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GROUND STATE SOLUTIONS FOR FRACTIONAL KIRCHHOFF TYPE EQUATIONS WITH CRITICAL GROWTH

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Abstract. We study the nonlinear fractional Kirchhoff problem

$$\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx\Big)(-\Delta)^su+u=f(x,u)+|u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^3,$$
$$u\in H^s(\mathbb{R}^3),$$

where $a,b>0$ are constants, $s(3/4,1)$, $2_s^*=6/(3-2s)$, $(-\Delta)^s$ is the fractional Laplacian. Under some relaxed assumptions on f, we prove the existence of ground state solutions.

1. Introduction

In this article, we consider the fractional Kirchhoff equation with critical growth

$$\begin{gathered}\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx\Big)(-\Delta)^su+u=f(x,u)+|u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^3,\\ u\in H^s(\mathbb{R}^3),\end{gathered} \tag{1.1}$$

where $a,b>0$, the fractional Laplacian $(-\Delta)^s$ is defined by $(-\Delta)^su=\mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u)$, $\mathcal{F}$ is the usual Fourier transform. The function f satisfies the following conditions:

- (A1) $f\in C(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$, there exist constants $C_0>0$ and $q\in(2,2_s^*)$ such that $|f(x,t)|\le C_0(1+|t|^{q-1})$, where $2_s^*=\frac{6}{3-2s}$ is the fractional critical Sobolev exponent.
- (A2) $f(x,t)\ge 0$ for $t\ge 0$ and $f(x,t)=o(|t|)$, $|t|\to 0$, uniformly on $\mathbb{R}^3$;
- (A3) For any $r>0$ and $\tau\in\mathbb{R}\backslash\{0\}$, f satisfies
$$\Big[\frac{f(x,\tau)}{\tau^3}-\frac{f(x,r\tau)}{(r\tau)^3}\Big]\operatorname{sign}(1-r)+\frac{|1-r^2|}{(r\tau)^2}\ge 0,\quad \forall x\in\mathbb{R}^3;$$
- (A4) There exists an open set $\Omega\subset\mathbb{R}^3$ satisfying $0\in\Omega$ and
$$\lim_{|t|\to\infty}\frac{F(x,t)}{|t|^{\frac{4s}{3-2s}}}=+\infty,\quad \forall x\in\Omega,$$
where $F(x,t)=\int_0^t f(x,s)ds$.

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When $s = 1$, equation (1.1) reduces to the Kirchhoff equation

$$-\Big(a + b\int_{\mathbb{R}^3} |\nabla u|^2\Big)\Delta u + u = f(x,u) + u^5. \tag{1.2}$$

Li and Ye [15] studied problem (1.2), under some assumptions on the sign-changing function $f(x,u)$, they proved the existence of positive solutions by variational methods.

In recent years, the Kirchhoff-type problem

$$\begin{gathered}-\Big(a + b\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^n,\\ u \in H^1(\mathbb{R}^n),\end{gathered} \tag{1.3}$$

has attracted much attention, where $V \in C(\mathbb{R}^n,\mathbb{R})$, $f \in C(\mathbb{R}^n \times \mathbb{R},\mathbb{R})$ and $a, b > 0$ are constants. Many results of existence, multiplicity of solutions, ground states and concentration phenomenon for problem (1.3) have been obtained when f satisfies various conditions; we refer the reader to [7, 9, 13, 17, 20, 22, 21, 24, 25]. One usually assumes that $f(x,u)$ is subcritical or superlinear at $u = 0$ and satisfies the Ambrosetti-Rabinowitz type condition:

- (AR) there exists $\mu > 4$ such that $0 < \mu F(x,t) \le tf(x,t)$ for all $t \in \mathbb{R}$, where $F(x,t) = \int_0^t f(x,s)ds$; or that 4-superlinear at $t = \infty$
- (F) $\lim_{|t|\to\infty} \frac{F(x,t)}{t^4} = \infty$ uniformly for $x \in \mathbb{R}^n$; or the following variant convex condition
- (VC) $f(x,t)/|t|^3$ is strictly increasing for $t \in \mathbb{R}\backslash\{0\}$.

Li and Ye [16] proved that (1.3) had a ground state solution in $\mathbb{R}^3$ when $f(x,u) = |u|^{p-1}u$ and $2 < p \le 3$. In this case, neither (AR) or (VC) was satisfied. Guo [12] generalized the result of [16] to (1.3) with general nonlinearity. Guo considered (1.3) with $f(x,u) = f(u)$ and proved that (1.3) had a positive ground state solution if f satisfies

- $f \in C^1(\mathbb{R}^+,\mathbb{R})$, $f(t) = o(t)$ as $t \to 0$;
- $\lim_{t\to+\infty} \frac{f'(t)}{t^4} = 0$;
- $\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty$;
- $\frac{f(t)}{t}$ is strictly increasing in $(0,+\infty)$.

Anello[2] studied the existence and multiplicity of solutions for the nonlocal perturbed Kirchhoff problem

$$\begin{gathered}-\Big(a + b\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)\Delta u = \lambda g(x,u) + f(x,u), \quad \text{in } \Omega,\\ u = 0, \quad \text{on } \partial\Omega,\end{gathered}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, $n > 4$, $a, b, \lambda > 0$, and $f, g : \Omega\times\mathbb{R} \to \mathbb{R}$ are Carathéodory functions, with f subcritical, and g of arbitrary growth.

