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EXISTENCE AND STABILIZATION FOR IMPULSIVE DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH MULTIPLE DELAYS

SANDRA PINELAS, OSMAN TUNÇ, ERDAL KORKMAZ, CEMIL TUNÇ

Abstract. Existence and stability of solutions are important parts in the qualitative study of delay differential equations. The stabilizing by imposing proper impulse controls are used in many areas of natural sciences and engineering. This article provides sufficient conditions for the existence and exponential stabilization of solutions to delay impulsive differential equations of second-order with multiple delays. The main tools in this article are the Schaefer fixed point theorem, fixed impulse effects, and Lyapunov-Krasovskii functionals. The outcomes extend earlier results in the literature.

1. Introduction

Impulsive differential equations have become a very important part in the study of differential equations with and without delays. This includes ordinary and partial differential equations, functional equations, integro-differential differential, etc. Various mathematical models of impulsive differential equations appear in real world applications, and they are very effective in modeling these problems. For applied impulsive mathematical models, see the book by Stamova and Stamov [34]. For recent advances in stability and control of impulsive delay systems, see the book by Li and Song [25]. For optimal impulsive control in cancer therapy medicine, see the book by Belfo and Lemos [4]. For periodic solutions and applications, see the article by Li et al. [20]. For applications of switched and impulsive systems, see the book by Li et al. [26]. For results on stability and nonlinear dynamics of high-order delayed cellular neural systems, see Huang et al. [13, 14] and Zhao et al. [51].

In the previous decades, significant progress has been made in the qualitative theory of impulsive differential equations. This progress includes areas such as networks, neutral-type functional differential equations, global asymptotic stability of periodic patch-constructed Nicholson's blowflies systems with time varying delays, etc. See the various works and interesting results mention in our references and the references therein.

An important fact about stability is that the effect of impulses can cause stable systems to become unstable, and inversely, unstable systems to become stable. The

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problem of stabilizing ordinary differential equations (ODEs) by imposing impulse controls can be used in several fields such as biotechnology, chemical technology, economics, medicine, physics, and population dynamics. From the mathematical point of view, also partial differential equations (PDEs) with delay can be stabilized by adding impulsive effects. For this, we mention the recent papers by Columbu et al. [7] and Li et al. [17, 18, 19].

Now we want to mention some papers that serve as source for the stabilization problems we study in this article.

In 2006, Gimenes and Federson [9] considered the impulsive delay differential equations

$$x'' + \sum_{i=1}^{N} a_i(t)x_i(t-\tau_i) + f(x,x') = 0, \quad t \geq t_0,\ t \neq t_k \tag{1.1}$$

and

$$x'' + \sum_{i=1}^{N} \int_{t-\tau_i}^{t} b_i(t-u)x(u)du + f(x,x') = 0, \quad t \geq t_0,\ t \neq t_k,\ k \in \mathbb{N}, \tag{1.2}$$

with suitable initial data and impulsive controls at $t = t_k$; see equation (2.1) below.

Gimenes and Federson [9] derived sufficient conditions for the existence of solutions on a closed time interval. The authors also show that the non-impulsive forms of (1.1) and (1.2) can be stabilized via impulse controls.

Later, in 2007, Gimenes et al. [10] dealt with the equations

$$x'' + f(t,x,x') + g(t,x,x(t-\tau)) = 0, \quad t \geq t_0, \quad t \neq t_k, \tag{1.3}$$

$$x'' + \sum_{i=1}^{N} a_i(t)x(t-\tau_i) + f(t,x,x') = 0, \quad t \geq t_0, \quad t \neq t_k \tag{1.4}$$

and

$$\begin{gathered} x'' + \sum_{i=1}^{N} \int_{t-\tau_i}^{t} b_i(t-u)x(u)du + f(t,x,x') = 0, \\ t \geq t_0,\ t \neq t_k,\ k = 1, 2, \dots, \end{gathered} \tag{1.5}$$

with suitable initial data and the impulsive controls at $t = t_k$; see equation (2.1) below.

Gimenes et al. [10] proved that non-impulsive forms of (1.3), (1.4), and (1.5) can be stabilized via impulsive controls. The basic tools utilized in the proofs of their results are Lyapunov functions, stability theory, and control by impulses.

Recently, Tunç et al. [43] dealt with the equations

$$\begin{gathered} x'' + c(t)f(x,x') + b(t)g(x) + \sum_{i=1}^{N} a_i(t)h_i(x(t-\tau_i)) + p(x,x') = 0, \\ t \geq t_0,\ t \neq t_k \end{gathered} \tag{1.6}$$

and

$$\begin{gathered} x'' + c(t)f(x,x') + b(t)g(x) + \sum_{i=1}^{N} \int_{t-\tau_i}^{t} d_i(t-u)h(x(u))du + q(x,x') = 0, \\ t \geq t_0,\ t \neq t_k,\ k \in \mathbb{N}, \end{gathered} \tag{1.7}$$

with suitable the initial data and the impulsive controls at $t = t_k$. The authors obtained results on the existence of solutions to (1.6) and (1.7).

In recent years, various qualitative results about impulsive differential equations have also been obtained. See, Graef and Tunç [12], Pinelas and Tunç [38], Tunç [39], Tunç and Tunç [36, 37, 41, 38], Tunç et al. [40, 42, 44], and their references. In the above publications the main tools are the Lyapunov-Razumikhin technique, fixed point theorems, direct method of Lyapunov, and impulsive perturbations. In particular, see Benchohra et al. [5], Smart [32], Tunç and Tunç [36], Xie [45].

In this article, $C([a,b],\mathbb{R})$ denotes the Banach space of continuous functions, endowed with the usual supremum norm; $k \in \mathbb{N} = \{1, 2, 3, \dots\}$; $i = 1, 2, \dots, N$; x denotes $x(t)$; x' denotes $x'(t)$.

