Electronic Journal of Differential Equations, Vol. 2024 (2024), No. 18, pp. 1–11.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2024.18

EXISTENCE OF HIGH ENERGY SOLUTIONS FOR SUPERLINEAR COUPLED KLEIN-GORDONS AND BORN-INFELD EQUATIONS

LIXIA WANG, PINGPING ZHAO, DONG ZHANG

Communicated by Marko Squassina

Abstract. In this article, we study the system of Klein-Gordon and Born-Infeld equations

$$
\begin{gathered}
-\Delta u+V(x)u-(2\omega+\phi)\phi u=f(x,u), \quad x\in\mathbb{R}^3,\\
\Delta\phi+\beta\Delta_4\phi=4\pi(\omega+\phi)u^2, \quad x\in\mathbb{R}^3,
\end{gathered}
$$

where $\Delta_4\phi=\operatorname{div}(|\nabla\phi|^2\nabla\phi)$, ω is a positive constant. Assuming that the primitive of $f(x,u)$ is of 2-superlinear growth in u at infinity, we prove the existence of multiple solutions using the fountain theorem. Here the potential V are allowed to be a sign-changing function.

1. Introduction and main results

In this article, we study Klein-Gordon equation using Born-Infeld theory

$$
\begin{gathered}
-\Delta u+V(x)u-(2\omega+\phi)\phi u=f(x,u), \quad x\in\mathbb{R}^3,\\
\Delta\phi+\beta\Delta_4\phi=4\pi(\omega+\phi)u^2, \quad x\in\mathbb{R}^3,
\end{gathered}
\tag{1.1}
$$

where ω is a positive constant, $V\in C(\mathbb{R}^3,\mathbb{R})$, and $f\in C(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$. By using the local linking theorem and the fountain theorem, we obtain multiple solutions for (1.1).

It is well known that Klein-Gordon equation can be used in theory of electrically charged fields [16]. The Born-Infeld theory is proposed by Born [7, 8, 9] to overcome the infinite energy problem associated with a point-charge source in the original Maxwell theory. The presence of the nonlinear term f simulates the interaction between many particles or external nonlinear perturbations. For more details in the physical aspects, we refer the readers to [5, 10, 17, 21, 30].

In recent years, the Born-Infeld nonlinear electromagnetism has become more important since its relevance in the theory of superstring and membranes. By using variational methods, several existence results for problem (1.1) have been found with constant potential $V(x)=m^2-\omega^2$. Next we recall some of them.

2020 *Mathematics Subject Classification.* 35B33, 35J65, 35Q55.
Key words and phrases. Klein-Gordon equation; Born-Infeld theory; superlinear; fountain theorem.

Submitted October 30, 2023. Published February 16, 2024.

In 2002, D'Avenia et al [15] considered for the Klein-Gordon equation on $\mathbb{R}^3$

$$\begin{gathered} -\Delta u+[m^2-(\omega+\phi)^2]\phi u=f(x,u), \quad x\in\mathbb{R}^3, \\ \Delta\phi+\beta\Delta_4\phi=4\pi(\omega+\phi)u^2, \quad x\in\mathbb{R}^3, \end{gathered} \tag{1.2}$$

with pure power nonlinearity, i.e., $f(x,u)=|u|^{p-2}u$, where ω and m are constants. By using the mountain pass theorem, they proved that (1.2) has infinitely many radially symmetric solutions under the assumptions that $|m|>\omega$ and $4<p<6$. Mugnai [21] covered the case $2<p\le 4$ assuming $\sqrt{\frac{p-2}{2}}|m|>\omega>0$. Later, for $f(x,u)=|u|^{p-2}u+|u|^{2^*-2}u$, i.e. the critical Sobolev case was studied in [23]. The authors obtained a nontrivial solution under the conditions $4<p<6$ and $m>\omega$. The authors in [19] improved the result of [23] and studied the existence of ground state solution. Zhang and Liu [32] considered the existence and multiplicity of sign-changing solutions by the method of invariant sets of descending flow.

Recently, for general potential $V(x)$, Chen and Song [14] obtained the existence of multiple nontrivial solutions for (1.1) with the nonlinearity $f(x,u)=\lambda k(x)|u|^{q-2}u+g(x)|u|^{p-2}u$; that is, the Klein-Gordon equation with concave and convex nonlinearities coupled with Born-Infeld equations on $\mathbb{R}^3$. Other related results about homogeneous Klein-Gordon equation with Born-Infeld equations can be found in [1, 11, 24, 25, 28, 31].

Next, we consider the non-homogeneous case, that is $f(x,u)$ is instead of $f(x,u)+h(x)$. Chen and Li [12] proved that (1.1) has two nontrivial radially symmetric solutions if $f(x,u)=|u|^{p-2}u$ and $h(x)$ is radially symmetric. In [26], the authors obtain the existence of two solutions by the Mountain Pass Theorem and the Ekeland's variational principle in critical point theory for general $f(x,u)$. In [27], the authors consider the existence of multiple solutions for nonhomogeneous Klein-Gordon equation with sign-changing potential coupled with Born-Infeld theory.

