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SIGNORINI'S PROBLEM FOR THE BRESSE BEAM MODEL WITH LOCALIZED KELVIN-VOIGT DISSIPATION

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Abstract. We prove the existence of a global solution to Signorini's problem for the localized viscoelastic Bresse beam model (circular arc) with continuous and discontinuous constitutive laws. We show that when the constitutive law is continuous, the solution decays exponentially to zero, and when the constitutive law is discontinuous the solution decays only polynomially to zero. The method we use for proving our result is different the others already used in Signorini's problem and is based on approximations through a hybrid model. Also, we present some numerical results using discrete approximations in time and space, based on the finite element method on the spatial variable and the implicit Newmark method to the discretized the temporal variable.

1. Introduction

In this work we consider the Signorini problem for Bresse model. The beam is configured over a circular arch of length ℓ over the interval $[0,\ell]\subset\mathbb{R}$,

$$\rho_1\varphi_{tt} = S_x + lN, \tag{1.1}$$

$$\rho_2\psi_{tt} = M_x - S, \tag{1.2}$$

$$\rho_1\omega_{tt} = N_x - lS, \tag{1.3}$$

where

$$\begin{gathered} S = \kappa(\varphi_x+\psi+l\omega) + K(x)(\varphi_{xt}+\psi_t+l\omega_t), \\ M = b\psi_x + B(x)\psi_{xt}, \\ N = \kappa_0(\omega_x - l\varphi) + K(x)(\omega_{xt} - l\varphi_t). \end{gathered} \tag{1.4}$$

The functions φ,ψ and ω are the transversal displacement, rotatory angle, and longitudinal displacement, respectively. The coefficient are $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\kappa = kGA$, $b = EI$, $\kappa_0 = EA$, $l = R^{-1}$. Where k is correction factor, E is the Young modulus, G is shear modulus. Moreover ρ, A, I, and R represent the density of the body, area of the cross-section, and radius of curvature of the beam, respectively. We assume the above coefficients are constant.

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We consider the initial conditions

$$\begin{aligned} \varphi(x,0)=\varphi_0(x), \quad \psi(x,0)=\psi_0(x), \quad \omega(x,0)=\omega_0(x), \quad \forall x\in(0,\ell) \\ \varphi_t(x,0)=\varphi_1(x), \quad \psi_t(x,0)=\psi_1(x), \quad \omega_t(x,0)=\omega_1(x) \quad \forall x\in(0,\ell). \end{aligned} \tag{1.5}$$

and Dirichlet boundary conditions

$$\varphi(0,t)=\psi(0,t)=\omega(0,t)=0, \quad \psi(\ell,t)=0 \quad \forall t>0. \tag{1.6}$$

On the other hand, at $x=\ell$ we consider the Signorini's conditions and φ:

$$\begin{aligned} \omega(\ell,t)\le g_1, \quad \forall t>0 \\ g_2\le\varphi(\ell,t)\le g_3, \quad \forall t>0. \end{aligned} \tag{1.7}$$

where g_1, g_2 and g_3 are the gaps to the obstacle, see Figure 1. We have the following conditions

$$S(\ell,t)\begin{cases}\ge 0 & \text{if } \varphi(\ell,t)=g_2,\\ =0 & \text{if } g_2<\varphi(\ell,t)<g_3,\\ \le 0 & \text{if } \varphi(\ell,t)=g_3,\end{cases} \quad N(\ell,t)\begin{cases}\le 0 & \text{if } \omega(\ell,t)=g,\\ =0 & \text{if } \omega(\ell,t)<g.\end{cases} \tag{1.8}$$

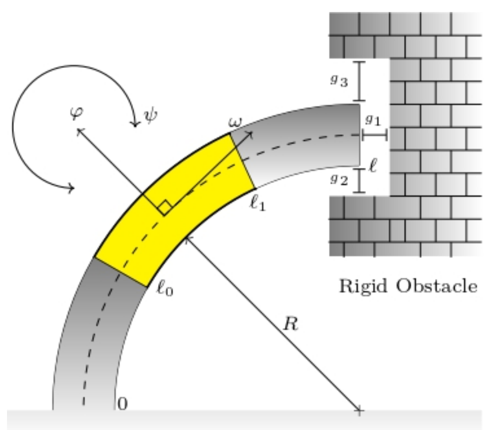

FIGURE 1. Beam subject to a constraint at the free $x=\ell-$end.

To ensure that in (1.8) only one condition occurs at the same time, we impose that

$$S(\ell,t)g_2-\varphi(\ell,t)]^+[\varphi(\ell,t)-g_3]^+=0 \quad \text{and} \quad N(\ell,t)[\omega(\ell,t)-g_1]^+=0, \tag{1.9}$$

where $h^+=\max\{h,0\}$ is the positive part of function h.

Here we consider two cases, first when the model (1.1)-(1.9) has a continuous constitutive law, and when the model has a discontinuous constitutive law. In the continuous case we assume that the functions $K,B\in C^1([0,\ell])$ are positive on the interval $]\ell_0,\ell_1[$ and vanish outside this interval. Furthermore, we assume that there are positive constants c, C_1 and C_2 such that

$$|B'|^2\le cB; \quad |K'|^2\le cK \tag{1.10}$$

$$C_1K\le B\le C_2K. \tag{1.11}$$

When the constitutive law is discontinuous we assume that $K,B\in C^1([\ell_0,\ell_1])$ are positive functions vanishing outside $]\ell_0,\ell_1[$. The typical graph of B and K for the continuous and the discontinuous case are given in figures 2 and 3. In both cases the viscoelastic component is localized over the interval $]\ell_0, \ell_1[$. This interval will be denoted by I_C when the constitutive law is continuous and I_D in the discontinue case.

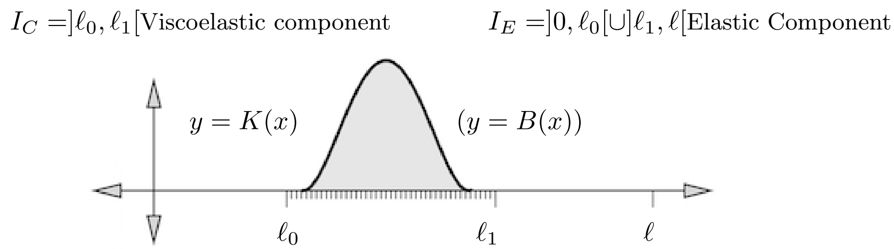

FIGURE 2. Typical example of $y = K(x)$ $(y = B(x))$.

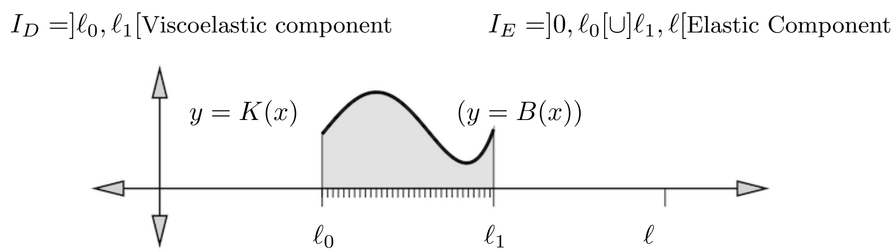

FIGURE 3. Typical example of $y = K(x)$ $(y = B(x))$.

The Signorini problem for the wave equation was studied by Kim [8], there the author proved the existence of at least one solution, by using the Divergent-Rotational Lemma. Similarly, Andrews et al [2] considered the one-dimensional contact problem for the Euler Bernoulli beam model. They showed the existence of a global solution. Kuttler and Shillor [9] considered the contact problem between two stops to viscoelastic Euler Bernoulli beam equation. Numerical aspects of the problem were considered in Dumont and Paoli [5] and Coppeti and Elliot [4]. Uniqueness has not been proven so far.

The main result of this article shows the existence of a global solution to the Signorini problem (1.1)-(1.9). Moreover we prove the exponential stability of the system provided the constitutive law is continuous, and satisfy conditions (1.10)-(1.11). When the constitutive law is discontinuous we show the lack of exponential stability and that the solution decays polynomially as $t^{-1/2}$.

The rest of this article is organized as follows. In section 2 introduce the semigroup associated with the hybrid model. In section 3, we show the exponential decay in case of continuous constitutive law and the polynomial stability in the discontinuous case. In section 4 we show the lack of exponential stability for the discontinuous case. In section 5 we introduce the penalized problem as a Lipschitz perturbation of the semigroup. In Section 6 we show the existence of solution of Signorini's problem. Finally, in section 7 we show some numerical results.

2. Existence: the hybrid-penalized method

To prove the existence of weak solutions to Signorin's problem we use the hybrid-penalized method introduced in [12]. That is given $\epsilon > 0$, we consider the linear hybrid model

$$\begin{aligned} \rho_1\varphi_{tt} - S_x - lN = 0, &\quad \text{in } (0,L)\times(0,\infty)\\ \rho_2\psi_{tt} - M_x + S = 0, &\quad \text{in } (0,L)\times(0,\infty)\\ \rho_1\omega_{tt} - N_x + lS = 0, &\quad \text{in } (0,L)\times(0,\infty) \end{aligned} \tag{2.1}$$

where S, M and N are given in (1.4). Here we consider dynamic boundary condition on φ and ω,

$$\varphi(\ell,t) = u(t), \quad \omega(\ell,t) = z(t),$$

where the functions u and z are defined by the coupled system of ordinary differential equations

$$\begin{aligned} \epsilon u_{tt} + \epsilon u_t + \epsilon u + S^\epsilon(\ell,t) = 0,\\ \epsilon z_{tt} + \epsilon z_t + \epsilon z + N^\epsilon(\ell,t) = 0, \end{aligned} \tag{2.2}$$

together with the stationary boundary condition

$$\varphi(0,t) = \psi(0,t) = \omega(0,t) = 0, \quad \psi(\ell,t) = 0, \quad \forall t > 0.$$

System (2.1) coupled with the ordinary differential equation (2.2) is called hybrid system. The initial conditions are given by

$$\begin{gathered} \varphi(x,0) = \varphi_0(x), \quad \psi(x,0) = \psi_0(x), \quad \omega(x,0) = \omega_0(x),\\ \varphi_t(x,0) = \varphi_1(x), \quad \psi_t(x,0) = \psi_1(x), \quad \omega_t(x,0) = \omega_1(x),\\ (u(0), u_t(0), z(0), z_t(0)) = (u_0, u_1, z_0, z_1) \quad \in \mathbb{C}^4. \end{gathered}$$

System (2.1)-(2.2) is the linear version of the penalized problem. This procedure allow us to arrive to the semi linear penalized problem (normal compliance) by using Lipstchitz perturbations to the hybrid model (2.1)-(2.2). To follows this ideas we introduce the notation

$$\Phi = \varphi_t, \quad \Psi = \psi_t, \quad W = \omega_t, \ u_t = U, \quad z_t = Z.$$

