Electronic Journal of Differential Equations, Vol. 2024 (2024), No. 13, pp. 1–16.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2024.13

FAILURE OF THE HOPF-OLEINIK LEMMA FOR A LINEAR ELLIPTIC PROBLEM WITH SINGULAR CONVECTION OF NON-NEGATIVE DIVERGENCE

LUCIO BOCCARDO, JESÚS ILDEFONSO DÍAZ, DAVID GÓMEZ-CASTRO

Abstract. In this article we study the existence, uniqueness, and integrability of solutions to the Dirichlet problem $-\operatorname{div}(M(x)\nabla u) = -\operatorname{div}(E(x)u) + f$ in a bounded domain of $\mathbb{R}^N$ with $N \geq 3$. We are particularly interested in singular E with $\operatorname{div} E \geq 0$. We start by recalling known existence results when $|E| \in L^N$ that do not rely on the sign of $\operatorname{div} E$. Then, under the assumption that $\operatorname{div} E \geq 0$ distributionally, we extend the existence theory to $|E| \in L^2$. For the uniqueness, we prove a comparison principle in this setting. Lastly, we discuss the particular cases of E singular at one point as $Ax/|x|^2$, or towards the boundary as $\operatorname{div} E \sim \operatorname{dist}(x, \partial\Omega)^{-2-\alpha}$. In these cases the singularity of E leads to u vanishing to a certain order. In particular, this shows that the Hopf-Oleinik lemma, i.e. $\partial u/\partial n < 0$, fails in the presence of such singular drift terms E.

1. Introduction

It is well known that many relevant applications lead to the presence of a convection term in the correspondent model which, in its simplest formulation, leads to a boundary value problem for linear elliptic second order equation of the type

$$\begin{gathered} -\operatorname{div}(M(x)\nabla u) = -\operatorname{div}(uE(x)) + f(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is an open, bounded set, and we assume that $M \in L^\infty(\Omega)^{N\times N}$ is elliptic

$$M(x)\xi\cdot\xi \geq \alpha|\xi|^2, \quad \forall \xi \in \mathbb{R}^N \text{ and a.e. } x \in \Omega.$$

According to the regularity of the right-hand side datum $f(x)$ it is natural to search the solution in the energy space $W_0^{1,2}(\Omega)$ (case of $f \in H^{-1}(\Omega)$: see, e.g. [21, 16, 1]), or in a larger Sobolev space if f is singular (see [1]); when $f \in L^1(\Omega)$, see, for instance, [8], or when $L^1(\Omega, \delta)$ with $\delta(x) = d(x, \partial\Omega)$, see, e.g., [7, 13].

In the mentioned references it assumed that the convection term is regular (for instance $E \in W^{1,\infty}(\Omega)$) and that it satisfies an additional condition which helps to

2020 *Mathematics Subject Classification.* 35J25, 35J75, 35B50, 35B60.
Key words and phrases. Linear elliptic equation; convection with singularity on the boundary; strong maximum principle; flat solutions.

Submitted October 31, 2023. Published January 31, 2024.

have a maximum principle:

$$\operatorname{div} E \geq 0 \quad \text{a.e. on } \Omega. \tag{1.2}$$

Recently, some effort has been devoted to get an existence and regularity theory under more general conditions on the convection term E by different authors (see, e.g. [1, 5] and their references). For instance, solutions in the energy space can be considered under the conditions $|E| \in L^N(\Omega)$ and $f \in L^{\frac{2N}{N+2}}(\Omega)$. In [13] and [12] the authors study the case in which $|E| \in L^N(\Omega)$ and $\operatorname{div} E = 0$ in Ω and $E \cdot n = 0$ on $\partial\Omega$, $f \in L^1(\Omega, \delta)$. See also [15, 20].

In this article, we show that (1.2) makes $\operatorname{div} E$ behave like a non-negative potential in the Schrödinger case, and we can apply techniques from that setting. See, for example, [12, 13, 14, 17]. We focus on the case where (1.2) holds in distributional sense.

The article is structured as follows. First, in Section 2 we review known results for the case $|E| \in L^N$ and $f \in L^{\frac{2N}{N+2}}(\Omega)$ which were published in [1], were shown there is a unique weak solution of (1.1) that can be constructed by approximation. In Section 3 we show that if $|E| \in L^2(\Omega)$, $\operatorname{div} E \geq 0$, and $f \in L^m(\Omega)$ for some $m > 1$ then the same approximation procedure converges to a weak solution of (1.1), and we give some a priori bounds for this solution. In Section 4 we show that, if we also assume $f \in L^{\frac{2N}{N+2}}(\Omega)$, then this constructed solution is the unique weak solution of (1.1).

Then we move to discussing interesting examples that fall in this setting. In Section 5 we focus on the case

$$E(x) = A \frac{x}{|x|^2}\,, \tag{1.3}$$

which is somehow in the limit of theory since it is not in $L^N(\Omega)$ but it is in $L^r(\Omega)$ for $r \in [1, N)$. In [5] the authors examined the more general class

$$|E| \leq \frac{|A|}{|x|}\,. \tag{1.4}$$

The authors show existence of solutions u, where the summability is reduced as $|A|$ is increased. Their results indicate that the sign of A should play a role, but the application of Hardy's inequality (which they use in a crucial way) is not able to detect this fact. In Theorem 5.2 we show that if $N > 1$, $f \in L^m(\Omega)$ for suitable m, and $A > 0$ then we can use the sign of $\operatorname{div} E$ to deduce that the solution u_A of (1.1) with $E = A\frac{x}{|x|^2}$ satisfies

$$u_A \to 0 \quad \text{in } L^1(\Omega) \text{ as } A \to +\infty\,.$$

By the contrary, when $A < 0$ we cannot improve the result in [5]. Notice that this is similar to the equation $L(u_B) + B\, u_B = f$, whereas $B \to \infty$ we have $u_B \to 0$.

Lastly, in Section 6, we discuss the case where E is suitably singular only on the boundary. We present an example showing that if $\operatorname{div} E$ behaves like $d(x, \partial\Omega)^{-2-\gamma}$ for some $\gamma > 0$ and f is bounded, then the solutions are flat on the boundary, i.e.

$$|u(x)| \leq C \operatorname{dist}(x, \partial\Omega)^{\alpha} \text{ for some } \alpha > 1\,.$$

In particular, this shows that the Hopf-Oleinik lemma, i.e. $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$, fails in the presence of such singular drift terms E. Our example can be easily extended to a more general class of E, as we comment in Section 7. Again, we use the fact that $\operatorname{div} E$ acts as a potential. However, in the Schrödinger equation it is sufficient that $V(x) \geq C\delta^{-2}$ to get flat solutions, whereas for E we need a strictly larger exponent (see Remark 6.6). Questions of this type are quite relevant in the framework of linear Schrödinger equations associated to singular potential since they can be understood as complements to the Heisenberg Incertitude Principle (see, e.g. [10, 11, 12, 13, 18, 14]). We conclude with some further comments and open problems in Section 7.

