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# EXISTENCE OF SOLUTIONS TO QUASILINEAR SCHRÖDINGER EQUATIONS WITH EXPONENTIAL NONLINEARITY

UBERLANDIO B. SEVERO, BRUNO H. C. RIBEIRO, DIOGO DE S. GERMANO

Abstract. In this article we study the existence of solutions to quasilinear Schrödinger equations in the plane, involving a potential that can change sign and a nonlinear term that may be discontinuous and exhibit exponential critical growth. To prove our existence result, we combine the Trudinger-Moser inequality with a fixed point theorem.

## 1. Introduction and main result

In this work we consider the quasilinear equation

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = f(x,u) + \lambda|u|^{p-2}u + h(x)g(u) \quad (1.1)$$

in $\mathbb{R}^2$, where $g:\mathbb{R}\to\mathbb{R}_+$ is a function of class $C^1$, $V:\mathbb{R}^2\to\mathbb{R}$ is a potential that can change sign, $f:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ is a measurable function, which may have exponential critical growth of Trudinger-Moser type, $\lambda\in\mathbb{R}$ is a parameter, $p\geq 2$ and $h\in L^q(\mathbb{R}^2)$ for some $1<q\leq 2$.

When $g(s)\equiv 1$ and $\lambda=0$, equation (1.1) becomes the nonhomogeneous semilinear Schrödinger equation

$$-\Delta u + V(x)u = f(x,u) + h(x) \quad \text{in } \mathbb{R}^2, \quad (1.2)$$

which has been studied by several researchers, see for example [1, 2, 5, 12, 14], [16] for a problem in a bounded domain. Usually, to obtain the existence and multiplicity of solutions, the authors require a restriction on the norm of the term $h(x)$ and thus (1.2) is regarded as a perturbation of the equation $-\Delta u + V(x)u = f(x,u)$, $x\in\mathbb{R}^2$. In (1.1), $h(x)g(u)$ can be viewed as the perturbation term.

The study of (1.1) is also related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$i\partial_t w = -\Delta w + W(x)w - \tilde{p}(x,|w|^2)w - \Delta[\rho(|w|^2)]\rho'(|w|^2)w, \quad (1.3)$$

where $w:\mathbb{R}\times\mathbb{R}^N\to\mathbb{C}$ is the unknown function, $W:\mathbb{R}^N\to\mathbb{R}$ is a given potential, $\rho:\mathbb{R}_+\to\mathbb{R}$ and $\tilde{p}:\mathbb{R}^N\times\mathbb{R}_+\to\mathbb{R}$ are real functions satisfying suitable conditions. Equation (1.3) has modeled many physical phenomena depending on the function $\rho$; for details see [3, 21, 22, 25].

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By considering standing wave solutions, i.e., solutions of the form $w(t,x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and $u$ is a real function, one knows that $w$ satisfies (1.3) if and only if $u(x)$ solves the elliptic equation

$$-\Delta u + V(x)u - \Delta[\rho(u^2)]\rho'(u^2)u = p(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $V(x) = W(x) - E$ and $p(x,u) = \tilde{p}(x,u^2)$. If we now use

$$g^2(u) = 1 + \frac{[(\rho(u^2))']^2}{2},$$

then (1.4) is transformed in the quasilinear elliptic equation (see [25])

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = p(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

which becomes (1.1) when $p(x,u) = f(x,u) + \lambda|u|^{p-2}u + h(x)g(u)$ and $N = 2$. Equation (1.5) have been extensively investigated in the literature depending on the function $g$. For example, if $g^2(s) = 1 + 2s^2$ then we obtain the superfluid film equation

$$-\Delta u + V(x)u - \Delta(u^2)u = p(x,u) \quad \text{in } \mathbb{R}^N,$$

which was studied for instance in [7, 13, 20, 21]. More generally, if we set $g^2(s) = 1 + 2\gamma^2(s^2)^{2\gamma-1}$, $\gamma > 1/2$, which corresponds to $\rho(s) = s^\gamma$, we obtain

$$-\Delta u + V(x)u - \gamma\Delta(|u|^{2\gamma})|u|^{2\gamma-2}u = p(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

which has been treated, for instance, in [19, 28].

Motivated by these physical and mathematical aspects, Equation (1.5) has attracted the attention of numerous researchers, leading to results of existence and multiplicity of solutions. Noteworthy contributions include the works [10, 15, 22, 25] in dimensions $N \geq 3$ and [23, 24] in the plane. In the later ones, the nonlinearity $p(x,u)$ is continuous and exhibits exponential critical growth in the sense of Trudinger-Moser inequality. Moreover, these studies assume that the potential $V(x)$ is continuous, bounded from below and the main results are obtained by exploiting variational methods. In [22], the authors studied (1.1), with $\lambda = 0$ and in dimension $N \geq 3$, considering potentials $V(x)$ that can be discontinuous and singular. Furthermore, they allow the nonlinearity to be discontinuous and to have critical growth. By applying a fixed point theorem, they prove that the problem has a weak solution by working in the space $D^{1,2}(\mathbb{R}^N)$.

In this article, under certain assumptions on $g(s)$, $V(x)$, $f(x,s)$, and $h(x)$, and by applying a fixed point theorem (see Lemma 2.6), as in [22] we show that (1.1) admits at least one weak solution. Here, the potential $V(x)$ has similar characteristics as in [22]. The nonlinear term $f(x,s)$ can be discontinuous and the critical exponential growth is allowed for it. We emphasize that the context in dimension two is more delicate because it makes no sense to work in the space $D^{1,2}(\mathbb{R}^2)$ and embedding available. Therefore, the strategy used to apply the fixed theorem is different from that of [22]. Our intention is to complement the study carried out in [22] and extend the results obtained in [23, 24].

Next, state the hypotheses on $g(s)$, $V(x)$ and $f(x,s)$. With respect to $g(s)$, we assume the following standard conditions:

- (A1) $g \in C^1(\mathbb{R}, \mathbb{R}_+)$ is even, $g'(s) \geq 0$ for all $s \geq 0$ and $g(0) = 1$;
- (A2) there exists $\alpha \geq 1$ such that $(\alpha - 1)g(s) \geq g'(s)s$ for all $s \geq 0$;
- (A3) $\lim_{s\to+\infty} \frac{g(s)}{s^{\alpha-1}} =: \beta > 0$.