Motivated by the above works, we consider system (1.1) with more general potential $V(x)$ and the primitive of $f(x,u)$ is of 2-superlinear growth in u at infinity. Precisely, we make the following assumptions.

- (A1) $V\in C(\mathbb{R}^3,\mathbb{R})$ is bounded below and, for every $C>0$, $\operatorname{meas}\{x\in\mathbb{R}^3: V(x)\le C\}<+\infty$, where meas denotes the Lebesgue measures;
- (A2) $f\in C(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$ and there exist $C>0$ and $p\in(2,6)$ such that
$$|f(x,t)|\le C(1+|t|^{p-1});$$
- (A3) $f(x,t)=o(t)$ uniformly in x as $t\to 0$;
- (A4) $f(x,t)/t\to+\infty$ uniformly in x as $|t|\to+\infty$;
- (A5) There exists $\theta>2$ and $b>0$ such that $\mathcal{F}(x,t):=\frac{1}{\theta}f(x,t)t-F(x,t)\ge -bt^2$, where $F(x,t):=\int_0^t(x,s)ds$.

The condition

- (AR) There exists $\mu>4$ such that $\mu F(x,t)\le tf(x,t)$, for all $(x,t)\in\mathbb{R}^3\times\mathbb{R}$

is widely used in the studies of elliptic problem by variational methods. Condition (AR) is used not only to prove that the Euler-Lagrange function associated has a mountain pass geometry, but also to guarantee that the Palais-Smale sequences, or Cerami sequences are bounded. Obviously, we can observe that the condition (AR) implies the following condition:

- (A6) There exist $\mu>4$ and $C_1,C_2>0$ such that $F(x,t)\ge C_1|t|^\mu-C_2$, for t sufficiently large.

Moreover, the condition (A6) implies condition (A4).

Another widely employed condition is the following condition, which is first introduced by Jeanjean [18].

- (A7) There exist $\theta_0 \geq 1$ such that $\theta_0 \mathcal{F}(x,t) \geq \mathcal{F}(x,st)$ for all $s \in [0,1]$ and $t \in \mathbb{R}$, where $\mathcal{F}(x,t)$ is given in (A5).

We can observe that when $s = 0$, then $\mathcal{F}(x,t) \geq 0$, but for our condition (A5), $\mathcal{F}(x,t)$ may assume negative values. Therefore, it is interesting to consider 2-superlinear problems under conditions (A4) and (A5).

Condition (A5) was used by Alves, Soares and Souto in [2]. With the additional conditions that

$$\alpha = \inf_{x \in \mathbb{R}^3} V(x) > 0 \tag{1.3}$$

and $b \in [0, \alpha)$, they proved that all Cerami sequences are bounded. Under the much weaker condition (A5), we can obtain the boundedness of Palais-Smale sequences, see Lemma 2.4. In 2015, Chen and Liu [13] also used conditions (A4) and (A5) to show the existence of infinitely many solutions for Schrödinger-Maxwell systems. In our case, many technical difficulties arise because of the presence of the non-local term ϕ, which is not homogeneous as it is in the Schrödinger-Maxwell systems. Hence, a more careful analysis of the interaction between the couple (u, ϕ) is required.

By (A1), we know that V is bounded from below, hence we may choose $V_0 > 0$ such that

$$\tilde{V}(x) := V(x) + V_0 > 1, \quad \forall x \in \mathbb{R}^3$$

and define a Hilbert space

$$E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x) u^2 \, dx < \infty \right\}$$

with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla v + \tilde{V}(x) uv \right) dx$$

and the norm $\|u\| = \langle u, u \rangle^{1/2}$. We also know that if V is coercive, then (A1) is satisfied.

Obviously, the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous, for each $s \in [2, 2^*]$. The norm on $L^s = L^s(\mathbb{R}^3)$ with $1 < s < \infty$ is $|u|_s = (\int_{\mathbb{R}^3} |u|^s \, dx)^{1/s}$. Consequently, for each $s \in [2, 6]$, there exists a constant $d_s > 0$ such that

$$|u|_s \leq d_s \|u\|, \quad \forall u \in E. \tag{1.4}$$

$D(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_D := |\nabla u|_2 + |\nabla u|_4.$$

$D(\mathbb{R}^3)$ is continuously embedded in $D^{1,2}(\mathbb{R}^3)$. By the Sobolev inequality, we know that $D^{1,2}(\mathbb{R}^3)$ is continuously embedded in $L^6 = L^6(\mathbb{R}^3)$ and $D(\mathbb{R}^3)$ is continuously embedded in $L^\infty = L^\infty(\mathbb{R}^3)$.

System (1.1) has a variational structure. In fact, we consider the functional $\mathcal{J} : E \times D(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{J}(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x) u^2 - (2\omega + \phi) \phi u^2) \, dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx$$

$$-\frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi|^4\,dx-\int_{\mathbb{R}^3}F(x,u)\,dx.$$

Solutions $(u,\phi)\in E\times D(\mathbb{R}^3)$ of system (1.1) are the critical points of $\mathcal{J}$. As it is pointed in [14], the functional $\mathcal{J}$ is strongly indefinite and is difficult to investigate. By the reduction method described in [6], we are led to the study of a new functional $I:E\to\mathbb{R}$ defined by $I(u)=\mathcal{J}(u,\phi_u)$. By Proposition 2.1 below, $I(u)$ as defined next does not present such strongly indefinite nature. Now we can state our main result.

