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LOCALIZED NODAL SOLUTIONS FOR SEMICLASSICAL CHOQUARD EQUATIONS WITH CRITICAL GROWTH

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Abstract. In this article, we study the existence of localized nodal solutions for semiclassical Choquard equation with critical growth

$$-\varepsilon^2\Delta v+V(x)v=\varepsilon^{\alpha-N}\Big(\int_{\mathbb{R}^N}\frac{|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}\,dy\Big)|v|^{2^*_\alpha-2}v+\vartheta|v|^{q-2}v,\ x\in\mathbb{R}^N,$$

where $\vartheta>0$, $N\geq 3$, $0<\alpha<\min\{4,N-1\},\max\{2,2^*-1\}<q<2^*$, $2^*_\alpha=\frac{2N-\alpha}{N-2}$, V is a bounded function. By the perturbation method and the method of invariant sets of descending flow, we establish for small ε the existence of a sequence of localized nodal solutions concentrating near a given local minimum point of the potential function V.

1. Introduction

In this article, we study localized nodal solutions of the nonlinear Choquard equation with critical exponent

$$\begin{gathered}-\varepsilon^2\Delta v+V(x)v=\varepsilon^{\alpha-N}\Big(\int_{\mathbb{R}^N}\frac{|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}\,dy\Big)|v|^{2^*_\alpha-2}v+\vartheta|v|^{q-2}v,\ x\in\mathbb{R}^N,\\ v(x)\to 0\quad\text{as } |x|\to\infty,\end{gathered}\tag{1.1}$$

where $\vartheta>0$, $N\geq 3$, $0<\alpha<\min\{4,N-1\},\max\{2,2^*-1\}<q<2^*$, $2^*_\alpha=\frac{2N-\alpha}{N-2}$ is the upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality, $\varepsilon>0$ is small parameter. The potential function V satisfies following assumptions:

(A1) $V\in C^1(\mathbb{R}^N,\mathbb{R})$ and there exist $b>a>0$ such that

$$a\leq V(x)\leq b,\quad \forall x\in\mathbb{R}^N\,.$$

(A2) There exists a bounded domain $\mathcal{M}\subset\mathbb{R}^N$ with the smooth boundary $\partial\mathcal{M}$ such that

$$\langle\overrightarrow{n}(x),\,\nabla V(x)\rangle>0\,,\quad \forall x\in\partial\mathcal{M}\,,$$

where $\overrightarrow{n}(x)$ is the outer normal of $\partial\mathcal{M}$ at x.

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Over the previous decades, the Choquard type equation has been widely studied. The Choquard equation

$$-\Delta u+u=\Big(\int_{\mathbb{R}^3}\frac{|u(y)|^2}{|x-y|}\,dy\Big)u\,,\quad x\in\mathbb{R}^3, \tag{1.2}$$

is the Choquard-Pekar equation which originated from the description of the quantum theory of a polaron at rest by Pekar in 1954 [35]. Choquard also used (1.2) to describe the Hartree-Fock [23] theory of one component plasma in 1976. If u is a solution of (1.2), then $\psi(t,x)=e^{it}u(x)$ is a solitary wave solution of the Hartree equation

$$i\psi_t=-\Delta\psi-\Big(\int_{\mathbb{R}^3}\frac{|\psi(y)|^2}{|x-y|}\,dy\Big)\psi\,,\quad x\in\mathbb{R}^3. \tag{1.3}$$

In 1996, Penrose [29] proposed equation (1.2) as a model of self gravitation. Lieb [21] proved the existence and uniqueness of the ground state solution for (1.2) by using the symmetric rearrangement inequality. Lions [24] showed that the equation

$$-\Delta u+\lambda u=\Big(\int_{\mathbb{R}^3}V(x-y)|u(y)|^2\,dy\Big)u\,,\quad x\in\mathbb{R}^3 \tag{1.4}$$

has a positive radial symmetric solution and infinitely many radial symmetric solutions, where $\lambda>0$, $V>0$ and V is radially symmetric. For semilinear Choquard equation

$$-\Delta u+V(x)u=\Big(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|x-y|^\alpha}\,dy\Big)f(x,u)\,,\quad x\in\mathbb{R}^N, \tag{1.5}$$

where $f(x,u)=\frac{\partial F(x,u)}{\partial u}$, a large number of research results have been obtained. We refer the reader to [2, 11, 12, 13, 14, 15, 30, 31, 33, 39, 40], and references therein.

For the semiclassical Choquard equation with subcritical growth

$$-\varepsilon^2\Delta u+V(x)u=\varepsilon^{\alpha-N}\Big(\int_{\mathbb{R}^N}\frac{F(u(y))}{|x-y|^\alpha}\,dy\Big)f(u),\quad x\in\mathbb{R}^N, \tag{1.6}$$

Wei and Winter [41] proved the existence of solution by using the Lyapunov-Schmidt reduction method [19] in 2009, where $N=3$, $\alpha=1$, $F(u)=|u|^2$, V satisfies

$$\inf_{x\in\mathbb{R}^3}V(x)>0,\ V(x)\in C^2(\mathbb{R}^3).$$

In 2015, Moroz and Van Schaftingen [32] constructed the single spike solution which concentrating around the local minimum of potential V by using the nonlocal penalization method, where $N\geq 1$, $0<\alpha<N$, $F(u)=|u|^p$, $p\in[2,\frac{2N-\alpha}{N-2})$, $V\in C(\mathbb{R}^N,[0,\infty))$. More results for the semiclassical Choquard equation with subcritical growth we refer [8, 17, 18, 27, 37, 44, 45, 48] and references therein.

For the semiclassical Choquard equation with critical growth

$$-\varepsilon^2\Delta u+V(x)u=\varepsilon^{\alpha-N}\Big(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|x-y|^\alpha}\,dy\Big)f(x,u),\quad x\in\mathbb{R}^N, \tag{1.7}$$

Cassani and Zhang [5] proved the existence and decays exponentially of positive solution for the semiclassical critical Choquard (1.7), where $N=3$, $0<\alpha<3$, V,F satisfies some suitable assumptions. In 2017, Alves and Gao [1] investigated the existence of ground state solutions, multiplicity and concentration of semiclassical solutions for the semiclassical Choquard (1.7) by using variational methods [3], where $N=3$, $0<\alpha<3$, V,F satisfy some suitable assumptions. In 2020, Gao, Yang and Zhou [10] obtained existence and multiplicity of solutions for (1.7), where

$N \geq 3$, $0 < \alpha < \min\{4, N\}$, V and F satisfy some suitable assumptions. Qi and Zou [36] investigated semiclassical critical Choquard equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{\alpha-N}\Big(\int_{\mathbb{R}^N}\frac{|u(y)|^{2_\alpha^*}}{|x-y|^\alpha}\,dy\Big)|u|^{2_\alpha^*-2}u + \lambda g(u), \quad x \in \mathbb{R}^N, \tag{1.8}$$

where $N \geq 3$, $0 < \alpha < \min\{4, N\}$, $2_\alpha^* = \frac{2N-\alpha}{N-2}$, $\lambda > 0$, $g \in C^1(\mathbb{R}^N, \mathbb{R})$ satisfies

- (A3) $\lim_{t\to 0^+}\frac{g(t)}{t} = 0$, and there exists $p \in (1, \frac{N+2}{N-2})$, such that $\lim_{t\to\infty}\frac{g(t)}{t^p} = 0$;
- (A4) There exists $\mu \in (2, \frac{2(2N-\alpha)}{N-2})$, such that $0 < \mu G(t) \leq g(t)t$ for all $t \in (0, +\infty)$;
- (A5) $\frac{g(t)}{t}$ is monotonic increasing on $(0, \infty)$.

and V satisfies:

There exists a bounded smooth domain $\mathcal{M} \subset \mathbb{R}^N$ such that

$$m = \min_{x\in\mathcal{M}} V(x) < \min_{x\in\partial\mathcal{M}} V(x).$$

They obtained a local solution concentrating around the local minimum of potential V by using Byeon-Wang [4] type penalization method. More results for the equation (1.7), one can see [9, 34, 42, 43, 46, 47] and references therein.