2. Known results when $|E| \in L^N$

We define the Sobolev conjugate exponent

$$m^* = \frac{mN}{N-m} \quad \text{if } m \leq N, \quad m^{**} = (m^*)^* = \frac{mN}{N-2m} \quad \text{if } m \leq \frac{N}{2}.$$

We have that $m^{**} \in [1, \infty]$ for $\frac{N}{N+2} \leq m \leq \frac{N}{2}$. Notice that $m^* \geq 2$ if and only if $m \geq \frac{2N}{N+2} = (2^*)'$. Notice that, since $m \geq 1$ we have $m^* \geq m$. In order to compute explicit a priori estimates, we use the Sobolev embedding constant S_p such that, for $1 < p < +\infty$

$$S_p \|u\|_{L^{p^*}(\Omega)} \leq \|\nabla u\|_{L^p(\Omega)}. \tag{2.1}$$

We point out the relevance of the constants, for $N > 2$ of $(2^*)' = \frac{2N}{N+2}$. This constant depends only of N. Since we are going to require the Sobolev embedding for $p = 2$, we assume that $N \geq 3$. In [1] the author proves the following existence theorem with a priori estimates.

Theorem 2.1 ([1]). *Let $f \in L^{\frac{2N}{N+2}}(\Omega)$ and $|E| \in L^N(\Omega)$. Then, there exists a unique weak solution u of (1.1) in the sense that*

$$u \in W_0^{1,2}(\Omega) \text{ is such that } \int_\Omega M(x) \nabla u \nabla v = \int_\Omega u\, E(x) \cdot \nabla v + \int_\Omega f(x)\, v(x),$$

for all $v \in W_0^{1,2}(\Omega)$. and it satisfies

(1) *Logarithmic estimate:*

$$\Big(\int_\Omega |\log(1 + |u|)|^2 \Big)^{2/2^*} \leq \frac{1}{S_2^2 \alpha^2} \int_\Omega |E|^2 + \frac{2}{S_2^2 \alpha} \int_\Omega |f|,$$

(2) *Gradient estimate: there exists $C = C(\alpha, N)$ such that*

$$\int_\Omega |\nabla u|^2 \leq C \left(\|E\|_{L^N}^2 + \|f\|^2_{L^{\frac{2N}{N+2}}} \right). \tag{2.2}$$

(3) *Stampacchia-type summability: For $m \in [\frac{2N}{N+2}, \frac{N}{2})$ there exists a constant $C = C(m, \alpha, N, \|E\|_{L^N})$ such that*

$$\|u\|_{m^{**}} \leq C \|f\|_m. \tag{2.3}$$

(4) *Stampacchia-type boudedness: Let $r > N$ and $m > \frac{N}{2}$. There exists C such that*

$$\|u\|_{L^\infty} \leq C(m, r, \alpha, \|f\|_{L^m}, \|E\|_{L^r}). \tag{2.4}$$

Remark 2.2. The natural theory for this problem in energy space is precisely $|E| \in L^N(\Omega)$, since in the weak formulation we need to justify a term of the form $E u \nabla v$, where $u, v \in W_0^{1,2}(\Omega)$. This means that $u \in L^{2^*}$ whereas $\nabla v \in L^2$. So we always have that $uE \in L^2(\Omega)$.

In [1] the main tool to study the linear problem (1.1) are the auxiliary non-linear Dirichlet problems

$$\begin{gathered} -\operatorname{div}(M(x)\nabla u_n) = -\operatorname{div}\big(\frac{u_n}{1+\frac{1}{n}u_n}E_n(x)\big) + f_n(x) \quad \Omega \\ u = 0 \quad \partial\Omega, \end{gathered} \tag{2.5}$$

where the author take $f_n = T_n(f)$ a truncation of f through the family

$$T_n(s) = \begin{cases} s & |s| \le k, \\ k \operatorname{sign}(s) & |s| > k, \end{cases}$$

and $E_n = \frac{E}{1+\frac{1}{n}|E|}$. We will take advantage of a similar approximation.

Remark 2.3. Since the problem is linear, for $t \in \mathbb{R}$ we have that tu is solution of

$$-\operatorname{div}(M(x)\nabla[t\,u]) = -\operatorname{div}([t\,u]E(x)) + t\,f(x),$$

and that E does not change. Thus, using (2.2)

$$t^2\int_\Omega |\nabla u|^2 \le C\big(\|E\|_{L^N}^2 + t^2\|f\|^2_{L^{\frac{2N}{N+2}}}\big).$$

Dividing by t^{-2} and taking the limit as $t \to \infty$ gives

$$\int_\Omega |\nabla u|^2 \le C\|f\|^2_{L^{\frac{2N}{N+2}}}. \tag{2.6}$$

Notice that in Theorem 4.1 we will prove this fact for the case $\operatorname{div} E \ge 0$.

3. Existence theory when $|E| \in L2$ and $\operatorname{div} E \ge 0$

The structural assumption in this section is the following:

$$\begin{gathered} E \text{ belongs to the Lebesgue space } (L^2(\Omega))^N, \\ \operatorname{div} E \ge 0 \text{ in } \mathcal{D}'(\Omega), \text{ that is } \int_\Omega E\cdot\nabla\phi \le 0, \quad \forall 0 \le \phi \in W_0^{1,2}(\Omega). \end{gathered} \tag{3.1}$$

Theorem 3.1. *Assume* (3.1) *and*

$$f \in L^m(\Omega), \quad 1 < m < \frac{N}{2}, \tag{3.2}$$

and let $p = \min\{2, m^*\}$. *Then, there exists a weak solution* u *of* (1.1) *in the sense that* $u \in W_0^{1,p}(\Omega)$ *is such that*

$$\int_\Omega M(x)\nabla u\nabla v = \int_\Omega u\,E(x)\cdot\nabla v + \int_\Omega f(x)\,v(x), \quad \forall v \in W_0^{1,\infty}(\Omega). \tag{3.3}$$

Furthermore, it satisfies

$$\begin{gathered} \|u\|_{W_0^{1,m^*}(\Omega)} \le C_m\|f\|_m, \quad \text{if } 1 < m < \frac{2N}{N+2}; \\ \|u\|_{W_0^{1,2}(\Omega)} + \|u\|_{m^{**}} \le \tilde{C_m}\,\|f\|_m, \quad \text{if } \frac{2N}{N+2} \le m < \frac{N}{2}. \end{gathered} \tag{3.4}$$

Remark 3.2. From the gradient estimates, we can extend (3.3) to all $v \in W_0^{1,q}(\Omega)$ by approximation, where $q = p'$.

Since the construction of solutions in the proof of Theorem 3.1 is achieved by approximation, we have the following result.

