R})\times H^{2,0}(\mathbb{R})\big)\cap C^1\big([0,S);H^{2,0}(\mathbb{R})\times H^{1,0}(\mathbb{R})\big) \tag{4.13}$$

satisfies

$$\begin{gathered}F_s-\frac12 yF_y=G,\\ \frac{e^{-s}}{b^2(t(s))}\big(G_s-\frac12 yG_y-G\big)+G=\alpha(s)a_0(ye^{s/2})\varphi_0'(y)+a(ye^{s/2})F_{yy}+H,\\ F(0,y)=\int_{-\infty}^y f(0,z)dz,\quad G(0,y)=\int_{-\infty}^y g(0,z)dz,\end{gathered}\tag{4.14}$$

where $s>0$, $y\in\mathbb{R}$, and

$$H(s,y)=\int_{-\infty}^y h(s,z)dz. \tag{4.15}$$

We define the following energies

$$E_0(s)=\frac{1}{2}\int_{\mathbb{R}}\Big(F_y^2+\frac{e^{-s}}{b^2(t(s))}\frac{G^2}{a(ye^{s/2})}\Big)dy+\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}\Big(\frac{1}{2}F^2+\frac{e^{-s}}{b^2(t(s))}FG\Big)dy,$$

$$\begin{aligned}E_1(s)&=\frac{1}{2}\int_{\mathbb{R}}\Big(a(ye^{\frac{s}{2}})f_y^2+\frac{e^{-s}}{b^2(t(s))}g^2\Big)dy+\alpha(s)\int_{\mathbb{R}}a_0(ye^{s/2})\varphi_0'f_ydy\\&\quad+\int_{\mathbb{R}}\Big(f^2+2\frac{e^{-s}}{b^2(t(s))}fg\Big)dy,\end{aligned}$$

$$\begin{aligned}E_2(s)&=\frac{1}{2}\int_{\mathbb{R}}\Big(y^2a(ye^{s/2})f_y^2+\frac{e^{-s}}{b^2(t(s))}y^2g^2\Big)dy+\alpha(s)\int_{\mathbb{R}}y^2a_0(ye^{s/2})\varphi_0'f_ydy\\&\quad+\int_{\mathbb{R}}\Big(\frac{1}{2}y^2f^2+\frac{e^{-s}}{b^2(t(s))}y^2fg\Big)dy,\end{aligned}$$

$$E_4(s)=C_0E_0(s)+C_1E_1(s)+E_2(s)+E_3(s),$$

where

$$E_3(s)=\frac{1}{2}\frac{e^{-s}}{b^2(t(s))}\Big(\frac{d\alpha(s)}{ds}\Big)^2+C_2e^{-2\lambda s}\alpha^2(s),$$

$1\ll C_1\ll C_0$, $C_2>0$ and $\lambda>0$ are given by Lemma 4.7 and Lemma 4.11, respectively. We have the following identities which will be used in the proof of Proposition 4.3.

**Lemma 4.6.** *(1)*

$$\frac{d}{ds}E_0(s)+\frac{1}{2}E_0(s)+L_0(s)=R_0(s),\tag{4.16}$$

*where*

$$L_0(s)=\int_{\mathbb{R}}\Big(\frac{1}{2}F_y^2+\frac{1}{a(ye^{s/2})}G^2\Big)dy,$$

$$\begin{aligned}R_0(s)&=\frac{3}{2}\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}G^2dy-\frac{1}{b^2(t(s))}\frac{db(t(s))}{dt}\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}\Big(G^2+2FG\Big)dy\\&\quad+\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}\Big(\big(\alpha(s)a_0(ye^{s/2})\varphi_0'+H\big)(F+G)\Big)dy.\end{aligned}$$

*(2)*

$$\frac{d}{ds}E_1(s)+\frac{1}{2}E_1(s)+L_1(s)=R_1(s),\tag{4.17}$$

*where*

$$L_1(s)=\int_{\mathbb{R}}\Big(a(ye^{s/2})f_y^2+g^2-f^2\Big)dy,$$

$$\begin{aligned}R_1(s)&=3\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}g^2dy+2\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}fg\,dy\\&\quad-\frac{1}{b^2(t(s))}\frac{db(t(s))}{dt}\int_{\mathbb{R}}(g^2+4fg)dy+\big(\dot{\alpha}(s)-\alpha(s)\big)\int_{\mathbb{R}}a_0(ye^{s/2})\varphi_0'f_y\,dy\\&\quad-\alpha(s)\int_{\mathbb{R}}\frac{y}{2}\varphi_0''(y)f_ya_0(ye^{s/2})dy+\int_{\mathbb{R}}(hg+2hf)dy.\end{aligned}$$

*(3)*

$$\frac{d}{ds}E_2(s)+\frac{1}{2}E_2(s)+L_2(s)=R_2(s),\tag{4.18}$$

*where*

$$\begin{aligned} L_2(s) &= \int_{\mathbb{R}} \Big(\frac{1}{2}y^2 a(ye^{s/2}) f_y^2 + y^2 g^2\Big) dy + 2\int_{\mathbb{R}} y a(ye^{s/2}) f_y (f+g) dy, \\ R_2(s) &= \big(\dot\alpha(s) - \alpha(s)\big) \int_{\mathbb{R}} y^2 a_0(ye^{s/2})\varphi_0'(y) f_y dy + \frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} \frac{3}{2} y^2 g^2 dy \\ &\quad - \alpha(s)\int_{\mathbb{R}} \frac{y^3}{2} a_0(ye^{s/2})\varphi_0''(y) f_y\, dy - 2\alpha(s)\int_{\mathbb{R}} y a_0(ye^{s/2})\varphi_0'(y)(f+g) dy \\ &\quad - \frac{1}{b^2(t(s))}\frac{db(t(s))}{dt}\int_{\mathbb{R}} y^2(g^2 + 2fg) dy + \int_{\mathbb{R}} y^2 (fh + gh) dy. \end{aligned}$$

*(4)*

$$\frac{d}{ds}E_3(s) + 2\lambda E_3(s) + \Big(\frac{d\alpha(s)}{ds}\Big)^2 = R_3(s), \tag{4.19}$$

*where*

$$\begin{aligned} R_3(s) &= \frac{1}{2}(2\lambda+1)\frac{e^{-s}}{b^2(t(s))}\Big(\frac{d\alpha(s)}{ds}\Big)^2 - \frac{1}{b^2(t(s))}\frac{db(t(s))}{dt}\Big(\frac{d\alpha(s)}{ds}\Big)^2 \\ &\quad + \frac{d\alpha(s)}{ds}\int_{\mathbb{R}} r(s,y) dy + 2C_2 e^{-2\lambda s}\alpha(s)\frac{d\alpha(s)}{ds}. \end{aligned}$$

*(5)*

$$\frac{d}{ds}E_4(s) + 2\lambda E_4(s) + L_4(s) = R_4(s), \tag{4.20}$$

*where*

$$\begin{gathered} L_4(s) = (\frac{1}{2} - 2\lambda)(C_0E_0 + C_1E_1 + E_2) + C_0L_0 + C_1L_1 + L_2 + \Big(\frac{d\alpha(s)}{ds}\Big)^2, \\ R_4(s) = C_0R_0(s) + C_1R_1(s) + R_2(s) + R_3(s). \end{gathered}$$

*Proof.* Clearly, (4.20) is the sum of (4.16)-(4.19), (4.19) can be derived by differentiating $E_3(s)$ directly, and (4.17), (4.18) are similar to (4.16), so it suffices to prove (4.16). Differentiating $E_0(s)$ gives

$$\begin{aligned} &\frac{d}{ds}E_0(s) \\ &= \frac{d}{ds}\Big(\frac{1}{2}\int_{\mathbb{R}} \frac{e^{-s}}{b^2(t(s))}\frac{G^2}{a(ye^{s/2})} dy\Big) \\ &\quad + \frac{d}{ds}\Big(\int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}(\frac{1}{2}F^2 + \frac{e^{-s}}{b^2(t(s))}FG) dy\Big) + \frac{1}{4}\int_{\mathbb{R}} F_y^2 dy + \int_{\mathbb{R}} F_y G_y dy, \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} \int_{\mathbb{R}} F_{ys}F_y dy &= \int_{\mathbb{R}} (\frac{y}{2}F_yF_{yy} + \frac{1}{2}F_y^2 + F_yG_y) dy \\ &= \int_{\mathbb{R}} \Big(\big(\frac{y}{4}F_y^2\big)_y + \frac{1}{4}F_y^2 + F_yG_y\Big) dy \\ &= \int_{\mathbb{R}} \frac{1}{4}F_y^2 dy + \int_{\mathbb{R}} F_yG_y dy \end{aligned}$$

has been used. For convenience, we define

$$\widetilde{F}(s,x) = F(s,y) = F(s, xe^{-s/2}), \quad \widetilde{G}(s,x) = G(s,y) = G(s, xe^{-s/2}).$$

Note $dy = e^{-s/2}dx$,

$$\widetilde{F}_s(s,x) = F_s - \frac{y}{2}F_y, \quad \widetilde{G}_s(s,x) = G_s - \frac{y}{2}G_y.$$