Let us denote by

$$\mathcal{U} := (\varphi, \Phi, \psi, \Psi, \omega, W, u, U, z, Z)^\top.$$

The phase space considered here is

$$\mathcal{H} := V_0 \times L^2(0,\ell) \times H_0^1(0,\ell) \times L^2(0,\ell) \times V_0 \times L^2(0,\ell) \times \mathbb{C}^4. \tag{2.3}$$

where

$$V_0 = \{u \in H^1(0,\ell); u(0) = 0\}.$$

$\mathcal{H}$ is a Hilbert space with the norm

$$\begin{aligned} \|\mathcal{U}\|_{\mathcal{H}}^2 = \int_0^\ell \Big[\kappa|\varphi_x + \psi + l\omega|^2 + \rho_1|\Phi|^2 + b|\psi_x|^2 + \rho_2|\Psi|^2 + \kappa_0|\omega_x\\ - l\varphi|^2 + \rho_1|W|^2\Big]dx + \epsilon(|u|^2 + |U|^2 + |z|^2 + |Z|^2). \end{aligned} \tag{2.4}$$

Let us denote the operator

$$\begin{aligned} \mathbb{A}\mathcal{U} = \Big(\Phi, \frac{S_x}{\rho_1} + \frac{lN}{\rho_1}, \Psi, \frac{M_x}{\rho_2} - \frac{S}{\rho_2}, W, \frac{N_x}{\rho_1} - \frac{lS}{\rho_1}, U, -[U + u + \frac{1}{\epsilon}S(\ell,t)],\\ Z, -[Z + z + \frac{1}{\epsilon}N(\ell,t)]\Big)^\top. \end{aligned} \tag{2.5}$$

The domain of $\mathbb{A}$ is

$$D(\mathbb{A}) := \Big\{\mathcal{U} \in \mathcal{H};\ \begin{pmatrix}\Phi\\ W\\ \Psi\end{pmatrix} \in V_0^2 \times H_0^1,\ \begin{pmatrix}\kappa\varphi_x + K\Phi_x,\\ b\psi_x + B\Psi_x,\\ \kappa_0\omega_x + KW_x\end{pmatrix} \in [H^1(0,\ell)]^3\Big\}. \tag{2.6}$$

The set $D(\mathbb{A})$ is the typical domain generated by the Kelvin-Voigh operators. Among its main properties we have that the domain depends on the differential operator and that the family of resolvent operators are not compact. System (2.1)-(2.2) can be rewritten as

$$\mathcal{U}_t = \mathbb{A}\mathcal{U}, \quad \mathcal{U}(0) = \mathcal{U}_0, \tag{2.7}$$

where $\mathcal{U}_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, u_0, u_1, z_0, z_1)^\top$. A straightforward calculation gives

$$\begin{aligned}\operatorname{Re}\langle \mathbb{A}\mathcal{U}, \mathcal{U}\rangle_{\mathcal{H}} = &-\int_0^\ell K|\Phi_x + \Psi + lW|^2 + B|\Psi_x|^2 + K|W_x - l\Phi|^2\,dx\\ &- \epsilon|U|^2 - \epsilon|Z|^2.\end{aligned} \tag{2.8}$$

Therefore $\mathbb{A}$ is a dissipative operator. The resolvent equation is

$$i\lambda\mathcal{U} - \mathbb{A}\mathcal{U} = \mathbf{F},$$

taking the inner product with $\mathcal{U}$ over $\mathcal{H}$ and then taking the real part we obtain

$$-\operatorname{Re}(\mathbb{A}\,\mathcal{U}, \mathcal{U})_{\mathcal{H}} = \operatorname{Re}(\mathbf{F}, \mathcal{U})_{\mathcal{H}}.$$

From (2.8) we obtain

$$\int_0^\ell K(x)|\Phi_x+\Psi+lW|^2+B(x)|\Psi_x|^2+K(x)|W_x-l\Phi|^2\,dx+\epsilon|U|^2+\epsilon|Z|^2 = \operatorname{Re}(\mathbf{F}, \mathcal{U})_{\mathcal{H}}. \tag{2.9}$$

The resolvent equation in terms of its components is

$$i\lambda\varphi - \Phi = f_1, \tag{2.10}$$
$$i\lambda\Phi - S_x - lN = f_2, \tag{2.11}$$
$$i\lambda\psi - \Psi = f_3, \tag{2.12}$$
$$i\lambda\Psi - M_x + S = f_4, \tag{2.13}$$
$$i\lambda\omega - W = f_5, \tag{2.14}$$
$$i\lambda W - N_x + lS = f_6, \tag{2.15}$$
$$i\lambda u - U = f_7, \tag{2.16}$$
$$i\lambda U + U + u + \frac{1}{\epsilon}S(\ell, t) = f_8, \tag{2.17}$$
$$i\lambda z - Z = f_9, \tag{2.18}$$
$$i\lambda Z + Z + z + \frac{1}{\epsilon}N(\ell, t) = f_{10}. \tag{2.19}$$

Our next step is to show that $\mathbb{A}$ is the infinitesimal generator of a contraction semigroup. To do that use the following result which is a consequence of Pazy result [16, Theorem 4.6] and the Lumer-Phillips Theorem.

Lemma 2.1. *Let $\mathcal{A}$ be dissipative with $0 \in \varrho(\mathcal{A})$. If $\mathcal{H}$ is reflexive then $\mathcal{A}$ is the infinitesimal generator of a semigroup of contractions.*

Proof. Since $\varrho(\mathcal{A})$ is an open set, there exists $\epsilon > 0$ such that $\epsilon \in \varrho(\mathcal{A})$. This implies that each $\lambda > 0$ belongs to $\varrho(\mathcal{A})$. In particular we have that $R(I-\mathcal{A}) = \mathcal{H}$. Using [16, Theorem 4.6], we conclude that $\overline{D(\mathcal{A})} = \mathcal{H}$. By using the Lumer-Phillips Theorem our conclusion follows. □

Theorem 2.2. *The operator $\mathbb{A}$ is the infinitesimal generator of a C_0 semigroup $\mathcal{T}$ of contractions.*

Proof. Since $\mathbb{A}$ is dissipative and because of Lemma 2.1 it is sufficient to show that $0 \in \varrho(\mathbb{A})$. In fact, we take $F \in \mathcal{H}$ and show that there exist only one $\mathcal{U} \in D(\mathbb{A})$ such that $-\mathbb{A}\mathcal{U} = F$. Let us denote

$$F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10})^\top,$$
$$\mathcal{U} = (\varphi, \Phi, \psi, \Psi, \omega, W, u, U, z, Z) \in D(\mathbb{A}).$$

For $\lambda = 0$ the resolvent system (2.10)-(2.19) can be written as

$$\begin{gathered}
\Phi = f_1 \quad, \Psi = f_3 \quad, W = f_5 \quad, U = f_7, \quad Z = f_9, \\
-\kappa(\varphi_x + \psi + l\omega)_x - l\kappa_0(\omega_x - l\varphi) = F_1, \\
-b\psi_{xx} + \kappa(\varphi_x + \psi + l\omega) = F_2, \\
-\kappa_0(\omega_x - l\varphi)_x + \kappa l(\varphi_x + \psi + l\omega) = F_3,
\end{gathered} \tag{2.20}$$

where

$$\begin{gathered}
F_1 = \rho_1 f_2 + [K(x)(f_{x,1} + f_3 + l f_4)]_x + l K_0(x)(f_{5,x} - l f_1), \\
F_2 = \rho_2 f_4 + (B(x) f_{3,x})_x - K(x)(f_{x,1} + f_3 + l f_4), \\
F_3 = \rho_1 f_6 + [K_0(x)(f_{5,x} - l f_1)]_x - l K(x)(f_{x,1} + f_3 + l f_4)
\end{gathered}$$

satisfying the following boundary conditions

$$\begin{gathered}
\varphi(0) = 0, \quad \varphi(\ell) + \frac{1}{\epsilon} S(\ell) = f_8 - f_7, \\
\psi(0) = 0, \quad \psi(\ell) = 0, \\
\omega(0) = 0, \quad \omega(\ell) + \frac{1}{\epsilon} N(\ell) = f_{10} - f_9.
\end{gathered}$$

Let us introduce the space $\mathcal{V} = V_0 \times H_0^1 \times V_0$. Denoting $\mathbf{U}^i = (\varphi^i, \psi^i, \omega^i) \in \mathcal{V}$ we conclude that the bilinear form $\mathbf{a} : \mathcal{V} \to \mathbb{C}$,

$$\begin{aligned}
&\mathbf{a}(\mathbf{U}^1, \mathbf{U}^2) \\
&= \int_0^\ell \kappa(\varphi_x^1 + \psi^1 + l\omega^1)(\varphi_x^2 + \psi^2 + l\omega^2) + b\psi_x^1\psi_x^2 + \kappa_0(\omega_x^1 - l\varphi^1)(\omega_x^2 - l\varphi^2)\,dx \\
&\quad + \epsilon\varphi^1(\ell)\varphi^2(\ell) + \epsilon\omega^1(\ell)\omega^2(\ell),
\end{aligned}$$

is coercive and continuous over $\mathcal{V}$. Note that the function

$$\mathbf{F}(\mathbf{U}) = \int_0^\ell F_1\overline{\varphi} + F_2\overline{\psi} + F_3\overline{\omega}\,dx + \epsilon(f_8 - f_7)\varphi(\ell) + \epsilon(f_{10} - f_9)\omega(\ell), \tag{2.21}$$

belongs to $\mathcal{V}^*$. So, Lax-Milgran's Lemma guarantees that there exists only one weak solution to problem

$$\mathbf{a}(\mathbf{U}, \tilde{\mathbf{U}}) = \mathbf{F}(\tilde{\mathbf{U}}), \quad \forall\, \tilde{\mathbf{U}} \in \mathbf{K}. \tag{2.22}$$

Using system (2.20) we conclude the solution $\mathcal{U} \in D(\mathbb{A})$. Hence $0 \in \varrho(\mathbb{A})$. □

As a consequence of Theorem 2.2 we have the following result.

Theorem 2.3. *For each $\mathcal{U}_0 \in \mathcal{H}$ there exist a unique mild solution to problem* (2.1)-(2.2). *Moreover if the initial data $\mathcal{U}_0 \in D(\mathbb{A})$ there exist a strong solution satisfying*

$$\mathcal{U} \in C^1(0,T;\mathcal{H}) \cap C(0,T;D(\mathbb{A})).$$

3. Asymptotic behavior

Here we show that the semigroup associated with the hybrid system (2.1)-(2.2) is exponentially stable provided the constitutive law is continuous and satisfies satisfies (1.10)-(1.11). When the constitutive law is discontinuous, the corresponding semigroup is polynomially stable, with rate $t^{-1/2}$. Our main tool is the characterization due to Prüss [18] and Borichev and Tomilov [3].

Theorem 3.1. *Let $S(t) = e^{\mathcal{A}}$ be a contraction C_0-semigroup over a Hilbert space $\mathcal{H}$. Then, in Prüss* [18] *is established that there exist C and $,\gamma > 0$ satisfying*

$$\|S(t)\| \le Ce^{-\gamma t} \;\Leftrightarrow\; i\mathbb{R} \subset \varrho(\mathcal{A}) \text{ and } \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant M,\ \forall \lambda \in \mathbb{R}. \tag{3.1}$$

For polynomial stability, Borichev and Tomilov [3] *results establish that there exists $C > 0$ such that*

$$\|\mathcal{S}(t)\mathcal{A}^{-1}\| \leqslant \frac{C}{t^{1/\alpha}} \;\Leftrightarrow\; i\mathbb{R} \subset \varrho(\mathcal{A}) \text{ and } \|(i\lambda I - \mathcal{A})^{-1}\| \leqslant M|\lambda|^{\alpha},\ \forall \lambda \in \mathbb{R}. \tag{3.2}$$

Our starting point to show the exponential stability is to show the strong stability.