Corollary 3.3. *The solutions constructed in Theorem 3.1 satisfy* (2.3) *and* (2.4)*.*

We say that u_n is a weak solution of (2.5) if $u \in W_0^{1,2}(\Omega)$ is such that

$$\int_\Omega M(x)\nabla u_n \nabla v = \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|}\, E_n(x)\cdot\nabla v + \int_\Omega f_n(x)\, v(x), \quad \forall v \in W_0^{1,2}(\Omega). \tag{3.5}$$

The existence of a weak solution if $E_n \in L^2(\Omega)^N$ is a consequence of the Schauder theorem. The proof of Theorem 3.1 is based on the following approximation lemma

Lemma 3.4. *Let u_n be any weak solution of* (3.5) *with $E_n = E$,* (3.1)*,* (3.2)*, and $f_n = T_n(f)$. Then, for any weak solution u_n of* (3.5) *we have that*

$$\begin{cases} \|u_n\|_{W_0^{1,m^*}(\Omega)} \le C_m\, \|f\|_m, & \text{if } 1 < m < \frac{2N}{N+2};\\ \|u_n\|_{W_0^{1,2}(\Omega)} + \|u_n\|_{m^{**}} \le \tilde{C_m}\, \|f\|_m, & \text{if } \frac{2N}{N+2} \le m \le \frac{N}{2}. \end{cases} \tag{3.6}$$

where

$$C_m \text{ does not depend on } E. \tag{3.7}$$

Hence, up to a subsequence, $\{u_n\}$ converges weakly in $L^{m^{**}}$.

Proof. Our proof is the same of [4], since we will see that the contribution of new term on E is a negative number. We use $T_k(u_n)|T_k(u_n)|^{2\gamma-2}$ as test function in (3.5), $\gamma = \frac{m^{**}}{2^*}$; we repeat it is possible since every $T_k(u_n)$ has exponential summability. Note that $2\gamma - 1 > 0$ since $m > 1$. Thus, we have

$$\begin{aligned} &\int_\Omega M(x)\nabla u_n \nabla(T_k(u_n)|T_k(u_n)|^{2\gamma-2}) \\ &= \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|}\, E(x)\cdot\nabla(T_k(u_n)|T_k(u_n)|^{2\gamma-2}) + \int_\Omega f_n(x)\, T_k(u_n)|T_k(u_n)|^{2\gamma-2}. \end{aligned}$$

To study the second integral, we define the function

$$H_\gamma(s) = \int_0^s \frac{t\,|t|^{2\gamma-2}}{1+\frac{1}{n}|t|}\, dt.$$

It is easy to check that $H_\gamma(s) \ge 0$ for all $s \in \mathbb{R}$. Thus, using the sign condition on $\operatorname{div} E$ we have that

$$\begin{aligned} &\int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|} E(x)\cdot\nabla(T_k(u_n)|T_k(u_n)|^{2\gamma-2}) \\ &= \int_\Omega (2\gamma-1)\frac{T_k(u_n)\,|T_k(u_n)|^{2\gamma-2}}{1+\frac{1}{n}|T_k(u_n)|}\, E(x)\cdot\nabla T_k(u_n) \\ &= \int_\Omega H_\gamma(T_k(u_n))\, E(x)\cdot\nabla T_k(u_n) \\ &= \int_\Omega E(x)\cdot\nabla[H_\gamma(T_k(u_n))] \le 0. \end{aligned}$$

Hence, we have that

$$\int_\Omega M(x)\nabla u_n \nabla(T_k(u_n)|T_k(u_n)|^{2\gamma-2}) \le \int_\Omega f_n(x)\, T_k(u_n)|T_k(u_n)|^{2\gamma-2},$$

which is the starting point of [4], and we obtain the estimates

$$\|T_k(u_n)\|_{W_0^{1,m^*}(\Omega)} \le C_m\, \|f\|_m, \quad \text{if } 1 < m < \frac{2N}{N+2};$$

$$\|T_k(u_n)\|_{W_0^{1,2}(\Omega)} + \|T_k(u_n)\|_{m^{**}} \le \tilde{C_m}\,\|f\|, \quad \text{if } \frac{2N}{N+2} \le m < \frac{N}{2}.$$

Letting $k \to \infty$ we recover (3.6). $\square$

With this lemma, we can pass to the limit to prove Theorem 3.1.

Proof of Theorem 3.1. Up to subsequences, $\{T_k(u_n)\}$, as constructed above, converges weakly (in $W_0^{1,m^*}(\Omega)$ or in $W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$) and it is possible to pass to some u (note that $u \in L^{m^{**}}(\Omega)$). Recall that $E \in (L^2)^N$. To pass to the limit in

$$\int_\Omega \frac{u_n}{1 + \frac{1}{n}|u_n|}\, E_n(x) \cdot \nabla v$$

in (3.5) we need

$$1 \ge \frac{1}{m^{**}} + \frac{1}{N} + \frac{1}{2}.$$

That is equivalent to $m \ge \frac{2N}{N+2}$. Thus we pass also to the limit in (3.6). $\square$

Remark 3.5. Note that, once more it is possible to develop an approximate method in order to prove the existence when $E \in L^r$. Indeed, let $E_0 \in L^r$, $r > 1$ and $E_n \in L^2$ converging to E_0 in L^r. Define now u_n in the corresponding way, we can use the statement of (3.7), so that we can say that estimates (3.4) still hold for this new sequence $\{u_n\}$ and once more we can pass to the limit, and we prove the existence if

$$1 \ge \frac{1}{r} + \frac{1}{m} - \frac{2}{N}$$

We can provide further a priori estimates when $\operatorname{div} E \ge 0$.

Proposition 3.6. *The solutions constructed in Theorem 3.1 satisfy the following additional estimates:*

(1) *(L^1 estimate) If* $\operatorname{div} E \in L^1(\Omega)$ *then we have that*

$$\int_\Omega |u| \operatorname{div} E \le \int_\Omega |f|. \tag{3.8}$$

(2) *(L^m estimate) If* $\operatorname{div} E \ge c_0 > 0$ *and* $m > 1$ *then*

$$\|u\|_{L^m} \le \frac{m}{m-1} \frac{\|f\|_{L^m}}{c_0}. \tag{3.9}$$

We will later take advantage of (3.8) and present several extensions. See, e.g., Lemma 6.3 where we extend the result to $\operatorname{div} E \in L^1_{\rm loc}$.

Remark 3.7. Notice that (3.9) blows up as $m \to 1$. In fact, it is known that the case $m = 1$ does not satisfy such an estimate.

We prove a priori estimates under the assumption of $\operatorname{div} E \ge 0$ for bounded (or even smooth) E, which we now know will hold for approximations.