In recent years, there are have been many results of localized nodal solution for the Choquard equation. In 2021, He and Liu [17] proved existence and concentration of infinitely many sign-changing solutions for the following Choquard equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{\alpha-N}\Big(\int_{\mathbb{R}^N}\frac{|u(y)|^p}{|x-y|^\alpha}\,dy\Big)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.9}$$

where $N \geq 3$, $0 < \alpha < \min\{4, N-1\}$, the potential function V satisfies (A1) and (A2). Zhang and Liu [45] investigated the semiclassical quasi-linear Choquard equation with subcritical growth, and obtained a conclusion similar to that of [17] in 2022.

For the semiclassical Choquard equation with critical growth

$$-\varepsilon^2\Delta v + V(x)v = \varepsilon^{\alpha-N}\Big(\int_{\mathbb{R}^N}\frac{|v(y)|^{2_\alpha^*}}{|x-y|^\alpha}\,dy\Big)|v|^{2_\alpha^*-2}v + \vartheta|v|^{q-2}v, \quad x \in \mathbb{R}^N, \tag{1.10}$$

where $\vartheta > 0$, $N \geq 3$, $0 < \alpha < \min\{4, N-1\}, \max\{2, 2^*-1\} < q < 2^*$, $2_\alpha^* = \frac{2N-\alpha}{N-2}$, there is no results on nodal solutions yet. Combining perturbation method, truncation method and the method of invariant sets of descending flow, we prove (1.10) possesses a sequence of localized nodal solutions. As for the method mentioned, we refer [49, 45, 16] and the references therein.

Under the assumption (A2), the critical set safisfies

$$\mathcal{A} = \{x \in \mathcal{M} | \nabla V(x) = 0\} \neq \varnothing,$$

and without loss of generality we assume $0 \in \mathcal{A}$. For any set $B \subset \mathbb{R}^N$ and any $\delta > 0$, we set

$$B_\delta = \{x \in \mathbb{R}^N | \delta x \in B\},$$
$$B^\delta = \{x \in \mathbb{R}^N | \operatorname{dist}(x, B) := \inf_{y\in B}|x-y| < \delta\}.$$

The main result of this paper is as follows.

Theorem 1.1. *Assume that* (A1) *and* (A2) *hold. Then for each positive integer k there exists $\varepsilon_k > 0$ such that if $0 < \varepsilon < \varepsilon_k$, equation* (1.1) *has at least k pairs of sign-changing solutions $\pm v_{j,\varepsilon}, j = 1, \dots, k$. Moreover, for each $\delta > 0$ there exist $\mu > 0$, $C = C_k > 0$ and $\varepsilon_k(\delta) > 0$ such that if $0 < \varepsilon < \varepsilon_k(\delta)$; then*

$$|v_{j,\varepsilon}(x)| \le C \exp\{-\frac{\mu}{\varepsilon} \operatorname{dist}(x, \mathcal{A}^\delta)\}, \quad \forall x \in \mathbb{R}^N,\ j = 1, \dots, k. \tag{1.11}$$

Denoting $u(x) = v(\varepsilon x)$, equation (1.1) is equivalent to

$$\begin{gathered} -\Delta u + V(\varepsilon x)u = \Big(\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}\,dy\Big)|u|^{2^*_\alpha-2}u + \vartheta|u|^{q-2}u, \quad x \in \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{gathered} \tag{1.12}$$

and the corresponding energy functional is

$$\begin{aligned} I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(\varepsilon x)u^2)\,dx - \frac{1}{2 \cdot 2^*_\alpha}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\alpha}|u(x)|^{2^*_\alpha}}{|x-y|^\alpha}\,dx\,dy \\ - \frac{\vartheta}{q}\int_{\mathbb{R}^N} |u|^q\,dx, \quad u \in H^1(\mathbb{R}^N). \end{aligned}$$

To obtain multiple localized nodal solutions for I_ε, we use the penalization method due to Byeon and Wang [4]. Let $\zeta \in C^\infty$ be a cut-off function, $\zeta(t) = 0$ for $t \le 0$; $\zeta(t) = 1$ for $t \ge 1$; $0 \le \zeta'(t) \le 2$ and $0 \le \zeta(t) \le 1$. We define

$$\chi_\varepsilon(x) = \begin{cases} 0, & \text{if } x \in \mathcal{M}_\varepsilon \\ \varepsilon^{-6}\zeta(\operatorname{dist}(x, \mathcal{M}_\varepsilon)), & \text{if } x \notin \mathcal{M}_\varepsilon. \end{cases}$$

We truncate the critical term to a subcritical term by truncation method. Now we define some auxiliary functions. Let $\xi(t) \in C^\infty(\mathbb{R}, [0,1])$ be a smooth, even function such that $\xi(t) = 1$ if $|t| \le 1$; $\xi(t) = 0$ if $|t| \ge 2$; $0 \le \xi(t) \le 1$ and ξ is decreasing in $[1, 2]$. For $\varepsilon \in (0, 1]$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, we define

$$\begin{gathered} b_\nu(t) = \xi(\nu t), \quad m_\nu(t) = \int_0^t b_\nu(\tau)\,d\tau, \\ f_\nu(t) = |m_\nu(t)|^{2^*_\alpha - r}|t|^{r-2}t, \quad F_\nu(t) = \int_0^t f_\nu(\tau)d\tau, \end{gathered}$$

where $2 < r < 2^*_\alpha$ and $r \le q$. We now consider the equation

$$\begin{gathered} -\Delta u + V(\varepsilon x)u = 2^*_\alpha\Big(\int_{\mathbb{R}^N} \frac{F_\nu(u(y))}{|x-y|^\alpha}\,dy\Big)f_\nu(u(x)) + \vartheta|u|^{q-2}u, \quad x \in \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{gathered} \tag{1.13}$$

and its corresponding energy functional

$$\begin{aligned} I_{\varepsilon,\nu}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(\varepsilon x)u^2)\,dx - \frac{2^*_\alpha}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\nu(u(y))F_\nu(u(x))}{|x-y|^\alpha}\,dx\,dy \\ - \frac{\vartheta}{q}\int_{\mathbb{R}^N} |u|^q\,dx, \quad u \in H^1(\mathbb{R}^N). \end{aligned}$$

Since the imbedding from $H^1(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)(2 \le p \le 2^*)$ is continuous but not compact, we need to choose a suitable function space as working space such that the functional $I_{\varepsilon,\nu}(u)$ recovers compactness. For this purpose, we denote $X_\varepsilon = H^1(\mathbb{R}^N) \cap L^m_\varepsilon(\mathbb{R}^N)$, where $L^m_\varepsilon(\mathbb{R}^N)$ is a weighted space defined as

$$L^m_\varepsilon(\mathbb{R}^N) = \big\{u \in L^m(R^N) : \int_{\mathbb{R}^N} \exp\{(m-2)\operatorname{dist}(\varepsilon x, \mathcal{M})\}|u|^m\,dx < +\infty\big\}$$

endowed with the norm

$$\|u\|_{L^m_\varepsilon(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} \exp\{(m-2)\operatorname{dist}(\varepsilon x, \mathcal{M})\}|u|^m\,dx\Big)^{1/m}.$$

We define

$$\|u\|_{H^1(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} (|\nabla u|^2 + E(\varepsilon x)u^2)\,dx\Big)^{1/2},$$
$$\|u\|_{X_\varepsilon} = \|u\|_{H^1(\mathbb{R}^N)} + \|u\|_{L^m_\varepsilon(\mathbb{R}^N)},$$

where $E(x) = V(x) - \sigma$, σ is small enough such that E satisfies the assumptions (A1) and (A2) (with a different constant $a' = a - \sigma > 0$).