Fiscella and Valdinoci [10] considered the Kirchhoff type problem

$$\begin{gathered}M\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx\Big)(-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^*-2}u, \text{ in } \Omega,\\ u = 0 \quad \text{in } \mathbb{R}^n\backslash\Omega,\end{gathered} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded set, M and f are two continuous functions. This equation models the nonlocal aspect of tension arising from nonlocal measurements of the fractional length of the string. Under some assumptions on M and f, they showed the existence of non-negative solutions. Autuori, Fiscella and Pucci [3] considered the problem (1.4) and obtained the existence and the asymptotic behavior of non-negative solutions when the Kirchhoff function M could be zero at zero. Ambrosio and Isernia [1] dealt with the fractional Kirchhoff equation

$$\Big(p+q(1-s)\int_{\mathbb{R}^n}|(-\Delta)^{s/2}u|^2dx\Big)(-\Delta)^s u=g(u)\quad \text{in } \mathbb{R}^n,$$

where $g:\mathbb{R}\to\mathbb{R}$ is an odd function satisfying Berestycki-Lions type assumptions. By using minimax arguments, they established a multiplicity result when q is sufficiently small. Zhang, Tang, and Chen [26] studied

$$\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx\Big)(-\Delta)^s u+V(x)u=f(x,u)+\lambda|u|^{p-2}u\ \ \text{in } \mathbb{R}^3,$$

where $a,b>0$, $s\in(3/4,1)$, $p\ge 2_s^*=\frac{6}{3-2s}$, V,f satisfies some conditions, in particular, f satisfies (VC). They showed the equation has a ground state solution and a signed-changing solution. Gu, Tang and Yang [11] considered the fractional Kirchhoff equation

$$\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx\Big)(-\Delta)^s u+V(x)u=Q(x)f(u)+\lambda|u|^{p-2}u\quad \text{in } \mathbb{R}^3,$$

where $a,b>0$, $s\in(\frac{3}{4},1)$, V vanishes at infinity, f satisfies some assumptions, which contains the generalized (AR) condition

$$tf(t)-4F(t)\ge \theta tf(\theta t)-4F(\theta t)$$

for all $t\in\mathbb{R}$ and $\theta\in[0,1]$, where $F(t):=\int_0^t f(s)ds$.

For any dimension $n>2s$, Jin and Liu [14] studied the fractional Kirchhoff equation

$$\Big(a+b\int_{\mathbb{R}^n}|(-\Delta)^{s/2}u|^2dx\Big)(-\Delta)^s u+u=f(u)\quad \text{in } \mathbb{R}^n,$$

with a critical nonlinearity. They proved the existence of solutions without the Ambrosetti-Rabinowitz condition when the parameter b is small and they obtained the asymptotic behavior of solutions as $b\to 0$. Liu, Squassina, Zhang [18] studied the following nonlinear fractional Kirchhoff equation

$$\begin{gathered}\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx\Big)(-\Delta)^s u+V(x)u=f(u)\quad \text{in } \mathbb{R}^n\\ u\in H^s(\mathbb{R}^n),\quad u>0\quad \text{in } \mathbb{R}^n,\end{gathered}\tag{1.5}$$

where $s\in(0,1)$ and $n>2s$. They proved the existence of ground states if f satisfies the following assumptions:

- (A5) $f\in C^1(\mathbb{R}^+,\mathbb{R})$, $f(t)=0$ for all $t\le 0$ and $\lim_{t\to 0}\frac{f(t)}{t}=0$;
- (A6) $\lim_{t\to\infty}\frac{f(t)}{t^{2_s^*-1}}=1$;
- (A7) there are $D>0$ and $2<q<2_s^*$ such that $f(t)\ge t^{2_s^*-1}+Dt^{q-1}$ for any $t\ge 0$, and the potential V satisfies some conditions.

Remark 1.1. It should be pointed out that $f\in C^1$, (A6) and (A7) are very crucial in [18, 14]. Similar to [4], the smoothness condition $f\in C^1$ is used to prove some Pohozaev type identity. In this paper, we only need $f\in C$ and f satisfying $(f_1)-(f_4)$. For example, if we take $f(x,t)=f(t)=(|t|^3+\alpha|t|^{\frac{3}{2}})t$, $0<\alpha\le 8\sqrt{2}$, $\forall t\in\mathbb{R}$, then after some calculations, f satisfies (A1)–(A4). It is easy to verify that f does not satisfy (AR), (VC) and the generalized (AR) condition in [11].

The main result of this paper is described as follows.

Theorem 1.2. *Assume that* (A1)–(A4) *hold. Then for any* $a, b > 0$, *equation* (1.1) *has a ground state solution.*

Throughout this paper, C denotes generic positive constants, which may change from line to line. The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we prove the main results.

2. Variational setting

For $p \in [1, \infty]$, we denote by $\|\cdot\|_p$ the usual norm of the space $L^p(\mathbb{R}^3)$. $B_\sigma(x)$ denotes the open ball in $\mathbb{R}^3$ of radius σ centred at x. We recall some definitions of fractional Sobolev spaces and the fractional Laplacian, for more details, we refer to [8]. For any $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^3)$ is defined as follows

$$H^s(\mathbb{R}^3) = \big\{u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (1 + |\xi|^{2s})|\mathcal{F}u(\xi)|^2 d\xi < \infty\big\}$$

with the norm

$$\|u\|_{H^s} = \Big(\int_{\mathbb{R}^3} (|\mathcal{F}u(\xi)|^2 + |\xi|^{2s}|\mathcal{F}u(\xi)|^2) d\xi\Big)^{1/2}, \tag{2.1}$$

where $\mathcal{F}u$ denotes the Fourier transform of u. By $\mathcal{S}(\mathbb{R}^n)$, we denote the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^n$. For $u \in \mathcal{S}(\mathbb{R}^n)$ and $s \in (0,1)$, $(-\Delta)^s$ is defined by

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u)), \quad \forall \xi \in \mathbb{R}^n.$$

By Plancherel's theorem, we have $|\mathcal{F}u|_2 = |u|_2$, $||\xi|^s \mathcal{F}u|_2 = |(-\Delta)^{s/2}u|_2$. Then by (2.1), we obtain the equivalent norm

$$\|u\|_{H^s} = \Big(\int_{\mathbb{R}^3} (|(-\Delta)^{s/2}u(x)|^2 + |u(x)|^2) dx\Big)^{1/2}.$$