Proof of Proposition 3.6. Assume first that $E \in (L^N)^N$, and $f \in L^m$ for $m \ge \frac{2N}{N+2}$. Then, we can deal with the unique solution $u \in W_0^{1,2}(\Omega)$ that exists by Theorem 2.1. Because of the construction by approximation in Theorem 3.1 the estimates pass to the limit in the construction. Take $h \in W^{1,\infty}(\mathbb{R})$ such that $h(0) = 0$. We take $v = h(u)$ as a test function we can write

$$\alpha \int_\Omega h'(u)|\nabla u|^2 \le \int_\Omega M(x)\nabla u \cdot \nabla h(u) = \int_\Omega uE \cdot \nabla h(u) + \int_\Omega f h(u).$$

We can write

$$u\nabla h(u) = uh'(u)\nabla u = \nabla F(u)$$

where $F(s) = \int_0^s \tau\, h'(\tau) d\tau$. Hence,

$$\alpha \int_\Omega h'(u)|\nabla u|^2 \le \int_\Omega E \cdot \nabla F(u) + \int_\Omega f h(u).$$

Now we prove both items

(1) Since $\operatorname{div} E \in L^1(\Omega)$ we can integrate by parts again to deduce

$$\alpha \int_\Omega h'(u)|\nabla u|^2 + \int_\Omega F(u) \operatorname{div} E \le \int_\Omega f h(u). \tag{3.10}$$

Let us consider $h_\varepsilon(s) = T_\varepsilon(s)/\varepsilon$. Then $h'_\varepsilon \ge 0$ and $|h_\varepsilon| \le 1$ and, hence, in (3.10)

$$\int_\Omega F_\varepsilon(u) \operatorname{div} E \le \int_\Omega |f|.$$

It is clear that $F_\varepsilon(s) \to |s|$ a.e. as $\varepsilon \to 0$. Then

$$\int_\Omega |u| \operatorname{div} E \le \int_\Omega |f|.$$

(2) Let us take, for $m > 1$, $h(s) = |s|^{m-1}$ then

$$F(s) = (m-1) \int_0^s |\tau|^{m-2} \operatorname{sign}(\tau) \tau d\tau = \frac{m-1}{m} s^m.$$

Hence, going back to (3.10),

$$c_0 \frac{m-1}{m} \|u\|_{L^m}^m \le \frac{m-1}{m} \int_\Omega |u|^m \operatorname{div} E \le \int_\Omega f|u|^{m-1} \le \|f\|_{L^m} \|u\|_{L^m}^{m-1}.$$

Hence, we simplify

$$\|u\|_{L^m} \le \frac{m}{m-1} \frac{\|f\|_{L^m}}{c_0}. \qquad \square$$

4. Comparison principle and uniqueness

To show uniqueness of solutions we prove a weak maximum principle.

Theorem 4.1. *Let $f \in L^{\frac{2N}{N+2}}(\Omega)$ and (3.1). Then, if $u \in W_0^{1,2}(\Omega)$ is a solution of (3.3) then*

$$\|\nabla u^+\|_2 \le \frac{1}{\alpha S_2} \|f^+\|_{\frac{2N}{N+2}}.$$

Hence, there is, at most, one solution of (3.3) in $W_0^{1,2}(\Omega)$. Furthermore, if $f \ge 0$ then $u \ge 0$.

We first prove the following lemma.

Lemma 4.2. *Let $m, r > 1$, $E \in L^{r'}(\Omega)$ with $0 \le \operatorname{div} E \in \mathcal{D}'(\Omega)$. Then, we have that*

$$-\int_\Omega E \nabla v \ge 0 \quad \forall\, 0 \le v \in W_0^{1,r}(\Omega). \tag{4.1}$$

Proof. By definition of having a sign in distributional sense, for $0 \le \varphi \in \mathcal{C}_c^\infty(\Omega)$, we have that

$$-\int_\Omega E\nabla\varphi = \langle \operatorname{div} E, \varphi\rangle \ge 0.$$

For $0 \le v \in W_0^{1,r}(\Omega)$, we can find a sequence $0 \le \varphi_n \in \mathcal{C}_c^\infty(\Omega)$, such that $\varphi_n \to v$ in $W_0^{1,r}(\Omega)$. In particular, $\nabla\varphi \to \nabla v$ in $L^r(\Omega)$. We can pass to the limit in the estimate. □

Proof of Theorem 4.1. Let u be a solution. Take ρ_n a family of non-negative mollifiers, and use $v_n = \rho_n * u^+$ as a test function. Passing to the limit in n and applying the previous lemma

$$\alpha\int_\Omega |\nabla u^+|^2 \le \int_\Omega E\nabla\frac{u_+^2}{2} + \int_\Omega fu^+ \le \|f\|_{(2^*)'}\|u^+\|_{2^*} \le \frac{1}{S}\|f\|_{(2^*)'}\|\nabla u^+\|_2.$$

We recover the estimate. □

Lemma 4.3. *Let $E \in L^r(\Omega)^N$ for $r > 1$ with $\operatorname{div} E \ge 0$ in $D'(\Omega)$. Then, there exists a sequence $E_n \in W^{1,\infty}(\Omega)$ with $\operatorname{div} E_n \ge 0$ such that $E \to E$ in $L^r(\Omega)^N$.*

Proof. We use a similar decomposition to [22, Theorem 1.5] (done there for $r = 2$). First, we define

$$\begin{gathered}-\Delta p^{(1)} = \operatorname{div} E \quad \text{in } \Omega,\\ p^{(1)} = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

By well-known results we obtain a unique solution $p^{(1)} \in W_0^{1,r}(\Omega)$. Take $E^{(1)} = \nabla p^{(1)}$. Lastly, take $E^{(2)} = E - E^{(1)} \in L^r(\Omega)$. Notice that $\operatorname{div} E^{(2)} = 0$. Due to [19], $E^{(2)}$ admits a divergence-free extension to $L^r(\mathbb{R}^d)$, which we denote $\widetilde{E}^{(2)}$. We can take a family of mollifiers ρ_n, and $E_n^{(2)} = \widetilde{E}^{(2)} * \rho_n \in W^{1,\infty}(\Omega)$. Now let $0 \le g^{(1)} = \operatorname{div} E^{(1)} \in W^{-1,r'}(\Omega)$. Let $g_n^{(1)} \ge 0$ be a sequence of $C_c^\infty(\Omega)$ functions with $g_n^{(1)} \to g^{(1)}$ in $L^{r'}(\Omega)$. Take $p_n^{(1)}$ the unique solution to

$$\begin{gathered}-\Delta p_n^{(1)} = g_n^{(1)} \quad \text{in } \Omega,\\ p_n^{(1)} = 0 \quad \text{in } \partial\Omega.\end{gathered}$$

Finally, define $E_n^{(1)} = \nabla p_n^{(1)} \in W^{1,\infty}(\Omega)$. It is now easy to see that $E_n^{(i)} \to E^{(i)}$ in $L^r(\Omega)^N$ for $i = 1, 2$, and the proof is complete. □

Theorem 4.4. *Let $f \in L^m(\Omega)$ and $E \in L^r(\Omega)$ such that $0 \le \operatorname{div} E \in \mathcal{D}'(\Omega)$ and*

$$\begin{gathered}\frac{1}{\min\{2^*, m^{**}\}} + \frac{1}{r} \le 1 \quad \text{if } \frac{2N}{N+2} \le m \le \frac{N}{2}\\ \frac{1}{2^*} + \frac{1}{r} \le 1 \quad \text{otherwise.}\end{gathered} \tag{4.2}$$

Then, taking $q = \min\{2, m^*\}$ (using formally $m^* = \infty$ for $m \ge N$) there exists a solution of: $u \in W_0^{1,q}(\Omega)$ such that

$$\int_\Omega M(x)\nabla u\nabla v = \int_\Omega uE\nabla v + \int_\Omega fv, \quad \forall v \in W_0^{1,q'}(\Omega). \tag{4.3}$$

Furthermore, if $m \ge \frac{2N}{N+2}$ and $r \ge N$ it is the unique solution of (3.3).