Hypotheses of this type have been considered in [8, 10, 22]. Since $g(s) \geq 1$ for all $s \in \mathbb{R}$, the primitive $G(s) := \int_0^s g(t)dt$ is increasing, $G(0) = 0$ and its inverse $G^{-1}$ is also increasing. Hereafter, for convenience we denote $G_1 := G^{-1}(1) > 0$. The following function $g$ satisfies (A1)–(A3):

(a) $g(s) \equiv 1$ ($\alpha = 1$ and $\beta = 1$);
(b) $g(s) = (1 + 2s^2)^{1/2}$ ($\alpha = 2$ and $\beta = \sqrt{2}$);
(c) $g(s) = (1 + 2\gamma^2(s^2)^{2\gamma-1})^{1/2}$ ($\alpha = 2\gamma$ and $\beta = \sqrt{2}\gamma$).

They appear in the context of mathematical physics, as previously mentioned.

The existence of solutiona for equations of the form (1.5) has been discussed under various conditions on the potential $V(x)$ and the nonlinear term $p(x, s)$, see for instance [10, 15, 25, 26]. It is usually assumed that the potential is continuous and another condition that guarantees some compactness result. Inspired by [6, 22], we focus here on the case where $V$ can have discontinuity and change sign without requiring any additional condition to obtain compactness.

Denoting $V^{\pm} = \max\{\pm V, 0\}$ and inspired by [22, 27], we consider the following hypotheses on $V$:

(A4) $V^+ \in L^1_{loc}(\mathbb{R}^2)$ and there exists a constant $R_0 > 0$ such that
$$0 < V_0 := \inf_{|x| \geq R_0} V^+(x) \leq \sup_{|x| \geq R_0} V^+(x) < \infty;$$

(A5) $V^- \in L^{p_0}(B_{R_0})$ for some $1 < p_0 \leq \infty$, where $B_{R_0}$ denotes the open ball centered at the origin in $\mathbb{R}^2$ with radius $R_0$.

In the following, for simplicity, we assume that $R_0 \geq 1$ and consider a constant
$$V_\infty \geq \max\{R_0, G_1^2\} \frac{2|\lambda| G_1^{p-2}}{p} \tag{1.7}$$
satisfying $V^+(x) \leq V_\infty$ for almost every $|x| \geq R_0$. Note that $V$ can change sign and singularities can appear in some points of $\mathbb{R}^2$. A simple example of a potential $V(x)$ satisfying (A4) and (A5) is
$$V_\delta(x) = -\frac{\delta}{|x|^{1/\gamma}} \text{ for } |x| \leq R_0, \quad \text{and} \quad C_1 \leq V_\delta(x) \leq C_2 \text{ for } |x| > R_0, \tag{1.8}$$
for some $\delta, R_0 > 0$, $\gamma > 1$, and $0 < C_1 \leq C_2 < \infty$.

Problems involving (1.5) with critical nonlinearities in dimension $N \geq 3$ have been addressed for instance in [10, 17, 18]. For dimension two, we can cite [23, 24], which consider the nonlinearity with exponential critical growth. However, in these works the authors suppose that the nonlinearity is continuous. In this article, we consider a more general class of nonlinearities $f(x, u)$, i.e., motivated by [22] we introduce the following hypotheses on $f$:

(A6) for each measurable function $u : \mathbb{R}^2 \to \mathbb{R}$, the Nemytskiǐ function $N_f : \mathbb{R}^2 \to \mathbb{R}$ given by $N_f(x) = f(x, u(x))$ is measurable;

(A7) for almost every $x \in \mathbb{R}^2$, the quotient $\frac{f(x,s)}{g(s)}$ is nondecreasing in $s$.

(A8) there exist $C_1, C_2 > 0$, $1 < \sigma \leq \infty$, $\varsigma_0 > 0$, $\rho, \mu > 1$ and $k \in L^\sigma(\mathbb{R}^2)$ such that
$$|f(x, s)| \leq C_1 k(x)|s|^\rho + C_2(e^{\varsigma_0 (s^2)^\alpha} - 1)|s|^\mu, \quad \text{for all } (x, s) \in \mathbb{R}^2 \times \mathbb{R}.$$

Condition (A8) is inspired by the Trudinger-Moser inequality in the whole space (see Lemma 2.3). Note that the growth (A8) allows $f(x, s)$ to behave as $e^{(s^2)^\alpha}$ at infinity, which is the exponential critical growth for this class of problems, for more details see [23, 24].

Let us now consider the subspace of $H^1(\mathbb{R}^2)$ defined by

$$W = \{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V^+(x)u^2 \mathrm{d}x < \infty\}$$

endowed with the norm

$$\|u\| = \Big[\int_{\mathbb{R}^2} (|\nabla u|^2 + V^+(x)u^2)\mathrm{d}x\Big]^{1/2}. \tag{1.9}$$

By [27, Lemma 2.1], there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^2} (|\nabla u|^2 + V^+(x)u^2)\mathrm{d}x \geq C \int_{\mathbb{R}^2} u^2 \mathrm{d}x, \quad \text{for all } u \in W.$$

This inequality guarantees the embedding $W \hookrightarrow H^1(\mathbb{R}^2)$ being continuous and consequently $W \hookrightarrow L^t(\mathbb{R}^2)$ begin also continuous for all $t \geq 2$. Moreover, $W$ is a Banach space with the norm introduced in (1.9). For each $t \geq 2$, we consider

$$S_t := \inf_{u \in W\setminus\{0\}} \frac{\int_{\mathbb{R}^2} (|\nabla u|^2 + V^+(x)|u|^2)\mathrm{d}x}{\big(\int_{\mathbb{R}^2} |u|^t \mathrm{d}x\big)^{2/t}} > 0, \tag{1.10}$$

which is the best constant of the embedding $W \hookrightarrow L^t(\mathbb{R}^2)$. Recalling that $G_1 = G^{-1}(1)$, in addition to the hypotheses on $V$ we assume the condition

- (A9) $\|V^-\|_{L^{p_0}(B_{R_0})} < \Lambda S_{t_0}$, where $t_0 := 2p_0/(p_0 - 1)$ if $1 < p_0 < \infty$, and $t_0 = 2$ if $p_0 = \infty$ and $\Lambda = \frac{2|\lambda| G_1^p}{\alpha p V_\infty}$.

This number $\Lambda$ is a kind of control for applying the fixed point theorem. Observe that in Example (1.8), (A9) will satisfied choosing a positive $\delta$ appropriately.