Lemma 3.2. *The operator $\mathbb{A}$ defined by* (2.5) *and* (2.6) *satisfies $i\mathbb{R} \subset \varrho(\mathbb{A})$.*

Proof. Let us consider the set

$$\mathcal{M} = \{s \in \mathbb{R}^+ :]-is, is[\subset \rho(\mathcal{A})\}.$$

Since $0 \in \rho(\mathcal{A})$, $\mathcal{M} \neq \varnothing$, the supremum $\sigma = \sup \mathcal{M}$ can be finite or infinite. If $\sigma = +\infty$ then $i\mathbb{R} \subseteq \rho(\mathcal{A})$ and we have nothing to prove. We will prove that the finite case is not possible. By contradiction, let us suppose that $\sigma < \infty$. Then, exists a sequence $\{\lambda_n\} \subseteq \mathbb{R}$ such that $\lambda_n \to \sigma < +\infty$ and

$$\|(i\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \to +\infty$$

Hence, there exists a sequence $\{f_n\} \subseteq \mathcal{H}$ such that $\|f_n\|_{\mathcal{H}} = 1$ and $\|(i\lambda_n I - \mathcal{A})^{-1} f_n\|_{\mathcal{H}} \to +\infty$. Noting

$$\tilde{\mathcal{U}_n} = (i\lambda_n I - \mathcal{A})^{-1} f_n \quad \Rightarrow \quad f_n = i\lambda_n \tilde{\mathcal{U}_n} - \mathcal{A}\tilde{\mathcal{U}_n}$$

and $\mathcal{U}_n = \frac{\tilde{\mathcal{U}_n}}{\|\tilde{\mathcal{U}_n}\|}$, $F_n = \frac{f_n}{\|\tilde{\mathcal{U}_n}\|}$ we obtain

$$i\lambda_n \mathcal{U}_n - \mathcal{A}\mathcal{U}_n = F_n \to 0. \tag{3.3}$$

Taking the inner product,

$$i\lambda_n \|\mathcal{U}_n\|^2 - \langle \mathcal{A}\mathcal{U}_n, \mathcal{U}_n\rangle = \langle F_n, \mathcal{U}_n\rangle \to 0$$

and taking real part,

$$\begin{aligned} -\operatorname{Re}\langle \mathcal{A}\mathcal{U}_n, \mathcal{U}_n\rangle = \int_0^\ell (B|\Psi_x^n|^2 + K|\Phi_x^n + \Psi^n + lW^n|^2 + K|W_x^n - l\Phi^n|^2)\,dx \\ + |U^n|^2 + |Z^n|^2 \to 0, \end{aligned} \tag{3.4}$$

$$\Psi_x^n; \quad \Phi_x^n + \Psi^n + lW^n; \quad W_x^n - l\Phi^n \to 0 \quad \text{strong in } L^2(]\ell_0, \ell_1[). \tag{3.5}$$

Therefore,

$$\psi_x^n;\quad \varphi_x^n+\psi^n+l\omega^n;\quad \omega_x^n-l\varphi^n\to 0\quad \text{strong in } L^2(]\ell_0,\ell_1[), \tag{3.6}$$
$$(U^n,Z^n)\to(0,0). \tag{3.7}$$

Since $\|\mathcal{A}\mathcal{U}_n\|\le C$, it follows that $\mathcal{U}_n$ is bounded in $D(\mathcal{A})$. This implies in particular that Φ_n, Ψ_n, and W_n are bounded in $H_0^1(0,\ell)$ and φ, ψ, w are bounded in $H^2(I_E)$. Then there exist subsequences such that

$$\Phi_n\to\Phi,\quad \Psi_n\to\Psi,\quad W_n\to W\quad \text{strong in } L^2(0,\ell) \tag{3.8}$$
$$\varphi_{n,x}+\psi_n+l\omega_n\to\varphi_x+\psi+l\omega, \tag{3.9}$$
$$\psi_{n,x}\to\psi_x,\omega_{n,x}-l\varphi_n\to\omega_x-l\varphi,\qquad \text{strong in } L^2(I_E), \tag{3.10}$$

where $I_E=]0,\ell_0[\cup]\ell_1,\ell[$. From (2.16) and (3.7) we obtain $u=z=0$; therefore $\varphi(0)=\omega(0)=0$. From (3.8), (3.10) and (3.6), it follows that $\mathcal{U}_n\to\mathcal{U}$ strongly in $\mathcal{H}$. Since $\mathcal{A}$ is closed, we conclude that $\mathcal{U}$ satisfies

$$i\sigma\mathcal{U}-\mathcal{A}\mathcal{U}=0.$$

Moreover, using the convergences (3.5)-(3.6) and the resolvent system, we conclude that the solution vanishes over $]\ell_0,\ell_1[$ so we have that

$$\varphi(\ell_0)=\psi(\ell_0)=\varphi_x(\ell_0)=\psi_x(\ell_0)=\omega(\ell_0)=\omega_x(\ell_0)=0.$$

So, over $]0,\ell_0[\cup]\ell_1,\ell[$ the solution satisfies

$$-\rho_1\sigma^2\varphi+\kappa(\varphi_x+\psi+l\omega)_x-l\kappa_0(\omega_x-l\varphi)=0,$$
$$-\rho_2\sigma^2\psi+b\psi_{xx}+\kappa(\varphi_x+\psi+l\omega)=0,$$
$$-\rho_1\sigma^2\omega+\kappa_0(\omega_x-l\varphi)_x+l\kappa(\varphi_x+\psi+l\omega)=0.$$

Looking the above equation as a second-order final-value problem over $]0,\ell_0[$, we obtain $\varphi=\psi=\omega=0$ over $]0,\ell_0[$. Using a similar argument we conclude that $\varphi=\psi=\omega=0$ over $]\ell_1,\ell[$. Hence $U\equiv 0$ on $\mathcal{H}$, which is a contradiction. This completes the proof. □

The next Lemma plays an important role in the proof of exponential stability.

Lemma 3.3. *Assume that hypothesis* (1.10) *and* (1.11) *hold. Then for each* $\epsilon>0$ *there exists* $C_\epsilon>0$ *such that the solution of* (2.10)-(2.19) *satisfies*

$$\int_{I_C}K|\lambda\Phi|^2+B|\lambda\Psi|^2\,dx+K|\lambda W|^2\,dx\le C_\varepsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+C_\varepsilon\|F\|_{\mathcal{H}}^2+\varepsilon|\mathcal{U}\|_{\mathcal{H}}^2.$$

Proof. Multiplying (2.11) by $\overline{i\lambda K\Phi}$ and integrating over $I_C=[\ell_0,\ell_1]$,

$$\int_{I_C}\rho_1K|\lambda\Phi|^2\,dx=\int_{I_C}(S_x+lN+f_2)\overline{i\lambda K\Phi}\,dx. \tag{3.11}$$

Recalling the definition of S and N over I_C we obtain

$$\begin{aligned}
&\int_{I_C} \rho_1 K|\lambda\Phi|^2\,dx \\
&= \int_{I_C} [\kappa(\varphi_x+\psi+l\omega)+K(\Phi_x+\Psi+lW)]_x\overline{i\lambda K\Phi}\,dx \\
&\quad + \int_{I_C} l[\kappa_0(\omega_x-l\varphi)+K(W_x-l\Phi)]\overline{i\lambda K\Phi}\,dx + \int_{I_C} f_2\overline{i\lambda K\Phi}\,dx, \\
&= \mathfrak{G}+\mathfrak{G}_0+\mathfrak{G}_1+\mathfrak{G}_2+\int_{I_C} f_2\overline{i\lambda K\Phi}\,dx.
\end{aligned} \tag{3.12}$$

Here

$$\begin{gathered}
\mathfrak{G} = \int_{I_C} [K(\Phi_x+\Psi+lW)]i\lambda\overline{(K'\Phi+K\Phi_x)}\,dx, \\
\mathfrak{G}_0 = \int_{I_C} [\kappa(\varphi_x+\psi+l\omega)]i\lambda\overline{(K'\Phi+K\Phi_x)}\,dx, \\
\mathfrak{G}_1 = \int_{I_C} \ell[\kappa_0(\omega_x-\ell\varphi)]i\lambda\overline{K\Phi}\,dx, \\
\mathfrak{G}_2 = \int_{I_C} \ell[K(W_x-\ell\Phi)]i\lambda\overline{K\Phi}\,dx,
\end{gathered}$$

where $I_C =]\ell_0,\ell_1[$ and $K(\ell_0)=K(\ell_1)=0$. Estimating $\mathfrak{G}$ we have

$$\begin{aligned}
\mathfrak{G} &= \int_{I_C} [K(\Phi_x+\Psi+lW)]i\lambda\overline{(K'\Phi+K(\Phi_x+\Psi+lW))}\,dx \\
&\quad - \int_{I_C} [K(\Phi_x+\Psi+lW)]i\lambda\overline{(K\Psi+KlW)}\,dx.
\end{aligned} \tag{3.13}$$

Taking the real part in (3.13) and using (2.9) we obtain

$$\begin{aligned}
\operatorname{Re}\mathfrak{G} &= \operatorname{Re}\int_{I_C} [K(\Phi_x+\Psi+lW)]i\lambda\overline{(K'\Phi)}\,dx \\
&\quad - \operatorname{Re}\int_{I_C} [K(\Phi_x+\Psi+lW)]i\lambda\overline{(K\Psi+\kappa_1 lW)}\,dx, \\
&\le \epsilon\|\lambda\Phi\|^2+\epsilon\|\lambda\Psi\|^2+\epsilon\|\lambda W\|^2+C_\epsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.
\end{aligned} \tag{3.14}$$

Similarly, using (2.10), (2.12), (2.14), and (2.9) we obtain

$$\begin{aligned}
\operatorname{Re}\mathfrak{G}_0 &= \operatorname{Re}\int_{I_C} [\kappa(\varphi_x+\psi+l\omega)]i\lambda\overline{(K'\Phi+K\Phi_x)}\,dx, \\
&\le \epsilon\int_{I_C} |\Phi|^2+|\Psi|^2+|W|^2\,dx+R,
\end{aligned} \tag{3.15}$$

$$\operatorname{Re}\mathfrak{G}_1 \le \epsilon\int_{I_C} |\Phi|^2+|\Psi|^2+|W|^2\,dx+R, \tag{3.16}$$

$$\operatorname{Re}\mathfrak{G}_2 \le \epsilon\int_{I_C} |\Phi|^2+|\Psi|^2+|W|^2\,dx+R. \tag{3.17}$$

Substituting (3.14) and (3.15) into (3.12) yields

$$\int_{I_C} \kappa_1|\lambda\Phi|^2\,dx \le \epsilon\|\lambda\Phi\|^2+\epsilon\|\lambda\Psi\|^2+C_\epsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} \tag{3.18}$$

for $|\lambda| > 1$. Repeating the above procedure multiplying equation (2.13) by $\overline{i\lambda b_1\Psi}$ and equation (2.15) by $\overline{i\lambda K_1 W}$ we arrive at

$$\int_{I_C} \rho_2 B|\lambda\Psi|^2\,dx \le \epsilon\|\lambda\Psi\|^2 + \epsilon\|\lambda W\|^2 + C_\epsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}},$$
$$\int_{I_C} \rho_1 K|\lambda W|^2\,dx \le \epsilon\|\lambda\Phi\|^2 + \epsilon\|\lambda W\|^2 + C_\epsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Summing the last three inequalities our conclusion follows. □

Remark 3.4. Lemma 3.3 is also valid for discontinuous constitutive law. That is for any θ vanishing out side of $I_D =]\ell_0, \ell_1[$ satisfying

$$|\theta'|^2 \le c|\theta| \tag{3.19}$$

it is valid that

$$\int_{I_D} \theta|\lambda\Phi|^2 + \theta|\lambda\Psi|^2\,dx + \theta|\lambda W|^2\,dx \le C_\varepsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|\mathcal{U}\|^2_{\mathcal{H}} + \varepsilon\|F\|^2_{\mathcal{H}}.$$

The proof is identical as Lemma 3.3.