Meanwhile, we introduce an additional coercive term such that $I_{\varepsilon,\nu}$ has necessary compactness property on X_ε. For this purpose, we need some auxiliary functions. For $\varepsilon \in (0,1]$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, we define

$$b_\varepsilon(x,t) = \xi(\varepsilon \exp\{\operatorname{dist}(\varepsilon x, \mathcal{M})\}t), \quad m_\varepsilon(x,t) = \int_0^t b_\varepsilon(x,\tau)\,d\tau,$$
$$k_\varepsilon(x,t) = (\frac{t}{m_\varepsilon(x,t)})^{m-2}t, \quad K_\varepsilon(x,t) = \int_0^t k_\varepsilon(x,\tau)\,d\tau,$$

where $2 < m < r$. We define the perturbed functional

$$\begin{aligned}\Gamma_{\varepsilon,\nu}(u) = &\frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + E(\varepsilon x)u^2)\,dx + \sigma\int_{\mathbb{R}^N} K_\varepsilon(x,u)\,dx \\ &+ \frac{1}{2\beta}\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u^2\,dx - 1\Big)_+^\beta - \frac{2^*_\alpha}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\nu(u(y))F_\nu(u(x))}{|x-y|^\alpha}\,dx\,dy \\ &- \frac{\vartheta}{q}\int_{\mathbb{R}^N} |u|^q\,dx\,, \quad u \in X_\varepsilon,\end{aligned}$$

where $2 < 2\beta < r$. Note that the method of invariant sets of descending flow [28] can not fit well for the functional $\Gamma_{\varepsilon,\nu}$, so we also use the perturbation method [17] to overcome this difficulty. For $t \in \mathbb{R}^+$, we define

$$b_\lambda(t) = \xi(\lambda t), \quad m_\lambda(t) = \int_0^t b_\lambda(\tau)\,d\tau,$$
$$g_\lambda(t) = \frac{m_\lambda(t)}{t}, \quad h_\lambda(t) = g_\lambda(t) + b_\lambda(t).$$

Now we define

$$\begin{aligned}\Gamma_{\varepsilon,\nu,\lambda}(u) = &\frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + E(\varepsilon x)u^2)\,dx + \sigma\int_{\mathbb{R}^N} K_\varepsilon(x,u)\,dx \\ &+ \frac{1}{2\beta}\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u^2\,dx - 1\Big)_+^\beta - \frac{2^*_\alpha}{2} g_\lambda(\varphi^{1/2}(u))\varphi(u) \\ &- \frac{\vartheta}{q}\int_{\mathbb{R}^N} |u|^q\,dx, \quad u \in X_\varepsilon,\end{aligned}$$

where
$$\varphi(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\nu(u(x))F_\nu(u(y))}{|x-y|^\alpha}\,dx\,dy.$$
By Hardy-Littlewood-Sobolev inequality and Sobolev inequality, we have
$$\varphi^{1/2}(u) \le C_0\|u\|_{H^1(\mathbb{R}^N)}^{2^*_\alpha}.$$
Note that if
$$|u(x)| \le \frac{1}{\varepsilon}\exp\{-\operatorname{dist}(\varepsilon x, \mathcal{M})\} \quad \text{for } x \in \mathbb{R}^N,$$
$$|u(x)| \le \frac{1}{\nu}, \quad \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u^2\,dx - 1\Big)_+ = 0, \quad \|u\|_{H^1(\mathbb{R}^N)} \le \Big(\frac{1}{C_0\lambda}\Big)^{1/2^*_\alpha}$$
for sufficiently small $\varepsilon, \nu, \lambda$, then $\Gamma_{\varepsilon,\nu,\lambda}(u) = I_\varepsilon(u)$, and $D\Gamma_{\varepsilon,\nu,\lambda}(u) = DI_\varepsilon(u)$. Hence we can obtain solutions of the equation (1.12) by researching $\Gamma_{\varepsilon,\nu,\lambda}$.

In the following c denotes various constants, c_ε denotes constants depending on ε and c, c_ε may be used from line to line for different constants but independent of the arguments.

This article organized as follows. In section 2 we prove preliminary results and verify the Palais-Smale condition for the function $\Gamma_{\varepsilon,\nu,\lambda}$. In section 3 we construct a sequence of nodal critical points of $\Gamma_{\varepsilon,\nu,\lambda}$ by using the invariant sets method. In Section 4, we prove uniform bound on the critical points obtained in Section 3. Section 5 is devoted to the proof of Theorem 1.1.

2. Preliminaries and Palais-Smale condition for $\Gamma_{\varepsilon,\nu,\lambda}$

In this section, we first collect some elementary results about the auxiliary functions involved in the perturbed functional $\Gamma_{\varepsilon,\nu,\lambda}$. Then, we prove that $\Gamma_{\varepsilon,\nu,\lambda}$ satisfies the (PS) condition.

Lemma 2.1 (Hardy-littlewood-Sobolev inequality [22])**.** *Suppose $\alpha \in (0, N)$, and $s, r > 1$ with $\frac{1}{s} + \frac{1}{r} = \frac{2N-\alpha}{N}$. Let $g \in L^s(\mathbb{R}^N), h \in L^r(\mathbb{R}^N)$, there exists a sharp constant $C(s, \alpha, r, N)$, independent of g, h, such that*
$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x-y|^\alpha}\,dx\,dy \le C(s,\alpha,r,N)\|g\|_{L^r(\mathbb{R}^N)}\|h\|_{L^s(\mathbb{R}^N)}.$$

By the above lemma we have the following result.

Lemma 2.2. *If $v \in L^s(\mathbb{R}^N)$ and $s \in (1, \frac{N}{N-\alpha})$, then $\int_{\mathbb{R}^N} \frac{v(y)}{|x-y|^\alpha}\,dy \in L^{\frac{Ns}{N-Ns+\alpha s}}(\mathbb{R}^N)$, and*
$$\Big(\int_{\mathbb{R}^N}\Big|\int_{\mathbb{R}^N} \frac{v(y)}{|x-y|^\alpha}\,dy\Big|^{\frac{Ns}{N-Ns+\alpha s}}dx\Big)^{\frac{N-Ns+\alpha s}{Ns}} \le c(s,N,\alpha)\|v\|_{L^s(\mathbb{R}^N)}.$$

We denote
$$D(f,g) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^\alpha}\,dx\,dy.$$

Lemma 2.3 ([22, Theorem 9.8])**.** *Let $N \ge 3$, $0 < \alpha < N$, and $D(f,f)$, $D(g,g) < \infty$. Then*
$$|D(f,g)|^2 \le D(f,f)D(g,g)$$
with equality for $g \ne 0$ if and only if $f = cg$ for some constant c.

Lemma 2.4. *For $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$ the following holds;*

(1) $0 \le b_\varepsilon(x,t) \le \frac{m_\varepsilon(x,t)}{t} \le 1$;
(2) $m_\varepsilon(x,t) = t$,*if* $|t| < \varepsilon^{-1}\exp\{-\operatorname{dist}(\varepsilon x,\mathcal{M})\}$;
(3) $\varepsilon^{-1}\exp\{-\operatorname{dist}(\varepsilon x,\mathcal{M})\} \le m_\varepsilon(x,t) \le C\varepsilon^{-1}\exp\{-\operatorname{dist}(\varepsilon x,\mathcal{M})\}$, *if* $\varepsilon^{-1}\exp\{-\operatorname{dist}(\varepsilon x,\mathcal{M})\} \le t \le 2\varepsilon^{-1}\exp\{-\operatorname{dist}(\varepsilon x,\mathcal{M})\}$;
(4) $m_\varepsilon(x,t) = C\varepsilon^{-1}\exp\{-\operatorname{dist}(\varepsilon x,\mathcal{M})\}$, *if* $t \ge 2\varepsilon^{-1}\exp\{-\operatorname{dist}(\varepsilon x,\mathcal{M})\}$, *where* $C = \int_0^\infty \xi(\tau)\,d\tau$;
(5) $C_1(1+\varepsilon^{m-2}\exp\{(m-2)\operatorname{dist}(\varepsilon x,\mathcal{M})\}|t|^{m-2})t \le k_\varepsilon(x,t) \le C_2(1+\varepsilon^{m-2}\exp\{(m-2)\operatorname{dist}(\varepsilon x,\mathcal{M})\}|t|^{m-2})t$;
(6) $\frac{1}{m}tk_\varepsilon(x,t) \le K_\varepsilon(x,t) \le \frac{1}{2}tk_\varepsilon(x,t)$;
(7) $(k_\varepsilon(x,t_1)-k_\varepsilon(x,t_2))(t_1-t_2) \ge C(1+\varepsilon^{m-2}\exp\{(m-2)\operatorname{dist}(\varepsilon x,\mathcal{M})\}|t_1-t_2|^{m-2})|t_1-t_2|^2$;
(8) $|k_\varepsilon(x,t_1)-k_\varepsilon(x,t_2)| \le C(1+\varepsilon^{m-2}\exp\{(m-2)\operatorname{dist}(\varepsilon x,\mathcal{M})\}(|t_1|^{m-2}+|t_2|^{m-2}))|t_1-t_2|$;
(9) $|F_\nu(t)| \le C_\nu|t|^r$;
(10) $|f_\nu(t)| \le C_\nu|t|^{r-1}$;
(11) $\frac{1}{2^*_\alpha}tf_\nu(t) \le F_\nu(t) \le \frac{1}{r}tf_\nu(t)$;