We say that a function $u : \mathbb{R}^2 \to \mathbb{R}$ is a weak solution of (1.1) if $u \in H^1(\mathbb{R}^2)$ and for all $\varphi \in C_0^\infty(\mathbb{R}^N)$ it holds

$$\begin{aligned} &\int_{\mathbb{R}^2} g^2(u)\nabla u \nabla \varphi \mathrm{d}x + \int_{\mathbb{R}^2} g(u)g'(u)|\nabla u|^2 \varphi \mathrm{d}x + \int_{\mathbb{R}^2} V(x)u\varphi \mathrm{d}x \\ &= \int_{\mathbb{R}^2} f(x,u)\varphi \mathrm{d}x + \lambda \int_{\mathbb{R}^2} |u|^{p-2}u\varphi \mathrm{d}x + \int_{\mathbb{R}^2} h(x)g(u)\varphi \mathrm{d}x. \end{aligned} \tag{1.11}$$

Our main result is stated as follows.

**Theorem 1.1.** *Suppose that* (A1)–(A9) *are satisfied and* $h \in L^q(\mathbb{R}^N)$ *for some* $1 < q \leq 2$. *Furthermore, assume that* $\alpha \leq 2$ *and* $p \geq 2\alpha$, *and for each* $\lambda < 0$ *there exists* $\delta_0 > 0$ *such that* $\|h\|_q \leq \delta_0$. *Then* (1.1) *has at least one weak solution.*

Note that if $h \neq 0$ then the solution obtained is nonzero because $g(0) = 1$. For the proof of the theorem, we adapt some arguments in [22]. However, the situation here is more delicate because the exponential critical growth of $f(x,s)$, and the fact that we can not work with $D^{1,2}(\mathbb{R}^2)$ as in [22]. As the potential $v(x)$ and the nonlinear term $f(x,s)$ can be discontinuous, the variational methods are not used for treating this class of problems, and we believe that the fixed point technique is more effective.

The outline of this article is as follows. The forthcoming section is the reformulation of the problem and some preliminary results, including the Trudinger-Moser inequality used in the fixed theorem. Section 3 is dedicated to the proof of our main result.

## 2. Preliminaries

In this section we obtain some technical results and establish the appropriate framework to prove Theorem 1.1. In the definition of weak solution for (1.1) (see (1.11)), we first face the problem that the integrals involving the function $g$, which, depending on $g$, are not well defined for functions $u \in H^1(\mathbb{R}^2)$. To overcome this difficulty, we follow the idea used in [25] (see also [9]), and consider the change of variable

$$v = G(u) = \int_0^u g(s)\mathrm{d}s.$$

From (A1), we have $g(t) \geq 1$ for all $t \in \mathbb{R}$. Thus, $G$ is strictly increasing and hence invertible. For an easy reference, we list below the main properties of the functions $g$ and $G^{-1}$.

**Lemma 2.1.** *Under conditions* (A1)–(A3)*, we have the following properties:*

(1) $G^{-1}$ *is increasing and* $G, G^{-1}$ *are odd functions;*
(2) $0 < [G^{-1}(t)]' = \frac{1}{g(G^{-1}(t))} \leq 1 = \frac{1}{g(0)}$ *for all* $t \in \mathbb{R}$*;*
(3) $|G^{-1}(t)| \leq |t|$ *for all* $t \in \mathbb{R}$*;*
(4) $\frac{G^{-1}(t)}{\alpha} \leq \frac{t}{g(G^{-1}(t))} \leq G^{-1}(t)$ *for all* $t \geq 0$ *and* $\frac{[G^{-1}(t)]^2}{\alpha} \leq \frac{G^{-1}(t)t}{g(G^{-1}(t))} \leq [G^{-1}(t)]^2$ *for all* $t \in \mathbb{R}$*;*
(5) $\frac{|G^{-1}(t)|^{\alpha-1}}{g(G^{-1}(t))} \leq \frac{1}{\beta}$ *for all* $t \in \mathbb{R}$*;*
(6) $|G^{-1}(t)|^\alpha \leq \frac{\alpha}{\beta}|t|$ *for all* $t \in \mathbb{R}$*;*
(7) *if* $1 \leq \alpha \leq 2$ *then* $\frac{G^{-1}(t)}{g(G^{-1}(t))}$ *is nondecreasing for* $t \in \mathbb{R}$*;*
(8) *if* $1 \leq \alpha \leq 2$ *then there exist* $C_1, C_2 > 0$ *such that*
$$|g'(t)| \leq C_1 \quad \textit{and} \quad g(t) \leq C_2 + (\beta+1)|t| \quad \textit{for all } t \in \mathbb{R};$$
(9) *if* $1 \leq \alpha \leq 2$ *then* $[G^{-1}(t)]^{2\alpha}$ *is convex.*
(10) $\frac{G^{-1}(t)}{t^{1/\alpha}} \to (\frac{\alpha}{\beta})^{1/\alpha}$ *as* $t \to +\infty$*;*
(11) *it holds that*
$$|G^{-1}(t)| \geq \begin{cases} G_1|t|, & |t| \leq 1 \\ G_1|t|^{1/\alpha}, & |t| \geq 1. \end{cases}$$

The proof of the above lemma can be found in [22, 23], so we omit it here.

Using the change of variable $v = G(u)$, a simple calculation shows that we can transform (1.1) into the nonhomogeneous semilinear equation

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))} + \lambda\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))} + h(x) \tag{2.1}$$

in $\mathbb{R}^2$.

We say that $v : \mathbb{R}^N \to \mathbb{R}$ is a weak solution of (2.1) if $v \in W$ and

$$\begin{aligned} &\int_{\mathbb{R}^2} \nabla v \nabla w \mathrm{d}x + \int_{\mathbb{R}^2} V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} w\mathrm{d}x \\ &= \int_{\mathbb{R}^2} \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))} w\mathrm{d}x + \lambda\int_{\mathbb{R}^2} \frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))} w\mathrm{d}x + \int_{\mathbb{R}^2} h(x)w\mathrm{d}x, \end{aligned} \tag{2.2}$$

for all $w \in W$. The next lemma relates weak solutions of (2.1) to weak solutions of (1.1).