Motivated by the articles of Gimenes and Federson [9, Theorem 3.1 and 3.2], Gimenes et al. [10], and Tunç et al. [43], we study the impulsive initial-value problem

$$\begin{gathered}
x'' + \sum_{i=1}^{N} a_i(t) F_i(t, x, x', x(t-\tau_i), x'(t-\tau_i)) + G(t, x, x') + b(t)H(x) \\
+ \sum_{i=1}^{N} G_i(t, x, x(t-\tau_i)) = 0, \quad t \ge t_0,\ t \ne t_k, \\
x(t_0) = \psi(t), \quad t_0 - \tau_N \le t \le t_0,\ x'(t_0) = y_0.
\end{gathered} \tag{1.8}$$

Letting $x'(t) = y(t)$, problem (1.8) can be rewritten as

$$\begin{gathered}
x' = y, \\
y' = -\sum_{i=1}^{N} a_i(t) F_i(t, x, y, x(t-\tau_i), y(t-\tau_i)) - G(t, x, y) \\
- b(t)H(x) - \sum_{i=1}^{N} G_i(t, x, x(t-\tau_i)) = 0, \quad t \ge t_0, \\
x(t_0) = \psi(t), \quad t_0 - \tau_N \le t \le t_0,\ y(t_0) = y_0,
\end{gathered} \tag{1.9}$$

where $0 < \tau_1 < \tau_2 < \dots < \tau_N$, $x(t) : [t_0 - \tau, +\infty) \to \mathbb{R}$, $\tau = \max\{\tau_i\}$, $F_i(t, 0, 0, 0, 0) = 0$, $F_i \in C(\mathbb{R}^+ \times \mathbb{R}^4, \mathbb{R})$, $\mathbb{R}^+ = [0, \infty)$, $G, G_i \in C(\mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R})$, $G(t, 0, 0) = 0$, $G_i(t, 0, 0) = 0$, $H \in C(\mathbb{R}, \mathbb{R})$, $H(0) = 0$, $a_i : [t_0, T] \to \mathbb{R}$ are piecewise continuous functions, $b \in C[\mathbb{R}^+, \mathbb{R}]$, $\{t_k\}_{k=0}^{\infty}$ is a monotone increasing unbounded sequence of real numbers and $\phi, \phi' : [-\tau, 0] \to \mathbb{R}$ have at most a finite number of discontinuity points such that all of them being of the first kind, and are right continuous at these points.

The remaining of this article is structured as follows: In Section 2, two new results with regard to the existence and exponentially stabilization of solutions of (1.8) are given. Section 3 deals with the contributions of the study, and the conclusion of this article.

2. Existence of solutions and exponentially stabilization

We now construct sufficient conditions for the existence of solutions of the problem (1.8), (2.1), and for exponentially stabilization of the ODE included by (1.8). See Theorems 2.1 and 2.2 below. Hence, for the existence result, we use the following conditions

- (A1) The functions $\phi(t)$ and $\phi'(t)$ are continuous on I_1, $I_1 = [t_0 - \tau_N, t_0]$, except at the most at a finite set Φ of points at which the lateral limits $\phi(t^-)$,

$\phi(t^+)$, $\phi'(t^-)$ and $\phi'(t^+)$ exist and ϕ and ϕ' are right continuous at these points;

(A2) Let a_0, $b_0 \in \mathbb{R}$, $a_0 > 0$, $b_0 > 0$ be such that
$$a_0 = \max_{1\le i\le N}\|a_i(t)\|, |b(t)| \le b_0, \quad \forall t \in \mathbb{R}^+;$$

(A3) Let $\alpha_1, \dots, \alpha_N$ be positive constants such that
$$\begin{gathered} |F_i(t,u,v,w,z)| \le \alpha_i|u|, \quad \forall t\in\mathbb{R}^+,\ \forall u,v,w,z\in\mathbb{R},\\ \sum_{i=1}^N |F_i(\cdot) - F_i(\cdot\cdot)| \le \sum_{i=1}^N \alpha_i|v_1-u_1| \le \alpha_0 N|v_1-u_1|, \end{gathered}$$
where
$$\begin{gathered} F_i(\cdot) = F_i(t,v_1,v_2,v_3,v_4), \quad F_i(\cdot\cdot) = F_i(t,u_1,u_2,u_3,u_4),\\ \alpha_0 = \max\{\alpha_i\}, \quad \forall t\in\mathbb{R}^+,\ \forall u_1,\dots,u_4,v_1,\dots,v_4\in\mathbb{R}; \end{gathered}$$

(A4) Let $\beta_1,\dots,\beta_N$ be positive constants such that
$$\begin{gathered} G_i(t,0,0) = 0, \quad |G_i(t,u,v)| \le \beta_i|u|,\\ \sum_{i=1}^N |G_i(t,u_n,v_n) - G_i(t,u,v)| \le \sum_{i=1}^N \beta_i|u_n-u| \le \beta_0 N|u_n-u|, \end{gathered}$$
where $\beta_0 = \max\{\beta_i : \forall t\in\mathbb{R}^+, \forall u,u_n,v,v_n\in\mathbb{R}\}$;

(A5) $\beta_0 N(T-t_0)^2 < 1$;

(A6) Let g_0, $h_0 \in \mathbb{R}$ be positive constants such that
$$\begin{gathered} G(t,0,0) = 0, \quad |G(t,u,v)| \le g_0|u|, \quad \forall t\in\mathbb{R}^+,\ \forall u,v\in\mathbb{R},\\ |G(t,v_1,v_2) - G(t,u_1,u_2)| \le g_0|v_1-u_1|, \quad \forall t\in\mathbb{R}^+,\ u_1,u_2,v_1,v_2\in\mathbb{R},\\ H(0)=0, \quad |H(u)| \le h_0|u|, \quad \forall u\in\mathbb{R}; \end{gathered}$$

According to (A1)–(A6), the impulses at times t_k, $k\in\mathbb{N}$, satisfy
$$\begin{gathered} x(t_k) = I_k(x(t_k^-)),\\ x'(t_k) = J_k(x'(t_k^-)); \end{gathered} \tag{2.1}$$

(A7) $I_k \in C[\mathbb{R},\mathbb{R}]$, $I_k(0)=0$, $J_k\in C[\mathbb{R},\mathbb{R}]$, $J_k(0)=0$, $k\in\mathbb{N}$, and there are non-negative constants c_k, $d_k\in\mathbb{R}$ which allow $I_k = I_k(x)$ and $J_k = J_k(x)$ to be bounded with upper bounds c_k, d_k, i.e.,
$$|I_k(x)| \le c_k, \quad |J_k(x)| \le d_k, \quad \forall k\in\mathbb{N},\ x\in\mathbb{R}.$$

Let $D\subset\mathbb{R}$, which is an open set such that $I = [t_0-\tau_N, T]\subset D$, and $x(t;t_0,\psi,y_0)$ represent the solutions of (1.8), (2.1) through a point (t_0,φ,y_0).