For any fixed constant $a > 0$, $H^s(\mathbb{R}^3)$ can be equipped with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^3} a(-\Delta)^{s/2}u(-\Delta)^{s/2}v \, dx + \int_{\mathbb{R}^3} uv \, dx$$

and the corresponding norm

$$\|u\| = \Big(\int_{\mathbb{R}^3} a|(-\triangle)^{s/2}u|^2 dx + \int_{\mathbb{R}^3} u^2 dx\Big)^{1/2}. \tag{2.2}$$

By Theorem 6.5 in [8], we know that $H^s(\mathbb{R}^3)$ is continuously embedded in $L^q(\mathbb{R}^3)$ for any $q \in [2, 2_s^*]$. It is easy to see that the norm in (2.2) is equivalent to $\|\cdot\|_{H^s}$. For $s \in (0,1)$, the fractional Sobolev space $D^{s,2}(\mathbb{R}^3)$ is defined as

$$D^{s,2}(\mathbb{R}^3) = \big\{u \in L^{2_s^*}(\mathbb{R}^3) : |\xi|^s \mathcal{F}u(\xi) \in L^2(\mathbb{R}^3)\big\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{s,2}} = \Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx\Big)^{1/2} = \Big(\int_{\mathbb{R}^3} |\xi|^{2s}|\mathcal{F}u(\xi)|^2 d\xi\Big)^{1/2}.$$

We define the best Sobolev constant

$$S_s := \inf_{u \in D^{s,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx}{\big(\int_{\mathbb{R}^3} |u|^{2_s^*} dx\big)^{2/2_s^*}} \,. \tag{2.3}$$

It is known that [5] S_s is attained by the functions $\tilde{u}(x) = \kappa\left(\mu^2 + |x - x_0|^2\right)^{-\frac{3-2s}{2}}$, $x \in \mathbb{R}^3$, where $\kappa \in \mathbb{R}\backslash\{0\}$, $\mu > 0$ and $x_0 \in \mathbb{R}^3$ are fixed constants. We define $\bar{u}(x) = \tilde{u}(x)/\|\tilde{u}\|_{2_s^*}$ and let $u^*(x) = \bar{u}(x/S_s^{\frac{1}{2s}})$, then by the [19, Claim 6], u^* is a solution of the problem

$$(-\Delta)^s u = |u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^3. \tag{2.4}$$

and $\|u^*\|_{2_s^*}^{2_s^*} = S_s^{\frac{3}{2s}}$. For any $\varepsilon > 0$, set $U_\varepsilon(x) = \varepsilon^{-\frac{3-2s}{2}} u^*(\frac{x}{\varepsilon})$. Then U_ε is a solution of (2.4) and $\|U_\varepsilon\|_{2_s^*}^{2_s^*} = S_s^{\frac{3}{2s}}$. Fix $r > 0$ and let $\varphi \in C_0^\infty(\mathbb{R}^3, [0,1])$ be such that $\operatorname{supp}\varphi \subset B_{2r}(0)$ and $\varphi(x) = 1$ for $x \in B_r(0)$. By Proposition 3.4 and Proposition 3.6 in [8], we have

$$\|(-\Delta)^{s/2}u\|_2^2 = \int_{\mathbb{R}^3} |\xi|^{2s}|\mathcal{F}u(\xi)|^2 d\xi = \frac{C(s)}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy, \tag{2.5}$$

where

$$C(s) = \Big(\int_{\mathbb{R}^3} \frac{1-\cos(\xi_1)}{|\xi|^{3+2s}} d\xi\Big)^{-1}.$$

Without loss of generality, we assume that $C(s) = 2$.

Let $v_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$. From (2.5), Proposition 21 and Proposition 22 in [19] and (24) in [23], it follows that

$$\int_{\mathbb{R}^3} |(-\Delta)^{s/2}v_\varepsilon(x)|^2 dx \le S_s^{\frac{3}{2s}} + O(\varepsilon^{3-2s}), \tag{2.6}$$

$$\int_{\mathbb{R}^3} |v_\varepsilon(x)|^{2_s^*} dx = S_s^{\frac{3}{2s}} + O(\varepsilon^3), \tag{2.7}$$

$$\int_{\mathbb{R}^3} |v_\varepsilon(x)|^p dx = \begin{cases} O(\varepsilon^{\frac{(3-2s)p}{2}}), & p < \frac{3}{3-2s}, \\ O(\varepsilon^{\frac{(2-p)3+2sp}{2}}|\log\varepsilon|), & p = \frac{3}{3-2s}, \\ O(\varepsilon^{\frac{(2-p)3+2sp}{2}}), & \frac{3}{3-2s} < p < \frac{6}{3-2s}. \end{cases} \tag{2.8}$$

The functional associated with (1.1), $I : H^s(\mathbb{R}^3) \to \mathbb{R}$ is

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\triangle)^{s/2}u|^2 dx\Big)^2 - \int_{\mathbb{R}^3} F(x,u)dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx, \tag{2.9}$$

where $F(x,u) = \int_0^u f(x,t)dt$.

It is easy to see that I is well defined, $I \in C^1(H^s(\mathbb{R}^3,\mathbb{R}))$ and

$$\begin{aligned} \langle I'(u), v\rangle &= \int_{\mathbb{R}^3} (a(-\Delta)^{s/2}u(-\Delta)^{s/2}v + uv)dx \\ &\quad + b\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx \int_{\mathbb{R}^3} (-\Delta)^{s/2}u(-\Delta)^{s/2}v\,dx \\ &\quad - \int_{\mathbb{R}^3} f(x,u)v\,dx - \int_{\mathbb{R}^3} |u|^{2_s^*-2}uv\,dx, \quad \forall v \in H^s(\mathbb{R}^3). \end{aligned} \tag{2.10}$$

A function $u \in H^s(\mathbb{R}^3)$ is a weak solution of (1.1) if for any $v \in H^s(\mathbb{R}^3)$,

$$\begin{aligned} &\Big(a + b\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx\Big)\int_{\mathbb{R}^3} (-\Delta)^{s/2}u(-\Delta)^{s/2}v \\ &= \int_{\mathbb{R}^3} f(x,u)v\,dx + \int_{\mathbb{R}^3} |u|^{2_s^*-2}uv\,dx. \end{aligned}$$

It is clear that the critical points of I are weak solutions of (1.1).