Proof. The proof is straightforward. We only prove (6). Let $f(x,t) = K_\varepsilon(x,t) - \frac{1}{2}tk_\varepsilon(x,t)$ and $g(x,t) = K_\varepsilon(x,t) - \frac{1}{m}tk_\varepsilon(x,t)$. Since $f(x,0)=0$, we have $\frac{\partial f(x,t)}{\partial t} \le 0$, if $t \ge 0$; and $\frac{\partial f(x,t)}{\partial t} \ge 0$, if $t \le 0$. $g(x,0)=0$; $\frac{\partial g(x,t)}{\partial t} \ge 0$, if $t \ge 0$; $\frac{\partial g(x,t)}{\partial t} \le 0$, if $t \le 0$. So (6) holds. □

Lemma 2.5. *For $t \in \mathbb{R}^+$ it holds*

(1) $g_\lambda(t) = 1, g'_\lambda(t) = 0$ *if* $0 < t < \frac{1}{\lambda}$;
(2) $b_\lambda(t)t \le g_\lambda(t)t \le c_\lambda$, *where* $c_\lambda = \frac{\int_0^\infty \xi(\tau)\,d\tau}{\lambda}$;
(3) $g'_\lambda(t)t + g_\lambda(t) = b_\lambda(t)$.

The proof of the above lemma is obviously, we omit it.

Lemma 2.6. *The imbedding $X_\varepsilon = H^1(\mathbb{R}^N) \cap L^m_\varepsilon(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $1 \le p < 2^*$.*

Proof. Let $\{u_n\}$ is bounded in X_ε and assume $u_n \rightharpoonup u$ in X_ε and $u_n \to u$ in $L^p_{loc}(R^N), 1 \le p < 2^*$. We first prove $u_n \to u$ in $L^1(R^N)$. For $R > 0$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N\setminus B(0,R)} |u|\,dx \\
&\le \Big(\int_{\mathbb{R}^N\setminus B(0,R)} \exp\{(m-2)\operatorname{dist}(\varepsilon x,\mathcal{M})\}|u|^m\,dx\Big)^{1/m} \\
&\quad\times \Big(\int_{\mathbb{R}^N\setminus B(0,R)} \exp\{-\frac{m-2}{m-1}\operatorname{dist}(\varepsilon x,\mathcal{M})\}\,dx\Big)^{\frac{m-1}{m}} \\
&\le \|u\|_{L^m_\varepsilon(\mathbb{R}^N)}\Big(\int_{\mathbb{R}^N\setminus B(0,R)} \exp\{-\frac{m-2}{m-1}\operatorname{dist}(\varepsilon x,\mathcal{M})\}\,dx\Big)^{\frac{m-1}{m}} = o_R(1).
\end{aligned}
$$

Hence $\int_{\mathbb{R}^N} |u_n-u|\,dx = \int_{B(0,R)} |u_n-u|\,dx + \int_{\mathbb{R}^N\setminus B(0,R)} |u_n-u|\,dx = o_n(1)+o_R(1) \to 0$ as $n \to \infty$. For $1 < p < 2^*$, we have

$$
\int_{\mathbb{R}^N} |u_n-u|^p\,dx
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N} |u_n - u|^{p\theta + (1-\theta)p}\,dx \\
&\le \Big(\int_{\mathbb{R}^N} |u_n - u|^{p\theta\cdot\frac{1}{p\theta}}\,dx\Big)^{p\theta}\Big(\int_{\mathbb{R}^N} |u_n - u|^{(1-\theta)p\cdot\frac{2^*}{(1-\theta)p}}\,dx\Big)^{\frac{(1-\theta)p}{2^*}} \\
&\le c\Big(\int_{\mathbb{R}^N} |u_n - u|\,dx\Big)^{p\theta},
\end{aligned}
$$

where $0 < \theta < 1$, $\frac{1}{p} = \theta + \frac{1-\theta}{2^*}$, so $u_n \to u$ in $L^p(\mathbb{R}^N)(1 \le p < 2^*)$. □

Lemma 2.7. *Let $\{u_n\}$ be a (PS) sequence of the functional $\Gamma_{\varepsilon,\nu,\lambda}$, then $\{u_n\}$ is bounded in X_ε.*

Proof. A direct computation shows that

$$
\begin{aligned}
\langle D\Gamma_{\varepsilon,\nu,\lambda}(u), v\rangle &= \int_{\mathbb{R}^N} \nabla u\nabla v + E(\varepsilon x)uv\,dx + \sigma\int_{\mathbb{R}^N} k_\varepsilon(x,u)v\,dx \\
&\quad + \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u^2\,dx - 1\Big)_+^{\beta-1}\int_{\mathbb{R}^N} \chi_\varepsilon(x)uv\,dx \\
&\quad - \frac{2^*_\alpha}{2}h_\lambda(\varphi^{1/2}(u))\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\nu(u(y))f_\nu(u(x))v(x)}{|x-y|^\alpha}\,dx\,dy \\
&\quad - \vartheta\int_{\mathbb{R}^N} |u|^{q-2}uv\,dx, \quad \text{for } v \in X_\varepsilon.
\end{aligned} \tag{2.1}
$$

By Lemma 2.4, we have

$$
\begin{aligned}
&\Gamma_{\varepsilon,\nu,\lambda}(u_n) - \frac{1}{r}\langle D\Gamma_{\varepsilon,\nu,\lambda}(u_n), u_n\rangle \\
&= (\frac{1}{2} - \frac{1}{r})\int_{\mathbb{R}^N} (|\nabla u_n|^2 + E(\varepsilon x)u_n^2)\,dx + \sigma\int_{\mathbb{R}^N} K_\varepsilon(x,u_n)\,dx \\
&\quad - \frac{\sigma}{r}\int_{\mathbb{R}^N} k_\varepsilon(x,u_n)u_n\,dx + \frac{1}{2\beta}\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u_n^2\,dx - 1\Big)_+^{\beta} \\
&\quad - \frac{1}{r}\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u_n^2\,dx - 1\Big)_+^{\beta-1}\int_{\mathbb{R}^N} \chi_\varepsilon(x)u_n^2\,dx \\
&\quad + \frac{2^*_\alpha}{2r}h_\lambda(\varphi^{1/2}(u_n))\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\nu(u_n(y))f_\nu(u_n(x))u_n(x)}{|x-y|^\alpha}\,dx\,dy \\
&\quad - \frac{2^*_\alpha}{2}g_\lambda(\varphi^{1/2}(u_n))\varphi(u_n) + \vartheta(\frac{1}{r} - \frac{1}{q})\int_{\mathbb{R}^N} |u_n|^q\,dx \\
&\ge (\frac{1}{2} - \frac{1}{r})\int_{\mathbb{R}^N} (|\nabla u_n|^2 + E(\varepsilon x)u_n^2)\,dx + (\frac{1}{m} - \frac{1}{r})\sigma\int_{\mathbb{R}^N} k_\varepsilon(x,u_n)u_n\,dx \\
&\quad + c\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u_n^2\,dx - 1\Big)_+^{\beta} - c \\
&\ge c\big(\|u_n\|^2_{H^1(\mathbb{R}^N)} + \|u_n\|^m_{L^m_\varepsilon(\mathbb{R}^N)}\big) + c\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u_n^2\,dx - 1\Big)_+^{\beta} - c.
\end{aligned}
$$

Combining $|\Gamma_{\varepsilon,\nu,\lambda}(u_n)| \le C$ and $D\Gamma_{\varepsilon,\nu,\lambda}(u_n) \to 0$, it follows that $\{u_n\}$ is bounded in X_ε. □