Therefore,

$$\begin{aligned}
&\frac{d}{ds}\Big(\frac{1}{2}\int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}F^2 dy\Big) \\
&= \frac{d}{ds}\Big(\frac{1}{2}\int_{\mathbb{R}} \frac{1}{a(x)}\widetilde{F}^2 e^{-s/2} dx\Big) \\
&= -\frac{1}{4}\int_{\mathbb{R}} \frac{1}{a(x)}\widetilde{F}^2 e^{-s/2}dx + \int_{\mathbb{R}} \frac{1}{a(x)}\widetilde{F}(x,s)\widetilde{F}_s(x,s)e^{-s/2}dx \\
&= -\frac{1}{4}\int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}F^2 dy + \int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}F(-\frac{y}{2}F_y + F_s)dy \\
&= \int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}\big(FF_s - \frac{y}{2}FF_y - \frac{1}{4}F^2\big)dy \\
&= \int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}\big(FG - \frac{1}{4}F^2\big)dy,
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{ds}\Big(\frac{1}{2}\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}G^2 dy\Big) \\
&= \frac{d}{ds}\Big(\frac{1}{2}\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} \frac{1}{a(x)}\widetilde{G}^2 e^{-s/2}dx\Big) \\
&= -\frac{3}{4}\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} \frac{1}{a(x)}\widetilde{G}^2 e^{-s/2}dx + \frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} \frac{1}{a(x)}\widetilde{G}\widetilde{G}_s e^{-s/2}dx \\
&\quad - \frac{1}{b^2(t(s))}\frac{db(t(s))}{dt}\int_{\mathbb{R}} \frac{1}{a(x)}\widetilde{G}^2 e^{-s/2}dx \\
&= \frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}\Big(GG_s - \frac{y}{2}GG_y - \frac{3}{4}G^2 dy\Big) \\
&\quad - \frac{1}{b^2(t(s))}\frac{db(t(s))}{dt}\int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}G^2 dy \\
&= \frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}\frac{1}{4}G^2 dy - \int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}G^2 dy + \int_{\mathbb{R}} F_{yy}G dy \\
&\quad + \int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}\Big(H + \alpha(s)a_0(ye^{s/2})\varphi'\Big)G dy - \frac{1}{b^2(t(s))}\frac{db}{dt}(t(s))\int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}G^2 dy,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{ds}\Big(\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}FG dy\Big) \\
&= \frac{d}{ds}\Big(\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} \frac{1}{a(x)}\widetilde{F}\widetilde{G}e^{-s/2}dx\Big) \\
&= \frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}G^2 dy - \frac{1}{2}\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} FG dy - \int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}FG dy \\
&\quad - \frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}FG dy - \frac{2}{b^2(t(s))}\frac{db(t(s))}{dt}\int_{\mathbb{R}} FG dy + \int_{\mathbb{R}} F_{yy}F dy
\end{aligned}$$

$$
+\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}FGdy+\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}\big(HF+\alpha(s)a_0(ye^{s/2})\varphi_0'F\big)dy.
$$

Substituting the three identities above into (4.21), we have

$$
\begin{aligned}
\frac{d}{ds}E_0(s)&=-\frac{1}{2}E_0(s)-\int_{\mathbb{R}}\Big(\frac{1}{2}F_y^2+\frac{1}{a(ye^{s/2})}G^2\Big)dy\\
&\quad-\frac{1}{b^2(t(s))}\frac{db(t(s))}{dt}\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}\Big(G^2+2FG\Big)dy\\
&\quad+\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}\Big((F+G)(\alpha a_0(ye^{s/2})\varphi_0'+H)\Big)dy+\frac{3}{2}\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}G^2dy\\
&=-\frac{1}{2}E_0(s)-L_0(s)+R_0(s).
\end{aligned}
$$

□

We define $\mu_0=\min(0,\mu)$,

$$
\begin{aligned}
A&=\underline{a}^{-1}\||x|^{\mu_0}a_0\|_{L^2(\mathbb{R})}^2\|\,|y|^{-\mu_0}\varphi_0'\|_{L^\infty(\mathbb{R})}^2\,,\\
B&=\underline{a}^{-1}\||x|^{\mu_0}a_0\|_{L^2(\mathbb{R})}^2\|\,|y|^{1-\mu_0}\varphi_0'\|_{L^\infty(\mathbb{R})}^2\,.
\end{aligned}
$$

**Lemma 4.7.** *If $0<\lambda\le\frac{1}{4}+\frac{\mu_0}{2}$, $C_2\ge 2(C_1A+B)$, then there exists a $s_1>0$ such that for each $s\ge s_1$,*

$$
E_4(s)\sim\|f\|_{H^{1,1}}^2+\frac{e^{-s}}{b^2(t(s))}\|g\|_{H^{0,1}}^2+\frac{e^{-s}}{b^2(t(s))}\Big(\frac{d\alpha(s)}{ds}\Big)^2+e^{-2\lambda s}\alpha^2(s). \tag{4.22}
$$

*Proof.* In view of the definition of $E_4$, it suffices to estimate $E_0$, $E_1$ and $E_2$. By Lemma 3.2, there exists a $s_1>0$ such that for each $s\ge s_1$, $\frac{e^{-s}}{b^2(t(s))}\le\frac{1}{8}$. As for the terms containing $FG$ and $fg$ in $E_0$, $E_1$ and $E_2$, for $s\ge s_1$, Young's inequality shows that

$$
\begin{aligned}
\Big|\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}FGdy\Big|&\le\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}\big(\frac{1}{8}G^2+2F^2\big)dy\\
&\le\frac{1}{4}\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}F^2dy+\frac{1}{8}\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}G^2dy,
\end{aligned}
$$

$$
\begin{aligned}
\Big|2\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}fgdy\Big|&\le 2\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}\big(2f^2+\frac{1}{8}g^2\big)dy\\
&\le\frac{1}{2}\int_{\mathbb{R}}f^2dy+\frac{1}{4}\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}g^2dy
\end{aligned}
$$

and

$$
\begin{aligned}
\Big|\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}y^2fgdy\Big|&\le\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}y^2\big(2f^2+\frac{1}{8}g^2\big)dy\\
&\le\frac{1}{4}\int_{\mathbb{R}}y^2f^2dy+\frac{1}{8}\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}y^2g^2dy.
\end{aligned}
$$

Hence

$$
E_0(s)\ge\frac{1}{2}\int_{\mathbb{R}}F_y^2dy+\frac{1}{4}\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}F^2dy+\frac{1}{4}\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}G^2dy. \tag{4.23}
$$

To obtain the lower bound estimates for $E_1$ and $E_2$, it suffices to estimate

$$
\alpha(s)\int_{\mathbb{R}}a_0(ye^{s/2})\varphi_0'f_ydy\quad\text{and}\quad\alpha(s)\int_{\mathbb{R}}y^2a_0(ye^{s/2})\varphi_0'f_ydy.
$$

It is easy to see that $|x|^{\mu_0}a_0(x)\in L^2(\mathbb{R})$ and $|y|^{-\mu_0}\varphi_0'(y)\in L^\infty(\mathbb{R})$ by assumption (A5), so

$$\begin{aligned}\big|\alpha(s)\int_{\mathbb{R}} a_0(ye^{s/2})\varphi_0' f_y dy\big| &\le \frac{1}{4}\int_{\mathbb{R}} a(ye^{s/2})f_y^2 dy + \alpha^2(s)\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}\big|a_0(ye^{s/2})\varphi_0'\big|^2 dy\\ &\le \frac{1}{4}\int_{\mathbb{R}} a(ye^{s/2})f_y^2 dy + \underline{a}^{-1}\alpha^2(s)\|a_0(ye^{s/2})\varphi_0'\|_{L^2(\mathbb{R})}^2\end{aligned}$$

and

$$\begin{aligned}\|a_0(ye^{s/2})\varphi_0'\|_{L^2(\mathbb{R})} &= \|a_0(ye^{s/2})(ye^{s/2})^{-\mu_0}(ye^{s/2})^{\mu_0}\varphi_0'\|_{L_y^2(\mathbb{R})}\\ &\le \|y^{-\mu_0}\varphi_0'\|_{L^\infty(\mathbb{R})}\|a_0(x)|x|^{\mu_0}\|_{L_x^2(\mathbb{R})}e^{-(\frac14+\frac{\mu_0}{2})s},\end{aligned}$$

then

$$\big|\alpha(s)\int_{\mathbb{R}} a_0(ye^{s/2})\varphi_0' f_y dy\big| \le \frac{1}{4}\int_{\mathbb{R}} a(ye^{s/2})f_y^2 dy + Ae^{-2(\frac14+\frac{\mu_0}{2})s}\alpha^2(s). \tag{4.24}$$

Similarly, we have

$$\begin{aligned}&\big|\alpha(s)\int_{\mathbb{R}} y^2 a_0(ye^{s/2})\varphi_0' f_y dy\big|\\ &\le \frac{1}{4}\int_{\mathbb{R}} y^2 a(ye^{s/2})f_y^2 dy + \alpha^2(s)\int_{\mathbb{R}}\frac{1}{a(ye^{s/2})}\big|ya_0(ye^{s/2})\varphi_0'\big|^2 dy\\ &\le \frac{1}{4}\int_{\mathbb{R}} y^2 a(ye^{s/2})f_y^2 dy + \underline{a}^{-1}\alpha^2(s)\|ya_0(ye^{s/2})\varphi_0'\|_{L^2(\mathbb{R})}^2\end{aligned}$$

and

$$\begin{aligned}\|ya_0(ye^{s/2})\varphi_0'\|_{L^2(\mathbb{R})} &= \|ya_0(ye^{s/2})\varphi_0'(ye^{s/2})^{\mu_0}(ye^{s/2})^{-\mu_0}\|_{L_y^2(\mathbb{R})}\\ &\le \||y|^{1-\mu_0}\varphi_0'\|_{L^\infty(\mathbb{R})}\|a_0(ye^{s/2})(ye^{s/2})^{\mu_0}(e^{s/2})^{-\mu_0}\|_{L^2(\mathbb{R})}\\ &\le \||y|^{1-\mu_0}\varphi_0'\|_{L^\infty(\mathbb{R})}\Big(\int_{\mathbb{R}} a_0(x)^2x^{2\mu_0}e^{-(\frac12+\mu_0)s}dx\Big)^{1/2}\\ &\le \||y|^{1-\mu_0}\varphi_0'\|_{L^\infty(\mathbb{R})}\|a_0(x)|x|^{\mu_0}\|_{L^2(\mathbb{R})}e^{-(\frac14+\frac{\mu_0}{2})s},\end{aligned}$$

then

$$\big|\alpha(s)\int_{\mathbb{R}} y^2a_0(ye^{s/2})\varphi_0' f_y dy\big| \le \frac{1}{4}\int_{\mathbb{R}} y^2a(ye^{s/2})f_y^2 dy + B\alpha^2(s)e^{-2(\frac14+\frac{\mu_0}{2})s}. \tag{4.25}$$