Theorem 1.1. *Assume that* (A1)—(A5) *are satisfied, and f is odd in u. If* 0 *is not an eigenvalue of* (2.2)*, then* (1.1) *has a sequence of solutions $(u_n,\phi_n)\in E\times D(\mathbb{R}^3)$ such that the energy $\mathcal{J}(u_n,\phi_n)\to+\infty$.*

We emphasize that unlike all previous results about system (1.1), see e.g. [1, 11, 14, 23, 25], we do not assume that the potential is the positive constant $V(x)=m^2-\omega^2$. We allow the potential V be sign changing. The author [20] considered the multiplicity of solutions for Klein-Gordon-Maxwell system. There the author assumed in addition that $\alpha=\inf_{x\in\mathbb{R}^3}V(x)>0$, and (AR) or (A6). When V is positive, the quadratic part of the functional I (see (1.3)) is positively definite, and I has a mountain pass geometry. Therefore, the mountain pass lemma [22] can be applied. In our case, the quadratic part may possesses a nontrivial negative space E^-, so I no longer possesses the mountain pass geometry. Therefore the methods in [20] cannot be applied here. To obtain our result, we adopt a technique developed in [13].

We denote by " $\rightharpoonup$ " weak convergence, and by " $\to$ " strong convergence. Also if we take a subsequence of a sequence $\{u_n\}$, we shall denote it again $\{u_n\}$.

2. Variational setting and compactness condition

Evidently, the properties of ϕ_u play an important role in the study of $\mathcal{J}$. So we need the following technical results.

Proposition 2.1. *For each $u\in H^1(\mathbb{R}^3)$, there exists a unique $\phi=\phi_u\in D(\mathbb{R}^3)$ which satisfies*

$$\Delta\phi+\beta\Delta_4\phi=4\pi(\phi+\omega)u^2\quad\text{in }\mathbb{R}^3.$$

Moreover, the map $\Phi:u\in H^1(\mathbb{R}^3)\mapsto\phi_u\in D(\mathbb{R}^3)$ is continuously differentiable, and

(i) $-\omega\le\phi_u\le0$ *on the set* $\{x\in\mathbb{R}^3|u(x)\neq0\}$;
(ii) $\int_{\mathbb{R}^3}(|\nabla\phi_u|^2+\beta|\nabla\phi_u|^4)\,dx\le4\pi\omega^2|u|_2^2$.

The first part of Proposition 2.1 was proved in [14], and the second part in [21].

After multiplying

$$\Delta\phi+\beta\Delta_4\phi=4\pi(\phi+\omega)u^2$$

by ϕ_u and integrating by parts, by the condition (i), we obtain that

$$\begin{aligned}\int_{\mathbb{R}^3}(|\nabla\phi_u|^2+\beta|\nabla\phi_u|^4)\,dx&=-4\pi\int_{\mathbb{R}^3}(\phi_u+\omega)\phi_u u^2\,dx\\&\le-4\pi\omega\int_{\mathbb{R}^3}\phi_u u^2\,dx\\&\le4\pi\omega^2|u|_2^2.\end{aligned}$$

By Proposition 2.1 and (1.1), if $u \in E$ is a critical point of I, then $(u, \phi_u) \in E \times D(\mathbb{R}^3)$ is a critical point of $\mathcal{J}$, that is, $(u, \phi_u) \in E \times D(\mathbb{R}^3)$ is a solution of (1.1). We can obtain a C^1 functional $I : E \to \mathbb{R}$ given by

$$\begin{aligned} I(u) = \mathcal{J}(u, \phi_u) &= \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2 - (2\omega + \phi_u)\phi_u u^2]\, dx \\ &\quad - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^2\, dx - \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^4\, dx - \int_{\mathbb{R}^3} F(x, u)\, dx \\ &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 + \phi_u^2 u^2)\, dx \\ &\quad + \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^2\, dx + \frac{3\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^4\, dx - \int_{\mathbb{R}^3} F(x, u)\, dx \\ &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2 - \omega \phi_u u^2)\, dx + \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^4\, dx \\ &\quad - \int_{\mathbb{R}^3} F(x, u)\, dx. \end{aligned} \tag{2.1}$$

We consider the map $\Phi : E \to D, u \to \phi_u$. By standard arguments, $\Phi \in C^1(E, D)$. The Gateaux derivative of I is

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv - (2\omega + \phi_u)\phi_u uv)\, dx - \int_{\mathbb{R}^3} f(x, u)v\, dx$$

for all $u, v \in E$.