(A8) We assume that
$$\begin{gathered} |a_i(t)| \le a_0, \quad |b(t)| \le b_0,\\ |F_i(t,x_1,y_1,x_2,y_2)| \le \beta_i|f_i(x_2)|\\ |f_i(x_2)| \le \sigma_i|x_2|,\\ |G(t,x_1,y_1)| \le \alpha_0|y_1|, \quad |H(x_1)| \le h_0|x_1|,\\ |G_i(t,x_1,x_2)| \le g_i|x_2|, \quad \forall x_1,y_1,x_2,y_2\in\mathbb{R},\ \forall t\in\mathbb{R}^+ \end{gathered}$$

where

$$a_0 \in \mathbb{R}, \quad a_0 > 0, \quad b_0 \in \mathbb{R}, \quad b_0 > 0, \quad \alpha_0 \in \mathbb{R}, \quad \alpha_0 > 0,$$
$$h_0 \in \mathbb{R}, \quad h_0 > 0, \sigma_i \in \mathbb{R}, \quad \sigma_i > 0, \quad \beta_i \in \mathbb{R}, \quad \beta_i > 0.$$

The first existence result of this article reads as follows.

Theorem 2.1. *If* (A1)–(A5), (A7) *are satisfied, then* (1.8) *with* (2.1) *admits a solution on* I.

Proof. Let the operator $Q_0 : C^1_\Psi(I,\mathbb{R}) \to C^1_\Psi(I,\mathbb{R})$ be defined by

$$Q_0(x) = \begin{cases} \psi(t), \quad \text{if } t \in I_1, \\ \psi(t_0) + (t-t_0)y_0 - \int_{t_0}^t (t-s)\Big[\sum_{i=1}^N a_i(s)F_i(s,x(s),x'(s), \\ x(s-\tau_i), x'(s-\tau_i))\Big]\,\mathrm{d}s \\ -\int_{t_0}^t (t-s)\Big[G(s,x(s),x'(s)) + b(s)H(x(s)) \\ +\sum_{i=1}^N G_i(s,x(s),x(s-\tau_i))\Big]\,\mathrm{d}s, \quad \text{if } t \in I_2, \end{cases}$$

where $I_1 = [t_0 - \tau_N, t_0]$, $I_2 = [t_0, T]$, and Ψ is the (finite) set, which includes the discontinuity points of ψ.

We show that the operator Q_0 has a fixed point. Note that ψ is known a fixed point of the restriction of the operator Q_0 to I_1, $Q_0\big|_{I_1}$. Hence, we will verify that $Q_0\big|_{I_2}$ admits a fixed point.

In the rest of the paper, let

$$F_i(s,x_n(s),\dots,{x'}_n(s-\tau_i)) = F_i(s,x_n(s),{x'}_n(s),x_n(s-\tau_i),{x'}_n(s-\tau_i))$$

and

$$F_i(s,x(s),\dots,x'(s-\tau_i)) = F_i(s,x(s),x'(s),x(s-\tau_i),x'(s-\tau_i))\,.$$

We consider the operator $Q_0 = Q_0\big|_{I_2}$. Then, the operator $Q_0 : C^1(I_2,\mathbb{R}) \to C^1(I_2,\mathbb{R})$ satisfies

$$\begin{aligned} Q_0(x) = \psi(t_0) + (t-t_0)y_0 - \int_{t_0}^t (t-s)[\sum_{i=1}^N a_i(s)F_i(s,x(s),\dots,x'(s-\tau_i))]\,\mathrm{d}s \\ - \int_{t_0}^t (t-s)\Big[G(s,x(s),x'(s)) + b(s)H(x(s)) \\ + \sum_{i=1}^N G_i(s,x(s),x(s-\tau_i))\Big]\,\mathrm{d}s. \end{aligned}$$

We will prove Theorem 2.1 at four steps using the operator Q_0.

Step 1: Operator Q_0 is continuous. Let $\{x_n\}$ be a sequence in $C^1(I_2,\mathbb{R})$ such that x_n tends to x. Then, ${x'}_n \to x'$ converges uniformly in $C^1(I_2,\mathbb{R})$. From Q_0 and (A1)–(A6), we derive

$$\begin{aligned} &|Q_0(x_n)(t) - Q_0(x)(t)| \\ &\le (T-t_0)\int_{t_0}^t \Big[\sum_{i=1}^N |a_i(s)||F_i(s,x_n(s),\dots,x'_n(s-\tau_i)) \\ &\quad - F_i(s,x(s),\dots,x'(s-\tau_i))|\Big]\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+ (T-t_0)\int_{t_0}^{t}[|G(s,x_n(s),x_n'(s)) - G(s,x(s),x'(s))|]\,\mathrm{d}s \\
&+ (T-t_0)\int_{t_0}^{t}|b(s)|[|H(x_n(s)) - H(x(s))|]\,\mathrm{d}s \\
&+ (T-t_0)\int_{t_0}^{t}\Big[\sum_{i=1}^{N}|G_i(s,x_n(s),x_n(s-\tau_i)) - G_i(s,x(s),x(s-\tau_i))|\Big]\,\mathrm{d}s \\
\le{}& (T-t_0)a_0\int_{t_0}^{t}[\sum_{i=1}^{N}\alpha_i|x_n(s)-x(s)|]\,\mathrm{d}s \\
&+ (T-t_0)g_0\int_{t_0}^{t}|x_n(s)-x(s)|\,\mathrm{d}s + (T-t_0)\int_{t_0}^{t}|b(s)|[|H(x_n(s)) - H(x(s))|]\,\mathrm{d}s \\
&+ (T-t_0)\int_{t_0}^{t}[\sum_{i=1}^{N}\beta_i|x_n(s)-x(s)|]\,\mathrm{d}s \\
\le{}& \alpha_0 a_0 N(T-t_0)^2\|x_n - x\| + g_0(T-t_0)^2\|x_n-x\| \\
&+ (T-t_0)b_0\int_{t_0}^{t}[\|H(x_n(s)) - H(x(s))|]\,\mathrm{d}s + \beta_0 N(T-t_0)^2\|x_n-x\|.
\end{aligned}
$$

Since H is a continuous function, the last inequality above allows that

$$|Q_0(x_n)(t) - Q_0(x)(t)| \to 0 \text{ as } n\to\infty.$$

Hence, the operator Q_0 is continuous.