So

$$\begin{aligned}E_1(s) &\ge \frac12\int_{\mathbb{R}} a(ye^{s/2})f_y^2 + \frac{e^{-s}}{b^2(t(s))}g^2 dy + \int_{\mathbb{R}} f^2 dy - \frac12\int_{\mathbb{R}} f^2 dy\\ &\quad - \frac14\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} g^2 dy + \alpha(s)\int_{\mathbb{R}} a_0(ye^{s/2})\varphi_0' f_y dy\\ &\ge \frac14\int_{\mathbb{R}} a(ye^{s/2})f_y^2 dy + \frac12\int_{\mathbb{R}} f^2 dy + \frac14\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} g^2 dy\\ &\quad - A\alpha^2(s)e^{-2(\frac14+\frac{\mu_0}{2})s}\end{aligned} \tag{4.26}$$

and

$$\begin{aligned} E_2(s) \geq & \frac{1}{2}\int_{\mathbb{R}} y^2(a(ye^{s/2}) + \frac{e^{-s}}{b^2(t(s))}g^2)dy + \frac{1}{4}\int_{\mathbb{R}} y^2 f^2 dy \\ & - \frac{1}{8}\int_{\mathbb{R}} \frac{e^{-s}}{b^2(t(s))} y^2 g^2 dy + \alpha(s)\int_{\mathbb{R}} y^2 a_0(ye^{s/2})\varphi_0' f_y dy \\ \geq & \ \frac{1}{4}\int_{\mathbb{R}} y^2 a(ye^{s/2}) f_y^2 dy + \frac{3}{8}\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} y^2 g^2 dy \\ & + \frac{1}{4}\int_{\mathbb{R}} y^2 f^2 dy - B\alpha^2(s) e^{-2(\frac{1}{4}+\frac{\mu_0}{2})s}. \end{aligned} \tag{4.27}$$

Note $C_2 \geq 2(C_1A + B)$ and $\lambda \leq \frac{1}{4} + \frac{\mu_0}{2}$, it is easy to see that

$$C_2 e^{-2\lambda s}\alpha^2(s) - AC_1 e^{-2(\frac{1}{4}+\frac{\mu_0}{2})s}\alpha^2(s) - Be^{-2(\frac{1}{4}+\frac{\mu_0}{2})s}\alpha^2(s) \geq \frac{1}{2}C_2 e^{-2\lambda s}\alpha^2(s),$$

then by combining (4.23), (4.26) and (4.27), we obtain the lower bound

$$\begin{aligned} E_4(s) \geq & \frac{1}{4}\min(1, \underline{a})\|f\|_{H^{1,1}}^2 + \frac{1}{4}\frac{e^{-s}}{b^2(t(s))}\|g\|_{H^{0,1}}^2 + \frac{1}{2}\frac{e^{-s}}{b^2(t(s))}\Big(\frac{d\alpha(s)}{ds}\Big)^2 \\ & + \frac{1}{2}C_2 e^{-2\lambda s}\alpha^2(s) \\ & \geq C(\|f\|_{H^{1,1}}^2 + \frac{e^{-s}}{b^2(t(s))}\|g\|_{H^{0,1}}^2 + \frac{e^{-s}}{b^2(t(s))}\Big(\frac{d\alpha(s)}{ds}\Big)^2 + e^{-2\lambda s}\alpha^2(s)), \end{aligned}$$

where $C = \frac{1}{4}\min(1, \underline{a})$.

The upper bound for $E_0(s)$, $E_1(s)$ and $E_2(s)$ are obtained similarly:

$$\begin{aligned} E_0(s) &\leq \frac{1}{2}\int_{\mathbb{R}} F_y^2 dy + \frac{3}{4\underline{a}}\int_{\mathbb{R}}\big(F^2 + \frac{e^{-s}}{b^2(t(s))}G^2\big)dy, \\ E_1(s) &\leq \frac{3}{4}\int_{\mathbb{R}} a(ye^{s/2}) f_y^2 dy + \frac{3}{2}\int_{\mathbb{R}} f^2 dy + \frac{3}{4}\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} g^2 dy \\ &\quad + A\alpha^2(s) e^{-2(\frac{1}{4}+\frac{\mu_0}{2})s} \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} E_2(s) \leq & \frac{3}{4}\int_{\mathbb{R}} a(ye^{s/2}) y^2 f_y^2 dy + \frac{3}{4}\int_{\mathbb{R}} y^2 f^2 dy + \frac{5}{8}\frac{e^{-s}}{b^2(t(s))}\int_{\mathbb{R}} y^2 g^2 dy \\ & + B\alpha^2(s) e^{-2(\frac{1}{4}+\frac{\mu_0}{2})s}. \end{aligned} \tag{4.29}$$

Therefore the upper bound for $E_4(s)$ can be formulated by Lemma 4.5 as

$$\begin{aligned} E_4(s) &= C_0E_0 + C_1E_1 + E_2 + E_3 \\ &\leq C_0\Big(\frac{1}{2}\int_{\mathbb{R}} F_y^2(y)dy + \frac{3}{4\underline{a}}\int_{\mathbb{R}}\big(F^2 + \frac{e^{-s}}{b^2(t(s))}G^2\big)dy\Big) \\ &\quad + C_1\Big(\frac{3}{4}\int_{\mathbb{R}} a(ye^{\frac{y}{2}}) f_y^2 dy + \frac{3}{4}\int_{\mathbb{R}}\frac{e^{-s}}{b^2(t(s))} g^2 dy + \frac{3}{2}\int_{\mathbb{R}} f^2 dy\Big) \\ &\quad + \frac{3}{4}\int_{\mathbb{R}} y^2 a(ye^{s/2}) f_y^2 dy + \frac{3}{4}\int_{\mathbb{R}} y^2 f^2 dy \\ &\quad + \frac{5}{8}\int_{\mathbb{R}}\frac{e^{-s}}{b^2(t(s))} y^2 g^2 dy + \frac{1}{2}\frac{e^{-s}}{b^2(t(s))}\Big(\frac{d\alpha(s)}{ds}\Big)^2 + e^{-2\lambda s}\alpha^2(s) \\ &\quad + C_1A\alpha^2(s) e^{-2(\frac{1}{4}+\frac{\mu_0}{2})s} + B\alpha^2(s) e^{-2(\frac{1}{4}+\frac{\mu_0}{2})s} \end{aligned}$$

$$\le C\Big(\|f\|_{H^{1,1}}^2 + \frac{e^{-s}}{b^2(t(s))}\|g\|_{H^{0,1}}^2 + \frac{e^{-s}}{b^2(t(s))}\Big(\frac{d\alpha(s)}{ds}\Big)^2 + e^{-2\lambda s}\alpha^2(s)\Big),$$

where $C = C(A, B, C_0, C_1, C_2, \|a\|_{L^\infty}, \underline{a})$. □

**Lemma 4.8.** *If $0 < \lambda \le \frac{1}{4} + \frac{\mu_0}{2}$, then*

$$L_4(s) \ge C\Big(\|f\|_{H^{1,1}}^2 + \|g\|_{H^{0,1}}^2 + \Big(\frac{d\alpha(s)}{ds}\Big)^2 - e^{-2\lambda s}\alpha^2(s)\Big) \tag{4.30}$$

*holds for $s \ge s_1$, where $L_4(s)$ and $s_1$ are given by Lemmas 4.6 and 4.7, respectively.*

*Proof.* (4.23) shows $E_0(s) \ge 0$. From (4.26) and (4.27), we have

$$\begin{aligned} C_1E_1 + E_2 &\ge -C_1A\alpha^2(s)e^{-2(\frac{1}{4}+\frac{\mu_0}{2})s} - B\alpha^2(s)e^{-2(\frac{1}{4}+\frac{\mu_0}{2})s}\\ &\ge -(C_1A + B)\alpha^2(s)e^{-2\lambda s}, \end{aligned}$$

here we have used $0 < \lambda \le \frac{1}{4} + \frac{\mu_0}{2}$. Note the definition of $L_4(s)$, it suffices to estimate $C_0L_0 + C_1L_1 + L_2$. In fact,

$$\begin{aligned} C_0L_0 + C_1L_1 + L_2 &= C_0\int_{\mathbb{R}}\Big(\frac{1}{2}F_y^2 + \frac{G^2}{a(ye^{s/2})}\Big)dy + C_1\int_{\mathbb{R}}\Big(a(ye^{s/2})f_y^2 + g^2 - f^2\Big)dy\\ &\quad + \int_{\mathbb{R}}\Big(\frac{1}{2}y^2a(ye^{s/2})f_y^2 + y^2g^2\Big)dy + 2\int_{\mathbb{R}} ya(ye^{s/2})f_y(f+g)dy\\ &\ge \frac{C_0}{2}\int_{\mathbb{R}} f^2dy + C_1\underline{a}\int_{\mathbb{R}} f_y^2dy + C_1\int_{\mathbb{R}} g^2dy - C_1\int_{\mathbb{R}} f^2dy\\ &\quad + \frac{\underline{a}}{2}\int_{\mathbb{R}} y^2f_y^2dy + \int_{\mathbb{R}} y^2g^2dy + 2\underline{a}\int_{\mathbb{R}} yf_y(f+g)dy. \end{aligned}$$

Young's inequality implies

$$2\underline{a}\Big|\int_{\mathbb{R}} yf_y(f+g)dy\Big| \le \frac{\underline{a}}{4}\int_{\mathbb{R}} y^2f_y^2dy + 8\underline{a}\int_{\mathbb{R}}(f^2+g^2)dy,$$

then

$$\begin{aligned} C_0L_0 + C_1L_1 + L_2 &\ge \big(\frac{C_0}{2} - C_1 - 8\underline{a}\big)\int_{\mathbb{R}} f^2dy + C_1\underline{a}\int_{\mathbb{R}} f_y^2dy + \frac{\underline{a}}{4}\int_{\mathbb{R}} y^2f_y^2dy\\ &\quad + (C_1 - 8\underline{a})\int_{\mathbb{R}} g^2dy + \int_{\mathbb{R}} y^2g^2dy\\ &\ge C\big(\|f\|_{H^{1,1}}^2 + \|g\|_{H^{0,1}}^2\big). \end{aligned}$$