Proof. Let $f_k = T_k(f)$ where T_k is the cut-off function. We consider E_k constructed in Lemma 4.3. By Proposition 3.6 there exists a unique weak u_k solution of (3.3). Since the $\cdot^*$ operation is monotone, then $q^* = \min\{2^*, m^{**}\}$. The sequence u_k is uniformly bounded in $W_0^{1,q}(\Omega)$. Therefore, by the Sobolev embedding theorem, it is uniformly bounded on $L^{q^*}(\Omega)$. Up to a subsequence, there exists $u \in W_0^{1,q}(\Omega)$ such that

$$\begin{gathered}\nabla u_k \rightharpoonup \nabla u \quad \text{in } L^q(\Omega)\\ u_k \rightharpoonup u \quad \text{in } L^{q^*}(\Omega).\end{gathered}$$

Since $M \in L^\infty(\Omega)^{N\times N}$, $E_k \to E \in L^r(\Omega)^N$ strongly and (4.2) we have that

$$\begin{gathered}M(x)\nabla u_k \rightharpoonup M(x)\nabla u \quad \text{in } L^q(\Omega)\\ u_k E_k \rightharpoonup uE \quad \text{in } L^1(\Omega).\end{gathered}$$

Therefore, we can pass to the limit in the weak formulation for $v \in W_0^{1,\infty}(\Omega)$. If $m \geq \frac{2N}{N+2}$ and $r \geq N$, then $uE \in L^2(\Omega)$, and it is a solution of (3.3) by approximation. □

5. Convection with singularity at one point

With the approach developed in this paper we are able to study the special situation

$$E = A\frac{x}{|x|^2} \quad \text{where } A > 0 \tag{5.1}$$

which is somehow in the limit of theory since it is not in $L^N(\Omega)$, but it is in $L^r(\Omega)$ for $r \in [1, N)$. In [5] the authors examined the framework of drifts such that

$$|E| \leq \frac{|A|}{|x|}, \tag{5.2}$$

The authors show existence of solutions u under (5.2), where the summability is reduced as $|A|$ is increased. They proved the following result.

Theorem 5.1 ([5]). *Let $f \in L^m(\Omega)$ and $|E| \leq |A|/|x|$. Then, there exists a solution u the solution of* (1.1) *and*

1. *If $|A| < \frac{\alpha(N-2m)}{m}$ and $m \in [\frac{2N}{N+2}, \frac{N}{2})$ then $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$.*
2. *If $|A| < \frac{\alpha(N-2m)}{m}$ and $m \in (1, \frac{2N}{N+2})$ then $u \in W_0^{1,m^*}(\Omega)$.*
3. *If $|A| < \alpha(N-2)$ and $m = 1$ then $\nabla u \in (M^{\frac{N}{N-1}}(\Omega))^N$ and $u \in W_0^{1,q}(\Omega)$, for every $q < \frac{N}{N-1}$.*

Above, $M^{\frac{N}{N-1}}$ denotes the Marcinkiewicz space (see [5] for the definition and some properties). The argument in [5] is based on Hardy's inequality

$$\Big(\frac{N-2}{N}\Big)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \leq \int_{\mathbb{R}^N} |\nabla u|^2. \tag{5.3}$$

We are able to extend this result to distinguish depending on the sign of A. Our result is the following theorem.

Theorem 5.2. *Let $f \in L^m(\Omega)$ for some $m > 1$ and (5.1). Then, there exists a solution u_A of (4.3), and it satisfies the estimates in Proposition 3.6. Furthermore, $u_A \to 0$ as $A \to \infty$ in the sense that*

$$\int_\Omega \frac{|u_A(x)|}{|x|^2} \le \frac{1}{A(N-2)} \int_\Omega |f|.$$

We point out that, if $m > \frac{2N}{N+2}$, we have furthermore $u_A E \in L^2(\Omega)$.

Proof. Since $N \ge 3$ we know that $|E| \in L^2(\Omega)$ and that

$$\operatorname{div} E = r^{1-N} \frac{\partial}{\partial r}(r^{N-1} A r^{-1}) = \frac{A(N-2)}{|x|^2} \tag{5.4}$$

is non-negative, and it is in $L^1(\Omega)$. Then, we have satisfied the existence theory of Theorem 3.1. Because of Proposition 3.6 and (5.4) the estimate follows. $\square$

6. Convection with singularity on the boundary

The aim of this section is to understand the case where E is regular inside Ω but blows up towards $\partial\Omega$. For the sake of simplicity we present an example, which as mentioned in Section 7 can be generalized, but the computations become quite technical. Let us consider φ_1 the first eigenfunction of $-\Delta$ with Dirichlet boundary conditions, i.e.,

$$\begin{gathered} -\Delta\varphi_1 = \lambda_1\varphi_1 \text{in } \Omega, \\ \varphi_1 = 0 \text{on } \partial\Omega. \end{gathered}$$

We normalize it so that $\|\nabla\varphi_1\|_{L^\infty} = 1$. It is known that there exists $C > 0$ such that

$$0 < C \operatorname{dist}(x, \partial\Omega) \le \varphi_1(x) \le C^{-1} \operatorname{dist}(x, \partial\Omega), \quad \forall x \in \Omega,$$

and near $\partial\Omega$ we have that

$$|\nabla\varphi_1(x)| \ge C > 0.$$

We focus our efforts on the particular case

$$E = -\varphi_1^{-1-\gamma}\nabla\varphi_1, \quad \text{for some } \gamma > 0, \tag{6.1}$$

and $f \in L^\infty_c(\Omega)$, the space of bounded functions with compact support in Ω.

The aim of this section is to prove the following theorem.

Theorem 6.1. *Let E be given by (6.1), $M = I$ and $f \in L^\infty_c(\Omega)$. Then there exists a unique $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$ such that $uE \in L^\infty(\Omega)$ and u is a weak solution in the sense that (3.3) holds. Furthermore, u is flat on the boundary in the sense that*

$$\text{for all } \alpha > 1 \text{ we have that } |u(x)| \le C_\alpha \operatorname{dist}(x, \partial\Omega)^\alpha, \text{ for a.e. } x \in \Omega. \tag{6.2}$$

We will give the proof below. First, we prove positivity in the interior.

Proposition 6.2. *In the assumptions of Theorem 6.1 if $f \ge 0$ and $\int_\Omega f > 0$, then $u > 0$ in Ω.*

Proof. Let $\Omega_\eta = \{x \in \Omega : d(x, \Omega) > \eta\}$. Consider u_η the solution of (1.1) with E given by (6.1) and $u_\eta = 0$ in $\partial\Omega_\eta$. Notice that E is smooth on Ω_η for $\eta > 0$. Since we already know from Theorem 4.1 that $u \ge 0$ in Ω, the classical comparison principle in Ω_η ensures that $u_\eta \le u$ for any $\eta \ge 0$. Take $\eta > 0$ small enough so that $\int_{\Omega_\eta} f > 0$. Then, by the “classical” strong maximum principle we obtain $u_\eta > 0$ in Ω_η, and the proof is complete. $\square$

It is immediate to compute that

$$\begin{aligned}\operatorname{div} E &= (1+\gamma)\varphi_1^{-2-\gamma}|\nabla\varphi_1|^2 - \varphi_1^{-1-\gamma}\Delta\varphi_1 \\ &= (1+\gamma)\varphi_1^{-2-\gamma}|\nabla\varphi_1|^2 + \lambda_1\varphi_1^{-\gamma}\,.\end{aligned}$$

Hence $\operatorname{div} E(x) \geq c\operatorname{dist}(x,\partial\Omega)^{-2-\gamma}$ near the boundary. Notice that E and $\operatorname{div} E$ are not in $L^1(\Omega)$. We start the proof with a lemma.