Furthermore, under the condition (A1), the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is compact for any $s \in [2, 6)$ (See [3]). By the compact embedding $E \hookrightarrow L^2(\mathbb{R}^3)$ and the standard elliptic theory [33], it is easy to see that the eigenvalue problem

$$-\triangle u + V(x)u = \lambda u, \quad u \in E \tag{2.2}$$

possesses a complete sequence of eigenvalues

$$-\infty < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots, \quad \lambda_j \to +\infty.$$

Each λ_j has finite multiplicity and $|\lambda_j|_2 = 1$. Denote e_j be the eigenfunction of λ_j. E^- is spanned by the eigenfunctions corresponding to negative eigenvalues. Note that the negative space E^- of the quadratic part of I is nontrivial if and only if some λ_j is negative.

If $\lambda_1 > 0$, we can easy to prove that I has the mountain pass geometry, so we omit this case. Since 0 is not an eigenvalue of (2.2), we assume that there exists $l \ge 1$ such that $0 \in (\lambda_l, \lambda_{l+1})$. Set

$$E^- = \operatorname{span}\{e_1, \dots, e_l\}, \quad E^+ = (E^-)^\perp. \tag{2.3}$$

Then E^- and E^+ are the negative space and positive space of the quadratic form

$$N(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\, dx$$

respectively, and $\dim E^- < \infty$. Moreover, there is a positive constant B such that

$$\pm N(u) \ge B\|u\|^2, \quad u \in E^\pm. \tag{2.4}$$

To prove Theorem 1.1, we shall use the fountain theorem by Bartsch [4]; see also [29, Theorem 3.6]. For $k = 1, 2, \dots,$ set

$$Y_k = \operatorname{span}\{e_1, \dots, e_k\}, \quad Z_k = \overline{\operatorname{span}\{e_{k+1}, \dots, \}}. \tag{2.5}$$

Proposition 2.2 (Fountain theorem). *Assume the even functional $I \in C^1(E, \mathbb{R})$ satisfies the (PS) condition. If there is a positive constant K such that for any $k \geq K$ there exist $\rho_k > r_k > 0$ such that*

(I) $a_k = \max_{u \in Y_k,\, \|u\|=\rho_k} I(u) \leq 0$,
(II) $b_k = \inf_{u \in Z_k,\, \|u\|=r_k} I(u) \to +\infty$ *as* $k \to +\infty$,

then I has a sequence of critical points $\{u_k\}$ such that $I(u_k) \to +\infty$.

Proposition 2.3. *Assume that $p_1, p_2 > 1, r, q \geq 1$ and $\Omega \subset \mathbb{R}^N$. Let g be a Caratheodory function on $\Omega \times \mathbb{R}$ that satisfies*

$$|g(x,t)| \leq a_1|t|^{(p_1-1)/r} + a_2|t|^{(p_2-1)/r}, \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

where $a_1, a_2 \geq 0$. If $u_n \to u$ in $L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$, and $u_n \to u$ a.e. $x \in \Omega$, then for each $v \in L^{p_1 q}(\Omega) \cap L^{p_2 q}(\Omega)$, we have

$$\lim_{n\to\infty} \int_\Omega |g(x,u_n) - g(x,u)|^r |v|^q \, dx = 0.$$

To study the functional I, we will write the functional I in a form in which the quadratic part is $\|u\|^2$. Let $h(x,t) = f(x,t) + V_0 t$. Then, by (A5) and computations, we obtain that

$$H(x,t) := \int_0^t h(x,s)ds \leq \frac{t}{\theta} h(x,t) + \tilde{V_0} t^2, \quad \tilde{V_0} := b + \frac{V_0}{2} - \frac{V_0}{\theta} > 0. \tag{2.6}$$

By (A4) we have

$$\lim_{|t|\to\infty} \frac{h(x,t)t}{t^2} = +\infty. \tag{2.7}$$

Furthermore, by (A3) we obtain

$$\lim_{|t|\to 0} \frac{h(x,t)t}{t^\theta} = \lim_{|t|\to 0} \Big(\frac{t^2}{t^\theta} \cdot \frac{f(x,t)t + V_0 t^2}{t^2} \Big) = +\infty.$$

Hence there exists $M > 0$ such that

$$h(x,t)t \geq -Mt^\theta, \quad \forall t \in \mathbb{R}. \tag{2.8}$$

With the modified nonlinearity h, the functional $I : E \to \mathbb{R}$ can be rewritten in the form

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\omega}{2} \int_{\mathbb{R}^3} \phi_u u^2 \, dx + \frac{\beta}{16\pi} \int_{\mathbb{R}^3} |\nabla \phi_u|^4 \, dx - \int_{\mathbb{R}^3} H(x,u) \, dx \tag{2.9}$$

with derivative

$$\langle I'(u), v \rangle = \langle u, v \rangle - \int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u uv \, dx - \int_{\mathbb{R}^3} h(x,u)v \, dx.$$

Lemma 2.4. *Assume* (A1)—(A5) *are satisfied, then the function I satisfies the (PS) condition.*