□

Let

$$E_5(s) = E_4(s) + \frac{1}{2}\alpha^2(s) + \frac{e^{-s}}{b^2(t(s))}\alpha(s)\frac{d\alpha(s)}{ds}.$$

**Lemma 4.9.** *Under the conditions of Lemma 4.8, there exists a $s_2 \ge s_1$ such that for each $s \ge s_2$,*

$$E_5(s) \sim \|f\|_{H^{1,1}}^2 + \frac{e^{-s}}{b^2(t(s))}\|g\|_{H^{0,1}}^2 + \alpha^2(s) + \frac{e^{-s}}{b^2(t(s))}\Big(\frac{d\alpha(s)}{ds}\Big)^2. \tag{4.31}$$

*Proof.* In view of the definition of $E_5(s)$, by Lemma 4.7, we only need to estimate $\frac{e^{-s}}{b^2(t(s))}\alpha(s)\frac{d\alpha(s)}{ds}$. Indeed, by Young's inequality, for each $\eta > 0$,

$$\Big|\frac{e^{-s}}{b^2(t(s))}\alpha(s)\frac{d\alpha(s)}{ds}\Big| \le C(\eta)\frac{e^{-s}}{b^2(t(s))}\alpha^2(s) + \eta\frac{e^{-s}}{b^2(t(s))}\Big(\frac{d\alpha(s)}{ds}\Big)^2.$$

Choose $\eta$ sufficiently small such that

$$\eta\frac{e^{-s}}{b^2(t(s))}\Big(\frac{d\alpha(s)}{ds}\Big)^2\le\frac12 E_4(s)$$

for each $s\ge s_1$. In view of Lemma 3.2, there exists a $s_2\ge s_1$ such that for each $s\ge s_2$, $C(\eta)\frac{e^{-s}}{b^2(t(s))}\alpha^2(s)\le\frac14\alpha^2(s)$. Then

$$E_5(s)\ge\frac12 E_4(s)+\frac14\alpha^2(s).$$

This and (4.22) show that

$$\|f\|^2_{H^{1,1}}+\frac{e^{-s}}{b^2(t(s))}\|g\|^2_{H^{0,1}}+\alpha^2(s)+\frac{e^{-s}}{b^2(t(s))}\Big(\frac{d\alpha(s)}{ds}\Big)^2\le CE_5(s).$$

The upper bound estimate for $E_5(s)$ can be similarly derived. □

**Lemma 4.10.** *We have*

$$\frac{d}{ds}E_5(s)+2\lambda E_4(s)+L_4(s)=R_5(s),\tag{4.32}$$

*where*

$$\begin{aligned}R_5(s)=R_4(s)&+\frac{e^{-s}}{b^2(t(s)))}\Big(\frac{d\alpha(s)}{ds}\Big)^2-\frac{2}{b^2(t(s))}\frac{db(t(s))}{dt}\alpha(s)\frac{d\alpha(s)}{ds}\\&+\alpha(s)\int_{\mathbb{R}}r(s,y)dy,\end{aligned}$$

*$r(s,y)$ and $R_4(s)$ are given by (3.4) and (4.20), respectively.*

*Proof.* (4.32) follows from Lemma 4.4, (4.20), and the direct calculation:

$$\begin{aligned}\frac{d}{ds}E_5(s)&=\frac{d}{ds}E_4(s)+\alpha(s)\dot\alpha(s)+\frac{e^{-s}}{b^2(t(s)))}\Big(\frac{d\alpha(s)}{ds}\Big)^2+\frac{e^{-s}}{b^2(t(s)))}\alpha(s)\frac{d^2\alpha(s)}{ds^2}\\&\quad-\frac{2}{b^2(t(s))}\frac{db(t(s))}{dt}\alpha(s)\frac{d\alpha(s)}{ds}-\frac{e^{-s}}{b^2(t(s)))}\alpha(s)\frac{d\alpha(s)}{ds}\\&=\frac{d}{ds}E_4(s)+\frac{e^{-s}}{b^2(t(s)))}\Big(\frac{d\alpha(s)}{ds}\Big)^2-\frac{2}{b^2(t(s))}\frac{db(t(s))}{dt}\alpha(s)\frac{d\alpha(s)}{ds}\\&\quad+\alpha(s)\int_{\mathbb{R}}r(s,y)dy.\end{aligned}$$

□

**Lemma 4.11.** *Let $\lambda=\min\{\frac14+\frac{\mu_0}{2},\lambda_0,\lambda_1\}$, where*

$$\begin{aligned}\lambda_0&=\min\{\frac{1-\beta}{1+\beta},\frac{\gamma}{1+\beta}-\frac12,\frac{\nu}{1+\beta}-1\},\\ \lambda_1&=\frac12\Big(p_1+2p_2+(3-\frac{2\beta}{1+\beta})p_3-3\Big),\end{aligned}\tag{4.33}$$

*then there exists a $s_0\ge s_2$ (as defined in Lemma 4.9), such that for each $s\ge s_0$, we have the following estimates*

$$\begin{aligned}|R_4(s)|\le{}&\eta\big(L_4(s)+e^{-2\lambda s}\alpha^2(s)\big)+C(\eta)e^{-2\lambda s}E_5(s)\\&+C(\eta)e^{-2\lambda_1 s}E_5(s)^{p_1+p_2}\big(E_5(s)^{p_3}+\big(L_4(s)+e^{-2\lambda s}\alpha^2(s)\big)^{p_3}\big),\end{aligned}$$

$$\begin{aligned}|R_5(s)| \le \eta\big(L_4(s)+e^{-2\lambda s}\alpha^2(s)\big)&+C(\eta)e^{-\lambda s}E_5(s)\\&+C(\eta)e^{-2\lambda_1 s}E_5(s)^{p_1+p_2}\big(E_5(s)^{p_3}+\big(L_4(s)+e^{-2\lambda s}\alpha^2(s)\big)^{p_3}\big)\\&+C(\eta)e^{-\lambda_1 s}E_5(s)^{\frac{p_1+p_2+p_3+1}{2}}.\end{aligned}$$

*Here, $\frac{1}{1+\beta}$ and $\frac{-2\beta p_3}{1+\beta}$ are regarded as sufficiently large numbers when $\beta=-1$ and $p_3\neq 0$, $\eta$ is any positive constants, $R_4(s)$ and $R_5(s)$ are given in Lemmas 4.6 and 4.10, respectively.*

We postpone the proof of Lemma 4.11 and now complete the proof of Proposition 4.3. The combination of Lemmas 4.10 and 4.11 gives

$$\begin{aligned}\frac{d}{ds}E_5(s)+L_4(s)&=R_5(s)-2\lambda E_4(s)\\&\le \eta\big(L_4(s)+e^{-2\lambda s}\alpha^2(s)\big)+C(\eta)e^{-\lambda s}E_5(s)\\&\quad+C(\eta)e^{-2\lambda_1 s}E_5(s)^{p_1+p_2}\big(E_5(s)^{p_3}+\big(L_4(s)+e^{-2\lambda s}\alpha^2(s)\big)^{p_3}\big)\\&\quad+C(\eta)e^{-\lambda_1 s}E_5(s)^{\frac{p_1+p_2+p_3+1}{2}}.\end{aligned}$$

Note that from Lemma 4.9, we know $\alpha^2(s)$ can be controlled by $E_5(s)$, so choose $\eta=1/2$, for each $s\ge s_0$, when $p_3=0$, we have

$$\frac{d}{ds}E_5(s)\le Ce^{-\lambda s}E_5(s)+C(\eta)e^{-2\lambda_1 s}E_5(s)^{p_1+p_2}+C(\eta)e^{-\lambda_1 s}E_5(s)^{\frac{p_1+p_2+1}{2}} \tag{4.34}$$

and when $p_3=1$, we have

$$\begin{aligned}\frac{d}{ds}E_5(s)\le Ce^{-\lambda s}E_5(s)&+C(\eta)e^{-2\lambda_1 s}E_5(s)^{p_1+p_2}\big(E_5(s)+L_4(s)\\&+e^{-2\lambda s}\alpha^2(s)\big)+C(\eta)e^{-\lambda_1 s}E_5(s)^{\frac{p_1+p_2+2}{2}}.\end{aligned} \tag{4.35}$$

We set

$$\Lambda(s)=\exp(-C\int_{s_0}^{s}e^{-\lambda\tau}d\tau).$$

Obviously, $e^{-\frac{Ce^{-\lambda s_0}}{\lambda}}\le\Lambda(s)\le 1$ for each $s\ge s_0$ and $\Lambda(s_0)=1$, where $s_0$ is given in Lemma 4.11. Multiplying $\Lambda(s)$ on both sides of (4.34) and integrating on the interval $[s_0,s]$, we obtain

$$\Lambda(s)E_5(s)\le E_5(s_0)+C\int_{s_0}^{s}\Lambda(\tau)\big[e^{-2\lambda_1\tau}E_5(\tau)^{p_1+p_2}+e^{-\lambda_1\tau}E_5(\tau)^{\frac{p_1+p_2+1}{2}}\big]d\tau. \tag{4.36}$$

Let

$$M(s)=\sup_{s_0\le\tau\le s}E_5(\tau).$$

As $\lambda_1>0$, (4.36) implies that, for each $s\ge s_0$,

$$M(s)\le CM(s_0)+C\Big(M(s)^{p_1+p_2}+M(s)^{\frac{p_1+p_2+1}{2}}\Big).$$

From the proof of Theorem 2.4, there exists a small $\tilde\varepsilon_0>0$, such that for each $\varepsilon\in(0,\tilde\varepsilon_0]$, $\|(v(s_0),w(s_0))\|_{H^{1,1}\times H^{0,1}}<2C(s_0)I_0\varepsilon$, then by Lemma 4.9, we obtain

$$\begin{aligned}M(s_0)&\le C(s_0)\big(\|(f(s_0),g(s_0))\|^2_{H^{1,1}\times H^{0,1}}+\alpha^2(s_0)+\dot\alpha(s_0)^2\big)\\&\le C(s_0)\|(v(s_0),w(s_0))\|^2_{H^{1,1}\times H^{0,1}}\le C(s_0)\varepsilon^2I_0^2,\end{aligned} \tag{4.37}$$

then

$$M(s) \le C_3\varepsilon^2 I_0^2 + C_3\big(M(s)^{p_1+p_2} + M(s)^{\frac{p_1+p_2+1}{2}}\big), \quad \forall\, s \ge s_0. \tag{4.38}$$