Step 2: Operator Q_0 maps bounded sets into bounded sets. Let $\omega \ge 0$, $\omega\in\mathbb{R}$, and $\upsilon \ge 0$. We define

$$B_\omega = \{y \in C^1(I_2,\mathbb{R}) : \|y\| \le \omega\},\ x\in B_\omega,$$

which implies that $\|Q_0(x)\| \le \upsilon$. Since $t\in I_2$, by the definition of Q_0 and (A1)–(A6), we derive that

$$
\begin{aligned}
&|Q_0(x)(t)| \\
&\le \|\psi(t_0)\| + (t-t_0)|y_0| + \int_{t_0}^{t}(t-s)\sum_{i=1}^{N}|a_i(s)|\,|F_i(s,x(s),\dots,x'(s-\tau_i))|\,\mathrm{d}s \\
&\quad + \int_{t_0}^{t}(t-s)|G(s,x(s),x'(s))|\,\mathrm{d}s + \int_{t_0}^{t}(t-s)b(s)|H(x(s))|\,\mathrm{d}s \\
&\quad + \int_{t_0}^{t}(t-s)\sum_{i=1}^{N}|G_i(s,x(s),x(s-\tau_i))|\,\mathrm{d}s \\
&\le \|\psi(t_0)\| + (T-t_0)|y_0| + (T-t_0)a_0\int_{t_0}^{t}\sum_{i=1}^{N}|F_i(s,x(s),\dots,x'(s-\tau_i))|\,\mathrm{d}s \\
&\quad + (T-t_0)\int_{t_0}^{t}|G(s,x(s),x'(s))|\,\mathrm{d}s + (T-t_0)b_0\int_{t_0}^{t}|H(x(s))|\,\mathrm{d}s \\
&\quad + (T-t_0)\int_{t_0}^{t}\sum_{i=1}^{N}|G_i(s,x(s),x(s-\tau_i))|\,\mathrm{d}s \\
&\le \|\psi(t_0)\| + (T-t_0)^2(\alpha_0 a_0 N + g_0 + h_0 b_0) + (T-t_0)|y_0| + \beta_0 N(T-t_0)^2\|x\|
\end{aligned}
$$

$$\leq \|\psi(t_0)\| + (T-t_0)^2(\alpha_0 a_0 N + g_0 + h_0 b_0) + (T-t_0)|y_0| + \omega\beta_0 N (T-t_0)^2.$$

Let

$$\upsilon = \|\psi(t_0)\| + (T-t_0)^2(\alpha_0 a_0 N + g_0 + h_0 b_0 + \omega\beta_0 N) + (T-t_0)|y_0|.$$

From the above inequality it follows that $\|Q_0(x)\| \leq \upsilon$, which completes Step 2.

Step 3: Operator Q_0 converts bounded sets to the equicontinuous sets, which are included in the space $C^1(I_2, \mathbb{R})$.

Let θ_1, $\theta_2 \in I_2$ with $\theta_1 < \theta_2$ and consider $x \in B_\omega$. Then, from the definition of Q_0 and (A1)–(A6), we have

$$\begin{aligned}
&|Q_0(x)(\theta_2) - Q_0(x)(\theta_1)| \\
&\leq |y_0(\theta_2-\theta_1)| + \Big| \int_{t_0}^{\theta_2} (\theta_2 - s) \sum_{i=1}^N a_i(s) F_i(s, x(s), \dots, x'(s-\tau_i))\, \mathrm{d}s \\
&\quad - \int_{t_0}^{\theta_1} (\theta_1 - s) \sum_{i=1}^N a_i(s) F_i(s, x(s), \dots, x'(s-\tau_i))\, \mathrm{d}s \Big| \\
&\quad + \Big| \int_{t_0}^{\theta_2} (\theta_2 - s) G(s, x(s), x'(s))\, \mathrm{d}s - \int_{t_0}^{\theta_1} (\theta_1 - s) G(s, x(s), x'(s))\, \mathrm{d}s \Big| \\
&\quad + \Big| \int_{t_0}^{\theta_2} (\theta_2 - s) b(s) H(x(s))\, \mathrm{d}s - \int_{t_0}^{\theta_1} (\theta_1 - s) b(s) H(x(s))\, \mathrm{d}s \Big| \\
&\quad + \Big| \int_{t_0}^{\theta_2} (\theta_2 - s) \sum_{i=1}^N G_i(s, x(s), x(s-\tau_i))\, \mathrm{d}s \\
&\quad - \int_{t_0}^{\theta_1} (\theta_1 - s) \sum_{i=1}^N G_i(s, x(s), x(s-\tau_i))\, \mathrm{d}s \Big| \\
&\leq |y_0|(\theta_2 - \theta_1) + (\theta_2 - \theta_1) \int_{t_0}^{\theta_2} \sum_{i=1}^N |a_i(s)|\, |F_i(s, x(s), \dots, x'(s-\tau_i))|\, \mathrm{d}s \\
&\quad + (\theta_2 - \theta_1) \int_{t_0}^{\theta_2} |G(s, x(s), x'(s))|\, \mathrm{d}s + (\theta_2 - \theta_1) \int_{t_0}^{\theta_2} |b(s)| \quad |H(x(s))|\, \mathrm{d}s \\
&\quad + (\theta_2 - \theta_1) \int_{t_0}^{\theta_2} \sum_{i=1}^N |G_i(s, x(s), x(s-\tau_i))|\, \mathrm{d}s,
\end{aligned}$$

which converges to 0 as $\theta_2 \to \theta_1$. As for the next step, the equicontinuity for the cases $\theta_1 < \theta_2 \leq 0$ and $\theta_1 \leq 0 \leq \theta_2$ can be confirmed similarly.

The outcomes of the steps above allow that $Q_0(B_\omega)$ be bounded and equicontinuous for all $\upsilon > 0$. Hence, using the Ascoli-Arzela theorem, it follows that $Q_0(B_\omega)$ is relatively compact and therefore the operator Q_0 is compact. This completes the proof of Step 3.