Lemma 2.8. *For every $\varepsilon, \nu, \lambda > 0$, $\Gamma_{\varepsilon,\nu,\lambda}$ satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence of the functional $\Gamma_{\varepsilon,\nu,\lambda}$. By Lemma 2.7, $\{u_n\}$ is bounded in X_ε. Up to a subsequence, we may assume $u_n \rightharpoonup u$ in X_ε and $u_n \to u$ in $L^r(\mathbb{R}^N)(2 \le r < 2^*)$. By Lemma 2.1, we have

$$
\begin{aligned}
o(1) &= \langle D\Gamma_{\varepsilon,\nu,\lambda}(u_n) - D\Gamma_{\varepsilon,\nu,\lambda}(u), u_n - u\rangle \\
&= \int_{\mathbb{R}^N} \big(|\nabla(u_n-u)|^2 + E(\varepsilon x)(u_n-u)^2\big)\,dx \\
&\quad + \sigma \int_{\mathbb{R}^N} (k_\varepsilon(x,u_n) - k_\varepsilon(x,u))(u_n-u)\,dx \\
&\quad + \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u_n^2\,dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x)u_n(u_n-u)dx \\
&\quad - \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u^2\,dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x)u(u_n-u)dx \\
&\quad - \frac{2_\alpha^*}{2} h_\lambda(\varphi^{1/2}(u_n)) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\nu(u_n(y))f_\nu(u_n(x))(u_n(x)-u(x))}{|x-y|^\alpha}\,dx\,dy \\
&\quad + \frac{2_\alpha^*}{2} h_\lambda(\varphi^{1/2}(u)) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\nu(u(y))f_\nu(u(x))(u_n(x)-u(x))}{|x-y|^\alpha}\,dx\,dy \\
&\quad - \vartheta \int_{\mathbb{R}^N} (|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n-u)\,dx \\
&\ge \int_{\mathbb{R}^N} \big(|\nabla(u_n-u)|^2 + E(\varepsilon x)(u_n-u)^2\big)\,dx \\
&\quad + c(1+\varepsilon^{m-2}\exp\{(m-2)\mathrm{dist}(\varepsilon x,\mathcal{M})\}|u_n-u|^{m-2})|u_n-u|^2 \\
&\quad - c\int_{\mathbb{R}^N} (|u_n|^{q-1} + |u|^{q-1})|u_n-u|\,dx + o(1) \\
&\ge c\big(\|u_n-u\|_{H^1(\mathbb{R}^N)}^2 + \|u_n-u\|_{L_\varepsilon^m(\mathbb{R}^N)}^m\big) + o(1), \quad \text{as } n\to\infty,
\end{aligned}
$$

which implies $\|u_n - u\|_{X_\varepsilon} \to 0$. Therefore, $\Gamma_{\varepsilon,\nu,\lambda}$ satisfies the (PS) condition. □

3. Existence of solutions for perturbed functions $\Gamma_{\varepsilon,\nu,\lambda}$

In this section, we construct a sequence of critical points of the functional $\Gamma_{\varepsilon,\nu,\lambda}$ by using the method of invariant sets with respect to a descending flow. Firstly we define an operator $A : X \to X$. The vector field $u - Au$ will be used as pseudo-gradient vector field of the functional $\Gamma_{\varepsilon,\nu,\lambda}$. To obtain multiple sign-changing critical points of $\Gamma_{\varepsilon,,\nu,\lambda}$, we introduce the abstract critical point theorem [26, Theorem 2.5], see also [7, Theorem 3.2].

Let X be a Banach space, f be an even C^1-functional on X. Let P, Q be two family of open convex sets of $X, Q = -P$. We set

$$
W = P \cup Q, \quad \Sigma = \partial P \cap \partial Q\,.
$$

Then we assume that

- (A6) f satisfies the (PS) condition.
- (A7) $c^* = \inf_{x\in\Sigma} f(x) > 0\,,$

and that there exists an odd continuous map $A : X \to X$ satisfying

- (A8) For each $c_0, b_0 > 0$, there exists $b = b(c_0, b_0) > 0$ such that if $\|Df(x)\| \ge b_0$, $|f(x)| \le c_0$, then
$$
\langle Df(x),\, x - Ax\rangle \ge b\|x - Ax\|_X > 0\,.
$$

(A9) $A(\partial P) \subset P$ and $A(\partial Q) \subset Q$.

We define

$$\Gamma_j = \{E \subset X : E \text{ is compact}, -E = E, \gamma(E \cap \eta^{-1}(\Sigma)) \geq j \text{ for } \eta \in \Lambda\},$$
$$\Lambda = \{\eta \in C(X, X) : \eta \text{ is odd}, \eta(P) \subset P, \eta(Q) \subset Q, \eta(x) = x \text{ if } f(x) < 0\}$$

where γ is the genus of symmetric sets, defined as

$$\gamma(E) = \inf \left\{n : \text{there exists an odd map } \eta : E \to \mathbb{R}^n \backslash \{0\}\right\}.$$

Now we assume

(A10) Γ_j is nonempty.

We define

$$c_j = \inf_{E \in \Gamma_j} \sup_{x \in E \backslash W} f(x), \quad j = 1, 2, \dots,$$
$$K_c = \{x : Df(x) = 0,\ f(x) = c\}, \quad K_c^* = K_c \setminus W.$$

Theorem 3.1 ([26, Theorem 3.1]). *Assume* (A6)–(A10) *hold. Then*

(1) $c_j \geq c^*$, $K_{c_j}^* \neq \emptyset$.
(2) $c_j \to \infty$ *as* $j \to \infty$.
(3) *If* $c_j = c_{j+1} = \dots = c_{j+k-1} = c$, *then* $\gamma(K_c^*) \geq k$.

We prove the existence of critical points of $\Gamma_{\varepsilon,,\nu,\lambda}$ by using the method of invariant sets of descending flow. First, we need to define the operator A.

Definition 3.2. *Given* $u \in X_\varepsilon$ *define* $v = Au$ *by the equation*

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\nabla v \nabla \eta + E(\varepsilon x) v \eta) dx + \sigma \int_{\mathbb{R}^N} k_\varepsilon(x, v) \eta\, dx \\
&+ \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2\, dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x) v \eta\, dx \\
&= \frac{2_\alpha^*}{2} h_\lambda(\varphi^{1/2}(u)) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F_\nu(u(y)) f_\nu(u(x)) \eta(x)}{|x-y|^\alpha}\, dx\, dy \\
&\quad + \vartheta \int_{\mathbb{R}^N} |u|^{q-2} u \eta\, dx, \quad \text{for } \eta \in X_\varepsilon.
\end{aligned} \tag{3.1}$$

Lemma 3.3 (Brezis-Libe type lemma [6]). *Assume* $0 < \alpha < \min\{4, N-1\}$ *and that* f *satisfies*

(1) *there exists a constant* $C > 0$ *such that*
$$|f(t)| \leq C(|t|^{\frac{N-\alpha}{N}} + |t|^{\frac{N+2-\alpha}{N-2}}), \quad \forall t \in \mathbb{R}.$$
(2) $\lim_{t \to 0^+} \frac{f(t)}{t} = 0$, $\lim_{t \to \infty} \frac{f(t)}{t^{\frac{N+2-\alpha}{N-2}}} = 1$.

Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ *be such that* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^N)$ *and* $u_n(x) \to u(x)$ *a.e.* $x \in \mathbb{R}^N$, *then up to a subsequence if necessary, it holds*

$$\begin{aligned}
&\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(u_n(x)) F(u_n(y))}{|x-y|^\alpha}\, dx\, dy \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(u_n(x) - u(x)) F(u_n(y) - u(y))}{|x-y|^\alpha}\, dx\, dy \\
&\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(u(x)) F(u(y))}{|x-y|^\alpha}\, dx\, dy + o_n(1),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u_n(x))f(u_n(y))v(y)}{|x-y|^\alpha}\,dx\,dy\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u_n(x)-u(x))f(u_n(y)-u(y))v(y)}{|x-y|^\alpha}\,dx\,dy\\
&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(x))f(u(y))v(y)}{|x-y|^\alpha}\,dx\,dy+o_n(1),
\end{aligned}\tag{3.3}$$

where $o_n(1)\to 0$ uniformly as $n\to\infty$ for any $v\in C_0^\infty(\mathbb{R}^N)$.