Multiplying by $\Lambda(s)$ on both sides of (4.35), similar to (4.36), we have

$$\begin{aligned}\Lambda(s)E_5(s) \le E_5(s_0) &+ \int_{s_0}^{s} \Lambda(\tau)e^{-2\lambda_1\tau}E_5(\tau)^{p_1+p_2}(L_4(\tau) + e^{-2\lambda\tau}\alpha^2(\tau))d\tau \\ &+ C\int_{s_0}^{s}[\Lambda(\tau)e^{-2\lambda_1\tau}E_5(\tau)^{p_1+p_2+1} + \Lambda(\tau)e^{-\lambda_1\tau}E_5(\tau)^{\frac{p_1+p_2+2}{2}}]d\tau,\end{aligned}$$

then, for each $s \ge s_0$,

$$\begin{aligned}M(s) \le M(s_0) &+ C\int_{s_0}^{s} M(\tau)^{p_1+p_2}L_4(\tau)d\tau + C(M(s)^{p_1+p_2+1} \\ &+ M(s)^{\frac{p_1+p_2+2}{2}}),\end{aligned} \tag{4.39}$$

so

$$\begin{aligned}M(s) \le\ & C_3\varepsilon^2 I_0^2 + C_3 M(s)^{p_1+p_2}\int_{s_0}^{s} L_4(\tau)d\tau \\ &+ C_3\big(M(s)^{p_1+p_2+1} + M(s)^{\frac{p_1+p_2+2}{2}}\big).\end{aligned} \tag{4.40}$$

We take $\tilde\varepsilon_0$ sufficiently small such that

$$2C_3\varepsilon^2 I_0^2 > C_3\varepsilon^2 I_0^2 + C_3[(4C_3\varepsilon^2 I_0^2)^{p_1+p_2} + (4C_3\varepsilon^2 I_0^2)^{(p_1+p_2+1)/2}]$$

and

$$\begin{aligned}2C_3\varepsilon^2 I_0^2 > C_3\varepsilon^2 I_0^2 &+ C_3(4C_3\varepsilon^2 I_0^2)^{p_1+p_2}\int_{s_0}^{s} L_4(\tau)d\tau \\ &+ C_3((4C_3\varepsilon^2 I_0^2)^{p_1+p_2+1} + (4C_3\varepsilon^2 I_0^2)^{\frac{p_1+p_2+2}{2}})\end{aligned}$$

hold for each $\varepsilon \in (0, \tilde\varepsilon_0]$. By the continuous induction method, from (4.37), (4.38) and (4.40), for each $s \ge s_0$ and $\varepsilon \in (0, \tilde\varepsilon_0)$, it follows that

$$M(s) \le 2C_3\varepsilon^2\|(v_0, w_0)\|^2_{H^{1,1}\times H^{0,1}}, \tag{4.41}$$

by Lemma 4.9, i.e.,

$$\|f\|^2_{H^{1,1}} + \frac{e^{-s}}{b^2(t(s))}\|g\|^2_{H^{0,1}} + \alpha^2(s) + \frac{e^{-s}}{b^2(t(s))}\Big(\frac{d\alpha(s)}{ds}\Big)^2 \le C\varepsilon^2\|(v_0, w_0)\|^2_{H^{1,1}\times H^{0,1}}.$$

Consequently, by (4.7), for each $s \ge s_0$ and $\varepsilon \in (0, \tilde\varepsilon_0)$, we have

$$\|v(s)\|^2_{H^{1,1}} + \frac{e^{-s}}{b^2(t(s))}\|w(s)\|^2_{H^{0,1}} \le C\varepsilon^2\|(v_0, w_0)\|^2_{H^{1,1}\times H^{0,1}},$$

which completes the proof of Proposition 4.3.

**Proof of Theorem 2.5.** For the proof we use Corollary 4.2 and Proposition 4.3.

**Step 1:** Global well-posedness. For $s_0 > 0$, as given in Lemma 4.11, it follows from Corollary 4.2 that there exists a $\varepsilon_0^* > 0$ such that for each $\varepsilon \in [0, \varepsilon_0^*)$, the mild solution $(v, w)$ uniquely exists on $[0, s_0]$ and $S(\varepsilon) > s_0$. Let $\varepsilon_1 = \min(\varepsilon_0^*, \tilde\varepsilon_0)$, where $\tilde\varepsilon_0$ is given by Proposition 4.3. It is claimed that for each $\varepsilon \in (0, \varepsilon_1]$, $S(\varepsilon) = \infty$. Indeed, if there exists a $\varepsilon_* \in (0, \varepsilon_1]$ such that $S(\varepsilon_*) < \infty$, let $(v, w)$ be the corresponding mild solution to (3.3), then by Proposition 4.3, for each $s \in [s_0, S(\varepsilon_*))$,

$$\|v(s)\|^2_{H^{1,1}} + \frac{e^{-s}}{b(t(s))^2}\|w(s)\|^2_{H^{0,1}} \le C_*\varepsilon_*^2\|(v_0, w_0)\|_{H^{1,1}\times H^{0,1}}. \tag{4.42}$$

However, it follows from Corollary 4.2 that $\lim_{s\to S(\varepsilon_*)} \|(v,w)(s)\|_{H^{1,1}(\mathbb{R})\times H^{0,1}(\mathbb{R})} = \infty$, which contradicts (4.42).

**Step 2:** Asymptotic profile. Taking $\eta = 1/4$ in Lemma 4.11 and using (4.20), we have

$$\begin{aligned}
&\frac{d}{ds}E_4(s) + 2\lambda E_4(s) + L_4(s)\\
&\le \frac{1}{4}L_4(s) + Ce^{-2\lambda s}E_5(s) + Ce^{-2\lambda_1 s}E_5(s)^{p_1+p_2+p_3} + Ce^{-2\lambda_1 s}E_5(s)^{p_1+p_2}L_4(s)^{p_3}\\
&\le \frac{1}{4}L_4(s) + Ce^{-2\lambda s}E_5(s) + Ce^{-2\lambda_1 s}E_5(s)^{p_1+p_2+p_3} + C(2C_3\varepsilon^2 I_0^2)^{p_1+p_2}L_4(s)^{p_3}.
\end{aligned}$$

We provide the proof only for the case when $p_3 = 1$ because the case of $p_3 = 0$ can be handled similarly. Take a small $\varepsilon_1$ such that $\frac{1}{4} + C(2C_3\varepsilon^2 I_0)^{p_1+p_2} \le \frac{1}{2}$ for each $\varepsilon \in (0, \varepsilon_1]$, then

$$\begin{aligned}
\frac{d}{ds}E_4(s) + 2\lambda E_4(s) + \frac{1}{2}L_4(s) &\le Ce^{-2\lambda s}E_5(s) + Ce^{-2\lambda_1 s}E_5(s)^{p_1+p_2+1}\\
&\le Ce^{-2\lambda s}\varepsilon^2\|(u_0,u_1)\|^2_{H^{1,1}\times H^{0,1}},
\end{aligned}$$

multiplying by $e^{2\lambda s}$ gives

$$\frac{d}{ds}(e^{2\lambda s}E_4(s)) + \frac{e^{2\lambda s}}{2}L_4(s) \le C\varepsilon^2\|(u_0,u_1)\|^2_{H^{1,1}\times H^{0,1}},$$

i.e.,

$$\begin{aligned}
&\frac{d}{ds}\big(e^{2\lambda s}E_4(s)\big) + \frac{e^{2\lambda s}}{2}\big(L_4(s) + e^{-2\lambda s}\alpha^2(s)\big)\\
&\le C\varepsilon^2\|(u_0,u_1)\|^2_{H^{1,1}\times H^{0,1}} + \alpha^2(s)\\
&\le C\varepsilon^2\|(u_0,u_1)\|^2_{H^{1,1}\times H^{0,1}},
\end{aligned} \tag{4.43}$$

where Lemma 4.9 has been used. Integrating (4.43) on $[s_0, s]$, multiplying by $e^{-2\lambda s}$ and using Lemma 4.8, we have

$$\begin{aligned}
&E_4(s) + \frac{1}{2}\int_{s_0}^{s} e^{-2\lambda(s-\tau)}(\|f\|^2_{H^{1,1}} + \|g\|^2_{H^{0,1}} + \dot{\alpha}^2(s))d\tau\\
&\le Ce^{-2\lambda s}\varepsilon^2(s-s_0)\|(u_0,u_1)\|^2_{H^{1,1}\times H^{0,1}}.
\end{aligned}$$

Note that when $s_0 \le \tilde{s} \le s$,

$$\begin{aligned}
|\alpha(s) - \alpha(\tilde{s})|^2 &= \Big(\int_{\tilde{s}}^{s} \frac{d\alpha}{d\tau}(\tau)d\tau\Big)^2\\
&\le \Big(\int_{\tilde{s}}^{s} e^{-2\lambda\tau}d\tau\Big)\Big(\int_{\tilde{s}}^{s} e^{2\lambda\tau}(\frac{d\alpha}{d\tau}(\tau))^2 d\tau\Big)\\
&\le Ce^{-2\lambda\tilde{s}}\varepsilon^2\|(u_0,u_1)\|^2_{H^{1,1}\times H^{0,1}},
\end{aligned}$$

so $\alpha^* = \lim_{s\to+\infty}\alpha(s)$ exists and $|\alpha(s) - \alpha^*|^2 \le Ce^{-2\lambda s}\varepsilon^2\|(u_0,u_1)\|^2_{H^{1,1}\times H^{0,1}}$. Therefore,

$$\begin{aligned}
\|v(s) - \alpha^*\varphi_0\|^2_{H^{1,1}} &\le 2\|f(s)\|^2_{H^{1,1}} + 2|\alpha(s) - \alpha^*|^2\|\varphi_0\|^2_{H^{1,1}}\\
&\le Ce^{-2\lambda s}\varepsilon^2\|(u_0,u_1)\|^2_{H^{1,1}\times H^{0,1}}.
\end{aligned}$$

By (3.2) and $\mathcal{G}(B(t)+1, x) = (B(t)+1)^{-1/2}\varphi_0((B(t)+1)^{-1/2}x)$, we infer that

$$\|u(t,\cdot) - \alpha^*\mathcal{G}(B(t)+1,\cdot)\|^2_{L^2} \le C\varepsilon^2(B(t)+1)^{-\frac{1}{2}-2\lambda}\|(u_0,u_1)\|^2_{H^{1,1}\times H^{0,1}}.$$

**Proof of Lemma 4.11.** To obtain the upper bound of $|R_4(s)|$ and $|R_5(s)|$, in view of their definitions, we need to estimate the $H^{0,1}$ norms of $r(s), h(s)$ and $H(s)$. Recall that $r(s)$ is defined by (3.4).