We introduce the following notation:

$$\mathcal{E}_\varphi = \frac{(\kappa q\rho_1)'}{2}|\Phi|^2 + \frac{q'}{2}|S|^2, \quad \mathcal{I}_\varphi = \frac{\kappa q\rho_1}{2}|\Phi|^2 + \frac{q}{2}|S|^2, \tag{3.20}$$
$$\mathcal{E}_\psi = \frac{(bq\rho_2)'}{2}|\Psi|^2 + \frac{q'}{2}|M|^2, \quad \mathcal{I}_\psi = \frac{bq\rho_2}{2}|\Phi|^2 + \frac{q}{2}|M|^2, \tag{3.21}$$
$$\mathcal{E}_w = \frac{(\kappa_0 q\rho_1)'}{2}|W|^2 + \frac{q'}{2}|N|^2, \quad \mathcal{I}_w = \frac{\kappa_0 q\rho_2}{2}|W|^2 + \frac{q}{2}|N|^2, \tag{3.22}$$
$$\mathcal{E} = \mathcal{E}_\varphi + \mathcal{E}_\psi + \mathcal{E}_\omega, \quad \mathcal{I} = \mathcal{I}_\varphi + \mathcal{I}_\psi + \mathcal{I}_\omega \tag{3.23}$$

and

$$\begin{aligned}\mathcal{L} &= \int_a^b \mathcal{E}(s)\,ds - \int_a^b (\rho_1\kappa q\Phi\overline{\Psi} + \rho_1\kappa ql\Phi\overline{W} + \rho_1 Kql W\overline{\Phi})dx\\ &\quad + \int_a^b (qlS\bar{N} + ql\bar{S}N + qS\bar{M})dx.\end{aligned}$$

Taking $q(x) = \frac{e^{nx}-e^{na}}{n}$ we have $q'(x) = e^{nx} \gg q(x)$, for n large, Hence

$$C_0\int_a^b \mathcal{E}\,dx \le \mathcal{L} \le C_1\int_a^b \mathcal{E}\,dx. \tag{3.24}$$

Remark 3.5. Recalling the definition of S and M we obtain

$$\int_a^b |S|^2\,dx \le c\int_a^b \kappa|\varphi_x + \psi + l\omega|^2\,dx + \int_a^b |K(\Phi_x + \Psi + lW)|^2\,dx.$$

Using the dissipative properties,

$$\int_a^b |S|^2\,dx \le c\int_a^b |\varphi_x + \psi + l\omega|^2\,dx + c\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Similarly

$$\int_a^b |M|^2\,dx \le c\int_a^b |\psi_x|^2\,dx + c\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}},$$

and

$$\int_a^b |N|^2\,dx \le c\int_a^b |\omega_x - l\psi|^2\,dx + c\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

From where it follows that, for large values of n,

$$\begin{aligned}&\int_a^b |\Phi|^2 + |\varphi_x+\psi+l\omega|^2 + |\Psi|^2 + |\psi_x|^2 + |W|^2 + |\omega_x - l\psi|^2\,dx\\&\le \int_a^b \mathcal{E}\,dx + c\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}},\end{aligned}$$

$$\begin{aligned}\int_a^b \mathcal{E}\,dx \le c\int_a^b |\Phi|^2 + |\varphi_x+\psi+l\omega|^2 + |\Psi|^2 + |\psi_x|^2 + |W|^2 + |\omega_x - l\psi|^2\,dx\\ + c\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.\end{aligned}$$

Lemma 3.6. *If the constitutive law is continuous, then for each $[a,b]\subset]0,\ell[$ we have*

$$\Big|\mathcal{L}(s) - \mathcal{I}(s)\Big|_a^b\Big| \le C_\varepsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|F\|_{\mathcal{H}} + \varepsilon\|\mathcal{U}\|^2_{\mathcal{H}}.$$

If the constitutive law in discontinuous, then for each $[a,b]\subset]0,\ell[$ we have

$$\Big|\mathcal{L} - \mathcal{I}(s)\Big|_a^b\Big| \le +C_\varepsilon|\lambda|^2\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|F\|_{\mathcal{H}} + \varepsilon\|\mathcal{U}\|^2_{\mathcal{H}}.$$

Proof. Multiplying (2.11) by $q\bar S$, (2.13) by $q\bar M$, and (2.15) by $q\bar N$, we obtain

$$\begin{aligned}&-\frac{\rho_1\kappa q}{2}\frac{d}{dx}|\Phi|^2 - \frac{q}{2}\frac{d}{dx}|S|^2\\&= R_1 + \rho_1\kappa q\Phi\overline{\Psi} + \rho_1\kappa q l\Phi\overline{W} - i\lambda\rho_1 qK\Phi\overline{(\Phi_x+\Psi+lW)} - ql\bar S N,\end{aligned}\tag{3.25}$$

$$-\frac{\rho_2 bq}{2}\frac{d}{dx}|\Psi|^2 - \frac{q}{2}\frac{d}{dx}|M|^2 = R_2 - i\lambda\rho_2 qB\Psi\overline{\Psi_x} - qS\bar M,\tag{3.26}$$

$$\begin{aligned}&-\frac{\rho_1 Kq}{2}\frac{d}{dx}|W|^2 - \frac{q}{2}\frac{d}{dx}|N|^2\\&= R_3 - \rho_1 Kql W\overline{\Phi} - i\lambda\rho_1 qKW\overline{(W_x - l\Phi)} - qlS\bar N.\end{aligned}\tag{3.27}$$

Summing identities (3.25), (3.26), (3.27) we arrive at

$$\begin{aligned}&-\frac{\rho_1\kappa q}{2}\frac{d}{dx}|\Phi|^2 - \frac{\rho_2 bq}{2}\frac{d}{dx}|\Psi|^2 - \frac{\rho_1 Kq}{2}\frac{d}{dx}|W|^2 - \frac{q}{2}\frac{d}{dx}|S|^2 - \frac{q}{2}\frac{d}{dx}|M|^2 - \frac{q}{2}\frac{d}{dx}|N|^2\\&= R_4 + \rho_1\kappa q\Phi\overline{\Psi} + \rho_1\kappa ql\Phi\overline{W} - \rho_1 Kql W\overline{\Phi} - qlS\bar N - ql\bar S N - qS\bar M + J(x),\end{aligned}$$

where

$$J(x) = -i\lambda\rho_1 qK\Phi\overline{(\Phi_x+\Psi+lW)} - i\lambda\rho_2 qB\Psi\overline{\Psi_x} - i\lambda\rho_1 qKW\overline{(W_x - l\Phi)}.\tag{3.28}$$

Note that J vanishes outside of $]\ell_0,\ell_1[$. Recalling the definition of $\mathcal{I}$ and $\mathcal{E}$ we obtain

$$\begin{aligned}&-\frac{d}{dx}\left(\mathcal{I}(x)\right) + \mathcal{E}(x))\\&= R_4 + \rho_1\kappa q\Phi\overline{\Psi} + \rho_1\kappa ql\Phi\overline{W} - \rho_1 Kql W\overline{\Phi} - qlS\bar N - ql\bar S N - qS\bar M + J(x).\end{aligned}$$

Hence, when the constitutive law is continuous, Lemma 3.3 implies

$$\Big|\int_0^\ell J(x)\,dx\Big| \le C_\varepsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|F\|_{\mathcal{H}} + \varepsilon\|\mathcal{U}\|^2_{\mathcal{H}}.\tag{3.29}$$

Instead when the constitutive law is discontinuous we obtain

$$\Big|\int_0^\ell J(x)\,dx\Big| \le C_\varepsilon|\lambda|^2\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|F\|_{\mathcal{H}} + \varepsilon\|\mathcal{U}\|^2_{\mathcal{H}}. \tag{3.30}$$

After an integration using the above inequalities our conclusion follows. □

Let us denote

$$\mathbb{E}(s) = \rho_1|\Phi|^2 + \rho_2|\Psi|^2 + \rho_1|W|^2 + b|\psi_x|^2 + \kappa|\varphi_x+\psi+l\omega|^2 + \kappa_0|\omega_x - l\varphi|^2.$$

Theorem 3.7. *The semigroup associated with system* (2.1)-(2.2) *is exponentially stable provided the constitutive law is continuous and satisfies* (1.10)-(1.11). *Also if the constitutive law is discontinuous, then solution decays polynomially to zero as*

$$\|S(t)\mathcal{U}_0\| \le Ct^{-1/2}\|\mathcal{U}_0\|_{D(\mathcal{A})}. \tag{3.31}$$

Proof. From (2.9), (2.16), and (2.18), we obtain

$$\epsilon(|u|^2+|U|^2+|z|^2+|Z|^2) \le C_\varepsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|F\|^2_{\mathcal{H}}. \tag{3.32}$$

From (2.9), (2.10), (2.12), and (2.14), we obtain

$$\int_{I_C} K|\varphi_x+\psi+l\omega|^2 + B|\psi_x|^2 + K|\omega_x - l\varphi|^2\,dx \le C_\varepsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|F\|^2_{\mathcal{H}}. \tag{3.33}$$

On the other hand, from Lemma 3.3 we have

$$\int_{I_C} K|\Phi|^2 + B|\Psi|^2\,dx + K|W|^2\,dx \le C_\varepsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|F\|^2_{\mathcal{H}} + \varepsilon\|\mathcal{U}\|^2_{\mathcal{H}}.$$

For λ large and each interval $]a,b[\subset I_C$ we have

$$\int_a^b \mathbb{E}(x)\,dx \le C_\varepsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|F\|^2_{\mathcal{H}} + \varepsilon\|\mathcal{U}\|^2_{\mathcal{H}}. \tag{3.34}$$

Using the observability Lemma 3.6 over the interval $]a,b[$ we obtain

$$\begin{aligned}\mathbb{E}(a)+\mathbb{E}(b) &\le c\int_a^b \mathbb{E}(x)\,dx + C_\varepsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|F\|^2_{\mathcal{H}} + \varepsilon\|U\|^2 \\ &\le C_\varepsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|F\|^2_{\mathcal{H}} + \varepsilon\|\mathcal{U}\|^2_{\mathcal{H}}.\end{aligned} \tag{3.35}$$

Using Lemma 3.6 over the interval $]0,a[$ and $]a,\ell[$ and the above inequalities we obtain

$$\begin{gathered}\int_0^a \mathbb{E}(x)\,dx \le C_\varepsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|F\|^2_{\mathcal{H}} + \varepsilon\|\mathcal{U}\|^2_{\mathcal{H}},\\ \int_a^\ell \mathbb{E}(x)\,dx \le C_\varepsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|F\|^2_{\mathcal{H}} + \varepsilon\|\mathcal{U}\|^2_{\mathcal{H}}.\end{gathered}$$

From the above inequalities we obtain

$$\|\mathcal{U}\|^2_{\mathcal{H}} = \int_0^\ell \mathbb{E}(x)\,dx + \epsilon(|u|^2+|U|^2+|z|^2+|Z|^2) \le C_\varepsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|F\|^2_{\mathcal{H}} + \varepsilon\|\mathcal{U}\|^2_{\mathcal{H}},$$

which implies that $\|\mathcal{U}\|_{\mathcal{H}} \le C\|F\|_{\mathcal{H}}$. From where the exponential stability holds.