Proof. It follows from $\frac{1}{\theta} tf(x,t) - F(x,t) \geq -bt^2$ that condition (A4) is equivalent to

$$\lim_{|t|\to+\infty} \frac{H(x,t)}{t^\theta} = +\infty.$$

Let $\{u_n\}$ be a (PS) sequence, i.e.,

$$I(u_n) \to c > 0, \quad \langle I'(u_n), u_n \rangle \to 0.$$

We first prove that $\{u_n\}$ is bounded in E. Arguing by contradiction, suppose that $\{u_n\}$ is unbounded, passing to a subsequence, by (2.6), we obtain

$$
\begin{aligned}
\theta \sup_n I(u_n) + \|u_n\| &\geq \theta I(u_n) - \langle I'(u_n), u_n\rangle \\
&= \big(\frac{\theta}{2} - 1\big)\|u_n\|^2 - \frac{\omega\theta}{2}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,dx + \frac{\theta\beta}{16\pi}\int_{\mathbb{R}^3} |\nabla\phi_{u_n}|^4\,dx \\
&\quad + \int_{\mathbb{R}^3} (2\omega + \phi_{u_n})\phi_{u_n} u_n^2\,dx + \int_{\mathbb{R}^3} (h(x,u_n)u_n - \theta H(x,u_n))\,dx \\
&\geq \big(\frac{\theta}{2} - 1\big)\|u_n\|^2 - \tilde{V_0}\int_{\mathbb{R}^3} u_n^2\,dx.
\end{aligned}
\tag{2.10}
$$

Let $v_n = u_n/\|u_n\|$. Then, going if necessary to a subsequence, by the compact embedding $E \hookrightarrow L^2(\mathbb{R}^3)$ we can assume that

$$
\begin{gathered}
v_n \rightharpoonup v_0 \quad \text{in } E, \\
v_n \to v_0 \quad \text{in } L^2(\mathbb{R}^3), \\
v_n(x) \to v_0(x) \quad \text{a. e. in } \mathbb{R}^3.
\end{gathered}
$$

Dividing both sides of (2.10) by $\|u_n\|^2$, we have

$$
\tilde{V_0}\int_{\mathbb{R}^3} v_0^2\,dx \geq 1 \quad \text{as } n \to \infty.
$$

Consequently, we have that $v_0 \neq 0$.

By (1.4) and (2.8), we have

$$
\begin{aligned}
\int_{v_0=0} \frac{h(x,u_n)u_n}{\|u_n\|^\theta}\,dx &= \int_{v_0=0} \frac{h(x,u_n)u_n}{u_n^\theta} v_n^\theta\,dx \\
&\geq -M\int_{v_0=0} v_n^\theta\,dx \\
&\geq -M\int_{\mathbb{R}^3} v_n^\theta\,dx \\
&= -M|v_n|_\theta^\theta \\
&\geq -Md_\theta^\theta > -\infty.
\end{aligned}
\tag{2.11}
$$

For $x \in \{x \in \mathbb{R}^3 | v_0 \neq 0\}$, we have $|u_n(x)| \to +\infty$ as $n \to \infty$. By (2.7) we have

$$
\frac{h(x,u_n(x))u_n(x)}{\|u_n\|^2} = \frac{h(x,u_n(x))u_n(x)}{u_n^2(x)} v_n^2(x) \to +\infty. \tag{2.12}
$$

Hence, by (2.11), (2.12) and Fatou's lemma we obtain

$$
\int_{\mathbb{R}^3} \frac{h(x,u_n)u_n}{\|u_n\|^\theta}\,dx \geq \int_{v_0\neq 0} \frac{h(x,u_n)u_n}{u_n^\theta} v_n^\theta(x)\,dx - Md_\theta^\theta \to +\infty. \tag{2.13}
$$

Hence

$$
\int_{\mathbb{R}^3} \frac{H(x,u_n)}{\|u_n\|^\theta}\,dx \to +\infty. \tag{2.14}
$$

Since $\{u_n\}$ is a (PS) sequence, using Proposition 2.1 and (2.13), for n large enough, we have

$$\begin{aligned} & c\omega+1 \\ & \geq \frac{1}{\|u_n\|^\theta}\Big(\frac{1}{2}\|u_n\|^2-\frac{\omega}{2}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx+\frac{3\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_{u_n}|^4\,dx-I(u_n)\Big) \\ & =\int_{\mathbb{R}^3}\frac{H(x,u_n)}{\|u_n\|^\theta}\,dx\to+\infty, \end{aligned} \tag{2.15}$$

which is a contradiction. It follows that $\{u_n\}$ is bounded in E.

Next we shall prove $\{u_n\}$ contains a convergent subsequence. Without loss of generality, passing to a subsequence if necessary, there exists $u\in E$ such that $u_n\rightharpoonup u$ in E. By using the embedding $E\hookrightarrow L^s(\mathbb{R}^3)$ are compact for any $s\in[2,6)$, $u_n\to u$ in $L^s(\mathbb{R}^3)$ for $2\leq s<6$ and $u_n(x)\to u(x)$ a.e. $x\in\mathbb{R}^3$. By (1.3) and the Gateaux derivative of I, we can obtain that

$$\begin{aligned} & \|u_n-u\|^2 \\ & =\langle I'(u_n)-I'(u),u_n-u\rangle+V_0\int_{\mathbb{R}^3}(u_n-u)^2\,dx+2\omega\int_{\mathbb{R}^3}(\phi_{u_n}u_n-\phi_u u)(u_n-u)\,dx \\ & \quad+\int_{\mathbb{R}^3}(h(x,u_n)-h(x,u))(u_n-u)\,dx+\int_{\mathbb{R}^3}(\phi_{u_n}^2u_n-\phi_u^2u)(u_n-u)\,dx \end{aligned}$$