**Lemma 4.12.** *Under assumptions* (2.1) *and* (2.3)*, we have*

$$\begin{aligned}
&\big\|e^{3s/2}N\big(e^{-s/2}v, e^{-s}v_y, b^{-1}(t(s))e^{-3s/2}w\big)\big\|^2_{H^{0,1}}\\
&\le Ce^{-2\lambda_1 s}\big(\|f\|^2_{H^{1,1}}+\alpha^2(s)\big)^{p_1+p_2}\Big(\|g\|^2_{H^{0,1}}+\alpha^2(s)+\big(\frac{d\alpha(s)}{ds}\big)^2\Big)^{p_3}
\end{aligned}\tag{4.44}$$

*holds for each $s\ge 0$, where $\lambda_1$ is defined by* (4.33).

*Proof.* There are two cases according to the value of $\beta$.

**Case 1:** $\beta\in(-1,1)$. By Lemma 3.2 and (2.3), we have

$$\begin{aligned}
&(1+y^2)e^{3s}N^2\big(e^{-s/2}v, e^{-s}v_y, b^{-1}(t(s))e^{-3s/2}w\big)\\
&\le C(1+y^2)e^{3s}e^{-p_1 s}|v|^{2p_1}e^{-2p_2 s}|v_y|^{2p_2}e^{-(3-\frac{2\beta}{1+\beta}p_3)s}|w|^{2p_3}\\
&= C(1+y^2)e^{-2\lambda_1 s}|v|^{2p_1}|v_y|^{2p_2}|w|^{2p_3}.
\end{aligned}$$

Sobolev's inequality shows that $\|v(s)\|_{L^\infty}\le C\|v(s)\|_{H^{1,0}}$, then

$$\begin{aligned}
&(1+y^2)e^{-2\lambda_1 s}|v|^{2p_1}|v_y|^{2p_2}|w|^{2p_3}\\
&= Ce^{-2\lambda_1 s}(1+y^2)^{1-p_2-p_3}(1+y^2)^{p_2}(1+y^2)^{p_3}|v^2|^{p_1+p_2+p_3-1}|v^2|^{1-p_2-p_3}|v_y^2|^{p_2}|w^2|^{p_3}\\
&\le Ce^{-2\lambda_1 s}\|v(s)\|^{2(p_1+p_2+p_3-1)}_{H^{1,0}}\big((1+y^2)v^2\big)^{1-p_2-p_3}\big((1+y^2)v_y^2\big)^{p_2}\big((1+y^2)w^2\big)^{p_3}.
\end{aligned}$$

So by Hölder's inequality,

$$\begin{aligned}
&\big\|e^{3s/2}N\big(e^{-s/2}v, e^{-s}v_y, b^{-1}(t(s))e^{-3s/2}w\big)\big\|^2_{H^{0,1}}\\
&\le Ce^{-2\lambda_1 s}\|v(s)\|^{2(p_1+p_2+p_3-1)}_{H^{1,0}}\|v(s)\|^{2(1-p_2-p_3)}_{H^{1,1}}\|v(s)\|^{2p_2}_{H^{1,1}}\|w(s)\|^{2p_3}_{H^{0,1}}\\
&\le Ce^{-2\lambda_1 s}\|v(s)\|^{2(p_1+p_2-1)}_{H^{1,0}}\|v(s)\|^2_{H^{1,1}}\|w(s)\|^{2p_3}_{H^{0,1}}\\
&\le Ce^{-2\lambda_1 s}\big(\|f\|_{H^{1,1}}+\alpha(s)\big)^{2(p_1+p_2)}\big(\|g\|_{H^{0,1}}+\alpha(s)+\dot\alpha(s)\big)^{2p_3}\\
&\le Ce^{-2\lambda_1 s}\big(\|f\|^2_{H^{1,1}}+\alpha^2(s)\big)^{p_1+p_2}\Big(\|g\|^2_{H^{0,1}}+\alpha^2(s)+\big(\frac{d\alpha(s)}{ds}\big)^2\Big)^{p_3}.
\end{aligned}$$

**Case 2:** $\beta=-1$, $p_3\ne 0$. By Lemma 3.2 and (2.3), we obtain

$$\begin{aligned}
&(1+y^2)e^{3s}N^2\big(e^{-s/2}v, e^{-s}v_y, b^{-1}(t(s))e^{-3s/2}w\big)\\
&\le C(1+y^2)e^{(3-p_1-2p_2-3p_3)s}b^{-p_3}(t(s))|v|^{2p_1}|v_y|^{2p_2}|w|^{2p_3}\\
&\le C(1+y^2)e^{-\lambda_* s}|v|^{2p_1}|v_y|^{2p_2}|w|^{2p_3}\\
&\le C(1+y^2)e^{-2\lambda_1 s}|v|^{2p_1}|v_y|^{2p_2}|w|^{2p_3},
\end{aligned}$$

here we have used $b^{-1}(t(s))\sim\exp(-e^s)$ by Lemma 3.2, and $\lambda_*=2\frac{e^s}{s}p_3+(p_1+2p_2+3p_3-3)>2\lambda_1$ is regarded as a sufficiently large number. Then (4.44) can be derived similarly to Case 1. $\square$

**Lemma 4.13.** *Under assumptions* (A1)–(A4)*, for each* $s \geq 0$*, we have*

$$
\begin{aligned}
&\|r(s)\|^2_{H^{0,1}} \\
&\le Ce^{-2\lambda_0 s}\Big(\|f\|^2_{H^{1,1}}+\|g\|^2_{H^{0,1}}+\alpha^2(s)+\Big(\frac{d\alpha(s)}{ds}\Big)^2\Big) \\
&\quad + Ce^{-2\lambda_1 s}\big(\|f\|^2_{H^{1,1}}+\alpha^2(s)\big)^{p_1+p_2}\Big(\|g\|^2_{H^{0,1}}+\alpha^2(s)+\big(\frac{d\alpha(s)}{ds}\big)^2\Big)^{p_3},
\end{aligned}
\tag{4.45}
$$

*where* $r(s)$ *is given by* (3.4)*,* $\lambda_0$ *and* $\lambda_1$ *are given by* (4.33).

*Proof.* By the definition of $r(s)$, it suffices to estimate

$$
\frac{1}{b^2(t(s))}\frac{db(t(s))}{dt}w+e^{s/2}c(t(s))v_y+e^s d(t(s))v.
$$

Combining (4.44) with

$$
\begin{aligned}
&\|\frac{1}{b^2(t(s))}\frac{db(t(s))}{dt}w\|^2_{H^{0,1}} \\
&\le C\Big(\|g\|^2_{H^{0,1}}+\alpha^2(s)+\big(\frac{d\alpha(s)}{ds}\big)^2\Big)\times\begin{cases} e^{-\frac{2(1-\beta)s}{1+\beta}}, & \beta\in(-1,1),\\ \exp(-4e^s), & \beta=-1,\end{cases}
\end{aligned}
$$

$$
\|e^{s/2}c(t(s))v_y\|^2_{H^{0,1}}\le C\big(\|f\|^2_{H^{1,1}}+\alpha^2(s)\big)\times\begin{cases} e^{-(\frac{2\gamma}{1+\beta}-1)s}, & \beta\in(-1,1),\\ \exp(-2\gamma e^s+s), & \beta=-1\end{cases}
$$

$$
\|e^s d(t(s))v\|^2_{H^{0,1}}\le C\big(\|f\|^2_{H^{1,1}}+\alpha^2(s)\big)\times\begin{cases} e^{-(\frac{2\nu}{1+\beta}-2)s}, & \beta\in(-1,1),\\ \exp(-2\nu e^s+2s). & \beta=-1,\end{cases}
$$

we obtain(4.45), where Lemma 3.2 and (2.2) have been used. $\square$

**Lemma 4.14.** *Under assumptions* (A1)–(A4),

$$
\begin{aligned}
&\|h(s)\|^2_{H^{0,1}} \\
&\le Ce^{-2\lambda_0 s}\Big(\|f\|^2_{H^{1,1}}+\|g\|^2_{H^{0,1}}+\alpha^2(s)+\Big(\frac{d\alpha(s)}{ds}\Big)^2\Big) \\
&\quad + Ce^{-2\lambda_1 s}\big(\|f\|^2_{H^{1,1}}+\alpha^2(s)\big)^{p_1+p_2}\big(\|g\|^2_{H^{0,1}}+\alpha^2(s)+\big(\frac{d\alpha(s)}{ds}\big)^2\big)^{p_3}
\end{aligned}
\tag{4.46}
$$

*holds for each* $s \geq 0$*, where* $h(s)$*,* $\lambda_0$*,* $\lambda_1$ *are given by* (4.11) *and* (4.33).