3. Ground state solutions

Lemma 3.1. *Under the assumptions* (A1)*, and* (A2) *we have*

(i) *there exist $\delta, \rho > 0$ such that $\|u\| = \rho$ implies that $I(u) \geq \delta > 0$;*
(ii) *there exists $e \in H^s(\mathbb{R}^3)$ such that $\|e\| > \rho$ implies that $I(e) < 0$.*

Proof. (i) From (A1) and (A2), it follows that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,u)| \leq \varepsilon |u| + C_\varepsilon |u|^{q-1}. \tag{3.1}$$

Then

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\int_{\mathbb{R}^3} |u|^2 dx - \frac{C_\varepsilon}{q}\int_{\mathbb{R}^3} |u|^q dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &\geq \Big(\frac{1}{2} - C\varepsilon\Big)\|u\|^2 - C\|u\|^q - C\|u\|^{2_s^*} \end{aligned}$$

For $\varepsilon \in (0, \frac{1}{2C})$, we can choose ρ and δ such that $I(u) \geq \delta > 0$ for $\|u\| = \rho$.

(ii) By (3.1), we have $F(x,u) \geq -\frac{\varepsilon}{2}|u|^2 - \frac{C_\varepsilon}{q}|u|^q$. Then for $t > 0$ and $u_0 \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned} I(tu_0) \leq \frac{t^2}{2}\|u_0\|^2 + \frac{bt^4}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_0|^2 dx\Big)^2 - \frac{\varepsilon t^2}{2}\int_{\mathbb{R}^3} |u_0|^2 dx \\ - \frac{C_\varepsilon t^q}{q}\int_{\mathbb{R}^3} |u_0|^q dx - \frac{t^{2_s^*}}{2_s^*}\int_{\mathbb{R}^3} |u_0|^{2_s^*} dx, \end{aligned}$$

since $2_s^* > 4$, it follows that $\lim_{t\to+\infty} I(tu_0) = -\infty$. We choose sufficiently large t_0 and set $e = t_0 u_0$ such that $\|e\| > \rho$ and $I(e) < 0$. □

We denote the mountain pass value by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C([0,1], H^s(\mathbb{R}^3)) : \gamma(0) = 0,\ I(\gamma(1)) < 0\}.$$

By Lemma 3.1, there exists a Palais-Smale sequence $\{u_n\} \subset H^s(\mathbb{R}^3)$ for I at the level c (or $(PS)_c$ sequence, for short):

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

Lemma 3.2. *Assume that* (A1)–(A3) *hold. Then*

$$0 < c < c^* := \frac{aS_s}{2} T^{3-2s} + \frac{bS_s^2}{4} T^{6-4s} - \frac{T^3}{2_s^*},$$

where $T > 0$ is the unique maximum point of the function $J(t) = \frac{aS_s}{2} t^{3-2s} + \frac{bS_s^2}{4} t^{6-4s} - \frac{t^3}{2_s^*}$, $t > 0$.

Proof. By (i) of Lemma 3.1, $c > 0$. Let $\gamma_\varepsilon(t) = v_\varepsilon(\cdot/t)$. From (2.9) and (2.5), it follows that

$$\begin{aligned} I(\gamma_\varepsilon(t)) = \frac{at^{3-2s}}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2} v_\varepsilon|^2 dx + \frac{bt^{6-4s}}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2} v_\varepsilon|^2 dx\Big)^2 \\ + \frac{t^3}{2}\int_{\mathbb{R}^3} |v_\varepsilon(x)|^2 dx - \int_{\mathbb{R}^3} F(x, v_\varepsilon(x/t)) dx - \frac{t^3}{2_s^*}\int_{\mathbb{R}^3} |v_\varepsilon(x)|^{2_s^*} dx. \end{aligned} \tag{3.3}$$

By [19, (4.18)], U_ε can be written as (with $x_0 = 0$)

$$U_\varepsilon(x) = \tilde{k}\varepsilon^{-\frac{3-2s}{2}}\Big(\mu^2 + \big|\frac{x}{\varepsilon S_s^{\frac{1}{2s}}}\big|^2\Big)^{-\frac{3-2s}{2}} \tag{3.4}$$

where $\tilde{k} \in \mathbb{R}\backslash\{0\}$. We choose $\tilde{k} > 0$, from (3.4),

$$U_\varepsilon(x) = \frac{C\varepsilon^{\frac{3-2s}{2}}}{\big(\mu^2 S_s^{1/s}\varepsilon^2 + |x|^2\big)^{\frac{3-2s}{2}}}\,, \tag{3.5}$$

where $C > 0$. Since $0 \in \Omega$ and Ω is an open set, then $B_\varepsilon(0) \subset \Omega$. Since $v_\varepsilon(x/t) \to +\infty$ as $t \to +\infty$, $\varepsilon \to 0^+$. By (A4), (3.5),

$$\begin{aligned}
\int_{\mathbb{R}^3} F(x, v_\varepsilon(x/t))dx &\geq \int_{B_\varepsilon(0)} F(x, v_\varepsilon(x/t))dx \\
&\geq M\int_{B_\varepsilon(0)} \Big[\frac{(t^2\varepsilon)^{\frac{3-2s}{2}}}{(\mu^2 t^2 S_s^{1/s}\varepsilon^2 + |x|^2)^{\frac{3-2s}{2}}}\Big]^{\frac{4s}{3-2s}}dx \\
&\geq M\int_{B_\varepsilon(0)} \Big(\frac{t^2\varepsilon}{\mu^2 t^2 S_s^{1/s}\varepsilon^2 + \varepsilon^2}\Big)^{2s}dx \\
&= \frac{Mt^{4s}}{(\mu^2 t^2 S_s^{1/s} + 1)^{2s}}\varepsilon^{3-2s},
\end{aligned} \tag{3.6}$$

where M is a large positive number.