Lemma 6.3. *In the assumptions of Theorem 4.4, assume furthermore that* $\operatorname{div} E \in L^1_{\rm loc}(\Omega)$*. Then* $u\operatorname{div} E \in L^1(\Omega)$*, still satisfying estimate* (3.8)*.*

Proof. We consider the approximating sequence for Theorem 4.4. For the approximation we know that

$$\int_\Omega |u_n| \operatorname{div} E_n \leq \int_\Omega |f|.$$

Let us fix $K \Subset \Omega$. We have that

$$\int_K |u_n| \operatorname{div} E_n \leq \int_\Omega |f|.$$

Since we know that $\operatorname{div} E_n \to \operatorname{div} E$ in $L^1(K)$, we have that, up to a further subsequence, $\operatorname{div} E_n$ converges a.e. in K. Hence, applying Fatou's lemma

$$\int_K |u| \operatorname{div} E \leq \int_\Omega |f|.$$

Since this estimate is uniform in K, we can take $K_h = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \geq h\}$ and deduce, as $h \to 0$, that (3.8) holds. $\square$

The solution found in Theorem 6.1 is unique in a certain class. We provide a uniqueness result extending Theorem 4.1, which can itself be generalized to a larger framework.

Lemma 6.4. *Assume that* $u \in H^1_0(\Omega)$*,* $E \in L^\infty_{\rm loc}(\Omega)$*,* $u|E| \in L^2(\Omega)$*,* $\operatorname{div} E \geq 0$ *distributionally, and* $f \in L^{\frac{2N}{N+2}}(\Omega)$*. Then*

$$\|\nabla u^+\|_2 \leq \frac{1}{\alpha S_2}\|f^+\|_{\frac{2N}{N+2}}\,.$$

In particular, there is at most one weak solution in $H^1_0(\Omega)$ *of* (1.1)*.*

Proof. We want to repeat the argument in Theorem 4.1, i.e., taking $v = u_+$ in the weak formulation and using that

$$-\int_\Omega uE\cdot\nabla u_+ \geq 0.$$

We prove this formula by approximation. Take $\eta \in C^\infty_c(\Omega)$. There exists $K \Subset \Omega$ and $\phi_m \in C^\infty_0(K)$ such that $\phi_m \to u_+\eta$ in $H^1_0(\Omega)$. We have that

$$-\int_\Omega \phi_m E\cdot\nabla\phi_m = \langle \operatorname{div} E, \frac{\phi_m^2}{2}\rangle \geq 0.$$

Since $E \in L^\infty(K)$, we pass to the limit to deduce

$$-\int_\Omega (u_+\eta)\cdot E\nabla(u_+\eta) \geq 0.$$

Now we expand

$$\int_\Omega (u_+\eta)E\cdot\nabla(u_+\eta) = \int_\Omega u_+^2\eta E\cdot\nabla\eta + \int_\Omega u_+\eta^2 E\cdot\nabla u_+$$

Now we take $\eta_m \nearrow 1$. In particular $\eta_m(x) = \eta_0(m\varphi_1(x))$ where η_0 is non-decreasing, $\eta_0(s) = 0$ if $s \le 1$ and $\eta_0(s) = 1$ if $s > 2$. Clearly $\|\nabla\eta_m\|_{L^\infty} \le Cm$. Since $u_+ \in H_0^1(\Omega)$, then $u_+(x)/\varphi_1(x) \in L^2(\Omega)$ by Hardy's inequality. And we compute

$$\Big|\int_\Omega u_+^2\eta_m\cdot E\nabla\eta_m\Big| \le \int_{\varphi_1(x)\le\frac{1}{m}} \frac{u_+}{\varphi_1}\frac{\varphi_1}{m}|uE|Cm \le C\int_{\varphi_1(x)\le\frac{1}{m}} \frac{u_+}{\varphi_1}|uE| \to 0$$

since $\frac{u_+}{\varphi_1}|uE| \in L^1(\Omega)$ and the size of the domain tends to zero. We conclude, by Dominated Convergence that

$$0 \ge \int_\Omega (u_+\eta_m)\cdot E\nabla(u_+\eta_m) \to \int_\Omega u_+E\cdot\nabla u_+ = \int_\Omega uE\cdot u_+. \qquad \square$$

We are finally ready to prove the main result.

Proof of Theorem 6.1. The uniqueness claim is proven in Lemma 6.4. We now prove the existence and bounds by approximation. We can assume, without loss of generality, that $f \ge 0$, and construct approximations of E given by

$$E_\ell = -(\varphi_1 + \frac{1}{\ell})^{-1-\gamma}\nabla\varphi_1.$$

Clearly $E_\ell \in L^\infty(\Omega)$. These satisfy the assumptions of Theorem 3.1. Hence, there exists a weak solution $u_\ell \in H_0^1(\Omega)$ of (1.1) where $E = E_\ell$. We compute

$$\operatorname{div} E_\ell = (1+\gamma)(\varphi_1 + \frac{1}{\ell})^{-2-\gamma}|\nabla\varphi_1|^2 + \lambda_1(\varphi_1 + \frac{1}{\ell})^{-1-\gamma}\varphi_1.$$

This is non-negative. Hence, from Theorem 4.1 we have that

$$\|\nabla u_\ell\|_{L^2} \le C\|f\|_{L^\infty}.$$

Splitting the behaviour near the boundary and away from the boundary, it is easy to see that $\operatorname{div} E_\ell \ge c_0 > 0$ uniformly. Therefore, by Proposition 3.6 we have that

$$\|u_\ell\|_{L^\infty} \le \frac{\|f\|_{L^\infty}}{c_0}. \tag{6.3}$$

Now we must construct barrier functions. Select a single $\alpha > 1$ and the barrier

$$U = \frac{1}{\alpha}(\varphi_1 + \frac{1}{\ell})^\alpha.$$

We drop the dependence on ℓ and α to make the presentation below more readable. Plugging it into the equation we obtain

$$\begin{aligned}
&-\Delta U + \operatorname{div}(UE_\ell)\\
&= -\Delta U + \nabla U\cdot E_\ell + U\operatorname{div} E_\ell\\
&= -(\alpha-1)(\varphi_1+\frac{1}{\ell})^{\alpha-2}|\nabla\varphi_1|^2 + \lambda_1(\varphi_1+\frac{1}{\ell})^{\alpha-1}\varphi_1 - (\varphi_1+\frac{1}{\ell})^{\alpha-2-\gamma}|\nabla\varphi_1|^2\\
&\quad + (1+\gamma)(\varphi_1+\frac{1}{\ell})^{\alpha-2-\gamma}|\nabla\varphi_1|^2 + \lambda_1(\varphi_1+\frac{1}{\ell})^{\alpha-1-\gamma}\varphi_1\\
&\ge \Big(\gamma(\varphi_1+\frac{1}{\ell})^{-\gamma} - (\alpha-1)\Big)(\varphi_1+\frac{1}{\ell})^{\alpha-2}|\nabla\varphi_1|^2.
\end{aligned}$$