*Proof.* Clearly,

$$
\begin{aligned}
&\Big\|\frac{e^{-s}}{b^2(t(s))}\Big(-2\frac{d\alpha(s)}{ds}\psi_0(y)+\alpha(s)\big(\frac{y}{2}\psi_0'(y)+\frac{3}{2}\psi_0(y)\big)\Big)\Big\|^2_{H^{0,1}} \\
&\le \Big\|\Big(-2\frac{d\alpha(s)}{ds}\psi_0(y)+\alpha(s)\big(\frac{y}{2}\psi_0'(y)+\frac{3}{2}\psi_0(y)\big)\Big)\Big\|^2_{H^{0,1}} \\
&\quad\times\begin{cases} e^{-2(\frac{1-\beta}{1+\beta})s}, & \beta\in(-1,1),\\ \exp(-2e^s-s), & \beta=-1\end{cases} \\
&\le Ce^{-2\lambda_0 s}\big(\dot\alpha^2(s)\|\psi_0(y)\|^2_{H^{0,1}}+\alpha^2(s)\|\psi_0(y)\|^2_{H^{0,1}}\big) \\
&\le Ce^{-2\lambda_0 s}\big(\dot\alpha^2(s)+\alpha^2(s)\big).
\end{aligned}
$$

Hölder's inequality gives

$$\begin{aligned} \big|\int_{\mathbb{R}} r(s,y)dy\big| &\leq \Big(\int_{\mathbb{R}} \frac{1}{1+y^2}dy\Big)^{1/2}\Big(\int_{\mathbb{R}}(1+y^2)r^2(s,y)dy\Big)^{1/2} \\ &\leq C\|r(s,\cdot)\|_{H^{0,1}}. \end{aligned} \tag{4.47}$$

Then in view of the definition of $h(s)$, (4.46) follows from Lemma 4.13. $\square$

**Lemma 4.15.** *Under assumptions* (A1)–(A4)*, for each $s \geq 0$, we have*

$$\begin{aligned} \|H(s)\|^2_{H^{0,1}} &\leq Ce^{-2\lambda_0 s}\big(\|f\|^2_{H^{1,1}} + \|g\|^2_{H^{0,1}} + \alpha^2(s) + \big(\frac{d\alpha(s)}{ds}\big)^2\big) \\ &\quad + Ce^{-2\lambda_1 s}\big(\|f\|^2_{H^{1,1}} + \alpha^2(s)\big)^{p_1+p_2}\Big(\|g\|^2_{H^{0,1}} + \alpha^2(s) + \big(\frac{d\alpha(s)}{ds}\big)^2\Big)^{p_3}, \end{aligned}$$

*where $H(s)$, $\lambda_0$, $\lambda_1$ are given by* (4.15) *and* (4.33).

*Proof.* By Lemma 4.5,

$$\int_{\mathbb{R}} H^2(s,y)dy \leq 4\int_{\mathbb{R}} y^2h^2(s,y)dy \leq 4\|h(s)\|^2_{H^{0,1}}.$$

A direct calculation and Young's inequality show that

$$\begin{aligned} \int_{\mathbb{R}} y^2H^2(s,y)dy &= \int_{|y|\leq 1} y^2H^2(s,y)dy + \int_{|y|>1} y^2H^2(s,y)dy \\ &\leq \int_{|y|\leq 1} H^2(s,y)dy + \int_{|y|>1} y^3H^2(s,y)dy \\ &\leq 4\|h(s)\|^2_{H^{0,1}} - 6\int_{|y|>1} y^2H(s,y)h(s,y)dy \\ &\leq 4\|h(s)\|^2_{H^{0,1}} + \frac{1}{2}\int_{|y|>1} y^2H^2dy + 72\int_{|y|>1} y^2h^2dy, \end{aligned}$$

so

$$\int_{\mathbb{R}} y^2H^2(s,y)dy \leq C\int_{\mathbb{R}} y^2h^2(s,y)dy \leq \|h(s)\|^2_{H^{0,1}}.$$

Then Lemma 4.15 can be deduced by Lemma 4.14. $\square$

*Proof of Lemma 4.11.* Recall that $R_4(s) = C_0R_0(s) + C_1R_1(s) + R_2(s) + R_3(s)$, where $R_i(s)$ $(i = 0,1,2,3)$ are defined by (4.16)-(4.19). Lemmas 4.7, 4.7 and Young's inequality show that there exists a $s_3 \geq s_2$ such that the terms that do not include $r(s,y)$, $h(s,y)$ and $H(s,y)$ can be controlled by $\eta(L_4(s) + e^{-2\lambda s}\alpha^2(s)) + C(\eta)e^{-2\lambda s}E_5(s)$. For example,

$$\begin{aligned} &\big|\int_{\mathbb{R}} \frac{1}{a(ye^{s/2})}(\alpha(s)a_0(ye^{s/2})\varphi_0'(F+G)dy\big| \\ &\leq \frac{1}{\underline{a}}(\|F\|_{L^2} + \|G\|_{L^2})(|\alpha|\|a_0(ye^{s/2})\varphi_0'\|_{L^2}) \\ &\leq \frac{1}{\underline{a}}\Big(\eta(\|f\|_{H^{1,1}} + \|g\|_{H^{0,1}}) + C(\eta)e^{-2(\frac{1}{4}+\frac{\mu_0}{2})s}\alpha^2(s)\Big) \\ &\leq C\Big(\eta(L_4(s) + e^{-2\lambda s}\alpha^2(s)) + C(\eta)e^{-2\lambda s}E_5(s)\Big). \end{aligned} \tag{4.48}$$

The other terms can be estimated similarly. The remainder terms consist of

$$\int_{\mathbb{R}}(F+G)Hdy, \quad \int_{\mathbb{R}}(1+|y|^2)(f+g)hdy, \quad \frac{d\alpha(s)}{ds}\int_{\mathbb{R}} r(s,y)dy.$$

Using Lemmas 4.5, 4.8, 4.9, 4.14, 4.15 and Young's inequality, the first two terms can be controlled by

$$\begin{aligned}&\eta(L_4+e^{-2\lambda s}\alpha^2(s))+C(\eta)e^{-2\lambda s}E_5(s)\\&+C(\eta)e^{-2\lambda_1 s}E_5(s)^{p_1+p_2}\Big(E_5(s)^{p_1}+(L_4+e^{-2\lambda s}\alpha^2(s))^{p_3}\Big).\end{aligned}$$

For example,

$$\begin{aligned}&\Big|\int_{\mathbb{R}}(F+G)Hdy\Big|\\&\le C(\|F\|_{L^2}+\|G\|_{L^2})\|H\|_{L^2}\\&\le C(\eta(\|f\|^2_{H^{1,1}}+\|g\|^2_{H^{0,1}})+C(\eta)\|h\|^2_{H^{0,1}})\\&\le \eta(L_4+e^{-2\lambda s}\alpha^2(s))+C(\eta)e^{-2\lambda s}E_5(s)\\&\quad+C(\eta)e^{-2\lambda_1 s}E_5(s)^{p_1+p_2}(E_5(s)^{p_3}+(L_4+e^{-2\lambda s}\alpha^2(s))^{p_3}).\end{aligned}\tag{4.49}$$

In addition, Young's inequality implies

$$\Big|\Big(\int_{\mathbb{R}}r(s,y)dy\Big)\frac{d\alpha(s)}{ds}\Big|\le\eta\Big(\frac{d\alpha(s)}{ds}\Big)^2+C(\eta)(\int_{\mathbb{R}}r(s,y)dy)^2,$$

then the third term can be estimated by using Lemmas 4.8, 4.9, 4.13, and (4.47). So far, the estimate for $R_4(s)$ has been obtained.

Note that $R_5(s)$ defined in Lemma 4.10 can be rewritten as $R_5(s)=\tilde R_5(s)+R_4(s)$, where

$$\tilde R_5(s)=\frac{e^{-s}}{b^2(t(s))}\Big(\frac{d\alpha(s)}{ds}\Big)^2-\frac{2}{b^2(t(s))}\alpha(s)\frac{db(t(s))}{dt}\frac{d\alpha(s)}{ds}+\alpha(s)\int_{\mathbb{R}}r(s,y)dy,$$

so it suffices to estimate $\tilde R_5(s)$. Obviously, there exists a $s_4\ge s_2$ such that for each $s\ge s_4$,

$$\begin{aligned}&\frac{e^{-s}}{b^2(t(s))}\Big(\frac{d\alpha(s)}{ds}\Big)^2-\frac{2}{b^2(t(s))}\alpha(s)\frac{db(t(s))}{dt}\frac{d\alpha(s)}{ds}\\&\le\eta(L_4+e^{-2\lambda s}\alpha^2(s))+C(\eta)e^{-2\lambda s}E_5(s).\end{aligned}$$

Using Lemma 4.13, (4.47) and $|\alpha(s)|\le CE_5(s)^{1/2}$ (by Lemma 4.9), there exists a $s_5\ge s_2$, such that for each $s\ge s_5$,

$$\begin{aligned}|\alpha(s)\int_{\mathbb{R}}r(s,y)dy|&\le|\alpha(s)|\|r(s)\|_{H^{0,1}}\\&\le CE_5(s)^{1/2}[(L_4(s)+e^{-2\lambda s}\alpha^2(s))^{1/2}+e^{-\lambda_0 s}E_5(s)^{1/2}]\\&\quad+Ce^{-\lambda_1 s}E_5(s)^{\frac{p_1+p_2+1}{2}}(E_5(s)^{p_3/2}+(L_4(s)+e^{-2\lambda s}\alpha^2(s))^{p_3/2})\\&\le\tilde\eta(L_4(s)+e^{-2\lambda s}\alpha^2(s))+Ce^{-\lambda s}E_5(s)\\&\quad+Ce^{-\lambda_1 s}E_5(s)^{\frac{p_1+p_2+1}{2}}(E_5(s)^{p_3/2}+(L_4(s)+e^{-2\lambda s}\alpha^2(s))^{p_3/2}).\end{aligned}$$