Step 4: We prove that the set

$$\Lambda(Q_0) = \{x \in C^1(I_2, \mathbb{R}) : x = \lambda Q_0(x), 0 < \lambda < 1\}$$

is bounded. Let $x \in \Lambda(Q_0)$. Then

$$x = \lambda Q_0(x), \quad 0 < \lambda < 1.$$

According to this equality and Q_0, for all $t \in I_2$, we have

$$\begin{aligned} x(t) = \lambda Q_0 x(t) &= \lambda\{\psi(t_0) + y_0(t-t_0)\} \\ &\quad - \lambda \int_{t_0}^t (t-s)\sum_{i=1}^N a_i(s)F_i(s,x(s),\dots,x'(s-\tau_i))\,\mathrm{d}s \\ &\quad - \lambda \int_{t_0}^t (t-s)[G(s,x(s),x'(s)) + b(s)H(x(s))]\,\mathrm{d}s \\ &\quad - \lambda \int_{t_0}^t (t-s)[\sum_{i=1}^N G_i(s,x(s),x(s-\tau_i))]\,\mathrm{d}s. \end{aligned}$$

Using the above equality, $0 < \lambda < 1$, and the outcomes of Step 2, we obtain

$$\begin{aligned} |x(t)| &\le (T-t_0)^2(\alpha_0 a_0 N + g_0 + h_0 b_0) \\ &\quad + \|\psi(t_0)\| + (T-t_0)|y_0| + \beta_0 N (T-t_0)^2 \|x\|. \end{aligned}$$

Then

$$\|x\| \le \frac{\|\psi(t_0)\| + (T-t_0)^2(\alpha_0 a_0 N + g_0 + h_0 b_0) + (T-t_0)|y_0|}{1 - \beta_0 N (T-t_0)^2}.$$

This result verifies that $\Lambda(Q_0)$ is bounded.

From steps four steps above, and the Schaefer fixed point theorem, the operator Q_0 has a fixed point, call it $y_1(t)$. In the light of this result,

$$x_1(t) = \begin{cases} \psi(t), & t_0 \le t \le t_0 - \tau_N \\ y_1(t), & t_0 \le t \le T \end{cases}$$

is a solution of (1.8), for all $t \in I$.

Let $I_1 = [t_1 - \tau_N, T]$. Consider the impulsive problem

$$\begin{gathered} x'' + \sum_{i=1}^N a_i(t)F_i(t,x,x',x(t-\tau_i),x'(t-\tau_i)) + G(t,x,x') \\ + b(t)H(x) + \sum_{i=1}^N G_i(t,x,x(t-\tau_i)) = 0, \quad t_1 < t \le T, \\ x = x_1(t), \quad t_1 - \tau_N \le t < t_1, \\ x(t_1) = I_1(x(t_1^-)), \quad x'(t_1) = J_1(x'(t_1^-)) \end{gathered} \tag{2.2}$$

and the operator $Q_1 : C^1_{K_1}(I_1,\mathbb{R}) \to C^1_{K_1}(I_1,\mathbb{R})$ defined by

$$Q_1(x)(t) = \begin{cases} x_1(t), \quad \text{if } t_1 - \tau_N \le t < t_1, \\ I_1(x(t_1^-)) + J_1(x'(t_1^-))(t-t_1) \\ -\int_{t_1}^t (t-s)\sum_{i=1}^N a_i(s)F_i(s,x(s),\dots,x'(s-\tau_i))\,\mathrm{d}s \\ -\int_{t_1}^t (t-s)[G(s,x(s),x'(s)) + b(s)H(x(s))]\,\mathrm{d}s \\ -\int_{t_0}^t (t-s)\sum_{i=1}^N G_i(s,x(s),x(s-\tau_i))\,\mathrm{d}s, \quad \text{if } t_1 \le t \le T, \end{cases}$$

where $K_1 = K_1(t_1) \subset \{t_1\} \cup \Psi$ including t_1.

We note that $x_1(t)$ is a fixed point of the restriction of Q_1 to $[t_1 - \tau_N, t_1)$, $N_1\big|_{[t_1-\tau_N,t_1)}$. Then, we will show that $Q_1\big|_{[t_1,T]}$ has a fixed point. Let $I_3 = [t_1, T]$.

We rename $Q_1\big|_{I_3}$ and represent it by the operator Q_1,

$$Q_1 : C^1(I_3,\mathbb{R}) \to C^1(I_3,\mathbb{R})$$

such that

$$\begin{aligned}
Q_1(x)(t) &= x(t_1) + x'(t_1)(t-t_1) \\
&\quad - \int_{t_0}^{t} (t-s) \sum_{i=1}^{N} a_i(s) \sum_{i=1}^{N} a_i(s) F_i(s, x(s), \dots, x'(s-\tau_i))\,\mathrm{d}s \\
&\quad - \int_{t_0}^{t} (t-s)[G(s,x(s),x'(s)) + b(s)H(x(s))]\,\mathrm{d}s \\
&\quad - \int_{t_0}^{t} (t-s) \sum_{i=1}^{N} G_i(s, x(s), x(s-\tau_i))\,\mathrm{d}s.
\end{aligned}$$

Following Steps 1–4 above, and using (A7), we easily obtain that Q_1 has a fixed point, call it $y_2(t)$. Hence, we have

$$x_2(t) = \begin{cases} x_1(t), & t \in [t_0-\tau_N, t_1), \\ y_2(t), & t \in [t_1, T] \end{cases} = \begin{cases} \psi(t), & t \in I_1, \\ y_1(t), & t \in [t_0, t_1), \\ y_2(t), & t \in I_3 \end{cases}$$

is a solution of the impulsive problem (2.2) on $I = [t_0 - \tau_N, T] \subset D$.

Let $\Upsilon_k = [t_k - \tau_N, T]$. Repeating a similar way as the above, for $t = t_k$, $k \in \mathbb{N}$, we now take into account the impulsive problem

$$\begin{gathered}
x'' + \sum_{i=1}^{N} a_i(t) F_i(t, x, x', x(t-\tau_i), x'(t-\tau_i)) + G(t,x,x') \\
+ b(t)H(x) + \sum_{i=1}^{N} G_i(t, x, x(t-\tau_i)) = 0, \quad t_k < t \le T, \\
x = x_k(t), \quad t_k - \tau_N \le t < t_k, \\
x(t_k) = I_k(x(t_k^-)), \quad x'(t_k) = J_k(x'(t_k^-))
\end{gathered} \tag{2.3}$$

and the operator

$$Q_k : C^1_{K_k}(\Upsilon_k, \mathbb{R}) \to C^1_{K_k}(\Upsilon_k, \mathbb{R}),$$

defined by

$$Q_k(x)(t) = \begin{cases} x_k(t), \quad \text{if } t_k - \tau_N \le t < t_k, \\
I_k(x(t_k^-)) + J_k(x'(t_k^-))(t-t_k) \\
- \int_{t_0}^{t} (t-s) \sum_{i=1}^{N} a_i(s) F_i(s, x(s), \dots, x'(s-\tau_i))\,\mathrm{d}s \\
- \int_{t_k}^{t} (t-s)[G(s,x(s),x'(s)) + b(s)H(x(s))]\,\mathrm{d}s \\
- \int_{t_k}^{t} (t-s) \sum_{i=1}^{N} G_i(s,x(s),x(s-\tau_i))\,\mathrm{d}s, \quad \text{if } t_k \le t \le T,
\end{cases}$$