This is non-negative if $\varphi_1(x) + \frac{1}{m} \le (\frac{\alpha-1}{\gamma})^{-1/\gamma}$. There exists $\eta_\alpha > 0$ small enough such that

$$f(x) = 0 \quad \text{and} \quad \varphi_1(x) \le \frac{1}{2}(\frac{\alpha-1}{\gamma})^{-1/\gamma}, \quad \forall x \text{ such that } \operatorname{dist}(x, \partial\Omega) \le \eta_\alpha.$$

We will use the neighborhood of the boundary $A_\alpha = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \eta_\alpha\}$. Also, we consider the candidate super-solution

$$\overline{u}(x) = U(x)\Big(\frac{\alpha}{\min_{\operatorname{dist}(x,\partial\Omega)=\eta_\alpha} \varphi_1(x)^\alpha} + \frac{\alpha}{c_0} \frac{\|f\|_{L^\infty}}{\min_{\operatorname{dist}(x,\partial\Omega)\ge\eta_\alpha} \varphi_1(x)^\alpha}\Big).$$

We denote the constant on the right-hand side as C_α. Using the first term of C_α, $\overline{u} \ge u$ when $\operatorname{dist}(x, \partial\Omega) = \eta_\alpha$. Also, $\overline{u} = \frac{1}{m^\alpha} \ge 0 = u$ on $\partial\Omega$. Let us call

$$\overline{f} = -\Delta\overline{u} + \operatorname{div}(\overline{u}E_\ell).$$

By the previous computations, if $\ell \ge 2(\frac{\alpha-1}{\gamma})^{\frac{1}{\gamma}}$, we have $\overline{f} \ge 0 = f$ in A_α, and clearly $\overline{f} \in L^\infty(A_\alpha)$. Hence, by Theorem 4.1 we have that

$$0 \le u_\ell(x) \le \overline{u}(x), \quad x \in A_\alpha.$$

Also, by (6.3) and the second part of C_α, we have that

$$0 \le u_\ell(x) \le \overline{u}(x), \quad x \in \Omega \setminus A_\alpha.$$

Eventually, we deduce that for any $\alpha > 1$ we have

$$0 \le u_\ell(x) \le \frac{C_\alpha}{\alpha}(\varphi_1 + \frac{1}{\ell})^\alpha, \quad \forall x \in \Omega \text{ and } \ell \ge 2(\frac{\alpha-1}{\gamma})^{\frac{1}{\gamma}}.$$

In particular, picking $\alpha = \gamma + 1$ we deduce that

$$|u_\ell E_\ell| \le \frac{C_{\gamma+1}}{\gamma+1}\|\nabla\varphi_1\|_{L^\infty} = \frac{C_{\gamma+1}}{\gamma+1}.$$

We deduce that, up to a subsequence,

$$u_\ell \to u \text{ a.e. and strongly in } L^2 \quad \text{and} \quad u_\ell \rightharpoonup u \text{ weakly in } H_0^1(\Omega).$$

This implies that $u_\ell E_\ell \to uE$ a.e. And hence uE is bounded. Passing to the limit in the weak formulation by the Dominated Convergence Theorem, the result is proven. $\square$

Remark 6.5. Notice that the construction of the super-solution above can be done in any dimension $N \ge 1$. However, most of the results in the rest of the paper are only available for $N \ge 3$.

Remark 6.6. For Schrödinger-type equations $-\Delta u + Vu = f$, it is known that if the potential V is greater than $\operatorname{dist}(x, \partial\Omega)^{-2}$ and f is compactly supported, then u is flat on the boundary, in the sense that $|u| \le C \operatorname{dist}(x, \partial\Omega)^{1+\varepsilon}$. This means that $\partial_n u = 0$ on $\partial\Omega$. This means that it satisfies Dirichlet and Neumann homogeneous boundary conditions. And it can be extended by 0 outside Ω with higher regularity than H^1. In contrast, the exponent γ in the above result can not be taken as $\gamma = 0$ in order to get flat solutions. Indeed, the convection term $E \cdot \nabla\varphi_1$, in the above computations, is more singular than the term $\varphi_1 \operatorname{div} E$. A very explicit example can be done in one dimension: if we consider $E = -Cx^{-1}$ then this drift does not generate flat solutions since if we take $U(x) = x^\alpha$ then

$$-U'' + (EU)' = (-\alpha x^{\alpha-1} - Cx^{\alpha-1})' = -(\alpha + C)(\alpha - 1)x^{\alpha-2},$$

and this is a supersolution only if $\alpha \le 1$.

Corollary 6.7. *In the hypothesis of Theorem 6.1 replace $f \in L^\infty_c(\Omega)$ by*
$$|f(x)| \le C \operatorname{dist}(x, \partial\Omega)^\omega \quad \text{for } 0 \le \omega \le \gamma + 1.$$
Then
$$|u(x)| \le \operatorname{dist}(x, \partial\Omega)^\alpha \quad \text{for all } \alpha \in (1, \gamma + 2 - \omega)\,.$$

Proof. We maintain the notation of the proof of Theorem 6.1. We have already shown that, on a neighborhood of the boundary,
$$-\Delta U + \operatorname{div}(U E_m) \ge \frac{\gamma}{2}(\varphi_1 + \frac{1}{m})^{\alpha-2-\gamma}|\nabla\varphi_1|^2 \ge c_1(\varphi_1 + \frac{1}{m})^{\alpha-2-\gamma} \ge c_2|f|.$$
For α in the range $(1, \gamma + 2 - \omega)$, we can take as a supersolutions for the approximating sequence
$$\overline{u}(x) = U(x)\Big(\frac{1}{c_2} + \frac{\alpha}{\min_{\operatorname{dist}(x,\partial\Omega)=\eta_\alpha} \varphi_1(x)^\alpha} + \frac{\alpha}{c_0}\frac{\|f\|_{L^\infty}}{\min_{\operatorname{dist}(x,\partial\Omega)\ge\eta_\alpha} \varphi_1(x)^\alpha}\Big).$$
And the rest of the proof remains as in Theorem 6.1. □

7. Further remarks, extensions, and open problems

(1) We point that the proofs of our estimates can be extended to many non-linear settings.