It follows from (2.6)-(2.8), (3.6) that for any $\varepsilon > 0$ small enough, $I(\gamma_\varepsilon(t)) \to -\infty$ as $t \to +\infty$. There exists $t_0 > 0$ such that $I(\gamma_\varepsilon(t_0)) < 0$. Since

$$\begin{aligned}
\|\gamma_\varepsilon(t)\|_{H^s}^2 &= \int_{\mathbb{R}^3}\Big(|(-\Delta)^{s/2}\gamma_\varepsilon(t)|^2 + |\gamma_\varepsilon(t)|^2\Big)dx \\
&= t^{3-2s}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}v_\varepsilon(x)|^2dx + t^3\int_{\mathbb{R}^3}|v_\varepsilon(x)|^2dx,
\end{aligned}$$

by (2.6) and (2.7), we see that $\lim_{t\to 0+}\|\gamma_\varepsilon(t)\|_{H^s}^2 = 0$ for sufficiently small $\varepsilon > 0$. We set $\gamma_\varepsilon(0) = 0$, thus $\gamma_\varepsilon(t_0\cdot) \in \Gamma$. Let $\gamma_0(\cdot) = \gamma_\varepsilon(t_0\cdot)$. Then

$$c \leq \max_{t\in[0,1]} I(\gamma_0(t)) = \max_{t\in[0,1]} I(\gamma_\varepsilon(t_0 t)) = \max_{t\in[0,t_0]} I(\gamma_\varepsilon(t)) \leq \sup_{t\geq 0} I(\gamma_\varepsilon(t)). \tag{3.7}$$

By (3.3), $\sup_{t\geq 0} I(\gamma_\varepsilon(t))$ is attainable at $t_\varepsilon = t(\gamma_\varepsilon) > 0$. By (2.6)-(2.8), (3.6) and (3.3), we have that $I(\gamma_\varepsilon(t)) \to 0^+$ as $t \to 0^+$ and $I(\gamma_\varepsilon(t)) \to -\infty$ as $t \to +\infty$ uniformly for $\varepsilon > 0$ small enough. Then there exist constants $t_1, t_2 > 0$, independent of ε, such that $0 < t_1 \leq t_\varepsilon \leq t_2 < \infty$.

Next we need to prove $\sup_{t\geq 0} I(\gamma_\varepsilon(t)) < c^*$. We define

$$\begin{aligned}
g_\varepsilon(t) := &\frac{at^{3-2s}}{2}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}v_\varepsilon(x)|^2dx + \frac{bt^{6-4s}}{4}\Big(\int_{\mathbb{R}^3}|(-\Delta)^{s/2}v_\varepsilon(x)|^2dx\Big)^2 \\
&- \frac{t^3}{2_s^*}\int_{\mathbb{R}^3}|v_\varepsilon(x)|^{2_s^*}dx.
\end{aligned}$$

Then by (2.8), we have

$$\sup_{t\geq 0} I(\gamma_\varepsilon(t)) \leq \sup_{t\geq 0} g_\varepsilon(t) + O(\varepsilon^{3-2s}) - \int_{\mathbb{R}^3} F(x, v_\varepsilon(x/t_\varepsilon))dx \tag{3.8}$$

By (3.6),

$$\sup_{t\geq 0} I(\gamma_\varepsilon(t)) \leq \sup_{t\geq 0} g_\varepsilon(t) + O(\varepsilon^{3-2s}) - MC\varepsilon^{3-2s}, \tag{3.9}$$

where $C > 0$ is a constant. By a similar argument as above, there exist constants $t_3, t_4 > 0$, independent of ε, such that $\sup_{t\geq 0} g_\varepsilon(t) = \sup_{t\in[t_3,t_4]} g_\varepsilon(t)$. Thus by (2.6), (2.7), we obtain

$$\sup_{t\geq 0} I(\gamma_\varepsilon(t)) \leq \sup_{t\geq 0} J(S_s^{\frac{1}{2s}} t) + O(\varepsilon^{3-2s}) - MC\varepsilon^{3-2s}, \tag{3.10}$$

where

$$J(t) = \frac{aS_s}{2} t^{3-2s} + \frac{bS_s^2}{4} t^{6-4s} - \frac{t^3}{2_s^*}. \tag{3.11}$$

Since M can be arbitrarily large, from (3.10), we see that

$$\sup_{t\geq 0} I(\gamma_\varepsilon(t)) < \sup_{t\geq 0} J(S_s^{\frac{1}{2s}} t). \tag{3.12}$$

By (3.11),

$$J'(t) = \frac{(3-2s)t^{2-2s}}{2} \tilde{J}(t),$$

where

$$\begin{gathered} \tilde{J}(t) = aS_s + bS_s^2 t^{3-2s} - t^{2s}, \\ \tilde{J}'(t) = t^{2-2s}(bS_s^2(3-2s) - 2st^{4s-3}). \end{gathered}$$

Since $s > 3/4$, there exists a unique $T > 0$ such that $\tilde{J}(t) > 0$ for $t \in (0, T)$ and $\tilde{J}(t) < 0$ for $t > T$. Then T is the unique maximum point of J. Therefore, by (3.7) and (3.12), we have $c < c^*$. $\square$

Lemma 3.3. *For any $u \in H^s(\mathbb{R}^3)$ and $t > 0$, the following inequality holds*

$$I(u) \geq I(tu) + \frac{1-t^4}{4} \langle I'(u), u\rangle + \frac{a(1-t^2)^2}{4} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 dx. \tag{3.13}$$