Lemma 3.4. *Function A is well defined, odd and continuous on X_ε.*

Proof. For simplicity, we denote

$$\psi_\varepsilon(u)=\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)_+^{\beta-1},$$

and define

$$B(v)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2+E(\varepsilon x)v^2)\,dx+\sigma\int_{\mathbb{R}^N}K_\varepsilon(x,v)\,dx+\frac{1}{2}\psi_\varepsilon(u)\int_{\mathbb{R}^N}\chi_\varepsilon(x)v^2\,dx.$$

Since

$$\langle DB(v_1)-DB(v_2),v_1-v_2\rangle\ge c(\|v_1-v_2\|^2_{H^1(\mathbb{R}^N)}+\|v_1-v_2\|^m_{L^m_\varepsilon(\mathbb{R}^N)}),\tag{3.4}$$

for all $v_1,v_2\in X_\varepsilon$, it follows that DB is strongly monotone. Then problem (3.1) has a unique solution $v=Au$, which can be obtained by solving the minimization problem

$$\inf\{B(v)-F(v)\big|v\in X_\varepsilon\},$$

where

$$F(v)=\frac{2^*_\alpha}{2}h_\lambda(\varphi^{1/2}(u))\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F_\nu(u(y))f_\nu(u(x))v(x)}{|x-y|^\alpha}\,dx\,dy+\vartheta\int_{\mathbb{R}^N}|u|^{q-2}uv\,dx.$$

So A is well defined. Moreover, it is easy to check the operator A is odd. Finally, let $u_n\to u$ in X_ε, and denote $v_n=Au_n$, $v=Au$. By choosing $\eta=v_n-v$ in (3.1), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}(|\nabla(v_n-v)|^2+E(\varepsilon x)|v_n-v|^2)\,dx+\sigma\int_{\mathbb{R}^N}(k_\varepsilon(x,v_n)-k_\varepsilon(x,v))(v_n-v)\,dx\\
&+\psi_\varepsilon(u)\int_{\mathbb{R}^N}\chi_\varepsilon(x)(v_n-v)^2\,dx\\
&=(\psi_\varepsilon(u_n))-\psi_\varepsilon(u))\int_{\mathbb{R}^N}\chi_\varepsilon(x)v_n(v-v_n)\,dx+\frac{2^*_\alpha}{2}h_\lambda(\varphi^{1/2}(u_n))\\
&\quad\times\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big(F_\nu(u_n(y))f_\nu(u_n(x))-F_\nu(u(y))f_\nu(u(x))\big)(v_n(x)-v(x))}{|x-y|^\alpha}\,dx\,dy\\
&\quad+\frac{2^*_\alpha}{2}(h_\lambda(\varphi^{1/2}(u_n))-h_\lambda(\varphi^{1/2}(u)))\\
&\quad\times\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F_\nu(u(y))f_\nu(u(x))(v_n(x)-v(x))}{|x-y|^\alpha}\,dx\,dy\\
&\quad+\vartheta\int_{\mathbb{R}^N}(|u_n|^{q-2}u_n-|u|^{q-2}u)(v_n-v)\,dx.
\end{aligned}\tag{3.5}$$

By Lemma 2.4, can estimate the two side of (3.5):

$$\text{LHS} \geq c(\|v_n - v\|^2_{H^1(\mathbb{R}^N)} + \|v_n - v\|^m_{L^m_\varepsilon(\mathbb{R}^N)}),$$
$$\text{RHS} = o_n(1)\|v_n - v\|_{H^1(\mathbb{R}^N)}.$$

So we obtain $\|Au_n - Au\|_{X_\varepsilon} \to 0$. This means A is continuous. □

Lemma 3.5. *Let $u \in X_\varepsilon$, $v = Au$. Then it holds:*

(1) $\langle D\Gamma_{\varepsilon,\nu,\lambda}(u), u - v\rangle \geq c(\|u - v\|^2_{H^1(\mathbb{R}^N)} + \|u - v\|^m_{L^m_\varepsilon(\mathbb{R}^N)})$;

(2) $\|D\Gamma_{\varepsilon,\nu,\lambda}(u)\| \leq c(1 + |\Gamma_{\varepsilon,\nu,\lambda}(u)| + \|u - v\|_{X_\varepsilon})^\gamma \|u - v\|_{X_\varepsilon} (\gamma > 1)$.

Proof. (1) By (3.1), for $\eta \in X_\varepsilon$, we have

$$\begin{aligned}
&\langle D\Gamma_{\varepsilon,\nu,\lambda}(u), \eta\rangle \\
&= \int_{\mathbb{R}^N} (\nabla u \nabla \eta + E(\varepsilon x) u\eta) dx + \sigma \int_{\mathbb{R}^N} k_\varepsilon(x, u)\eta\, dx \\
&\quad + \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2\, dx - 1\Big)^{\beta-1}_+ \int_{\mathbb{R}^N} \chi_\varepsilon(x) u\eta\, dx \\
&\quad - \frac{2^*_\alpha}{2} h_\lambda(\varphi^{1/2}(u)) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\nu(u(y)) f_\nu(u(x))\eta(x)}{|x - y|^\alpha}\, dx\, dy - \vartheta \int_{\mathbb{R}^N} |u|^{q-2} u\eta\, dx \\
&= \int_{\mathbb{R}^N} \big(\nabla(u - v)\nabla\eta + E(\varepsilon x)(u - v)\eta\big) dx + \sigma \int_{\mathbb{R}^N} \big(k_\varepsilon(x, u) - k_\varepsilon(x, v)\big)\eta\, dx \\
&\quad + \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2\, dx - 1\Big)^{\beta-1}_+ \int_{\mathbb{R}^N} \chi_\varepsilon(x)(u - v)\eta\, dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\langle D\Gamma_{\varepsilon,\nu,\lambda}(u), u - v\rangle \\
&= \int_{\mathbb{R}^N} (|\nabla(u - v)|^2 + E(\varepsilon x)(u - v)^2)\, dx + \sigma \int_{\mathbb{R}^N} \big(k_\varepsilon(x, u) - k_\varepsilon(x, v)\big)(u - v)\, dx \\
&\quad + \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2\, dx - 1\Big)^{\beta-1}_+ \int_{\mathbb{R}^N} \chi_\varepsilon(x)(u - v)^2\, dx \\
&\geq c\Big(\|u - v\|^2_{H^1(\mathbb{R}^N)} + \|u - v\|^m_{L^m_\varepsilon(\mathbb{R}^N)}\Big).
\end{aligned}$$

(2) We define

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + E(\varepsilon x) u^2)\, dx + \sigma \int_{\mathbb{R}^N} K_\varepsilon(x, u)\, dx.$$

On the one hand,

$$\begin{aligned}
&\Gamma_{\varepsilon,\nu,\lambda}(u) - \frac{1}{r}\langle DJ_\varepsilon(u) - DJ_\varepsilon(v), u\rangle \\
&= (\frac{1}{2} - \frac{1}{r})\int_{\mathbb{R}^N} (|\nabla u|^2 + E(\varepsilon x) u^2)\, dx + \sigma \int_{\mathbb{R}^N} K_\varepsilon(x, u)\, dx - \frac{\sigma}{r}\int_{\mathbb{R}^N} k_\varepsilon(x, u) u\, dx \\
&\quad + \frac{1}{2\beta}\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2\, dx - 1\Big)^\beta_+ - \frac{1}{r}\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2\, dx - 1\Big)^{\beta-1}_+ \int_{\mathbb{R}^N} \chi_\varepsilon(x) uv\, dx \\
&\quad + \frac{2^*_\alpha}{2r} h_\lambda(\varphi^{1/2}(u)) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\nu(u(y)) f_\nu(u(x)) u(x)}{|x - y|^\alpha}\, dx\, dy - \frac{2^*_\alpha}{2} g_\lambda(\varphi^{1/2}(u))\varphi(u) \\
&\quad + \vartheta(\frac{1}{r} - \frac{1}{q})\int_{\mathbb{R}^N} |u|^q\, dx.
\end{aligned} \tag{3.6}$$

By Hölder's inequality and the Young's inequality, we have

$$
\begin{aligned}
&\frac{1}{2\beta}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)_+^{\beta}-\frac{1}{r}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)uv\,dx\\
&=\frac{1}{2\beta}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)_+^{\beta}-\frac{1}{r}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx\\
&\quad+\frac{1}{r}\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)u(u-v)\,dx\\
&\geq c\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)_+^{\beta}-c\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u-v||u|\,dx\Big)^{\beta}-c\\
&\geq c\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)_+^{\beta}-c\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u-v|^2\,dx\Big)^{\beta}-c.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\Gamma_{\varepsilon,\nu,\lambda}(u)-\frac{1}{r}\langle DJ_\varepsilon(u)-DJ_\varepsilon(v),u\rangle\\
&\geq C\big(\|u\|^2_{H^1(\mathbb{R}^N)}+\|u\|^m_{L^m_\varepsilon(\mathbb{R}^N)}\big)+C\big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2dx-1\big)_+^{\beta}\\
&\quad-c\big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)(u-v)^2dx\big)^{\beta}-c.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\Gamma_{\varepsilon,\nu,\lambda}(u)-\frac{1}{r}\langle DJ_\varepsilon(u)-DJ_\varepsilon(v),u\rangle\\
&\leq|\Gamma_{\varepsilon,\nu,\lambda}(u)|+c\|u\|_{H^1(\mathbb{R}^N)}\|u-v\|_{H^1(\mathbb{R}^N)}\\
&\quad+c(\|u\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}+\|v\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)})\|u-v\|_{L^m_\varepsilon(\mathbb{R}^N)}\|u\|_{L^m_\varepsilon(\mathbb{R}^N)}.
\end{aligned}
\tag{3.7}
$$