**Lemma 2.2.** *If $v \in W$ is a weak solution of* (2.1)*, then $u = G^{-1}(v)$ is a weak solution of* (1.1)*.*

*Proof.* First, since $\nabla u = \frac{1}{g(G^{-1}(v))}\nabla v$, $g(t) \geq 1$ for all $t \in \mathbb{R}$ and $|u| = |G^{-1}(v)| \leq |v|$, it follows that $u \in H^1(\mathbb{R}^2)$. Note that for each $\varphi \in C_0^\infty(\mathbb{R}^2)$, the function $w := g(G^{-1}(v))\varphi$ belongs to $H^1(\mathbb{R}^2)$. Indeed, we have

$$\nabla w = \varphi \frac{g'(G^{-1}(v))}{g(G^{-1}(v))}\nabla v + g(G^{-1}(v))\nabla\varphi.$$

Setting $\mathcal{K} = \operatorname{supp}(\varphi)$ and using properties (3) and (8) of Lemma 2.1 we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla w|^2 \mathrm{d}x &\leq 2\int_{\mathcal{K}} |\varphi|^2 \frac{[g'(G^{-1}(v))]^2}{[g(G^{-1}(v)]^2}|\nabla v|^2 \mathrm{d}x + 4C_2^2 \int_{\mathcal{K}} |\nabla\varphi|^2 \mathrm{d}x \\
&\quad + 4(\beta+1)^2 \int_{\mathcal{K}} v^2 |\nabla\varphi|^2 \mathrm{d}x \\
&\leq 2C_1^2\|\varphi\|_\infty^2 \int_{\mathcal{K}} |\nabla v|^2 \mathrm{d}x + 4C_2^2 \int_{\mathcal{K}} |\nabla\varphi|^2 \mathrm{d}x + 4(\beta+1)^2\|\nabla\varphi\|_\infty^2 \|v\|_2^2,
\end{aligned}$$

and so $|\nabla w| \in L^2(\mathbb{R}^2)$. Moreover, according to item (8) of Lemma 2.1 and since $V^+ \in L^\infty_{loc}(\mathbb{R}^2)$, we also have $(V^+)^{1/2} w \in L^2(\mathbb{R}^2)$. Therefore $w \in W$ and taking $w = g(G^{-1}(v))\varphi = g(u)\varphi$ in (2.2), it follows that (1.11) holds for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. Hence, $u = G^{-1}(v)$ is a weak solution to (1.1). $\square$

As a consequence of the previous lemma, to obtain weak solutions of (1.1), it is sufficient to look for weak solutions for (2.1). Next, we recall a version of the Trudinger-Moser inequality that holds in the whole space (see [4, 11]).

**Lemma 2.3.** *If $\varsigma > 0$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} (e^{\varsigma u^2} - 1)\mathrm{d}x < \infty. \tag{2.3}$$

*Moreover, if $0 < \varsigma < 4\pi$ and $\|u\|_2 \leq M$, then there exists a constant $C = C(\varsigma, M) > 0$ such that*

$$\sup_{\|\nabla u\|_2 \leq 1} \int_{\mathbb{R}^2} (e^{\varsigma u^2} - 1)\mathrm{d}x \leq C. \tag{2.4}$$

In many arguments, we will need of the following lemma.

**Lemma 2.4.** *Let $\varsigma > 0$ and $r \geq 1$. Then*

$$(e^{\varsigma s^2} - 1)^r \leq e^{r\varsigma s^2} - 1, \quad \textit{for all } s \in \mathbb{R}.$$

To prove the above lemma we only need to apply the inequality $(1+t)^r \geq 1 + t^r$ with $t = e^{\varsigma s^2} - 1 \geq 0$. As a consequence of Lemma 2.3 we establish an estimate which will be essential for our argument.

**Lemma 2.5.** *Suppose that* (A4) *holds. Let $v, \varphi \in W$ and $\varsigma, \mu > 0$. If $\|v\| \leq M$ and $\varsigma M^2 < 4\pi$ then there exists a constant $C = C(\varsigma, M, \mu) > 0$ such that*

$$\int_{\mathbb{R}^2} (e^{\varsigma v^2} - 1)|v|^\mu |\varphi| \mathrm{d}x \leq C\|v\|^\mu \|\varphi\|.$$

*Proof.* First, we choose $q_1 > 1$ sufficiently close to 1 such that

$$q_1 \varsigma M^2 < 4\pi \quad \text{and} \quad \sigma_1 := \frac{2q_1}{q_1 - 1} > \max\big\{2, \frac{2}{\mu}\big\}.$$

Thus, $1/q_1 + 1/\sigma_1 + 1/\sigma_1 = 1$ and applying the generalized Hölder inequality and the embedding $W \hookrightarrow L^s(\mathbb{R}^2)$, for $s \in [2, +\infty)$, one obtains

$$\begin{aligned}
\int_{\mathbb{R}^2} (e^{\varsigma v^2} - 1)|v|^{\mu}|\varphi| \mathrm{d}x &\le \Big( \int_{\mathbb{R}^2} (e^{q_1 \varsigma v^2} - 1) \mathrm{d}x \Big)^{1/q_1} \|v\|^{\mu}_{\mu\sigma_1} \|\varphi\|_{\sigma_1} \\
&\le C_1 \Big( \int_{\mathbb{R}^2} (e^{q_1 \varsigma v^2} - 1) \mathrm{d}x \Big)^{1/q_1} \|v\|^{\mu} \|\varphi\| \\
&\le C_1 \Big( \int_{\mathbb{R}^2} (e^{q_1 \varsigma M^2 \frac{v^2}{\|\nabla v\|_2^2}} - 1) \mathrm{d}x \Big)^{1/q_1} \|v\|^{\mu} \|\varphi\|.
\end{aligned}$$

Since $q_1 \varsigma M^2 < 4\pi$, the lemma is proved because of the Trudinger-Moser inequality (2.4). $\square$

For the convenience of the reader, some basic notions and notation are reproduced below. Let $X$ be a real Banach space. A nonempty subset $X_+ \neq \{0\}$ of $X$ is called an *order cone* if the following holds:

(i) $X_+$ is closed and convex;
(ii) if $u \in X_+$ and $\tau \ge 0$, then $\tau u \in X_+$;
(iii) if $u \in X_+$ and $-u \in X_+$, then $u = 0$.

We observe that an order cone $X_+$ naturally induces a partial order in $X$ as follows: $x \preceq y$ if and only if $y - x \in X_+$, and $(X, \preceq)$ is called an *ordered Banach space*. Moreover, if $\inf\{x, y\}$ and $\sup\{x, y\}$ exist for all $x, y \in X$ with respect to $\preceq$, then we say that $(X, \|\cdot\|)$ is a *lattice*. Furthermore, if $\|x^{\pm}\| \le \|x\|$ for all $x \in X$, with $x^+ := \sup\{0, x\}$ and $x^- := -\inf\{0, x\}$ then $(X, \|\cdot\|)$ is called a *Banach semilattice*.