Proof. By (A3), for any $r \geq 0$ and $\tau \in \mathbb{R}\backslash\{0\}$, we have

$$\begin{aligned} &\frac{(1-r^4)\tau f(x,\tau)}{4} + F(x, r\tau) - F(x,\tau) + \frac{1}{4}(1-r^2)^2\tau^2 \\ &= \int_r^1 \Big(\frac{f(x,\tau)}{\tau^3} - \frac{f(x,\nu\tau)}{(\nu\tau)^3} + \frac{1-\nu^2}{(\nu\tau)^2}\Big)\nu^3\tau^4 d\nu \geq 0. \end{aligned} \tag{3.14}$$

Then, for any $u \in H^s(\mathbb{R}^3)$ and $t \geq 0$,

$$
\begin{aligned}
& I(u) - I(tu) - \frac{1-t^4}{4}\langle I'(u), u\rangle \\
& = \frac{(1-t^2)^2}{4}\|u\|^2 + \frac{1-t^4}{4}\int_{\mathbb{R}^3} f(x,u)u\,dx + \int_{\mathbb{R}^3} F(x,tu)dx - \int_{\mathbb{R}^3} F(x,u)dx \\
& \quad + \Big(-\frac{1}{2_s^*} + \frac{t^{2_s^*}}{2_s^*} + \frac{1-t^4}{4}\Big)\int_{\mathbb{R}^3} |u|^{2_s^*}dx \\
& \geq \frac{a(1-t^2)^2}{4}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx \\
& \quad + \int_{\mathbb{R}^3} \big[\frac{1-t^4}{4} f(x,u)u + F(x,tu) - F(x,u) + \frac{(1-t^2)^2}{4}|u|^2\big]dx \\
& \quad + \Big(-\frac{1}{2_s^*} + \frac{t^{2_s^*}}{2_s^*} + \frac{1-t^4}{4}\Big)\int_{\mathbb{R}^3} |u|^{2_s^*}dx.
\end{aligned}
\tag{3.15}
$$

Next we prove $\xi(t) = -\frac{1}{2_s^*} + \frac{t^{2_s^*}}{2_s^*} + \frac{1-t^4}{4} \geq 0$ for all $t \geq 0$. In fact, since $2_s^* = \frac{6}{3-2s} > 4$, it is easy to see that $\xi(t)$ decreases on $(0,1)$ and increases on $(1,\infty)$, then $\min_{t\geq 0}\xi(t) = \xi(1) = 0$ and $\xi(t) \geq 0$. This together with (3.14) and (3.15) yield that

$$
I(u) - I(tu) - \frac{1-t^4}{4}\langle I'(u), u\rangle \geq \frac{a(1-t^2)^2}{4}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx. \qquad \square
$$

Lemma 3.4. *Any sequence satisfying (3.2) is bounded in $H^s(\mathbb{R}^3)$. There exists $u \in H^s(\mathbb{R}^3)$, such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H^s(\mathbb{R}^3)$ and $\langle I'(u), u\rangle \leq 0$.*

Proof. Let $\{u_n\} \subset H^s(\mathbb{R}^3)$ be a sequence satisfying (3.2). By (A3), for any $\tau \in \mathbb{R}\backslash\{0\}$,

$$
\begin{aligned}
\frac{1}{4}\tau f(x,\tau) - F(x,\tau) + \frac{1}{4}\tau^2 &= \int_0^1 \Big(\frac{f(x,\tau)}{\tau^3} - \frac{f(x,\nu\tau)}{(\nu\tau)^3} + \frac{1-\nu^2}{(\nu\tau)^2}\Big)\nu^3\tau^4 d\nu \\
&\geq 0.
\end{aligned}
\tag{3.16}
$$

For $\tau = 0$, it is obvious that $\frac{1}{4}\tau f(x,\tau) - F(x,\tau) + \frac{1}{4}\tau^2 = 0$. Since $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$, it follows that

$$
\begin{aligned}
& c + 1 + o(1) \\
& \geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\
& = \frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3} \big[\frac{1}{4}u_n f(x,u_n) - F(x,u_n)\big]dx + (\frac{1}{4} - \frac{1}{2_s^*})\int_{\mathbb{R}^3} |u_n|^{2_s^*}dx \\
& \geq \frac{a}{4}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 dx.
\end{aligned}
\tag{3.17}
$$

By (3.17) and (2.3), there exists a constant $M_c = M(a, c, s) > 0$ such that $|(-\Delta)^{s/2}u_n|_2 \le M_c$ and $|u_n|_{2_s^*} \le M_c$. It follows from $I(u_n) = c + o_n(1)$ that

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^3} |u_n|^2 dx &= c + o_n(1) + \int_{\mathbb{R}^3} F(x,u_n)dx + \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u_n|^{2_s^*}dx \\ &\quad - \frac{a}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2dx - \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2dx\Big)^2 \\ &\le c + o_n(1) + \varepsilon|u_n|_2^2 + C_\varepsilon|u_n|_q^q + \frac{M_c}{2_s^*}. \end{aligned} \tag{3.18}$$

By an L^p interpolation inequality and (2.3),

$$|u_n|_q^q \le |u_n|_2^{q\theta}|u_n|_{2_s^*}^{q(1-\theta)} \le C|u_n|_2^{q\theta}|(-\Delta)^{s/2}u_n|_2^{q(1-\theta)}, \tag{3.19}$$

where $\theta \in (0,1)$, $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{2_s^*}$. Since $q\theta \in (0,2)$, by Young inequality and (3.19), we have

$$C_\varepsilon|u_n|_q^q \le \varepsilon|u_n|_2^2 + \widetilde{C_\varepsilon}|(-\Delta)^{s/2}u_n|_2^{\frac{2q(1-\theta)}{2-q\theta}}. \tag{3.20}$$

If we take $\varepsilon = 1/5$, from (3.18) and (3.20), it follows that $|u_n|_2$ is bounded. Therefore $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$.