By an easy computation, we obtain that

$$\begin{gathered} \langle I'(u_n)-I'(u),u_n-u\rangle\to0\quad\text{as } n\to\infty, \\ \int_{\mathbb{R}^3}[(\phi_{u_n}u_n-\phi_u u)(u_n-u)\,dx+\int_{\mathbb{R}^3}[(\phi_{u_n}^2u_n-\phi_u^2u)(u_n-u)\,dx\to0 \end{gathered}$$

as $n\to+\infty$. Indeed, by the Hölder inequality, the Sobolev inequality and Proposition 2.1, we obtain

$$\begin{aligned} \Big|\int_{\mathbb{R}^3}(\phi_{u_n}-\phi_u)(u_n-u)u_n\,dx\Big| & \leq|(\phi_{u_n}-\phi_u)(u_n-u)|_2|u_n|_2 \\ & \leq|\phi_{u_n}-\phi_u|_6|u_n-u|_3|u_n|_2 \\ & \leq C\|\phi_{u_n}-\phi_u\||u_n-u|_3|u_n|_2, \end{aligned}$$

where C is a positive constant. Since $u_n\to u$ in $L^s(\mathbb{R}^3)$ for $2\leq s<6$, we obtain

$$\begin{gathered} \Big|\int_{\mathbb{R}^3}(\phi_{u_n}-\phi_u)(u_n-u)u_n\,dx\Big|\to0\quad\text{as } n\to+\infty, \\ \Big|\int_{\mathbb{R}^3}\phi_u(u_n-u)(u_n-u)\,dx\Big|\leq|\phi_u|_6|u_n-u|_3|u_n-u|_2\to0\quad\text{as } n\to+\infty. \end{gathered}$$

Thus we obtain

$$\begin{aligned} & \int_{\mathbb{R}^3}[(\phi_{u_n}u_n-\phi_u u)(u_n-u)\,dx \\ & =\int_{\mathbb{R}^3}(\phi_{u_n}-\phi_u)(u_n-u)u_n\,dx+\int_{\mathbb{R}^3}\phi_u(u_n-u)(u_n-u)\,dx\to0 \end{aligned}$$

as $n\to+\infty$. Since the sequence $\{\phi_{u_n}^2u_n\}$ is bounded in $L^{3/2}(\mathbb{R}^3)$, we have

$$|\phi_{u_n}^2u_n|_{3/2}\leq|\phi_{u_n}|_6^2|u_n|_3,$$

so

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^3}[(\phi_{u_n}^2 u_n-\phi_u^2 u)(u_n-u)\,dx\Big| &\le |\phi_{u_n}^2 u_n-\phi_u^2 u|_{3/2}|u_n-u|_3\\
&\le (|\phi_{u_n}^2 u_n|_{3/2}+|\phi_u^2 u|_{3/2})|u_n-u|_3\to 0,
\end{aligned}
$$

as $n\to+\infty$.

By Proposition 2.3 and $u_n\to u$ in $L^s(\mathbb{R}^3)$ for $2\le s<6$, we have

$$
\int_{\mathbb{R}^3}(h(x,u_n)-h(x,u))(u_n-u)\,dx\to 0 \quad \text{as } n\to+\infty.
$$

Since $u_n\to u$ in $L^2(\mathbb{R}^3)$, we obtain that $V_0\int_{\mathbb{R}^3}(u_n-u)^2\,dx\to 0$ as $n\to+\infty$.

Therefore $\|u_n-u\|\to 0$ in E as $n\to\infty$. The proof is complete. □

Lemma 2.5. *Let X be a finite dimensional subspace of E, then I is anti-coercive on X, i.e.*

$$
I(u)\to-\infty, \quad \text{as } \|u\|\to\infty,\ u\in X.
$$

Proof. If this were not true, we can choose a sequence $\{u_n\}\subset X$ and ξ is a real number such that

$$
\|u_n\|\to\infty, \quad I(u_n)\ge\xi. \tag{2.16}
$$

Let $v_n=\frac{u_n}{\|u_n\|}$. Since $\dim X<\infty$, going if necessary to a subsequence we have

$$
\|v_n-v_0\|\to 0, \quad v_n(x)\to v_0(x) \quad \text{a.e. in } \mathbb{R}^3
$$

for every $v_0\in X$, with $\|v_0\|=1$. Since $v_0\neq 0$, similar to (2.14) we obtain that

$$
\int_{\mathbb{R}^3}\frac{H(x,u_n)}{\|u_n\|^\theta}\,dx\to+\infty.
$$

Arguing similar to (2.15), it follows from $\sup_n|I(u_n)|<\infty$ that

$$
\begin{aligned}
I(u_n)=\|u_n\|^\theta\Big(&\frac{\|u_n\|^2}{2\|u_n\|^\theta}-\frac{\omega}{2\|u_n\|^\theta}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx\\
&+\frac{\beta}{16\pi\|u_n\|^\theta}\int_{\mathbb{R}^3}|\nabla\phi_{u_n}|^4\,dx-\int_{\mathbb{R}^3}\frac{H(x,u_n)}{\|u_n\|^\theta}\,dx\Big)\to-\infty,
\end{aligned}
$$

which is contradicts $I(u_n)\ge\xi$. The proof is complete. □

Proof of Theorem 1.1. We will find a sequence of critical points $\{u_n\}$ of I such that $I(u_n)\to+\infty$. Since $f(x,t)$ is odd in t, I is an even function. From Lemma 2.4 it follows that I satisfies the (PS) condition. Therefore, it suffices to verify (I) and (II) of Proposition 2.2.