Special examples of Banach semilattices are the spaces $L^q(\mathbb{R}^N)$, $W^{1,q}(\mathbb{R}^N)$ and $D^{1,2}(\mathbb{R}^N)$, if one considers the natural partial order $u \preceq v$ when $u \le v$ almost everywhere in $\mathbb{R}^N$.

Let $(X, \preceq)$ and $(\widetilde{X}, \lhd)$ be ordered Banach spaces. We say that an operator $G : X \to \widetilde{X}$ is increasing if and only if for all $x, y \in X$, $x \preceq y$ implies that $Gx \lhd Gy$.

A subset $B$ of $X$ has the fixed point property if every increasing operator $S : B \to B$ has a fixed point.

We now present a version of the fixed point result due to Carl and Heikkilä [6, Corollary 2.2], which we use for proving Theorem 1.1.

**Lemma 2.6.** *Let $X$ be a Banach semilattice which is reflexive. Then every closed ball of $X$ has the fixed point property.*

For more details in terms of definitions and results about ordered Banach spaces, we refer the reader to [6] and the references therein.

## 3. Proof of Theorem 1.1

We need to introduce some suitable operators to apply Lemma 2.6. First, we consider the operator $L : W \to W^*$ defined by

$$\begin{aligned}
\langle L(v), \varphi \rangle = &\int_{\mathbb{R}^2} \nabla v \nabla \varphi + \int_{\mathbb{R}^2} V^+(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \varphi \mathrm{d}x - \lambda \int_{\mathbb{R}^2} \frac{|G^{-1}(v)|^{p-2} G^{-1}(v)}{g(G^{-1}(v))} \varphi \mathrm{d}x \\
&- \int_{\mathbb{R}^2} h(x) \varphi \mathrm{d}x,
\end{aligned}$$

for $v, \varphi \in W$, where $W^*$ is the dual space of $W$ and its norm is denoted by $\|\cdot\|_*$. It is clear that for each $v \in W$, $L(v)$ is a linear mapping. Moreover, it follows from Hölder's inequality, items (2)-(3) of Lemma 2.1, and Sobolev's embedding that

$$\begin{aligned}
|\langle L(v), \varphi\rangle| &\leq \int_{\mathbb{R}^2} |\nabla v \nabla \varphi| \mathrm{d}x + \int_{\mathbb{R}^2} V^+(x) \frac{|G^{-1}(v)|}{g(G^{-1}(v))} |\varphi| \mathrm{d}x \\
&\quad + |\lambda| \int_{\mathbb{R}^2} \frac{|G^{-1}(v)|^{p-1}}{g(G^{-1}(v))} |\varphi| \mathrm{d}x + \int_{\mathbb{R}^2} |h||\varphi| \mathrm{d}x \\
&\leq \|v\|\|\varphi\| + \Big(\int_{\mathbb{R}^2} V^+(x) v^2 \mathrm{d}x\Big)^{1/2} \Big(\int_{\mathbb{R}^2} V^+(x) \varphi^2 \mathrm{d}x\Big)^{1/2} \\
&\quad + \|v\|_p \|\varphi\|_p + \|h\|_q \|\varphi\|_{q'} \\
&\leq \Big(2\|v\| + \frac{1}{S_p^{1/2}} \|v\|_p + \frac{1}{S_{q'}^{1/2}} \|h\|_q\Big) \|\varphi\|, \quad \text{for all } \varphi \in W,
\end{aligned}$$

which justifies that $L(v) \in W^*$.

**Lemma 3.1.** *Under the hypothesis* (A4)*, the operator $L : W \to W^*$ is invertible.*

*Proof.* We must show that for every $\Psi \in W^*$ there exists a unique $v_0 \in W$ such that $L(v_0) = \Psi$, i.e.,

$$\begin{aligned}
\langle L(v_0), \varphi\rangle &= \int_{\mathbb{R}^2} \nabla v_0 \nabla \varphi \mathrm{d}x + \int_{\mathbb{R}^2} V^+(x) \frac{G^{-1}(v_0)}{g(G^{-1}(v_0))} \varphi \mathrm{d}x \\
&\quad - \lambda \int_{\mathbb{R}^2} \frac{|G^{-1}(v_0)|^{p-2} G^{-1}(v_0)}{g(G^{-1}(v_0))} \varphi \mathrm{d}x - \int_{\mathbb{R}^2} h(x) \varphi \mathrm{d}x = \langle \Psi, \varphi\rangle,
\end{aligned} \tag{3.1}$$

for all $\varphi \in W$. This is equivalent to show that for each $\Psi \in W^*$, the functional $I : W \to \mathbb{R}$ defined by

$$\begin{aligned}
I(v) &= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla v|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^2} V^+(x) [G^{-1}(v)]^2 \mathrm{d}x - \frac{\lambda}{p} \int_{\mathbb{R}^2} |G^{-1}(v)|^p \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^2} h(x) v \mathrm{d}x - \langle \Psi, v\rangle
\end{aligned}$$

has a unique critical point. It is not difficult to verify that $I$ is well-defined and differentiable on $W$, where the derivative is given by

$$\begin{aligned}
\langle I'(v), \varphi\rangle &= \int_{\mathbb{R}^2} \nabla v \nabla \varphi \mathrm{d}x + \int_{\mathbb{R}^2} V^+(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \varphi \mathrm{d}x \\
&\quad - \lambda \int_{\mathbb{R}^2} \frac{|G^{-1}(v)|^{p-2} G^{-1}(v)}{g(G^{-1}(v))} \varphi \mathrm{d}x - \int_{\mathbb{R}^2} h(x) \varphi \mathrm{d}x - \langle \Psi, \varphi\rangle,
\end{aligned}$$

for $v, \varphi \in W$. First, we show that $I$ is coercive. By Lemma 2.1-(11), (A4) and since $\lambda < 0$, $p \geq 2\alpha$, we have

$$-\frac{\lambda}{p} \int_{\mathbb{R}^2} |G^{-1}(v)|^p \mathrm{d}x \geq \frac{|\lambda| G_1^p}{p} \int_{|v| \geq R_0} v^2 \mathrm{d}x \geq \frac{|\lambda| G_1^p}{p V_\infty} \int_{|v| \geq R_0} V^+(x) v^2 \mathrm{d}x \tag{3.2}$$

and

$$\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}^2} V^+(x) [G^{-1}(v)]^2 \mathrm{d}x &\geq \frac{1}{2} \int_{|v| \leq R_0} V^+(x) \big[G^{-1}\big(\frac{|v|}{R_0}\big)\big]^2 \mathrm{d}x \\
&\geq \frac{G_1^2}{2R_0^2} \int_{|v| \leq R_0} V^+(x) v^2 \mathrm{d}x.
\end{aligned} \tag{3.3}$$