Finally, we consider the discontinuous case. Note that (3.33) is also valid for discontinuous functions B and K. From Remark 3.4,

$$\int_{I_D} \theta|\lambda\Phi|^2 + \theta|\lambda\Psi|^2\,dx + \theta|\lambda W|^2\,dx \le C_\varepsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon\|F\|^2_{\mathcal{H}} + \varepsilon\|\mathcal{U}\|^2_{\mathcal{H}}.$$

Therefore for each interval $]a,b[\subset I_D$ we have

$$\int_a^b \mathbb{E}(x)\,dx \le C_\varepsilon \|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon \|F\|_{\mathcal{H}}^2 + \varepsilon\|\mathcal{U}\|_{\mathcal{H}}^2. \tag{3.36}$$

Using the observability Lemma 3.6 for discontinuous constitutive law over the interval $]a,b[$ we obtain

$$\mathbb{E}(a)+\mathbb{E}(b) \le C_\varepsilon|\lambda|^2\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon \|F\|_{\mathcal{H}}^2 + \varepsilon\|\mathcal{U}\|_{\mathcal{H}}^2. \tag{3.37}$$

Reasoning as above we conclude that

$$\begin{aligned}\|\mathcal{U}\|_{\mathcal{H}}^2 &= \int_0^\ell \mathbb{E}(x)\,dx + \epsilon(|u|^2+|U|^2+|z|^2+|Z|^2)\\ &\le C_\varepsilon|\lambda|^2\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\varepsilon \|F\|_{\mathcal{H}}^2 + \varepsilon\|\mathcal{U}\|_{\mathcal{H}}^2.\end{aligned}$$

From where we obtain that $\|\mathcal{U}\|_{\mathcal{H}} \le |\lambda|^2\|F\|_{\mathcal{H}}$, which implies the polynomial stability. □

4. Lack of exponential stability

In this section we prove that the semigroup associated with system (2.1)-(2.2) is not exponentially stable when the viscoelastic constitutive law is discontinuous. To do that we use [14, Theorem 5.1].

Theorem 4.1. *Let $\mathcal{T}$ be a contractions semigroup over $\mathcal{H}$ and $\mathcal{T}_0$ a group with unitary norm, that is $\|\mathcal{T}_0(t)\mathbf{U}\| = \|\mathbf{U}_0\|$, defined in $\mathcal{H}_0 \subset \mathcal{H}$. If the operator $\mathcal{T}(t)-\mathcal{T}_0(t)$ is a compact operator, then the semigroup $\mathcal{T}(t)$ is not exponentially stable.*

Another key result for our purpose is given in the following Lemma.

Lemma 4.2 (Lions-Aubin [10, Theorem 5.1])**.** *Let be V, H, V_0 Banach spaces such that $V \subseteq V_0 \subseteq H$, where the first embedding is compact. Let $\varphi \in L^p([a,b];V)$, $\varphi_t \in L^p([a,b];H)$. Denoting*

$$W = \{\varphi \in L^p([a,b);V) \colon \varphi_t \in L^p([a,b];H\}.$$

Then the embedding $W \subseteq L^p([a,b];V_0)$ is compact.

Finally, we establish the observability inequality to the evolution Bresse system. To do it let us denote

$$\begin{gathered}\mathcal{I}_\varphi(x,t) = |\tilde\varphi_t(x,t)|^2 + |\tilde\varphi_x(x,t)+\tilde\psi(x,t)+l\tilde\omega(x,t)|^2,\\ \mathcal{I}_\psi(x,t) = |\tilde\psi_t(x,t)|^2 + |\tilde\psi_x(x,t)|^2,\\ \mathcal{I}_\omega(\cdot,t) = |\tilde\omega_t(\cdot,t)|^2 + |\tilde\omega_x(x,t)-l\tilde\varphi(x,t)|^2.\end{gathered}$$

Lemma 4.3 ([19, Lemma 2.1])**.** *Let us suppose that there exists a solution to Bresse system, bounded by the initial energy associated with the model*

$$\begin{gathered}\rho_1\tilde\varphi_{tt} - \kappa(\tilde\varphi_x+\tilde\psi+l\tilde\omega)_x - lK(\tilde\omega_x - l\tilde\varphi) = 0 \quad \text{in }]a,b[\times]0,T[,\\ \rho_2\tilde\psi_{tt} - b\tilde\psi_{xx} + \kappa(\tilde\varphi_x+\tilde\psi+l\tilde\omega) = 0 \quad \text{in }]a,b[\times]0,T[,\\ \rho_1\tilde\omega_{tt} - lK(\tilde\omega_x - l\tilde\varphi)_x + l\kappa(\tilde\varphi_x+\tilde\psi+l\tilde\omega) = 0 \quad \text{in }]a,b[\times]0,T[.\end{gathered} \tag{4.1}$$

Then there exists a positive constant satisfying

$$\int_0^T \mathcal{I}_\varphi(a,t)+\mathcal{I}_\varphi(b,t)+\mathcal{I}_\psi(a,t)+\mathcal{I}_\psi(b,t)+\mathcal{I}_\omega(a,t)+\mathcal{I}_\omega(b,t)\,dt \le cE(0)$$

where $E(t) = \int_a^b \mathcal{I}_\varphi(x,t) + \mathcal{I}_\psi(x,t) + \mathcal{I}_\omega(x,t)\,dx$.

Now we are in a position to show the lack of exponential stability.

Theorem 4.4. *If the constitutive law is discontinuous then the semigroup* $\mathcal{T}(t)$ *is not exponentially stable.*

Proof. Let us denote by $\widetilde{\mathcal{T}}_0$ the group associated with system (4.1) for $]a,b[=]0,\ell_0[$ satisfying the boundary conditions

$$\tilde\varphi(0,t) = \tilde\varphi(\ell_0,t) = \tilde\psi(0,t) = \tilde\psi(\ell_0,t) = \tilde\omega(0,t) = \tilde\omega(\ell_0,t) = 0 \tag{4.2}$$

over the phase space

$$\widetilde{\mathcal{H}} = H_0^1(0,\ell_0)\times L^2(0,\ell_0)\times H_0^1(0,\ell_0)\times L^2(0,\ell_0).$$

Note that $\|\widetilde{\mathcal{T}}_0(t)\mathcal{U}_0\|^2 = \|\mathcal{U}_0\|^2$ for all $\mathcal{U}_0\in\widetilde{\mathcal{H}}$. Let us consider the spaces

$$\mathbb{L}_0 = \{f\in L^2(0,\ell)\colon f\Big|_{[\ell_0,\ell]} = 0\},\quad V_0 = H_0^1(0,\ell)\cap\mathbb{L}_0,$$
$$\mathcal{H}_0 = V_0\times\mathbb{L}_0\times V_0\times\mathbb{L}_0\times V_0\times\mathbb{L}_0\times\{\mathbf{0}\}\times\{\mathbf{0}\}\times\{\mathbf{0}\}\times\{\mathbf{0}\}.$$

Let us denote by $\mathcal{T}_0$ the group over $\mathcal{H}_0$ (null extensions on $[\ell_0,\ell]$) associated with (4.1)-(4.2) that is

$$\mathcal{T}_0(t)\tilde U_0 = (\tilde\varphi,\tilde\varphi_t,\tilde\psi,\tilde\psi_t,\tilde w,\tilde w_t,\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}).$$

So we have

$$\|\mathcal{T}_0(t)\tilde{\mathcal{U}}_0\|^2 = \|\tilde{\mathcal{U}}_0\|^2,\quad \forall\tilde{\mathcal{U}}_0\in\mathcal{H}_0 \tag{4.3}$$

Now, we prove that $\mathcal{T}(t)-\mathcal{T}_0(t):\mathcal{H}_0\to\mathcal{H}$ is a compact operator, where

$$\mathcal{T}\mathcal{U}_0^m = (\varphi^m,\varphi_t^m,\psi^m,\psi_t^m,\omega^m,\omega_t^m,u,u_t,\ z,z_t)\in\mathcal{H},$$
$$\mathcal{T}_0(t)\mathcal{U}_0^m = (\tilde\varphi^m,\tilde\varphi_t^m,\tilde\psi^m,\tilde\psi_t^m,\tilde\omega^m,\tilde\omega_t^m,\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}))\in H_0$$

Let $v^m := \varphi^m - \tilde\varphi^m$, $y^m := \psi^m-\tilde\psi^m$, $\zeta^m := \omega^m-\tilde\omega^m$. By definition we have

$$v^m(x,t) = \begin{cases}\varphi^m-\tilde\varphi^m, & \text{if } x\in[0,\ell_0]\\ \varphi^m, & \text{if } x\notin[0,\ell_0],\end{cases}$$
$$y^m(x,t) = \begin{cases}\psi^m-\tilde\psi^m, & \text{if } x\in[0,\ell_0]\\ \psi^m, & \text{if } x\notin[0,\ell_0],\end{cases}$$
$$\zeta^m(x,t) = \begin{cases}\omega^m-\tilde\omega^m, & \text{if } x\in[0,\ell_0]\\ \omega^m, & \text{if } x\notin[0,\ell_0].\end{cases}$$

Moreover v, y and ζ satisfies

$$\begin{aligned}\rho_1 v_{tt} - \kappa(v_x+y+lz)_x - K(v_{xt}+y_t+l\zeta_t)_x\\ - lK(\zeta_x-lv) - lK(\zeta_{xt}-lv_t) = 0,\end{aligned} \tag{4.4}$$

$$\rho_2 y_{tt} - by_{xx} - (By_{xt})_x + \kappa(v_x+y+l\zeta) + K(v_{xt}+y_t+l\zeta_t) = 0, \tag{4.5}$$

$$\begin{aligned}\rho_1\zeta_{tt} - \kappa_0(\zeta_x-lv)_x - lK(\zeta_{xt}-lv_t)_x\\ + l\kappa(v_x+y+l\zeta) + lK(v_{xt}+y_t+l\zeta_t) = 0.\end{aligned} \tag{4.6}$$

Multiplying (4.4) by v_t, (4.5) by w_t, and integrating over $[0,\ell]$, we obtain

$$\begin{aligned}&\frac{d}{dt}\|\mathfrak{U}^m(t)\|_{\mathcal{H}}^2+\int_{\ell_0}^{\ell}K|v_{xt}+y_t+l\zeta_t|^2+B|y_{xt}|^2+K|\zeta_{xt}-lv_t|^2\,dx+\epsilon|u_t|^2+\epsilon|z_t|^2\\&=-\tilde{S}^m(\ell_0^-,t)\varphi_t(\ell_0^-,t)-\tilde{M}^m(\ell_0^-,t)\psi_t(\ell_0^-,t)-\tilde{N}^m(\ell_0^-,t)w_t(\ell_0^-,t)\end{aligned}\tag{4.7}$$

where

$$\mathfrak{U}^m(t)=[\mathcal{T}(t)-\mathcal{T}_0(t)]U_0^m=(v^m,v_t^m,y^m,y_t^m,\zeta^m,\zeta_t^m,u,u_t,z,z_t).$$