where $K_k \subset \{t_1, t_2, \dots, t_k\} \cup \Psi$. For $i = 1, 2, \dots, k-1$, we note that $t_k \in K_k$. However, whether $t_i \in K_k$, for example, depends upon to the magnitude of the delay τ_N. We write that $Q_k = Q_k\big|_{[t_k,T]}$ and have that

$$Q_k : C^1([t_k,T],\mathbb{R}) \to C^1([t_k,T],\mathbb{R}),$$

is defined by

$$\begin{aligned}Q_k(x)(t) &= x(t_k) + x'(t_k)(t-t_k) - \int_{t_0}^{t} (t-s) \sum_{i=1}^{N} a_i(s) F_i(s, x(s), \dots, x'(s-\tau_i))\, \mathrm{d}s \\ &\quad - \int_{t_0}^{t} (t-s)[G(s,x(s),x'(s)) + b(s)H(x(s))]\, \mathrm{d}s \\ &\quad - \int_{t_0}^{t} (t-s) \sum_{i=1}^{N} G_i(s, x(s), x(s-\tau_i))\, \mathrm{d}s.\end{aligned}$$

Then, using the four steps above, and using (A7), we conclude that Q_k has a fixed point, call it $y_k(t)$. Then

$$x_k(t) = \begin{cases} x_{k-1}(t), & t_0 - \tau_N, t_{k-1}), \\ y_k(t), & t \in [t_{k-1}, T] \end{cases} = \begin{cases} \psi(t), & t \in I_1, \\ y_1(t), & t \in [t_0, t_1), \\ y_2(t), & t \in [t_1, t_2), \\ \dots \\ y_k(t), & t \in [t_{k-1}, T] \end{cases}$$

is a solution of (2.3) on I_1.

Next, in the same way as above, it follows that

$$x(t) = \begin{cases} x_m(t), & t_0 - \tau_N, t_m), \\ y_{m+1}(t), & t \in [t_m, T] \end{cases} = \begin{cases} \psi(t), & t \in I_1, \\ y_1(t), & t \in [t_0, t_1), \\ y_2(t), & t \in [t_1, t_2), \\ \dots \\ y_k(t), & t \in [t_{k-1}, t_k), \\ \dots \\ y_{m+1}(t), & t \in [t_m, T] \end{cases}$$

is a solution of (1.8) satisfying (2.1) on I, where $m = \max\{k \in \mathbb{N} : t_k \le T\}$. The proof of Theorem 2.1 is complete. $\square$

The second result of this article reads as follows.

Theorem 2.2. *Let* (A1), (A7), (A8) *be satisfied and*

$$\sum_{i=1}^{N} \tau_i (g_i + a_0 \sigma_i) < \exp\big[-\sum_{i=1}^{N} (\tau_i K_2)\big] \tag{2.4}$$

with

$$K_2 = \max\big\{\sum_{i=1}^{N} (g_i + a_0 \sigma_i) + b_0 h_0,\ 1 + \alpha_0\big\}.$$

Then the zero solution of (1.8) *is exponentially stabilized by impulses.*

Proof. Assume that (2.4) is satisfied. Then by (2.4) there exist $\bar{\alpha} \in \mathbb{R}$, $\bar{\alpha} > 0$, and $\bar{\ell} \ge \sum_{i=1}^{N} (\tau_i)$ such that

$$\sum_{i=1}^{N} \tau_i (g_i + a_0 \sigma_i) \le \exp\Big[-\bar{\alpha}\Big(\bar{\ell} + \sum_{i=1}^{N} (\tau_i)\Big)\Big] \exp[-K_2 \bar{\ell}]. \tag{2.5}$$

Let $\bar{\alpha} > 0$ and $\bar{\ell} \geq \sum_{i=1}^{N}(\tau_i)$ be as in (2.5). Consider the sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $t_0 < t_1 < \ldots < t_k < \ldots$, with $\lim_{k\to\infty} t_k = \infty$ and

$$\sum_{i=1}^{N}(\tau_i) \leq t_k - t_{k-1} \leq \bar{\ell}.$$

Let

$$I_k(u) = \bar{d}_k u, J_k(u) = \bar{d}_k u,$$
$$\bar{d}_k = \bar{p}_k - \sum_{i=1}^{N}\tau_i(g_i + a_0\sigma_i),$$
$$\bar{p}_k = \exp\Big[-\bar{\alpha}(t_{k+1} - t_k) - \bar{\alpha}\sum_{i=1}^{N}\tau_i(g_i + a_0\sigma_i)\Big]\exp[-K_2(t_{k+1} - t_k)].$$

Then, $\bar{d}_k \geq 0$, $\bar{d}_k \in \mathbb{R}$, since $\bar{p}_k \geq \sum_{i=1}^{N}\tau_i(g_i + a_0\sigma_i)$ by (2.5).

For a given $\varepsilon > 0$, let

$$\delta = \frac{\varepsilon}{1 + \sum_{i=1}^{N}\tau_i(g_i + a_0\sigma_i)}\exp[-(\bar{\alpha} + K_2)(t_1 - t_0)].$$

We now prove that, for every solution $x(t) = x(t; t_0, \psi, y_0)$ of (1.8) satisfying (2.1), if

$$\|\psi(t_0)\| + |y(t_0)| \leq \delta,$$

then

$$|x(t)| + |y(t)| \leq \varepsilon\exp[-\alpha(t - t_0)], t_0 \leq t \leq T.$$

Let $z_t = (x_t, y_t)$ be a solution of (1.8). We define a new Lyapunov-Krasovskii functional

$$\begin{aligned} W_0(z_t) &= W_0(t) \\ &= |x| + |y| + \sum_{i=1}^{N} g_i\int_{t-\tau_i}^{t}|x(s)|\,\mathrm{d}s \\ &\quad + \sum_{i=1}^{N}\gamma_i\int_{t-\tau_i}^{t}|a_i(s+\tau_i)|\,|f_i(x(s))|\,\mathrm{d}s, \end{aligned} \tag{2.6}$$

where $\gamma_i \in \mathbb{R}$, $\gamma_i > 0$ are arbitrary constants.