In view of (1.7), we obtain

$$\frac{1}{2}\int_{\mathbb{R}^2} V^+(x)[G^{-1}(v)]^2\mathrm{d}x - \frac{\lambda}{p}\int_{\mathbb{R}^2} |G^{-1}(v)|^p\mathrm{d}x \geq \frac{|\lambda|G_1^p}{pV_\infty}\int_{\mathbb{R}^2} V^+(x)v^2\mathrm{d}x \tag{3.4}$$

and therefore

$$\begin{aligned} I(v) &\geq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2\mathrm{d}x + \frac{|\lambda|G_1^p}{pV_\infty}\int_{\mathbb{R}^2} V^+(x)v^2\mathrm{d}x - \int_{\mathbb{R}^2} |h(x)||v|\mathrm{d}x - \|\Psi\|_*\|v\| \\ &\geq \frac{|\lambda|G_1^p}{pV_\infty}\|v\|^2 - \frac{1}{S_{q'}^{1/2}}\|h\|_q\|v\| - \|\Psi\|_*\|v\|, \end{aligned}$$

which guarantees that the functional $I$ is coercive.

On the other hand, it follows from Lemma 2.1-(9) that the functionals $\Phi(v) := \int_{\mathbb{R}^2} V^+(x)[G^{-1}(v)]^2\mathrm{d}x$ and $\widehat{\Phi}(v) := \int_{\mathbb{R}^2} |G^{-1}(v)|^p\mathrm{d}x$ are convex and it is not hard to see that $\Phi$ and $\widehat{\Phi}$ are strongly continuous. Therefore, $\Phi$ and $\widehat{\Phi}$ are weakly lower semicontinuous. Consequently, if $v_n \rightharpoonup v$ in $W$ then

$$\begin{aligned} \liminf_{n\to\infty} I(v_n) &\geq \liminf_{n\to\infty}\frac{1}{2}\int_{\mathbb{R}^2} |\nabla v_n|^2\mathrm{d} + \liminf_{n\to\infty}\frac{1}{2}\int_{\mathbb{R}^N} V^+(x)[G^{-1}(v_n)]^2\mathrm{d}x \\ &\quad + \liminf_{n\to\infty}\frac{-\lambda}{p}\int_{\mathbb{R}^2} |G^{-1}(v_n)|^p\mathrm{d}x + \liminf_{n\to\infty}\Big(-\int_{\mathbb{R}^N} h(x)v_n\mathrm{d}x\Big) \\ &\quad + \liminf_{n\to\infty}\Big(-\langle\Psi, v_n\rangle\Big) \\ &\geq I(v), \end{aligned}$$

showing that $I$ is weakly lower-semicontinuous in $W$. Since $W$ is a Hilbert space, there exists $v_0 \in W$ such that

$$I(v_0) = \inf_{v\in W} I(v).$$

Once $I$ is differentiable, we have $I'(v_0) = 0$ and the strict convexity of $I$ implies that the critical point $v_0$ is unique. Therefore, there exists a unique $v_0 \in W$ satisfying (3.1) and the lemma is proved. □

At this point, we consider another operator $T : W \to W^*$, which is given by

$$\langle T(v), \varphi\rangle = \int_{\mathbb{R}^2} V^-(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi\mathrm{d}x + \int_{\mathbb{R}^2}\frac{f(x, G^{-1}(v))}{g(G^{-1}(v))}\varphi\mathrm{d}x, \quad v, \varphi \in W.$$

It is clear that for each $v \in W$, $T(v)$ is a linear mapping. The next result shows that $T$ is well defined and we obtain an estimate for the norm of $T(v)$.

**Lemma 3.2.** *Assume* (A1)–(A6), (A8) *and Let $M > 0$ be such that $\varsigma_0\big(\frac{\alpha}{\beta}\big)^2M^2 < 4\pi$. Then there exist constants $C_3, C_4 > 0$ such that if $\|v\| \leq M$, then*

$$|\langle T(v), \varphi\rangle| \leq \Big(S_{t_0}^{-1}\|V^-\|_{L^{p_0}(B_{R_0})}\|v\| + C_3\|v\|^\rho + C_4\|v\|^\mu\Big)\|\varphi\|, \quad \textit{for all } \varphi \in W.$$

*Specifically,*

$$\|T(v)\|_* \leq S_{t_0}^{-1}\|V^-\|_{L^{p_0}(B_{R_0})}\|v\| + C_3\|v\|^\rho + C_4\|v\|^\mu.$$

*Proof.* We consider here $1 < p_0 < \infty$ and $1 < \sigma < \infty$. The cases $p_0 = \infty$ or $\sigma = \infty$ are simpler and are treated similarly. Note that

$$\frac{1}{p_0} + \frac{1}{t_0} + \frac{1}{t_0} = 1 \;\Leftrightarrow\; t_0 = \frac{2p_0}{p_0 - 1} > 2.$$

Since $V^-(x) = 0$ for $|x| \geq R_0$, using the generalized Hölder inequality together with (1.10) and Lemma 2.1-(2),(3), one has that

$$\Big|\int_{\mathbb{R}^2} V^-(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi \mathrm{d}x\Big| \leq \frac{1}{S_{t_0}}\|V^-\|_{L^{p_0}(B_{R_0})}\|v\|\|\varphi\|. \tag{3.5}$$

Analogously, we see that

$$\frac{1}{\sigma} + \frac{\rho}{t_1} + \frac{1}{t_1} = 1 \ \Leftrightarrow \ t_1 := \frac{\sigma(\rho+1)}{\sigma - 1} > 2,$$

from which it follows that

$$\Big|\int_{\mathbb{R}^2} k(x)|v|^\rho \varphi \mathrm{d}x\Big| \leq \|k\|_\sigma \|v\|_\lambda^\rho \|\varphi\|_\lambda \leq S_{t_1}^{-(\rho+1)/2}\|k\|_\sigma \|v\|^\rho \|\varphi\|. \tag{3.6}$$

On the other hand, using Lemma 2.5 with $\varsigma = \varsigma_0\big(\frac{\alpha}{\beta}\big)^2$ yields

$$\int_{\mathbb{R}^2}\Big(e^{\varsigma_0\left(\frac{\alpha}{\beta}\right)^2 v^2} - 1\Big)|v|^\mu|\varphi|\mathrm{d}x \leq C\|v\|^\mu\|\varphi\|. \tag{3.7}$$