Integrating (4.7) over $[0,t]$, we obtain

$$\begin{aligned}&\|\mathfrak{U}^m(t)\|_{\mathcal{H}}^2+\int_0^t\int_{\ell_0}^{\ell}\tilde{\kappa}|v_{xt}^m+y_t^m+l\zeta_t^m|^2+\tilde{b}|y_{xt}^m|^2+\tilde{K}|\zeta_{xt}^m-lv_t^m|^2\,dx\,dt\\&=-\int_0^t(\tilde{S}^m(\ell_0^-,t)\varphi_t(\ell_0^-,t)+\tilde{M}^m(\ell_0^-,t)\psi_t(\ell_0^-,t)+\tilde{N}^m(\ell_0^-,t)w_t(\ell_0^-,t))\,dt.\end{aligned}\tag{4.8}$$

Using Lemma 4.3 on the interval $]0,\ell_0[$, we conclude that $\tilde{S}^m$. $\tilde{M}^m$, $\tilde{N}^m$ are bounded, therefore there exists a subsequence (we still denote in the same way) such that

$$\begin{gathered}\tilde{S}^m(\ell_0^-,t)\to\tilde{S}(\ell_0^-,t)\quad\text{weak in }L^2(0,T),\\\tilde{M}^m(\ell_0^-,t)\to\tilde{M}(\ell_0^-,t)\quad\text{weak in }L^2(0,T),\\\tilde{N}^m(\ell_0^-,t)\to\tilde{N}(\ell_0^-,t)\quad\text{weak in }L^2(0,T).\end{gathered}$$

We only need to prove that

$$\big(\varphi_t^m(\ell_0^-,t),\psi_t^m(\ell_0^-,t),\omega_t^m(\ell_0^-,t)\big)\to\big(\varphi_t(\ell_0^-,t),\psi_t(\ell_0^-,t),\omega_t(\ell_0^-,t)\big)\tag{4.9}$$

strong in $[L^2(0,T)]^3$, which implies the norm convergence in (4.8). To do that we use (2.9) and system (2.1) to obtain

$$\varphi_t^m,\psi_t^m,\omega_t^m\in L^2(0,T;H^1(I_D)),\quad\varphi_{tt}^m,\psi_{tt}^m,\omega_{tt}^m\in L^2(0,T;H^{-1}(I_D)).$$

Since $H^1\subset H^{1-\delta}\subset H^{-1}$ for $0<\delta<\frac{1}{2}$ where the first inclusion is a compact embedding, then Lemma 4.2 implies that there exists a subsequence (we still denote in the same way) such that

$$(\varphi_t^m,\psi_t^m,\omega_t^m)\to(\varphi_t,\psi_t,\omega_t)\quad\text{strong in }L^2(0,T;H^{1-\delta}(I_D)\times H^{1-\delta}(I_D)\times H^{1-\delta}(I_D)),$$

and since the embedding $H^{1-\delta}(I_D)\subset C(\overline{I_D})$ is compact, we have

$$(\varphi_t^m,\psi_t^m,\omega_t^m)\to(\varphi_t,\psi_t,\omega_t)\quad\text{strong in }L^2(0,T;C(\overline{I_D})\times C(\overline{I_D})\times C(\overline{I_D})).$$

Since $\overline{I_D}=[\ell_0,\ell_1]$ the above convergence implies (4.9). Hence (4.8) implies the convergence in norm of $\mathfrak{U}^m$, since $\mathcal{H}$ is a Hilbert space we obtain that $\mathcal{T}(t)-\widetilde{\mathcal{T}_0}(t)$ is a compact operator. The proof is now complete. □

5. Penalized problem

Here we establish the well possedness and the asymptotic behavior of the abstract semi linear problem. We introduce a local Lipschitz $\mathcal{F}$ function defined over a Hilbert space $\mathcal{H}$. We assume that for each ball $B_R=\{W\in\mathcal{H}:\|W\|_{\mathcal{H}}\le R\}$, there exists a function globally of Lipschitz type $\widetilde{\mathcal{F}_R}$, such that

$$\mathcal{F}(0)=0,\quad\mathcal{F}(U)=\widetilde{\mathcal{F}_R}(U),\quad\forall U\in B_R\tag{5.1}$$

and additionally, that there exists a positive constant κ_0 such that

$$\int_0^t \big(\widetilde{\mathcal{F}_R}(U(s)), U(s)\big)_{\mathcal{H}}\, ds \le \kappa_0 \|U(0)\|^2_{\mathcal{H}}, \quad \forall U \in C([0,T];\mathcal{H}). \tag{5.2}$$

Under the above conditions we have the following theorem proved in [13].

Theorem 5.1. *Let $\{T(t)\}_{t\ge 0}$ be a C_0 semigroup of contraction, exponentially or polynomially stable semigroup with infinitesimal generator $\mathbb{A}$ over the phase space $\mathcal{H}$. Let $\mathcal{F}$ locally Lipschitz on $\mathcal{H}$ satisfying conditions* (5.1) *and* (5.2)*. Then there exists a global solution to*

$$U_t - \mathbb{A}U = \mathcal{F}(U), \quad U(0) = U_0 \in \mathcal{H}, \tag{5.3}$$

that decays exponentially or polynomially respectively.

Let us consider the semilinear system

$$\begin{gathered}
\rho_1\varphi_{tt} - S_x - lN = 0,\\
\rho_2\psi_{tt} - M_x + S = 0,\\
\rho_1\omega_{tt} - N_x + lS = 0,\\
\epsilon u_{tt} + \epsilon u_t + \epsilon u + S^\epsilon(\ell,t) = -\frac{1}{\epsilon}\Big[(u-g_3)^+ - (g_2-u)^+\Big],\\
\epsilon z_{tt} + \epsilon z_t + \epsilon z + N^\epsilon(\ell,t) = -\frac{1}{\epsilon}(z-g_1)^+.
\end{gathered} \tag{5.4}$$

Let us denote

$$\mathcal{F}(U) = (0,0,0,0,0,0,f_1(u),0,f_2(z))$$

where

$$f_1(u) = -\frac{1}{\epsilon}\Big[(u-g_3)^+ - (g_2-u)^+\Big] \quad \text{and} \quad f_2(z) = -\frac{1}{\epsilon}(z-g_1)^+.$$

Since f_1, f_2 are Lipschitz functions, it follows that $\mathcal{F}$ is Lipschitz on $\mathcal{H}$ satisfying condition (5.2). Under these conditions we have the following result.

Theorem 5.2. *The semigroup defined by* (5.4) *is exponentially or polynomially stable provided $K(x)$, $B(x)$ and $K_0(x)$ are differentiable or discontinuous functions, respectively.*

Proof. Note that the above system can be written as

$$\mathcal{U}_t - \mathbb{A}\mathcal{U} = \mathcal{F}(\mathcal{U}), \quad \mathcal{U}(0) = \mathcal{U}_0.$$

Where $\mathbb{A}$ is given by (2.5). From Theorem 3.7 and Theorem 5.1 our conclusion follows. □

6. Signorini's problem

Here we show the main result of this article. First we introduce the space

$$\mathcal{H}_0 = \mathcal{K} \times L^2(0,\ell) \times H^1_0(0,\ell) \times L^2(0,\ell) \times \mathcal{V} \times L^2(0,\ell)$$

where

$$\mathcal{K} = \{u \in V_0; g_2 \le u(\ell) \le g_3\}, \quad \mathcal{V} = \{w \in V_0; w(\ell) \le g_1\}.$$

Theorem 6.1. *For any initial data $(\varphi_0,\varphi_1,\psi_0,\psi_1,\omega_0,\omega_1) \in \mathcal{H}$ there exist a weak solution to Signorin's problem* (1.1)-(1.8) *which decays as establish in Theorem 5.2.*

Proof. Multiplying, equation $(5.4)_1$-$(5.4)_6$ by φ_t^ϵ, ψ_t^ϵ, ω_t^ϵ, u_t, v_t, and z_t, respectively. Integrating over $(0,\ell)$, we obtain

$$\frac{d}{dt}\mathbf{E}_\epsilon(t) = -\int_0^\ell K|\varphi_{xt}^\epsilon+\psi_t^\epsilon+l\omega_t^\epsilon|^2+B|\psi_{xt}^\epsilon|^2+K|\omega_{tx}^\epsilon-l\varphi_t^\epsilon|^2\,dx-\epsilon|u_t^\epsilon|^2-\epsilon|z_t^\epsilon|^2 \quad (6.1)$$

where

$$\begin{aligned}
2\mathbf{E}_\epsilon(t) &= E_\epsilon(t) + \frac{1}{\epsilon}\mathcal{N}_\epsilon(t) + \epsilon(|u^\epsilon|^2 + |u_t^\epsilon|^2 + |z^\epsilon|^2 + |z_t^\epsilon|^2),\\
2E_\epsilon(t) &= \int_0^\ell \Big[\rho_1|\varphi_t^\epsilon|^2 + \rho_2|\psi_t^\epsilon|^2 + \rho_1|\omega_t^\epsilon|^2 + \kappa|\varphi_x^\epsilon + \psi^\epsilon + l\omega^\epsilon|^2 + b|\psi_x^\epsilon|^2\\
&\quad + \kappa_0|\omega_x^\epsilon - l\varphi^\epsilon|^2\Big]dx\\
\mathcal{N}_\epsilon(t) &:= |(\omega^\epsilon(\ell,t) - g_1)^+|^2 + |(\varphi^\epsilon(\ell,t) - g_3)^+|^2 + |(g_2 - \varphi^\epsilon(\ell,t))^+|^2.
\end{aligned}$$

Taking $(\varphi_0,\omega_0)\in\mathcal{K}\times\mathcal{V}$, and integrating we obtain

$$E_\epsilon(t) + \frac{1}{\epsilon}\mathcal{N}_\epsilon(t) + \epsilon(|u^\epsilon|^2 + |u_t^\epsilon|^2 + |z^\epsilon|^2 + |z_t^\epsilon|^2) \le \mathbf{E}_\epsilon(0), \quad (6.2)$$

$$\mathcal{N}_\epsilon(t) \le \epsilon\mathbf{E}_\epsilon(0) \to 0, \quad \text{as } \epsilon\to 0. \quad (6.3)$$

Hence denoting the limit $(\varphi^\epsilon,\psi^\epsilon,\omega^\epsilon)\to(\varphi,\psi,\omega)$ we obtain

$$\omega(\ell,t)\le g_1, \quad g_2\le\varphi(\ell,t)\le g_3. \quad (6.4)$$

Using Lemma 4.3 we obtain that $\varphi_t^\epsilon(\ell,t)$, $\omega_t^\epsilon(\ell,t)$, $S^\epsilon(\ell,t)$, and $N^\epsilon(\ell,t)$ are bounded in $L^2(0,T)$, so is u_{tt} and z_{tt}. Using $(5.4)_4$ we obtain

$$\int_0^T \Big[\epsilon u_{tt}^\epsilon+\epsilon u_t^\epsilon+\epsilon u^\epsilon+S^\epsilon(L,t)\Big][v-u^\epsilon]\,dt = -\frac{1}{\epsilon}\int_0^T \Big[(u^\epsilon-g_2)^+-(g_1-u^\epsilon)^+\Big][v-u^\epsilon]\,dt.$$