(2) Theorem 6.1 admits many generalizations. For instance, one can consider the case $|E| \le c_0 \operatorname{dist}(x, \partial\Omega)^{-\gamma-1}$ with $\operatorname{div} E \ge c_1 \operatorname{dist}(x, \partial\Omega)^{-\gamma-2}$ up to suitable conditions on the constants. Also, the techniques in this paper could be extended to the situation where $\operatorname{dist}(x, \partial\Omega)$ is replaced by $\operatorname{dist}(x, \Gamma)$ with a suitable part $\Gamma \subset \partial\Omega$. The case Γ an interior manifold can also be studied.

(3) Including a non-negative potential. The same analysis can be performed on the equation
$$-\operatorname{div}(M(x)\nabla u) + a(x)u = -\operatorname{div}(uE(x)) + f(x)$$
when $a \ge 0$. As above, our approach allows for less regularity in a than most previous literature, e.g. $a \in L^1_{\rm loc}(\Omega)$. Furthermore, one will then obtain
$$\int_\Omega |u|(a + \operatorname{div} E) \le \int_\Omega |f|.$$
Hence, one can reduce the hypothesis to $a + \operatorname{div} E \ge 0$ in the whole analysis.

(4) The study of $a \equiv 1$ is useful in the study of the evolution problem
$$u_t - \operatorname{div}(M(x)\nabla u) + \operatorname{div}(uE(x)) = 0.$$
For the study of this problem one can write $u_t + Au = 0$ where
$$Au = -\operatorname{div}(M(x)\nabla u) + \operatorname{div}(uE(x)).$$
To obtain solutions in semigroup form in L^p (where $1 \le p \le +\infty$), following the theory of accretive operators, it is sufficient that,
$$\|u\|_{L^p} \le \|u + \lambda Au\|_{L^p}.$$
Letting $f = u + \lambda Au$, this is precisely what we have proven above, where $M = \lambda I$ and $a \equiv 1$. See also [6].

(5) We point out that when $|E| \le |A|/|x|$, we have that, if $m > \frac{2N}{N+2}$ then $u|E| \in L^2(\Omega)$. It seems possible to extend the uniqueness result (6.4) to this setting.

Acknowledgements. This article was started during the doctoral course of LB in Madrid in 2019, funded by the Instituto de Matemática Interdisciplinar. The research of J. I. Díaz was partially supported by the project PID2020-112517GB-I00 of the DGISPI (Spain) and the Research Group MOMAT (Ref. 910480) of the UCM. The research of D. Gomez-Castro was supported by the Advanced Grant Nonlocal-CPD (Nonlocal PDEs for Complex Particle Dynamics: Phase Transitions, Patterns and Synchronization) of the European Research Council Executive Agency (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 883363) and RYC2022-037317-I from the Spanish Government.

References

[1] L. Boccardo; Some developments on Dirichlet problems with discontinuous coefficients. *Bolletino dell Unione Matematica Italiana*, 2(1):285–297, 2009.

[2] L. Boccardo; Dirichlet problems with singular convection term and applications; *J. Differential Equations*, 258 (2015), 2290-2314.

[3] L. Boccardo; The impact of the zero order term in the study of Dirichlet problems with convection or drift terms. *Revista Matemática Complutense* https://doi.org/10.1007/s13163-022-00434-1

[4] L. Boccardo, T. Gallouët; Nonlinear elliptic equations with right hand side measures; *Comm. Partial Differential Equations*, 17 (1992), 641–655.

[5] L. Boccardo, L. Orsina; Very singular solutions for linear Dirichlet problems with singular convection terms. *Nonlinear Analysis* 194, 2019.

[6] L. Boccardo, L. Orsina, M. M. Porzio: Regularity results and asymptotic behavior for a noncoercive parabolic problem; *J. Evol. Equ.*, 21 (2021), 2195-2211.

[7] H. Brezis, T. Cazenave, Y. Martel, A. Ramiandrisoa; Blow up for $u_t - \Delta u = g(u)$ revisited, Advances in Diff. Eq., 1 (1996), 73–90.

[8] H. Brezis, W. Strauss; Semilinear second order elliptic equations in L^1, *J. Math.Soc. Japan*, 25 (1974), 831-844.

[9] J. I. Díaz; *Nonlinear Partial Differential Equations and Free Boundaries*. Pitman, London, 1985.

[10] J. I. Díaz; On the ambiguous treatment of the Schrödinger equation for the infinite potential well and an alternative via singular potentials: the multi-dimensional case. *SeMA-Journal*, 74 3 (2017), 225-278.

[11] J. I. Díaz; Correction to: On the ambiguous treatment of the Schrödinger equation for the infinite potential well and an alternative via singular potentials: the multi-dimensional case. *SeMA-Journal*, 75 (2018), no. 3, 563–568.

[12] J. I. Díaz, D. Gómez-Castro, J.-M. Rakotoson; Existence and uniqueness of solutions of Schrödinger type stationary equations with very singular potentials without prescribing boundary conditions and some applications. *Differential Equations & Applications*, 10(1):47–74, 2018.

[13] J. I. Díaz, D. Gómez-Castro, J. M. Rakotoson, R. Temam; Linear diffusion with singular absorption potential and/or unbounded convective flow: The weighted space approach. *Discrete and Continuous Dynamical Systems*, 38(2) (2018), 509–546.

[14] J. I. Díaz, D. Gómez-Castro, J. L. Vázquez; The fractional Schrödinger equation with general nonnegative potentials. The weighted space approach. *Nonlinear Analysis*, 177 (2018), 325–360.

[15] J. Droniou; Non-coercive linear elliptic problems. *Potential Analysis*, 17(2) (2002), 181-203.

[16] D. Gilbarg, N.S. Trudinger; *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, 1977.

[17] D. Gómez-Castro, J. L. Vázquez; The fractional Schrödinger equation with singular potential and measure data. *Discrete & Continuous Dynamical Systems - A*, 39(12) (2019), 7113–7139.

[18] L. Orsina, A. C. Ponce; Hopf potentials for Schrödinger operators, *Anal PDE*, 11(8) (2018), 2015–2047.

[19] T. Kato, M. Mitrea, G. Ponce, M. Taylor; Extension and representation of divergence-free vector fields on bounded domains. *Mathematical Research Letters*, 7(5) (200), 643-650.

[20] H. Kim, T. P. Tsai; Existence, uniqueness, and regularity results for elliptic equations with drift terms in critical weak spaces. *SIAM Journal on Mathematical Analysis*, 52(2) (2020), 1146-1191.

[21] G. Stampacchia: Le probléme de Dirichlet pour les equations elliptiques du second ordre a coefficients discontinus. *Ann. Inst. Fourier (Grenoble)*, 15 (1965) 189-258.

[22] R. Temam. *Navier-Stokes equations: theory and numerical analysis* (Vol. 343). American Mathematical Soc., 1979.

Lucio Boccardo
Istituto Lombardo & Sapienza Università di Roma, Italy
Email address: boccardo@mat.uniroma1.it

Jesús Ildefonso Díaz
Inst. de Matemática Interdisciplinar, Fac. de Matemáticas, Univ. Complutense de Madrid, Spain
Email address: jidiaz@ucm.es

David Gómez-Castro
Inst. de Matemática Interdisciplinar, Fac. de Matemáticas, Univ. Complutense de Madrid, Spain
Email address: dgcastro@ucm.es