With the condition (A8), the estimates (3.5)-(3.7) and Lemma 2.1-(6), we arrive at

$$\begin{aligned}
|\langle T(v), \varphi\rangle| &\leq \frac{1}{S_{t_0}}\|V^-\|_{L^{p_0}(B_{R_0})}\|v\|\|\varphi\| + C_3\|k\|_\sigma\|v\|^\rho\|\varphi\| \\
&\quad + C_2\int_{\mathbb{R}^2}\Big(e^{\varsigma_0[G^{-1}(v)]^{2\alpha}} - 1\Big)|G^{-1}(v)|^\mu|\varphi|\mathrm{d}x \\
&\leq \frac{1}{S_{t_0}}\|V^-\|_{L^{p_0}(B_{R_0})}\|v\|\|\varphi\| + C_3\|k\|_\sigma\|v\|^\rho\|\varphi\| \\
&\quad + C_2\int_{\mathbb{R}^2}\Big(e^{\varsigma_0\left(\frac{\alpha}{\beta}\right)^2 v^2} - 1\Big)|v|^\mu|\varphi|\mathrm{d}x \\
&\leq \Big(\frac{1}{S_{t_0}}\|V^-\|_{L^{p_0}(B_{R_0})}\|v\| + C_3\|k\|_\sigma\|v\|^\rho + C_4\|v\|^\mu\Big)\|\varphi\|
\end{aligned}$$

which proves the first estimate. The second is immediate. □

To apply Lemma 2.6, we consider the following partial order in $W$:

$$v_1, v_2 \in W, v_1 \preccurlyeq v_2 \ \Leftrightarrow \ v_1 \leq v_2 \text{ a.e. in } \mathbb{R}^2. \tag{3.8}$$

It is clear that $(W, \preccurlyeq)$ is an ordered Banach space and for all $u, v \in W$, there exist $\sup\{u, v\}$ and $\inf\{u, v\}$ with respect to the order $\preccurlyeq$. Moreover, recalling that $v^+ = \sup\{v, 0\}$ and $v^- = -\inf\{v, 0\}$, we have that $v^+$ and $v^-$ are the positive and negative parts of $v$. Since $|\nabla v^\pm| \leq |\nabla v|$ almost everywhere in $\mathbb{R}^2$, we see that $\|v^\pm\| \leq \|v\|$. Consequently, $(W, \preccurlyeq)$ is a Banach semilattice which is reflexive. We also observe that the dual space $W^*$, endowed with the order

$$\Phi_1, \Phi_2 \in W^*, \Phi_1 \lhd \Phi_2 \ \Leftrightarrow \ \langle \Phi_1, \varphi\rangle \leq \langle \Phi_2, \varphi\rangle, \text{ for all } \varphi \in W_+, \tag{3.9}$$

where $W_+ = \{v \in W; v \geq 0 \text{ a.e. in } \mathbb{R}^2\}$ is also an ordered Banach space. Now, we need to check the monotonicity of the operators $T$ and $L^{-1}$.

**Lemma 3.3.** $T : (W, \preccurlyeq) \to (W^*, \lhd)$ *is an increasing operator.*

*Proof.* Let $v_1, v_2 \in W$ be such that $v_1 \preccurlyeq v_2$, i.e., $v_1 \leq v_2$ a.e. in $\mathbb{R}^2$. By Lemma 2.1-(7) and (A7), we obtain

$$\langle T(v_1), \varphi\rangle = \int_{\mathbb{R}^2} V^-(x)\frac{G^{-1}(v_1)}{g(G^{-1}(v_1))}\varphi \mathrm{d}x + \int_{\mathbb{R}^2}\frac{f(x, G^{-1}(v_1))}{g(G^{-1}(v_1))}\varphi \mathrm{d}x$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}^2} V^-(x)\frac{G^{-1}(v_2)}{g(G^{-1}(v_2))}\varphi \mathrm{d}x + \int_{\mathbb{R}^2} \frac{f(x,G^{-1}(v_2))}{g(G^{-1}(v_2))}\varphi \mathrm{d}x \\
&= \langle T(v_2),\varphi\rangle
\end{aligned}
$$

for all $\varphi \in W_+$ and this proves that $T(v_1) \lhd T(v_2)$. $\square$

**Lemma 3.4.** *The operator $L^{-1} : (W^*, \lhd) \to (W, \preccurlyeq)$ is increasing.*

*Proof.* Let $\Phi_1, \Phi_2 \in W$ such that $\Phi_1 \lhd \Phi_2$, that is,

$$
\langle \Phi_1,\varphi\rangle \le \langle \Phi_2,\varphi\rangle, \quad \text{for all} \quad \varphi \in W_+.
$$

Setting $v_1 = L^{-1}(\Phi_1)$ and $v_2 = L^{-1}(\Phi_2)$, for $\varphi \in W_+$ one has $\langle L(v_1),\varphi\rangle \le \langle L(v_2),\varphi\rangle$ and thus

$$
\begin{aligned}
0 \le &\int_{\mathbb{R}^2} (\nabla v_2 - \nabla v_1)\nabla\varphi \mathrm{d}x + \int_{\mathbb{R}^2} V^+(x)\big[\frac{G^{-1}(v_2)}{g(G^{-1}(v_2))} - \frac{G^{-1}(v_1)}{g(G^{-1}(v_1))}\big]\varphi \mathrm{d}x \\
&- \lambda\int_{\mathbb{R}^2} \big[\frac{|G^{-1}(v_2)|^{q-2}G^{-1}(v_2)}{g(G^{-1}(v_2))} - \frac{|G^{-1}(v_1)|^{q-2}G^{-1}(v_1)}{g(G^{-1}(v_1))}\big]\varphi \mathrm{d}x.
\end{aligned}
$$

Now, taking $\varphi = (v_2 - v_1)^- = \max\{v_1 - v_2, 0\} \in W_+$ and using Lemma 2.1-(9) we reach

$$
\begin{aligned}
0 &\le -\int_{\mathbb{R}^2} |\nabla(v_2 - v_1)^-|^2 + \int_{\{v_2<v_1\}} V^+(x)\Big[\frac{G^{-1}(v_2)}{g(G^{-1}(v_2))} - \frac{G^{-1}(v_1)}{g(G^{-1}(v_1))}\Big](v_1 - v_2)\mathrm{d}x \\
&\quad + \int_{\{v_2<v_1\}} V^+(x)\Big[\frac{|G^{-1}(v_2)|^{p-2}G^{-1}(v_2)}{g(G^{-1}(v_2))} - \frac{|G^{-1}(v_1)|^{p-2}G^{-1}(v_1)}{g(G^{-1}(v_1))}\Big](v_1 - v_2)\mathrm{d}x \\
&\le -\int_{\mathbb{R}^2} |\nabla(v_2 - v_1)^-|^2 \mathrm{d}x
\end{aligned}
$$

from which it follows that $\int_{\mathbb{R}^2} |\nabla(v_2 - v_1)^-|^2 \mathrm{d}x = 0$. So $(v_2 - v_1)^- = 0$ and thus $v_1 \le v_2$ a.e. in $\mathbb{R}^2$, i.e., $L^{-1}(\Phi_1) \preccurlyeq L^{-1}(\Phi_2)$. $\square$