For all $v\in L^2(0,T;\mathcal{K})\cap H^1(0,T;L^2(0,L))$. Where $\mathcal{K}=\{w\in H^1(0,L): g_1\le w(L)\le g_2\}$. It is no difficult to show that

$$\lim_{\epsilon\to0}\int_0^T \Big[\epsilon u_{tt}^\epsilon + \epsilon u_t^\epsilon + \epsilon u^\epsilon\Big][v-u^\epsilon]\,dt = 0.$$

In fact, from $(5.4)_4$ ϵu_{tt}^ϵ is bounded by a constant depending on ϵ, in $L^2(0,T)$. Moreover, Lemma 4.3 implies that $u_t^\epsilon = \varphi_t^\epsilon(\ell,t)$ is also uniformly bounded in $L^2(0,T)$. Therefore u_t^ϵ is a continuous function, uniformly bounded in $L^\infty(0,T)$. Integrating by parts we obtain

$$\int_0^T \epsilon u_{tt}^\epsilon[u(t)-u^\epsilon]\,dt = \epsilon\, u_t^\epsilon[v(t)-u^\epsilon]\big|_0^T - \int_0^T \epsilon u_t^\epsilon[v_t(t)-u_t^\epsilon]\,dt \to 0.$$

Hence,

$$\begin{aligned}
&\lim_{\epsilon\to0}\int_0^T S^\epsilon(L,t)[v(t)-u^\epsilon(t)]\,dt\\
&= \lim_{\epsilon\to0}\int_0^T -\frac{1}{\epsilon}\Big[(u^\epsilon-g_2)^+ - (g_1-u^\epsilon)^+\Big][v(t)-u^\epsilon(t)]\,dt.
\end{aligned}$$

Since

$$\int_0^T (u^\epsilon-g_2)^+[v(t)-u^\epsilon(t)]\,dt$$

$$= \int_0^T (u^\epsilon - g_2)^+[u(^\epsilon(t) - g_2]\,dt - \int_0^T (u^\epsilon - g_2)^+(u^\epsilon - g_2)\,dt,$$
$$= \int_0^T (v^\epsilon - g_2)^+[u^\epsilon(t) - g_2]\,dt - \int_0^T (u^\epsilon - g_2)^+(u^\epsilon - g_2)^+\,dt \le 0.$$

For all $g_1 \le v(L,t) \le g_2$. Similarly we obtain

$$-\int_0^T [(g_1 - u^\epsilon)^+[v(t) - u^\epsilon(t)]\,dt \le 0.$$

Therefore, from the last two inequalities we obtain

$$\int_0^T \frac{1}{\epsilon}\big[(u^\epsilon - g_2)^+ - (g_1 - u^\epsilon)^+\big][v(t) - u^\epsilon(t)]\,dt \le 0, \quad \forall \epsilon > 0.$$

For all $v \in H^1(0,T;L^2(0,L))$ such that $g_1 \le v(L,t) \le g_2$. Taking the limit $\epsilon \to 0$ we obtain

$$\int_0^T S(L,t)[v(L,t) - \varphi(L,t)]\,dt \ge 0, \quad \forall v \in L^2(0,T;\mathcal{K}). \tag{6.5}$$

Inequality (6.5) implies condition (1.8). Using similar ideas we arrive at

$$\int_0^T N(L,t)[w(L,t) - \omega(L,t)]\,dt \ge 0, \quad \forall w \in L^2(0,T;\mathcal{V}).$$

From where condition (1.8) follows. Then the proof of existence is now complete.

To show the asymptotic behavior we use Theorem 5.2 and obtain

$$E(t) \le CE(0)e^{-\gamma t},$$

where

$$E(t) = \frac{1}{2}\int_0^\ell \Big[\rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + \rho_1|\omega_t|^2 + \kappa|\varphi_x + \psi + l\omega|^2 + b|\psi_x|^2 + \kappa_0|\omega_x - l\varphi|^2\Big]\,dx.$$

So, using the semicontinuity of the norm and noting that $\mathcal{N}(0) = 0$, we obtain

$$E(t) \le \liminf_{\epsilon\to 0^+} \mathbf{E}_\epsilon(t) \le C\big\{\lim_{\epsilon\to 0^+} \mathbf{E}_\epsilon(0)\big\}e^{-\gamma t} \le CE(0)e^{-\gamma t}$$

where C is a positive constant independent of parameter ϵ. Thus, we conclude the exponential stability of the Signorini's problem. Similarly we obtain the polynomial stability. □

7. Numerical results

In this section we show numerical results for the penalized system (2.1)-(2.2). Here, we use the well-known Newmark's methods [6, 15].

7.1. **Variational formulation.** Letting $\mathbf{u} = [\varphi, \psi, \omega]^\top$, from (2.1) we obtain the variational problem

$$(\mathbf{u}_{tt}^\epsilon(t), \tilde{\mathbf{u}}) + a_1(\mathbf{u}^\epsilon(t), \tilde{\mathbf{u}}) + a_2(\mathbf{u}_t^\epsilon(t), \tilde{\mathbf{u}}) = 0, \quad \forall \tilde{\mathbf{u}} \in \mathcal{V} \tag{7.1}$$

where $\mathcal{V} = V_0 \times H_0^1 \times V_0$ and $\mathbf{u}^\epsilon$ satisfy the initial conditions

$$(\mathbf{u}^\epsilon(0), \tilde{\mathbf{u}}) = (\mathbf{u}_0^\epsilon, \tilde{\mathbf{u}}), \quad (\mathbf{u}_t^\epsilon(0), \tilde{\mathbf{u}}) = (\mathbf{u}_1^\epsilon, \tilde{\mathbf{u}}). \tag{7.2}$$

Here $a_i : \mathcal{V} \times \mathcal{V} \mapsto \mathbb{R}$ are functionals defined by

$$(\mathbf{u}_{tt}^\epsilon(t), \tilde{\mathbf{u}}) = \rho_1(\varphi_t^\epsilon, u_1) + \rho_2(\psi_t^\epsilon, u_2) + \rho_1(\omega_t^\epsilon, u_3),$$

$$
\begin{aligned}
a_1(\mathbf{u}^\epsilon(t),\tilde{\mathbf{u}}) &= k(\varphi^\epsilon_x+\psi^\epsilon+l\omega^\epsilon, u_{1,x}+u_2+lu_3)+b(\psi^\epsilon_x,u_{2,x})\\
&\quad + k_0(\omega^\epsilon_x - l\varphi^\epsilon, u_{3,x}-lu_1) - S^\epsilon(\ell,t)u_1(L) - N^\epsilon(\ell,t)u_3(L),\\
a_2(\mathbf{u}^\epsilon_t(t),\tilde{\mathbf{u}}) &= \int_0^\ell K(x)(\varphi^\epsilon_{tx}+\psi^\epsilon_t+l\omega^\epsilon_t)(u_{1,x}+u_2+lu_3)\,dx + \int_0^\ell B(x)\psi^\epsilon_{tx}u_{2,x}\,dx\\
&\quad + \int_0^\ell K_0(x)(\omega^\epsilon_{tx}-l\varphi^\epsilon_t)(u_{3,x}-lu_1)\,dx.
\end{aligned}
$$

Here $(\cdot,\cdot)$ is the inner product in $L^2(0,\ell)$.

7.2. **Algorithms and numerical approximation.** To have the full discretization of (7.1)–(7.2) we first consider a partition of the interval $\Omega=(0,\ell)$, $X_h=\{0=x_0<x_1<\dots<x_N=\ell\}$, $\Omega_{j+1}=(x_j,x_{j+1})$, and $\Omega_i\cap\Omega_j=\emptyset$ if $i\neq j$, and $\Omega=\cup_{e=1}^{Ne}\overline{\Omega}_e$, where N_e is the number of the elements of the partition. Then we define the finite-dimensional subspaces $S^h_1=\{u\in C(0,\ell); u\big|_{\Omega_e}\in P_1(\Omega_e)\}$ where P_1 is the set linear polinomials defined in Ω_e,

$$
\mathcal{V}^h=\{v^h\in S^h_1; v^h(0)=0\}\quad\text{and}\quad \mathcal{U}^h=\{u^h\in S^h_1; u^h(0)=u^h(\ell)=0\}.
$$

Considering $\mathbf{u}^h=[\varphi^h,\psi^h,\omega^h]^\top$, the approximation is characterized as the finite-dimensional problem in $\mathbb{R}^{3N_e}$,

$$
(\mathbf{u}^{h,\epsilon}_{tt}(t),\tilde{\mathbf{u}}^h)+a_1(\mathbf{u}^{h,\epsilon}(t),\tilde{\mathbf{u}}^h)+a_2(\mathbf{u}^{h,\epsilon}_t(t),\tilde{\mathbf{u}}^h)=0,\quad \forall\tilde{\mathbf{u}^h}\in\mathcal{V}^h\times\mathcal{U}^h\times\mathcal{V}^h, \tag{7.3}
$$

where $\mathbf{u}^{h,\epsilon}(t)$ satisfies the initial conditions

$$
(\mathbf{u}^{h,\epsilon}(0),\tilde{\mathbf{u}}^h)=(\mathbf{u}^{h,\epsilon}_0,\tilde{\mathbf{u}}^h),\quad (\mathbf{u}^{h,\epsilon}_t(0),\tilde{\mathbf{u}}^h)=(\mathbf{u}^{h,\epsilon}_1,\tilde{\mathbf{u}}^h). \tag{7.4}
$$

In matrix form, dynamical problem (7.3)-(7.4) can be written as

$$
\begin{gathered}
\mathbf{M}\ddot{\mathbf{d}}(t)+\mathbf{K}(\mathbf{d}(t))+\mathbf{C}\dot{\mathbf{d}}(t)=0,\\
\mathbf{d}(0)=\mathbf{d}_0,\quad \dot{\mathbf{d}}(0)=\mathbf{d}_1,
\end{gathered}
$$

where, $\mathbf{d}(t)$ is the vector of displacement nodal generalized at time t. $\mathbf{M}$, $\mathbf{C}$ are matrices associated with the functionals: $(\mathbf{u}^\epsilon_{tt}(t),\tilde{\mathbf{u}})$ and $a_2(\mathbf{u}^\epsilon_t(t),\tilde{\mathbf{u}})$ respectively. $\mathbf{K}(\mathbf{d}(t))$ is the vector of consistent nodal elastic stiffness at time t.