If $p_3=1$ and

$$\begin{aligned}&e^{-\lambda_1 s}E_5(s)^{\frac{p_1+p_2+1}{2}}(L_4+e^{-2\lambda s}\alpha^2(s))^{1/2}\\&\le\eta(L_4+e^{-2\lambda s}\alpha^2(s))+C(\eta)e^{-2\lambda_1 s}E_5(s)^{p_1+p_2+1},\end{aligned}$$

then

$$
\begin{aligned}
|\alpha(s)\int_{\mathbb{R}} r(s,y)dy| &\le \eta(L_4(s)+e^{-2\lambda s}\alpha^2(s))+Ce^{-\lambda s}E_5(s)\\
&\quad + Ce^{-\lambda_1 s}E_5(s)^{\frac{p_1+p_2+p_3+1}{2}}+C(\eta)e^{-2\lambda_1 s}E_5(s)^{p_1+p_2+p_3}.
\end{aligned}
$$

The case of $p_3=0$ can be estimated similarly. Setting $s_0=\max\{s_3,s_4,s_5\}$, the estimate for $R_5(s)$ can be obtained. □

## 5. Proof of Theorem 2.6

For each $s_*>0$, let $M(s)=\sup_{s_*\le\tau\le s}E_5(\tau)$, $\Lambda(s)=\exp(-C\int_{s_*}^s e^{-\lambda\tau}d\tau)$, where $E_5(s)$ is defined in Lemma 4.9. Then (4.31) implies that

$$
M(s)\sim \sup_{s_*\le\tau\le s}\Big(\|v(\tau,\cdot)\|_{H^{1,1}}^2+\frac{e^{-\tau}}{b^2(t(\tau))}\|w(\tau,\cdot)\|_{H^{0,1}}^2\Big).
$$

**Lemma 5.1.** *Under assumptions* (A1)–(A5)*, where* (2.3) *is replaced by* (2.10)*, there exist $s_*>0$, $C>0$ such that for each $s\ge s_*$ and each solution $(v,w)$ to* (3.3)*, we have*

$$
M(s)\le M(s_*)+Ce^{(3-p_1)s}M(s)^{p_1}+Ce^{\frac{3-p_1}{2}s}M(s)^{\frac{p_1+1}{2}}. \tag{5.1}
$$

*Proof.* As in the proof of Proposition 4.3, by Lemmas 4.10 and 4.11 we have

$$
\begin{aligned}
\frac{d}{ds}E_5(s)+L_4(s) &\le \eta\big(L_4(s)+e^{-2\lambda s}\alpha^2(s)\big)+C(\eta)e^{-\lambda s}E_5(s)\\
&\quad + C(\eta)e^{-2\lambda_1 s}E_5(s)^{p_1}+C(\eta)e^{-\lambda_1 s}E_5(s)^{\frac{p_1+1}{2}},
\end{aligned}
$$

letting $\eta=1/2$ and using Lemma 4.8, we deduce that

$$
\begin{aligned}
\frac{d}{ds}E_5(s) &\le Ce^{-\lambda s}E_5(s)+Ce^{-2\lambda_1 s}E_5(s)^{p_1}\\
&\quad + Ce^{-2\lambda_1 s}E_5(s)^{p_1}+Ce^{-\lambda_1 s}E_5(s)^{\frac{p_1+1}{2}}.
\end{aligned} \tag{5.2}
$$

In the following, it suffices to estimate the upper bound for $L_4(s)$. Indeed, note that

$$
\begin{aligned}
L_4(s) &= (\frac{1}{2}-2\lambda)(C_0E_0(s)+C_1E_1(s)+E_2(s))+C_0L_0(s)+C_1L_1(s)\\
&\quad + L_2(s)+\dot{\alpha}^2(s).
\end{aligned}
$$

From (4.28), (4.29), and (4.31), it follows that

$$
C_0E_0(s)+C_1E_1(s)+C_2E_2(s)\le CE_5(s).
$$

Lemma 4.5 gives

$$
\begin{aligned}
&C_0L_0(s)+C_1L_1(s)+L_2(s)+\dot{\alpha}^2(s)\\
&= C_0\int_{\mathbb{R}}\big(\frac{1}{2}F_y^2+\frac{1}{a(e^{s/2}y)}G^2\big)dy+C_1\int_{\mathbb{R}}\big(a(e^{s/2}y)f_y^2+g^2-f^2\big)dy\\
&\quad + \int_{\mathbb{R}}\big(\frac{1}{2}y^2a(e^{s/2}y)f_y^2+y^2g^2\big)dy+2\int_{\mathbb{R}} ya(e^{s/2}y)f_y(f+g)dy+\dot{\alpha}^2(s)\\
&\le C\big(\|f\|_{L^2}^2+\|G\|_{L^2}^2+\|f_y\|_{L^2}^2+\|g\|_{L^2}^2+\|yf_y\|_{L^2}^2+\|yg\|_{L^2}^2+\dot{\alpha}^2(s)\big)\\
&\le C\big(\|g\|_{H^{0,1}}^2+\dot{\alpha}^2(s)\big)+CE_5(s),
\end{aligned}
$$

while from Lemma 3.2 it follows that

$$\begin{aligned}\|g\|_{H^{0,1}}^2+\dot\alpha^2(s)&=e^s b^2(t(s))\Big(\frac{e^{-s}}{b^2(t(s))}\big(\|g\|_{H^{0,1}}^2+\dot\alpha^2(s)\big)\Big)\\&\le Ce^s b^2(t(s))E_5(s)\le Ce^{\frac{1-\beta}{1+\beta}s}E_5(s);\end{aligned}$$

therefore,

$$L_4(s)\le Ce^{\frac{1-\beta}{1+\beta}s}E_5(s). \tag{5.3}$$

Substituting (5.3) into (5.2) gives

$$\begin{aligned}\frac{d}{ds}E_5(s)&\le Ce^{-\lambda s}E_5(s)+Ce^{-2\lambda_1 s}E_5(s)^{p_1}\\&\quad+Ce^{-2\lambda_1 s}E_5(s)^{p_1}+Ce^{-\lambda_1 s}E_5(s)^{\frac{p_1+1}{2}}.\end{aligned}$$

By (2.10), it is easy to see that $-2\lambda_1=3-p_1>0$, so

$$\begin{aligned}\frac{d}{ds}E_5(s)&\le Ce^{-\lambda s}E_5(s)+Ce^{-2\lambda_1 s}E_5(s)^{p_1}\\&\quad+Ce^{(3-p_1)s}E_5(s)^{p_1}+Ce^{-\lambda_1 s}E_5(s)^{\frac{p_1+1}{2}}.\end{aligned}$$

Then

$$\begin{aligned}\frac{d}{ds}\big(E_5(s)\Lambda(s)\big)&\le Ce^{-2\lambda_1 s}E_5(s)^{p_1}\Lambda(s)+Ce^{(3-p_1)s}E_5(s)^{p_1}\Lambda(s)\\&\quad+Ce^{-\lambda_1 s}E_5(s)^{\frac{p_1+1}{2}}\Lambda(s)\\&\le Ce^{(3-p_1)s}E_5(s)^{p_1}\Lambda(s)+Ce^{\frac{3-p_1}{2}s}E_5(s)^{\frac{p_1+1}{2}}\Lambda(s).\end{aligned} \tag{5.4}$$

As the indexes satisfy $3-p_1>0$, after integrating, there exists a $s_*>0$ sufficiently large such that (5.1) holds for each $s\ge s_*$. □

*Proof of Theorem 2.6.* By Corollary 4.2, the lifespan $S(\varepsilon)$ of the solution $(v,w)$ to (3.3) satisfies $S(\varepsilon)>s_*$ provided $\varepsilon$ is sufficiently small, where $s_*$ is given by Lemma 5.1. Moreover, in view of (4.37), $M(s_*)\le C\varepsilon^2 I_0^2$. If $M(s)$ cannot reach $2C\varepsilon^2I_0^2$ at any time, then $\lim_{s\to S(\varepsilon)}M(s)\le 2C\varepsilon^2I_0^2$, which contradicts (2.8). So let $S\ge s_*$ be the first time such that $M(S)=2C\varepsilon^2I_0^2$. By Lemma 5.1, we have

$$2C\varepsilon^2I_0^2\le C\varepsilon^2I_0^2+Ce^{(3-p_1)S}\big(C\varepsilon^2I_0^2\big)^{p_1}+Ce^{\frac{3-p_1}{2}S}\big(C\varepsilon^2I_0^2\big)^{\frac{p_1+1}{2}},$$

so

$$C\varepsilon^2\le Ce^{\frac{3-p_1}{2}S}\varepsilon^{p_1+1}.$$

Then

$$\varepsilon^{\frac{2(1-p_1)}{3-p_1}}\le Ce^S\le Ce^{S(\varepsilon)}\le C\big(B(T(\varepsilon))+1\big),$$

namely,

$$B(T(\varepsilon))+1\ge C\varepsilon^{\frac{2(1-p_1)}{3-p_1}}.$$

□

**Remark 5.2.** We aim to establish an upper bound for (1.1), analogous to the lower bound found in (2.11). For simplicity, let us consider the equation

$$\begin{gathered}\partial_t^2u-\partial_x(a(x)\partial_xu)+b(t)\partial_tu=|u|^p,\quad t>0,\ x\in\mathbb{R},\\u(0,x)=\varepsilon u_0(x),\quad \partial_tu(0,x)=\varepsilon u_1(x),\quad x\in\mathbb{R}.\end{gathered} \tag{5.5}$$

If $a(x) = 1$, we note that the lower bound (2.11) for (5.5) coincides with [8, Theorem 1.2], which suggests that (2.11) is sharp for (5.5). By employing a test function argument, we can establish the following upper bound for (5.5)

$$B(T(\varepsilon)) \lesssim \begin{cases} \varepsilon^{-\frac{2(p-1)}{3-2p}}, & 1 < p < 3/2, \\ e^{\varepsilon^{-(p-1)}}, & p = 3/2. \end{cases} \tag{5.6}$$

However, because of the variable-coefficient diffusion, we still lack information about the sharpness of estimate (5.6). We have not yet established an upper bound estimate for (1.1).

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