Firstly, we show that (2.6) fulfill the following steps.

Step 5: From (2.6), it follows that $W_0(t) \geq |x| + |y|$.

Step 6: According to (A8), from (2.6), we obtain

$$\begin{aligned} W_0(t) &\leq |x| + |y| + \sum_{i=1}^{N}(\tau_i g_i)\|x_t\| + \sum_{i=1}^{N}\gamma_i\int_{t-\tau_i}^{t}(a_0\sigma_i)|x(s)|\,\mathrm{d}s \\ &\leq |x| + |y| + \sum_{i=1}^{N}(\tau_i g_i)\|x_t\| + \sum_{i=1}^{N}(a_0\gamma_i\tau_i\sigma_i)\|x_t\| \\ &\leq (1 + \sum_{i=1}^{N}\tau_i(g_i + a_0\gamma_i\sigma_i))\|x_t\| + |y|, \end{aligned}$$

where $\|x_t\| = \sup_{t-\tau_N\leq s\leq t}|x(s)|$.

Step 7: Let $t_0 < t < t_1$. Calculating the right upper derivative of $W_0(t)$ along solutions of (1.8) and using (A1), (A7), and (A8), we have

$$
\begin{aligned}
&\frac{d}{dt}W_0(t)\\
&= x' \operatorname{sgn} x + y' \operatorname{sgn} y + \sum_{i=1}^{N}(g_i)|x| - \sum_{i=1}^{N}(g_i)|x(t-\tau_i)|\\
&\quad + \sum_{i=1}^{N}\gamma_i|a_i(t+\tau_i)| \quad |f_i(x)| - \sum_{i=1}^{N}\gamma_i|a_i(t)| \quad |f_i(x(t-\tau_i))|\\
&= y \operatorname{sgn} x - \operatorname{sgn} y\Big[\sum_{i=1}^{N} a_i(t)F_i(t,x,y,x(t-\tau_i),y(t-\tau_i)) + G(t,x,y)\Big]\\
&\quad - \operatorname{sgn} y\Big[b(t)H(x) + \sum_{i=1}^{N} G_i(t,x,x(t-\tau_i))\Big]\\
&\quad + \sum_{i=1}^{N}(g_i)|x| - \sum_{i=1}^{N}(g_i)|x(t-\tau_i)|\\
&+ \sum_{i=1}^{N}\gamma_i|a_i(t+\tau_i)|\,|f_i(x)| - \sum_{i=1}^{N}\gamma_i|a_i(t)|\,|f_i(x(t-\tau_i))|\\
&\le |y| + \sum_{i=1}^{N}|a_i(t)| \quad |F_i(t,x,y,x(t-\tau_i),y(t-\tau_i))| + |G(t,x,y)|\\
&\quad + |b(t)| \quad |H(x)| + \sum_{i=1}^{N}|G_i(t,x,x(t-\tau_i))|\\
&\quad + \sum_{i=1}^{N}(g_i)|x| - \sum_{i=1}^{N}(g_i)|x(t-\tau_i)|\\
&\quad + \sum_{i=1}^{N}\gamma_i|a_i(t+\tau_i)|\,|f_i(x)| - \sum_{i=1}^{N}\gamma_i|a_i(t)|\,|f_i(x(t-\tau_i))|\\
&\le (1+\alpha_0)|y| + \sum_{i=1}^{N}\gamma_i|a_i(t)|\,|f_i(x(t-\tau_i))|\\
&\quad + (b_0h_0)|x| + \sum_{i=1}^{N}(g_i)|x(t-\tau_i)| + \sum_{i=1}^{N}(g_i)|x| - \sum_{i=1}^{N}(g_i)|x(t-\tau_i)|\\
&\quad + \sum_{i=1}^{N}\gamma_i|a_i(t+\tau_i)|\,|f_i(x)| - \sum_{i=1}^{N}\gamma_i|a_i(t)| \quad |f_i(x(t-\tau_i))|.
\end{aligned}
$$

Let $\gamma_1 = \gamma_2 = \cdots = \gamma_N = 1$. It follows that

$$
\frac{d}{dt}W_0(t) \le \Big(\sum_{i=1}^{N}(g_i + a_0\sigma_i) + b_0h_0\Big)|x| + (1+\alpha_0) \quad |y| \le K_2(|x|+|y|), \tag{2.7}
$$

where

$$K_2 = \max\Big\{\sum_{i=1}^{N}(g_i + a_0\sigma_i) + b_0 h_0,\ 1+\alpha_0\Big\}.$$

From (2.6) and (2.7), we derive that $\frac{d}{dt}W_0(t) \le K_2 W_0(t)$. Integrating this inequality, we have

$$W_0(t) \le W_0(t_0)\exp[K_2(t-t_0)], \quad t_0 < t < t_1. \tag{2.8}$$

According to Steps 5 and 6, (2.8), and the above inequalities, we obtain

$$\begin{aligned}
|x| + |y| &\le W_0(t)\\
&\le W_0(t_0)\exp[K_2(t-t_0)]\\
&\le W_0(t_0)\exp[K_2(t_1-t_0)]\\
&\le \Big(1 + \sum_{i=1}^{N}\tau_i(g_i + a_0\sigma_i)\Big)[\|x_{t_0}\| + |y(t_0)|]\exp[K_2(t_1-t_0)]\\
&\le \Big(1 + \sum_{i=1}^{N}\tau_i(g_i + a_0\sigma_i)\Big)\delta\exp[K_2(t_1-t_0)]\\
&\le \varepsilon\exp[-\bar{\alpha}(t_1-t_0)]\\
&\le \varepsilon\exp[-\bar{\alpha}(t-t_0)].
\end{aligned}$$

Hence, we conclude that

$$|x| + |y| \le \varepsilon\exp[-\bar{\alpha}(t-t_0)], \quad t \in (t_0, t_1). \tag{2.9}$$

Since the right continuity of x and x' holds on $[t_0, t_1)$, from (2.9), it follows that

$$|x| + |y| \le \varepsilon\exp[-\bar{\alpha}(t-t_0)], \quad t \in [t_0, t_1).$$

Secondly, we will now show that (2.6) satisfies the following step:

Step 7: For $t_1 < t < t_2$ we have the results in Steps 5 and 6. We repeat similar calculations as in the lines above. When we consider

$$\sum_{i=1}^{N}(\tau_i) \le t_k - t_{k-1} \le \bar{\ell}$$

and the definition of $\bar{p}_k$. It follows that

$$\sum_{i=1}^{N}(\tau_i) \le t_1 - t_0 \le \bar{\ell},$$

$$\bar{d}_1 = \bar{p}_1 - \sum_{i=1}^{N}\tau_i(g_i + a_0\sigma_i).$$

Hence, repeating similar calculations as above, for $t_1 < t < t_2$, we obtain

$$\begin{aligned}
W_0(t) &\le W_0(t_1^+)\exp[K_2(t_2-t_1)]\\
&\le \Big(|x(t_1^+)| + |y(t_1^+)| + \sum_{i=1}^{N} g_i\int_{t_1-\tau_i}^{t_1}|x(s)|\,\mathrm{d}s\\
&\quad + \sum_{i=1}^{N}\int_{t_1-\tau_i}^{t_1}|a_i(s+\tau_i)|\,|f_i(x(s))|ds\Big)\exp[K_2(t_2-t_1)]
\end{aligned}$$

$$
\begin{aligned}
&= \Big(|x(t_1)| + |y(t_1)| + \sum_{i=1}^{N} g_i \int_{t_1-\tau_i}^{t_1} |x(s)|\,\mathrm{d}s \\
&\quad + \sum_{i=1}^{N} \int_{t_1-\tau_i}^{t_1} |a_i(s+\tau_i)|\,|f_i(x(s))|ds\Big) \exp[K_2(t_2-t_1)] \\
&= \Big(|I_1(x(t_1^-))| + |J_1(y(t_1^-))| + \sum_{i=1}^{N} g_i \int_{t_1-\tau_i}^{t_1} |x(s)|\,\mathrm{d}s \\
&\quad + \sum_{i=1}^{N} \int_{t_1-\tau_i}^{t_1} |a_i(s+\tau_i)| \quad |f_i(x(s))|ds\Big) \exp[K_2(t_2-t_1)] \\
&\le \Big(\bar{d}_1[|x(t_1^-)| + |y(t_1^-)|] + \sum_{i=1}^{N} g_i \int_{t_1-\tau_i}^{t_1} |x(s)|\,\mathrm{d}s \\
&\quad + \sum_{i=1}^{N} \int_{t_1-\tau_i}^{t_1} |a_i(s+\tau_i)|\,|f_i(x(s))|ds\Big) \exp[K_2(t_2-t_1)] \\
&\le \bar{d}_1 \sup_{t_1-\tau_i\le t\le t_1} [|x(t)| + |y(t)|] \exp[K_2(t_2-t_1)] \\
&\quad + \sup_{t_1-\tau_i\le t\le t_1} |x| \sum_{i=1}^{N} (\tau_i g_i) \exp[K_2(t_2-t_1)] \\
&\quad + \sup_{t_1-\tau_i\le t\le t_1} |x| \sum_{i=1}^{N} (a_0\tau_i\sigma_i) \exp[K_2(t_2-t_1)] \\
&\le (\bar{d}_1 + \sum_{i=1}^{N} \tau_i(g_i + a_0\sigma_i)) \sup_{t_1-\tau_i\le t\le t_1} [|x(t)| + |y(t)|] \exp[K_2(t_2-t_1)] \\
&\le \Big(\bar{d}_1 + \sum_{i=1}^{N} \tau_i(g_i + a_0\sigma_i)\Big) \varepsilon \exp[-\bar{\alpha}(t_1 - t_0 + \sum_{i=1}^{N}(\tau_i))] \\
&\quad \times \exp[K_2(t_2-t_1)] \\
&= \bar{p}_1 \varepsilon \exp\big[-\bar{\alpha}(t_1 - t_0 + \sum_{i=1}^{N}(\tau_i))\big] \exp[K_2(t_2-t_1)] \\
&\le \varepsilon \exp[-\bar{\alpha}(t_2-t_0)] \\
&\le \varepsilon \exp[-\bar{\alpha}(t-t_0)].
\end{aligned}
$$

From the calculations above, we have

$$|x| + |y| \le W_0(t) \le \varepsilon \exp[-\bar{\alpha}(t-t_0)], \quad t \in (t_1, t_2).$$

It follows that

$$|x| + |y| \le \varepsilon \exp[-\bar{\alpha}(t-t_0)], \quad t \in (t_1, t_2). \tag{2.10}$$

From the right continuity of x and x' on $[t_0, t_1)$, inequality (2.10) holds for $t \in [t_1, t_2)$, i.e., we have

$$|x| + |y| \le \varepsilon \exp[-\bar{\alpha}(t-t_0)], t \in [t_1, t_2).$$

From this results, we obtain

$$|x| + |y| \le \varepsilon \exp[-\bar{\alpha}(t-t_0)], t \in [t_0, t_2).$$

Repeating these calculations, for $k \in \mathbb{N}$, we arrive at

$$|x| + |y| \le \varepsilon \exp[-\bar{\alpha}(t - t_0)], t \in [t_0, t_k).$$

Then, we conclude that

$$|x| + |y| \le \varepsilon \exp[-\bar{\alpha}(t - t_0)], t \ge t_0.$$

Then, the proof of Theorem 2.2 is complete. □

3. Conclusion

The new results in this paper, Theorems 2.1 and 2.2, are derived from concise derivations and provide strong conditions. The impulsive problem (1.8) includes N constant delays and has a more general nonlinear form than the ones found in the literature. Theorems 2.1 and 2.2 extend earlier results from particular cases to general multiple-delays equations. It is expected naturally that Theorems 2.1 and 2.2 can be extended to some stronger results.

Additionally, we would like to suggest the study of existence and stabilization of nonlinear impulsive differential systems. Also suggest the study of impulsive integro-differential equations of high order and fractional order including multiple constant or variable time delays.

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Sandra Pinelas

Departamento de Ciências Exatas e Engenharia, Academia Militar, Av. Conde Castro Guimarães, 2720-113 Amadora, Portugal.

Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

Email address: `sandra.pinelas@gmail.com`

Osman Tunç
Department of Computer Programing, Baskale Vocational School, Van Yuzuncu Yil University, Campus, 65080 Van-Turkey
Email address: osmantunc89@gmail.com

Erdal Korkmaz
Department of Mathematics, Faculty of Arts and Sciences, Muş Alparslan University, 49250, Muş, Turkey
Email address: korkmazerdal36@hotmail.com

Cemil Tunç
Department of Mathematics, Faculty of Sciences, Van Yuzuncu Yil University, 65080, Campus, Van, Turkey
Email address: cemtunc@yahoo.com