Taking a partition P on the interval $[0,T]$ of M intervals of length Δt such that $0=t_0<t_1<\dots<t_M=T$, with $t_{n+1}-t_n=\Delta t$ and considering the non-linearity the numerical scheme becomes

$$
\begin{gathered}
\mathbf{M}\ddot{\mathbf{d}}_{n+1}+\mathbf{C}\dot{\mathbf{d}}_{n+1}+\mathbf{K}\mathbf{d}_{n+1}=\tilde{\mathbf{K}}(\mathbf{d}_{n+1}),\\
\mathbf{d}_{n+1}=\mathbf{d}_n+\Delta t\dot{\mathbf{d}}_n+\frac{\Delta t^2}{2}[(1-2\beta)\ddot{\mathbf{d}}_n+2\beta\ddot{\mathbf{d}}_{n+1}],\\
\dot{\mathbf{d}}_{n+1}=\dot{\mathbf{d}}_n+\Delta t[(1-\gamma)\ddot{\mathbf{d}}_n+\gamma\ddot{\mathbf{d}}_{n+1}]
\end{gathered}
$$

with

$$
\tilde{\mathbf{K}}(\mathbf{d}_{n+1})=-\frac{1}{\epsilon}\Big(0,\dots,0,(d_{3N_e-2}(t)-g_3)^+-(g_2-d_{3N_e-2}(t))^+,0,(d_{3N_e}(t)-g_1)^+)\Big)^\top.
$$

Here, β and γ are two parameters that govern the stability and accuracy of the method. The matrices, from the above system, are obtained from the finite element method standard assembly, $\mathbf{M}=\cup_{e=1}^{Ne}\mathbf{m}^e$, $\mathbf{K}=\cup_{e=1}^{Ne}(\mathbf{k}^e_S+\mathbf{k}^e_M+\mathbf{k}^e_N)$, for instance, considering linear functions in the interpolation functions, we obtain the elementary matrices

$$\mathbf{m}^e = \begin{bmatrix} \rho_1 h/3 & 0 & 0 & \rho_1 h/6 & 0 & 0 \\ 0 & \rho_2 h/3 & 0 & 0 & \rho_2 h/6 & 0 \\ 0 & 0 & \rho_1 h/3 & 0 & 0 & \rho_1 h/6 \\ \rho_1 h/6 & 0 & 0 & \rho_1 h/3 & 0 & 0 \\ 0 & \rho_2 h/6 & 0 & 0 & \rho_2 h/3 & 0 \\ 0 & 0 & \rho_1 h/6 & 0 & 0 & \rho_1 h/3 \end{bmatrix},$$

$$\mathbf{k}_M^e = \frac{b}{h} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{k}_S^e = \begin{bmatrix} k/h & -k/2 & -kl/2 & -k/h & -k/2 & -kl/2 \\ -k/2 & kh/3 & klh/3 & k/2 & kh/6 & klh/6 \\ -kl/2 & klh/3 & kl^2h/3 & kl/2 & klh/6 & kl^2h/6 \\ -k/h & k/2 & kl/2 & k/h & k/2 & kl/2 \\ -k/2 & kh/6 & klh/6 & k/2 & kh/3 & klh/3 \\ -kl/2 & klh/6 & kl^2h/6 & kl/2 & klh/3 & kl^2h/2 \end{bmatrix}.$$

Remark 7.1. Generally to penalized models, in particular to Bresse beams, occurs a typical numerical problem, the *shear locking*, for more details see [11]. To overcome this problem alternatives was performed in Hughes et al [7], Prathap and Bhashyam [17] and Abimael et al [11] and references therein.

To obtain computational results, we use the implemented code in the Language C. The graphics were developed using GNUplot. In all experiments we use the following parameters to Newmark's method: $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$. The finite element mesh $h = 0.01$ m and length of the beam $\ell = 1$ m.

Let I_v viscoelastic component in $I = [0, \ell]$. Here we are consider the following case.

7.3. **Cases** I_v**.** We consider the localized viscoleastic damping functions $I_v = [\ell_0, \ell_1]$ and $I_v = [\ell_0, \ell]$.

$$f_1(x) = \begin{cases} c_0(x-\ell_0)^4(x-\ell_1)^4 & \text{if } x \in I_v, \\ 0 & \text{if } x \in I \setminus I_v. \end{cases}$$

$$f_2(x) = \begin{cases} c_0(x-\ell_0)^2 & \text{if } x \in I_v \\ 0 & \text{if } x \in I \setminus I_v. \end{cases}$$

7.4. **Linear case - without contact:** $g_1 = +\infty$**,** $g_2 = -\infty$**,** $g_3 = +\infty$**.**

7.4.1. *Experiment.* We take a rectangular arch beam with $I_v = [0.3, 0.6]$, thickness and width equal 0.08 m, $\rho = 2700$ kg/m^3, $\kappa = 5/6$, $r = 0.3$ (Poisson ratio),

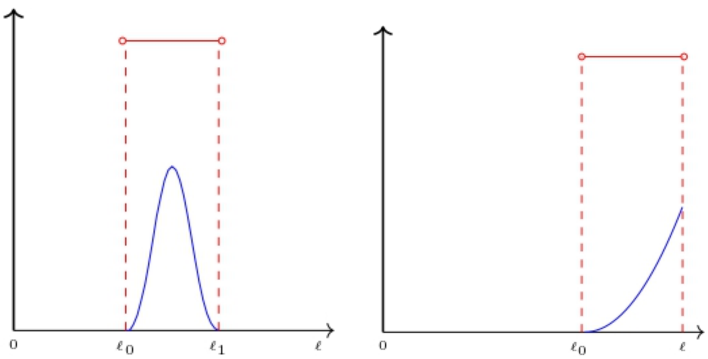

FIGURE 4. Dissipation functions: continuous (blue lines) and discontinuous (red lines) cases.

$E = 69 \cdot 10^9$ N/m^2, $R = 2$ m and $\Delta t = 1$ s. We use the initial conditions

$$\varphi_0 = \begin{cases} x^2, & x \in [0, 0.15], \\ -3(x-0.15)^2 - 0.3(x-0.15) - 0.0225, & x \in [0.15, 0.3] \\ 2(x-0.3)^2 - 0.6(a-0.3), & x \in [0.3, 0.6] \\ -2(x-0.6)^2 - 0.8(a-0.6), & x \in [0.6, 1], \end{cases}$$

$$\varphi_1 = \begin{cases} 1 & x \in (0, 0.15), \\ 0 & x \in [0.15, 1]. \end{cases}$$

and $\psi_0 = \psi_1 = \omega_0 = \omega_1 = 0$. See Figures 5 and 6.

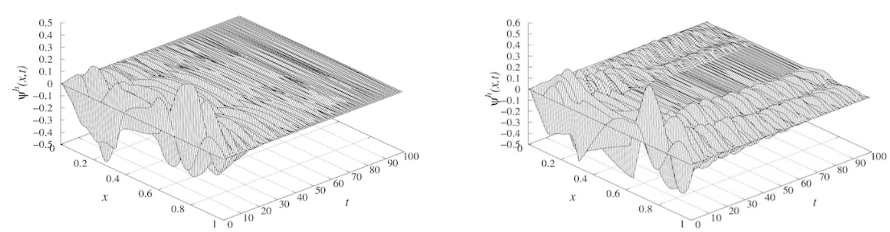

FIGURE 5. Evolution of the rotation angle of filaments: $\psi^{h,\epsilon}(x,t)$.

7.5. **Nonlinear case - penalized problem.**

7.5.1. *Experiment: contact problem:* $g_2 = -0.3$, $g_3 = 0.3$. We consider a rectangular arch beam with $\rho_1 = 0.2$, $\rho_2 = 0.16$, $\rho_3 = 0.3$, $k = 0.064$, $b = 1.0$, and $\kappa_0 = 0.2$. The initial conditions $\varphi(x,0) = 0$, $\varphi_t(x,0) = x$, $\psi(x,0) = 0$, $\psi_t(x,0) = \sin(\frac{\pi}{2}x)$, $\omega(x,0) = \omega_t(x,0) = 0$. Furthermore, we consider $\Delta t = 10^{-2}$ s. and the penalization parameter $\epsilon = 10^{-1}$. See Figure 7.

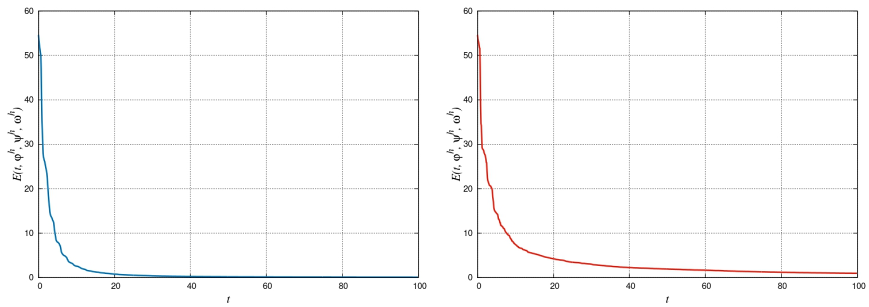

FIGURE 6. Asymptotic behavior of the numerical energy $E^h(t, \varphi^{h,\epsilon}, \psi^{h,\epsilon}, \omega^{h,\epsilon})$ at time 100 s. Here we compared the numerical experiments to continuous case (left side) versus discontinuous case (right side), where we obtain the exponential and polynomial stability, respectively.

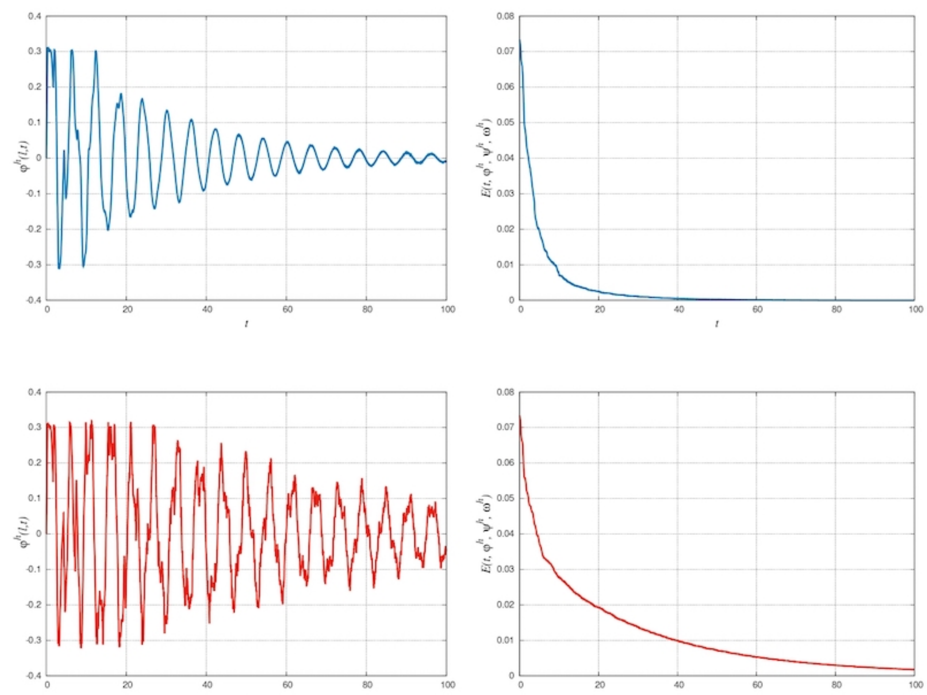

FIGURE 7. Beam's oscillations at the end $x = \ell$: $\varphi^{h,\epsilon}(\ell, t)$ and the asymptotic behavior of the energy at 100 s we performed the experiment for $I_v = [0.6, 1]$ and viscoelastic damping function differentiable (blue line and discontinuous (red line) cases, respectively, see Theorem 5.2.

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