Let $A^2 := \lim_{n\to\infty}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2dx$. By (2.5) and Fatou's lemma,

$$\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2dx \le A^2. \tag{3.21}$$

Since $u_n \rightharpoonup u$, then u is a weak solution of

$$(a + bA^2)(-\Delta)^{s/2}u + u = f(x,u) + |u|^{2_s^*-2}u. \tag{3.22}$$

Thus,

$$\begin{aligned} \langle I'(u), u\rangle &= \|u\|^2 + b\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}|^2dx\Big)^2 - \int_{\mathbb{R}^3} f(x,u)u\,dx - \int_{\mathbb{R}^3} |u|^{2_s^*}dx \\ &\le (a + bA^2)\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2dx + \int_{\mathbb{R}^3} |u|^2dx - \int_{\mathbb{R}^3} f(x,u)u\,dx \\ &\quad - \int_{\mathbb{R}^3} |u|^{2_s^*}dx = 0. \end{aligned} \tag{3.23}$$

□

Lemma 3.5. *For any sequence $\{u_n\}$ satisfying* (3.2)*, there exists $\delta_0 > 0$ such that*

$$\delta_0 = \lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)} u_n^2dx > 0.$$

Proof. By contradiction, we assume that $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)} u_n^2dx = 0$. By [6, lemma 2.3], for any $2 < r < 2_s^*$,

$$u_n \to 0 \quad \text{in } L^r(\mathbb{R}^3) \text{ for } r \in (2, 2_s^*).$$

By (A1) and (A2),

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} u_nf(x,u_n)dx \to 0 \quad \text{and}quad \lim_{n\to\infty}\int_{\mathbb{R}^3} F(x,u_n)dx \to 0.$$

This and (3.2) yield

$$\frac{1}{2}\|u_n\|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2dx\Big)^2 - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u_n|^{2_s^*}dx = c + o(1), \tag{3.24}$$

and

$$\|u_n\|^2 + b\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 dx\Big)^2 - \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx = o(1). \tag{3.25}$$

Up to a subsequence, we assume that there exist $l_i \geq 0$ $(i = 1, 2, 3)$ such that

$$\|u_n\|^2 \to l_1, \quad b\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 dx\Big)^2 \to l_2, \quad \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \to l_3^3. \tag{3.26}$$

by (3.24) and (3.25), we have

$$l_1 + l_2 = l_3^3, \tag{3.27}$$

$$\Big(\frac{1}{2} - \frac{1}{2_s^*}\Big)l_1 + \Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)l_2 = c. \tag{3.28}$$

From (2.3), it follows that

$$\|u_n\|^2 \geq aS_s\Big(\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx\Big)^{\frac{2}{2_s^*}}, \tag{3.29}$$

$$b\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 dx\Big)^2 \geq bS_s^2\Big(\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx\Big)^{\frac{4}{2_s^*}}. \tag{3.30}$$

Then by (3.25),

$$aS_s\Big(\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx\Big)^{2/2_s^*} + bS_s^2\Big(\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx\Big)^{4/2_s^*} \leq \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx + o_n(1). \tag{3.31}$$

By (3.31),

$$J'(l_3) = \frac{(3-2s)l_3^{-1}}{2}(aS_s l_3^{3-2s} + bS_s^2 l_3^{6-4s} - l_3^3) \leq 0, \tag{3.32}$$

where J is defined in (3.11). Since T is the unique maximum point of J, then $l_3 \geq T$.

From (3.26), (3.28), (3.29), (3.30) and note that $J'(T) = 0$, we obtain

$$\begin{aligned} c &\geq \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big)aS_s l_3^{3-2s} + \Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)bS_s^2 l_3^{6-4s} \\ &\geq \Big(\frac{1}{2} - \frac{1}{2_s^*}\Big)aS_s T^{3-2s} + \Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)bS_s^2 T^{6-4s} \\ &= \frac{1}{2}aS_s T^{3-2s} + \frac{1}{4}bS_s^2 T^{6-4s} - \frac{1}{2_s^*}T^3 = c^*. \end{aligned}$$

This contradicts $c < c^*$. The proof is complete. □

Lemma 3.6. *Any Palais-Smale sequence for I at the level $c \in (0, c^*)$ possesses a strongly convergent subsequence.*

Proof. From [8, Corollary 7.2], it is known that the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ locally compact for $q \in [1, 2_s^*)$. Suppose $\{u_n\}$ is the $(PS)_c$ sequence for I, then $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$. By Lemma 3.4, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^3)$. Up to a subsequence, there exists $u \in H^s(\mathbb{R}^3)$ such that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } H^s(\mathbb{R}^3), \\ u_n \to u \text{ in } L^q_{loc}(\mathbb{R}^3), \quad q \in [1, 2_s^*), \\ u_n \to u \quad \text{a.e. in } \mathbb{R}^3. \end{gathered} \tag{3.33}$$

By Lemma 3.5 and (3.33),

$$\int_{B_1(y)} u^2 dx \geq \frac{\delta_0}{2} > 0.$$

Then $\tilde{u}(x) := u(x+y) \neq 0$. Without loss of generality, we assume that $u \neq 0$. By (3.15) and Fatou's lemma, for n large, we have

$$\begin{aligned} c + o_n(1) &= I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2 + \int_{\mathbb{R}^3} \big[\frac{1}{4}u_n f(x,u_n) - F(x,u_n)\big]dx + \big(\frac{1}{4} - \frac{1}{2_s^*}\big)\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \\ &= \frac{a}{4}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 dx + \int_{\mathbb{R}^3} \big[\frac{1}{4}u_n f(x,u_n) - F(x,u_n) + \frac{1}{4}u_n^2\big]dx \\ &\quad + \big(\frac{1}{4} - \frac{1}{2_s^*}\big)\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \\ &\geq \frac{a}{4}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx + \int_{\mathbb{R}^3} \big[\frac{1}{4}u f(x,u) - F(x,u) + \frac{1}{4}u^2\big]dx \\ &\quad + \big(\frac{1}{4} - \frac{1}{2_s^*}\big)\int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &= I(u) - \frac{1}{4}\langle I'(u), u\rangle. \end{aligned} \tag{3.34}$$