By (3.6), (3.7) and Young's inequality, we have

$$
\begin{aligned}
&\|u\|^2_{H^1(\mathbb{R}^N)}+\|u\|^m_{L^m_\varepsilon(\mathbb{R}^N)}+\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)_+^{\beta}\\
&\leq c(1+|\Gamma_{\varepsilon,\nu,\lambda}(u)|+\|u-v\|^2_{H^1(\mathbb{R}^N)}+\|u-v\|^m_{L^m_\varepsilon(\mathbb{R}^N)}+\|u-v\|^{2\beta}_{H^1(\mathbb{R}^N)}).
\end{aligned}
$$

By (3.1), we have

$$
\begin{aligned}
&|\langle D\Gamma_{\varepsilon,\nu,\lambda}(u),\eta\rangle|\\
&\leq c\|u-v\|_{H^1(\mathbb{R}^N)}\|\eta\|_{H^1(\mathbb{R}^N)}+\Big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\Big)_+^{\beta-1}\|u-v\|_{H^1(\mathbb{R}^N)}\|\eta\|_{H^1(\mathbb{R}^N)}\\
&\quad+c\int_{\mathbb{R}^N}\varepsilon^{m-2}\exp\{(m-2)\operatorname{dist}(\varepsilon x,\mathcal{M})\}(|u|^{m-2}+|u|^{m-2})|u-v||\eta|\,dx\\
&\leq c(\|u\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}+\|v\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)})\|u-v\|_{L^m_\varepsilon(\mathbb{R}^N)}\|\eta\|_{L^m_\varepsilon(\mathbb{R}^N)}\\
&\quad+c\Big(1+\big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\big)_+^{\beta-1}\Big)\|u-v\|_{H^1(\mathbb{R}^N)}\|\eta\|_{H^1(\mathbb{R}^N)}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\|D\Gamma_{\varepsilon,\nu,\lambda}(u)\|\\
&\leq c\ (\|u\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}+\|v\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)})\|u-v\|_{L^m_\varepsilon(\mathbb{R}^N)}\\
&\quad+c\Big(1+\big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\big)_+^{\beta-1}\Big)\|u-v\|_{H^1(\mathbb{R}^N)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq c\Big(1+\|u\|^m_{L^m_\varepsilon(\mathbb{R}^N)}+\|u-v\|^m_{L^m_\varepsilon(\mathbb{R}^N)}+\big(\int_{\mathbb{R}^N}\chi_\varepsilon(x)u^2\,dx-1\big)^\beta_+\Big)\|u-v\|_{X_\varepsilon}\\
&\leq c(1+|\Gamma_{\varepsilon,\nu,\lambda}(u)|+\|u-v\|^2_{H^1(\mathbb{R}^N)}+\|u-v\|^m_{L^m_\varepsilon(\mathbb{R}^N)}+\|u-v\|^{2\beta}_{H^1(\mathbb{R}^N)})\|u-v\|_{X_\varepsilon}\\
&\leq c(1+|\Gamma_{\varepsilon,\nu,\lambda}(u)|+\|u-v\|_{X_\varepsilon})^\gamma\|u-v\|_{X_\varepsilon},
\end{aligned}
$$

where $\gamma>1$. □

Corollary 3.6. *Given b_0,c_0, there exist $b=b(b_0,c_0)$ such that if $|\Gamma_{\varepsilon,\nu,\lambda}(u)|\leq c_0$ and $\|D\Gamma_{\varepsilon,\nu,\lambda}(u)\|\geq b_0$, then $u-Au\neq 0$ and*

$$\langle D\Gamma_{\varepsilon,\nu,\lambda}(u),u-Au\rangle\geq b\|u-Au\|>0.$$

For $\delta>0$, we define the convex open sets

$$
\begin{aligned}
P&=\{u\big|u\in X_\varepsilon(\mathbb{R}^N),\|u_-\|_{H^1(\mathbb{R}^N)}<\delta\},\\
Q&=\{u\big|u\in X_\varepsilon(\mathbb{R}^N),\|u_+\|_{H^1(\mathbb{R}^N)}<\delta\}.
\end{aligned}
$$

Lemma 3.7. *There exists $\delta_\lambda>0$ such that for $0<\delta<\delta_\lambda$,*

$$A(\partial P)\subset P,\quad A(\partial Q)\subset Q.$$

Proof. We only prove $A(\partial Q)\subset Q$. Similarly, $A(\partial P)\subset P$. For $u\in\partial Q$, let $v=Au$. By Lemma 2.3 and Lemma 2.5, we have

$$
\begin{aligned}
&\|v_+\|^2_{H^1(\mathbb{R}^N)}\\
&\leq c\int_{\mathbb{R}^N}(\nabla v\nabla v_++E(\varepsilon x)vv_+)\,dx+c\int_{\mathbb{R}^N}k_\varepsilon(x,v)v_+\,dx\\
&\leq ch_\lambda(\varphi^{1/2}(u))\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F_\nu(u(y))f_\nu(u(x))v_+(x)}{|x-y|^\alpha}\,dx\,dy\\
&\quad+c\int_{\mathbb{R}^N}|u|^{q-2}uv_+\,dx\\
&\leq ch_\lambda(\varphi^{1/2}(u))\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F_\nu(u(y))f_\nu(u_+(x))v_+(x)}{|x-y|^\alpha}\,dx\,dy+c\int_{\mathbb{R}^N}|u_+|^{q-1}v_+\,dx\\
&\leq ch_\lambda(\varphi^{1/2}(u))\varphi^{1/2}(u)\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f_\nu(u_+(x))v_+(x)f_\nu(u_+(y))v_+(y)}{|x-y|^\alpha}\,dx\,dy\Big)^{1/2}\\
&\quad+c\|u_+\|^{q-1}_{H^1(\mathbb{R}^N)}\|v_+\|_{H^1(\mathbb{R}^N)}\\
&\leq c_\lambda\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_+(x)|^{2^*_\alpha-1}v_+(x)|u_+(y)|^{2^*_\alpha-1}v_+(y)}{|x-y|^\alpha}\,dx\,dy\Big)^{1/2}\\
&\quad+c\|u_+\|^{q-1}_{H^1(\mathbb{R}^N)}\|v_+\|_{H^1(\mathbb{R}^N)}\\
&\leq c_\lambda(\|u_+\|^{2^*_\alpha-1}_{H^1(\mathbb{R}^N)}+\|u_+\|^{q-1}_{H^1(\mathbb{R}^N)})\|v_+\|_{H^1(\mathbb{R}^N)}.
\end{aligned}
$$

Taking $\delta_\lambda=\min\{\frac{1}{2}c_\lambda^{-\frac{1}{2^*_\alpha-2}},\frac{1}{2}c_\lambda^{-\frac{1}{q-2}}\}$, it is easy to get that $\|v_+\|_{H^1(\mathbb{R}^N)}<\delta$ for $0<\delta<\delta_\lambda$. Consequently, the conclusion follows. □

Lemma 3.8. *There exist $\delta_0>0$ and $0<c^*=c^*(\delta)$, such that for $0<\delta<\delta_0$ and $u\in\partial P\cap\partial Q$, we have $\Gamma_{\varepsilon,\nu,\lambda}(u)\geq c^*$.*