Now we define the operator $S : W \to W$ by $S = L^{-1} \circ T$ and our goal is to show that there exists a ball in $W$ which is invariant by $S$. We will use the following notation:

$$
\mathbf{B}_W[0,R] = \{v \in W : \|v\| \le R\}.
$$

**Lemma 3.5.** *Under the hypotheses of Theorem 1.1, there exists $0 < R_1 \le M$ and $\delta_0 > 0$ such that if $\|h\|_q \le \delta_0$ then*

$$
S(\mathbf{B}_W[0,R_1]) \subset \mathbf{B}_W[0,R_1].
$$

*Proof.* Let $v \in W$ and $w = S(v) = L^{-1}(T(v))$. By Lemma 2.1-(4) and the estimates (3.2)-(3.3), we have

$$
\begin{aligned}
\langle L(w),w\rangle &= \int_{\mathbb{R}^2} |\nabla w|^2 + \int_{\mathbb{R}^2} V^+(x)\frac{G^{-1}(w)w}{g(G^{-1}(w))}\mathrm{d}x \\
&\quad - \lambda\int_{\mathbb{R}^2} \frac{|G^{-1}(w)|^{p-2}G^{-1}(w)w}{g(G^{-1}(w))}\mathrm{d}x - \int_{\mathbb{R}^2} h(x)w\mathrm{d}x \\
&\ge \int_{\mathbb{R}^2} |\nabla w|^2 \mathrm{d}x + \frac{1}{\alpha}\Big[\int_{\mathbb{R}^2} V^+(x)[G^{-1}(w)]^2 \mathrm{d}x - \lambda\int_{\mathbb{R}^2} |G^{-1}(w)|^p \mathrm{d}x\Big] \\
&\quad - \|h\|_q\|w\|_{q'}
\end{aligned}
$$

$$\geq \Lambda\|w\|^2 - \frac{1}{S_{q'}^{1/2}}\|h\|_q\|w\|,$$

where $\Lambda = \frac{2|\lambda|G_1^p}{\alpha p V_\infty}$ according to (A9). From this it follows that

$$\begin{aligned}\Lambda\|S(v)\|^2 &= \Lambda\|w\|^2\\ &\leq \langle L(w), w\rangle + \frac{1}{S_{q'}^{1/2}}\|h\|_q\|w\|\\ &\leq \langle T(v), S(v)\rangle + \frac{1}{S_{q'}^{1/2}}\|h\|_q\|S(v)\|\\ &\leq \Big(\|T(v)\|_* + \frac{1}{S_{q'}^{1/2}}\|h\|_q\Big)\|S(v)\|.\end{aligned}$$

Thus, if $\|v\| \leq M$ then by Lemma 3.2 one has

$$\|S(v)\| \leq \frac{1}{\Lambda}\Big(\frac{1}{S_{t_0}}\|V^-\|_{L^{p_0}(B_{R_0})}\|v\| + C_3\|k\|_\sigma\|v\|^\rho + C_4\|v\|^\mu + \frac{1}{S_{q'}^{1/2}}\|h\|_q\Big).$$

Hence, if $M \geq R > 0$ and $\|v\| \leq R$, then

$$\frac{\|S(v)\|}{R} \leq \frac{1}{\Lambda}\Big(\frac{1}{S_{t_0}}\|V^-\|_{L^{p_0}(B_{R_0})} + C_3R^{\rho-1} + C_4R^{\mu-1} + \frac{1}{S_{q'}^{1/2}R}\|h\|_q\Big). \tag{3.10}$$

Next, we choose $M \geq R_1 > 0$ sufficiently small so that

$$\frac{1}{\Lambda}\left(C_3R_1^{\rho-1} + C_4R_1^{\mu-1}\right) \leq \frac{1 - \Lambda^{-1}S_{t_0}^{-1}\|V^-\|_{L^{p_0}(B_{R_0})}}{2}.$$

By considering

$$\delta_0 := \frac{S_{q'}^{1/2}R_1\big(1 - \Lambda^{-1}S_{t_0}^{-1}\|V^-\|_{L^{p_0}(B_{R_0})}\big)}{2} > 0$$

and taking $R = R_1$ in (3.10), we deduce that if $\|h\|_q \leq \delta_0$, then $\frac{\|S(v)\|}{R_1} \leq 1$. Therefore, $S(\mathbf{B}_W[0, R_1]) \subset \mathbf{B}_W[0, R_1]$ and the proof is complete. $\square$

Finally, let us conclude the proof of Theorem 1.1. From the definition of the operator $S$ and by invoking Lemmas 3.3 and 3.4, it follows that $S$ is increasing. In view of Lemma 3.5, $\mathbf{B}_W[0, R]$ is invariant by $S$ and by Lemma 2.6, $\mathbf{B}_W[0, R]$ has the fixed point property. Therefore, there exists $v \in \mathbf{B}_W[0, R]$ such that $S(v) = v$. Since $S = L^{-1} \circ T$ we have

$$\langle L(v), w\rangle = \langle T(v), w\rangle, \quad \text{for all} \quad w \in W$$

and according to (2.2) $v$ is a weak solution of Equation (2.1). Using now Lemma 2.2, we see that $u = G^{-1}(v)$ is a weak solution for (1.1) and Theorem 1.1 is proved.

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Uberlandio B. Severo
Universidade Federal da Paraíba, Departamento de Matemática, CEP 58051-900, João Pessoa - PB, Brazil
*Email address*: uberlandio@mat.ufpb.br

Bruno H. C. Ribeiro
Universidade Federal da Paraíba, Departamento de Matemática, CEP 58051-900, João Pessoa - PB, Brazil
*Email address*: bruno@mat.ufpb.br

Diogo de S. Germano
Universidade Federal de Campina Grande, Unidade Académica de Matemática, CEP 58109-970, Campina Grande - PB, Brazil
*Email address*: diogosg@mat.ufcg.edu.br