Since $\dim Y_k<\infty$, by Lemma 2.5, we obtain the conclusion of (I).

By (A2) and (A3), we have

$$
|f(x,t)|\le\epsilon|t|+C_\epsilon|t|^{p-1}, \quad |F(x,t)|\le\frac{\epsilon}{2}t^2+\frac{C_\epsilon}{p}|t|^p,
$$

where $\epsilon>0$ is very small. Then we have

$$
|F(x,t)|\le\frac{B}{2d_2^2}t^2+\frac{CB}{p}|t|^p, \tag{2.17}
$$

where B is defined in (2.4). We assume that $0\in[\lambda_l,\lambda_{l+1})$. Then if $k>l$, we have that $Z_k\subset E^+$, where E^+ is defined in (2.3). Now we have

$$
N(u)\ge B\|u\|^2, \quad u\in Z_k \tag{2.18}
$$

and, as in the proof of [29, Lemma3.8],

$$\beta_k = \sup_{u\in Z_k, \|u\|=1} |u|_p \to 0, \quad \text{as } k\to\infty. \tag{2.19}$$

Let $r_k = (Cp\beta_k^p)^{1/(2-p)}$, where C is chosen as in (2.17). For $u \in Z_k \subset E^+$ with $\|u\| = r_k$, $\phi_u \le 0$, by (2.18) we deduce that

$$\begin{aligned}
I(u) &= N(u) - \frac{1}{2}\omega\int_{\mathbb{R}^3}\phi_u u^2\,dx + \frac{\beta}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi_u|^4\,dx - \int_{\mathbb{R}^3}F(x,u)\,dx \\
&\ge B\|u\|^2 - \frac{B}{2d_2^2}|u|_2^2 - \frac{CB}{p}|u|_p^p \\
&\ge B\Big(\frac{1}{2}\|u\|^2 - \frac{C\beta_k^p}{p}\|u\|^p\Big) \\
&= B\big(\frac{1}{2} - \frac{1}{p^2}\big)(Cp\beta_k^p)^{2/(2-p)}.
\end{aligned}$$

Since $\beta_k \to 0$ and $p > 2$, it follows that

$$b_k = \inf_{u\in Z_k, \|u\|=r_k} I(u) \to +\infty.$$

We obtain the conclusion of (II). The proof is complete. □

Acknowledgments. This research was partially supported by the Scientific Research Program of Tianjin Education Commission (2023KJ216).

References

[1] F. S. B. Albuquerque, S. J. Chen, L. Li; *Solitary wave of ground state type for a nonlinear Klein-Gordon equation coupled with Born-Infeld theory in* $\mathbb{R}^2$, Electronic Journal of Qualitative Theory of Differential Equations, 12, 2020, 1-18.

[2] C. O. Alves, M. A. S. Souto, S. H. M. Soares; *Schrödinger-Poisson equations without Ambrosetti-Rabinowitz condition*, Journal of Mathematical Analysis and Applications, 377(2), 2011, 584-592.

[3] T. Bartsch, Z-Q. Wang; *Existence and multiplicity results for some superlinear elliptic problem on* $\mathbb{R}^N$, Communications in Partial Differential Equations, 20, 1995, 1725-1741.

[4] T. Bartsch; *Infinitely many solutions of a symmetric Dirichlet problem*, Nonlinear Analysis: Theory, Methods and Applications, 20, 1993, 1205-1216.

[5] V. Benci, D. Fortunato, A. Masiello, L. Pisani; *Solitons and the electromagnetic field*, Mathematische Zeitschrift, 3, 2012, 299-301.

[6] V. Benci, D. Fortunato; *Solitary waves of the nonlinear Klein-Gordon equation coupled with the Maxwell equations*, Reviews in Mathematical Physics, 14(4), 2002, 409-420.

[7] M. Born; *Modified field equations with a finite radius of the electron*, Nature, 132, 1933, 282.

[8] M. Born; *Quantum theory of the electromagnetic field*, Proceedings of the Royal Society of London Series A, 143(849), 1934, 410-437.

[9] M. Born, L. Infeld; *Foundations of the new field theory*, Nature, 144(852), 1934, 425-451.

[10] M. Carmeli; *Field theory on $R\times S$ 3 topology I: The Klein-Gordon and Schrödinger equations*, Foundations of Physics, 15, 1985, 175-184.