By (3.13) and (3.23), we have

$$\begin{aligned} I(u) - \frac{1}{4}\langle I'(u), u\rangle &\geq \max_{t\geq 0}\big[I(tu) + \frac{1-t^4}{4}\langle I'(u), u\rangle\big] - \frac{1}{4}\langle I'(u), u\rangle \\ &= \max_{t\geq 0}\big[I(tu) - \frac{t^4}{4}\langle I'(u), u\rangle\big] \geq \max_{t\geq 0} I(tu). \end{aligned} \tag{3.35}$$

Since $u \neq 0$, for any $u \in H^s(\mathbb{R}^3)\backslash\{0\}$, by (ii) of Lemma 3.1, for sufficiently large $\tilde{t} > 0$, $I(\tilde{t}u) < 0$. Let $\gamma_1(t) = t\tilde{t}u$. Then $\gamma_1 \in C([0,1], H^s(\mathbb{R}^3))$ and $\gamma_1 \in \Gamma$. We have

$$c \leq \max_{t\in[0,1]} I(\gamma_1(t)) = \max_{t\in[0,1]} I(t\tilde{t}u) = \max_{t\in[0,\tilde{t}]} I(tu) \leq \max_{t\geq 0} I(tu). \tag{3.36}$$

From (3.35) and (3.36), it follows that

$$I(u) - \frac{1}{4}\langle I'(u), u\rangle \geq c. \tag{3.37}$$

By (3.34) and (3.36), it is easy to see that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 dx = \int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx, \tag{3.38}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} u_n^2 dx = \int_{\mathbb{R}^3} u^2 dx. \tag{3.39}$$

By (3.38) and (3.39), we have

$$\|u_n\| \to \|u\| \text{ as } n \to \infty.$$

This and $u_n \rightharpoonup u$ yield $u_n \to u$ in $H^s(\mathbb{R}^3)$. □

4. Proof of Theorem 1.2

We define $c_1 := \inf_{u\in\mathfrak{U}} I(u)$, where $\mathfrak{U} = \{u \in H^s(\mathbb{R}^3)\backslash\{0\} : I'(u) = 0\}$.

Lemma 4.1. *$0 < c_1 < c^*$.*

Proof. For $I(\gamma_\varepsilon(t))$, there exists $t_\varepsilon = t(\gamma_\varepsilon) > 0$ such that $I(\gamma_\varepsilon(t_\varepsilon)) = \sup_{t>0} I(\gamma_\varepsilon(t))$, where $I(\gamma_\varepsilon(t))$ is the same as in Lemma 3.2. By Lemma 3.2, $c_1 \le \sup_{t\ge 0} I(\gamma_\varepsilon(t)) < c^*$. Next we show $c_1 > 0$. For any $u \in \mathfrak{U}$,

$$\begin{aligned}\|u\|^2 &\le \|u\|^2 + b\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2dx\Big)^2\\ &= \int_{\mathbb{R}^3} uf(x,u)dx + \int_{\mathbb{R}^3} |u|^{2_s^*}dx\\ &= \frac{\varepsilon}{2}\|u\|^2 + C\|u\|^q + C\|u\|^{2_s^*}.\end{aligned}$$

We take $\varepsilon = 1/2$, then there exists $\alpha > 0$ such that

$$\|u\| \ge \alpha > 0. \tag{4.1}$$

For any $u \in \mathfrak{U}$, it follows that $\langle I'(u), u\rangle = 0$, then by (3.15), we have

$$\begin{aligned}I(u) &= I(u) - \frac{1}{4}\langle I'(u), u\rangle\\ &= \frac{1}{4}\|u\|^2 + \int_{\mathbb{R}^3} \Big[\frac{1}{4}uf(x,u) - F(x,u)\Big]dx + \big(\frac{1}{4} - \frac{1}{2_s^*}\big)\int_{\mathbb{R}^3} |u|^{2_s^*}dx \\ &\ge \frac{a}{4}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2dx.\end{aligned} \tag{4.2}$$

Thus $c_1 \ge 0$. We need to prove $c_1 \ne 0$. Assume that $c_1 = 0$ and $\{u_n\}$ is the corresponding minimizing sequence, that is, $\{u_n\} \subset \mathfrak{U}$, and $I(u_n) \to 0$ as $n \to \infty$. By (4.2), $\lim_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2dx = 0$. This and (2.3) yield $\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^{2_s^*}dx = 0$. Then by (3.18)-(3.20), we obtain $\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^2dx = 0$. Therefore, we have $\lim_{n\to\infty} \|u_n\|^2 = 0$. Since (4.1) holds for any $u \in \mathfrak{U}$, we obtain a contradiction. The proof is complete. □

Proof of Theorem 1.2. Suppose $\{u_n\} \subset \mathfrak{U}$ is the minimizing sequence for $c_1 = \inf_{u\in\mathfrak{U}} I(u)$. Then $\{u_n\}$ is the $(PS)_{c_1}$ sequence for I. By Lemma 4.1 and Lemma 3.6, there exists $u \in H^s(\mathbb{R}^3)$ such that $I'(u) = 0$ and $I(u) = c_1$. Then u is a ground state solution of (1.1). □

5. Conclusion

In this article, we have considered a nonlinear fractional Kirchhoff problem with critical growth. We have proved the existence of ground state solutions by variational methods. The main feature of the result is that we only need the nonlinear term $f \in C$ and f satisfying $(f_1) - (f_4)$. In fact, such f may not satisfy the condition $f \in C^1$, which is used to prove some Pohozaev type identity, see [18, 14, 4]. And such f may not satisfiy the classical (AR) condition or (VC) condition in [12] or the generalized (AR) condition in [11], see Remark 1.1.

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