[11] G. F. Che, H. B. Chen; *Infinitely many solutions for the Klein-Gordon equation with sublinear nonlinearity coupled with Born-Infeld theory*, Bulletin of the Iranian Mathematical Society, 46, 2019, 1083-1100.

[12] S. J. Chen, L. Li; *Multiple solutions for the nonhomogeneous Klein-Gordon equation coupled with Born-Infeld theory on* $\mathbb{R}^3$, Journal of Mathematical Analysis and Applications, 400(2), 2013, 517-524.

[13] H. Y. Chen, S. B. Liu; *Standing waves with large frequency for 4-superlinear Schrödinger-Poisson systems*, Annali di Matematica, 194, 2015, 43-53.

[14] S. J. Chen, S. Z. Song; *The existence of multiple solutions for the Klein-Gordon equation with concave and convex nonlinearities coupled with Born-Infeld theory on* $\mathbb{R}^3$, Nonlinear Analysis Real World Applications, 38, 2017, 78-95.

[15] P. D'Avenia, L. Pisani; *Nonlinear Klein-Gordon equations coupled with Born-Infeld type equations*, Electronic Journal of Differential Equations, Vol. 2002(2002), No. 26, pp. 1-13.

[16] B. Felsager, Geometry; *Particles and Fields*, Odense University Press, Odense, American Journal of Physics, 52, 573, 1984.

[17] D. Fortunato, L. Orsani, L. Pisina; *Born-Infeld type equations for electrostatic fields*, Journal of Mathematical Physics, 43(11), 2002, 5698-5706.

[18] L. Jeanjean; *On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on* $\mathbb{R}^N$, Proceedings of The Royal Society of Edinburgh, 129, 1999, 787-809.

[19] C. M. He, L. Li, S. J. Chen, D. O'Regan; *Ground state solution for the nonlinear Klein-Gordon equation coupled with Born-Infeld theory with critical exponents*, Analysis and Mathematical Physics, 12, 2022, 48.

[20] X. M. He; *Multiplicity of solutions for a nonlinear Klein-Gordon-Maxwell system*, Acta Applicandae Mathematicae, 130, 2014, 237-250.

[21] D. Mugnai; *Coupled Klein-Gorndon and Born-Infeld type equations: looking for solitary waves*, Proceedings of the Royal Society of London A Mathematical Physical and Engineering Sciences, 460(2045), 2004, 1519-1527.

[22] P. H. Rabinowitz; *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Ser. Math. 65, American Mathematical Society, Providence, 1986.

[23] K. M. Teng, K. Zhang; *Existence of solitary wave solutions for the nonlinear Klein-Gordon equation coupled with Born-Infeld theory with critical Sobolev exponent*, Nonlinear Analysis: An International Multidisciplinary Journal, 74(12), 2011, 4241-4251.

[24] K. M. Teng; *Existence and multiple of the solutions for the nonlinear Klein-Gordon equation coupled with Born-Infeld theory on boundary domain*, Differential Equations and Applications, 4(3), 2012, 445-457.

[25] F. Z. Wang; *Solitary waves for the coupled nonlinear Klein-Gordon and Born-Infeld type equations*, Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 82, pp. 1-12.

[26] L. X. Wang, C. L. Xiong, P. P. Zhao; *Two solutions for the nonhomogeneous Klein-Gordon equation coupled with Born-Infeld theory on* $\mathbb{R}^3$, Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 74, pp. 1-11.

[27] L. X. Wang, C. L. Xiong, D. Zhang; *Multiple solutions for nonhomogeneous Klein-Gordon equation with sign-changing potential coupled with Born-Infeld theory*, Journal of Applied Analysis and Computation, 14 (1), 2024, 84-105.

[28] L. X. Wen, X. H. Tang, S. T. Chen; *Infinitely many solutions and least energy solutions for Klein-Gordon equation coupled with Born-Infeld theory*, Complex Variables and Elliptic Equations, 2019, 1572124

[29] M. Willem; *Minimax theorems*, Progress in Nonlineäuser Boston Inc., Boston, MA, 1996.

[30] Y. Yang; *Classical solutions in the Born-Infeld theory*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 456, 1995, 615-640.

[31] Y. Yu; *Solitary waves for nonlinear Klein-Gordon equations coupled with Born-Infeld theory*, Annales de l'Institut Henri Poincare (C) Non Linear Analysis, 27(1), 2010, 351-376.

[32] Z. H. Zhang, J. L. Liu; *Existence and multiplicity of sign-changing solutions for Klein-Gordon equation coupled with Born-Infeld Theory with subcritical exponent*, Qualitative Theory of Dynamical Systems, 22(1), 2023, 1-9.

[33] W. M. Zou, M. Schechter; *Critical Point Theory and Its Applications*, Springer, New York, 2006.

Lixia Wang
School of Sciences, Tianjin Chengjian University, Tianjin 300384, China
Email address: `wanglixia0311@126.com`

Pingping Zhao
School of Sciences, Tianjin Chengjian University, Tianjin 300384, China
Email address: `ppzhao_math@126.com`

Dong Zhang (corresponding author)
School of Sciences, Tianjin Chengjian University, Tianjin 300384, China
Email address: zhangdongtg@126.com