Proof. For $u\in\partial P\cap\partial Q$, we have $\|u\|_{H^1(\mathbb{R}^N)}\geq\delta$, $\|u\|_{L^{2^*}(\mathbb{R}^N)}\leq c\delta$, and $\|u\|_{L^q(\mathbb{R}^N)}\leq c\delta$. Hence

$$\Gamma_{\varepsilon,\nu,\lambda}(u)\geq\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+E(\varepsilon x)u^2)\,dx-\frac{2^*_\alpha}{2}g_\lambda(\varphi^{1/2}(u))\varphi(u)-\frac{\vartheta}{q}\int_{\mathbb{R}^N}|u|^q\,dx$$

$$
\begin{aligned}
&\geq c_1\|u\|^2_{H^1(\mathbb{R}^N)} - c_2\big(\|u\|^{2\cdot 2^*_\alpha}_{L^{2^*}(\mathbb{R}^N)} + \|u\|^q_{L^q(\mathbb{R}^N)}\big)\\
&\geq c_1\|u\|^2_{H^1(\mathbb{R}^N)} - c_2(\delta^{2\cdot 2^*_\alpha - 2} + \delta^{q-2})\|u\|^2_{H^1(\mathbb{R}^N)}.
\end{aligned}
$$

Taking $\delta_0 = \min\{(\frac{c_1}{4c_2})^{\frac{1}{2\cdot 2^*_\alpha - 2}}, (\frac{c_1}{4c_2})^{\frac{1}{q-2}}\}$, then for $0 < \delta < \delta_0$, we have

$$
\Gamma_{\varepsilon,\nu,\lambda}(u) \geq \frac{c_1}{2}\|u\|^2_{H^1(\mathbb{R}^N)} \geq \frac{c_1}{2}\delta^2 := c^*. \qquad \square
$$

Assume $B(0,R) \subset \mathcal{M}$. Let $\{e_n\}_{n=1}^\infty$ be a family of linearly independent functions in $C_0^\infty(B(0,R))$. There exists an increasing sequence R_n such that

$$
J_0(u) < 0,\ \forall u \in H_n,\ \|u\| \geq R_n.
$$

where $H_n := span\{e_1, \dots, e_n\}$ and

$$
J_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + bu^2)\,dx + \sigma\int_{\mathbb{R}^N} e^{(m-2)|x|}|u|^m\,dx - \frac{\vartheta}{q}\int_{\mathbb{R}^N} |u|^q\,dx.
$$

We define $\varphi_n \in C(B_n, C_0^\infty(B(0,R)))$,

$$
\varphi_n(t) = R_n\sum_{i=1}^n t_i e_i, \quad t = (t_1, \dots, t_n) \in B_n = \{t \in \mathbb{R}^n : |t| \leq 1\}.
$$

Let

$$
\begin{gathered}
\Gamma_j = \{E \subset X_\varepsilon : E \text{ is compact},\ E = E,\ \gamma(E \cap \eta^{-1}(\Sigma)) \geq j \text{ for } \eta \in \Lambda\},\\
\Lambda = \{\eta \in C(X_\varepsilon, X_\varepsilon) : \eta \text{is odd},\ \eta(P) \subset P,\ \eta(Q) \subset Q,\ \eta(u) = u \text{ if } \Gamma_\varepsilon(u) \leq 0\}.
\end{gathered}
$$

Similarly from [25, Lemma 5.6], we obtain the following Lemma.

Lemma 3.9. *Γ_j is nonempty, for $j = 1, 2, \dots$.*

Theorem 3.10. *Assume that conditions* (A1) *and* (A2) *hold, then there exist $0 < \tilde\varepsilon < 1$, $0 < \tilde\nu < 1$, and $0 < \tilde\lambda < 1$, such that if $0 < \varepsilon < \tilde\varepsilon$, $0 < \nu < \tilde\nu$, and $0 < \lambda < \tilde\lambda$, then the functional $\Gamma_{\varepsilon,\nu,\lambda}$ has infinitely many sign-changing critical points; the corresponding critical values are*

$$
c_j(\varepsilon,\nu,\lambda) = \inf_{E\in\Gamma_j}\sup_{u\in E\setminus W} \Gamma_{\varepsilon,\nu,\lambda}(u), \quad j = 1, 2, \dots. \tag{3.8}
$$

Moreover

1. *there exist m_j, $j = 1, \dots$, independent of $\varepsilon, \nu, \lambda$ such that*
$$
c_j(\varepsilon,\nu,\lambda) \leq m_j, \quad j = 1, 2, \dots. \tag{3.9}
$$
2. *If $c_j(\varepsilon,\nu,\lambda) = \dots = c_{j+k}(\varepsilon,\nu,\lambda) = c$, then $\gamma(K_c^*) \geq k + 1$, where*
$$
K_c^* = K_c \setminus W, \quad K_c = \{x : D\Gamma_{\varepsilon,\nu,\lambda}(u) = 0,\ \Gamma_{\varepsilon,\nu,\lambda}(u) = c\}.
$$

Proof. For the functional $\Gamma_{\varepsilon,\mu,\lambda}$, it is easy to check that $\Gamma_{\varepsilon,\nu,\lambda}$ satisfies the assumptions of Theorem 3.1. Therefore, we only need to prove (3.9). Note that $E_j = \varphi_{j+1}(B_{j+1}) \in \Gamma_j$. It is easy to know that there exist $0 < \tilde\varepsilon < 1$, $0 < \tilde\nu < 1$, and $0 < \tilde\lambda < 1$, such that if $0 < \varepsilon < \tilde\varepsilon$, $0 < \nu < \tilde\nu$, and $0 < \lambda < \tilde\lambda$, then $\Gamma_{\varepsilon,\nu,\lambda}(u) \leq J_0(u)$ for $u \in \varphi_{j+1}(B_{j+1})$ and $\big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p dx - 1\big)_+^\beta = 0$. Hence

$$
c_j(\varepsilon,\nu,\lambda) \leq m_j := \sup_{u\in E_j} J_0(u). \qquad \square
$$

4. Uniform bounds

In this section, we the following theorem that gives uniform bounds needed for proving Theorem 1.1.

Theorem 4.1. *(1) Assume $\Gamma_{\varepsilon,\nu,\lambda}(u) \le L$ and $D\Gamma_{\varepsilon,\nu,\lambda}(u) = 0$. Then there exists a constant $H = H(L)$ such that*

$$\|u\|_{H^1(\mathbb{R}^N)} \le H.$$

(2) Assume $\Gamma_{\varepsilon,\nu}(u) \le L$ and $D\Gamma_{\varepsilon,\nu}(u) = 0$. Then there exist constants $\mu > 0$, $C = C(L)$ such that, for any $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta) > 0$, for $0 < \varepsilon < \varepsilon(\delta)$,

$$|u(x)| \le C \exp\{-\mu\, \mathrm{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\} \quad \text{for } x \in \mathbb{R}^N.$$

(3) Assume $I_{\varepsilon,\nu}(u) \le L$ and $DI_{\varepsilon,\nu}(u) = 0$. Then there exists a positive constant $M = M(L)$ such that $|u(x)| \le M$ for $x \in \mathbb{R}^N$.

Similar to Lemma 2.7, it is easy to obtain Theorem 4.1(1). Before proving parts (2) and (3), we need establish some preliminary lemmas.

Lemma 4.2. *Assume $D\Gamma_{\varepsilon,\nu}(u) = 0$ and $\Gamma_{\varepsilon,\nu}(u) \le L$. Then*
(1) *there exist $c_{\nu,L}$, such that $|u(x)| \le c_{\nu,L}$ for $x \in \mathbb{R}^N$;*
(2) *there exist $b_{\nu,L}$, such that $\int_{\mathbb{R}^N} \frac{F_\nu(u(y))}{|x-y|^\alpha}\, dy \le b_{\nu,L}$ for $x \in \mathbb{R}^N$;*
(3) *for any $\delta > 0$ there exist $c = c(\delta, \nu, L)$ such that $|u(x)| \le c\varepsilon^3$ for $x \in \mathbb{R}^N \backslash (\mathcal{M}_\varepsilon)^\delta$.*

Proof. (1) Assume $D\Gamma_{\varepsilon,\nu}(u) = 0$ and $\Gamma_{\varepsilon,\nu}(u) \le L$, it is easy to show that u is bounded in $H^1(\mathbb{R}^N)$ and $(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2\, dx - 1)_+^\beta$ is bounded. Choose $\phi = |u_T|^{2k-2} u$ as test function in $\langle D\Gamma_{\varepsilon,\nu}(u), \phi\rangle = 0$, where $k \ge 1, T > 0$ and $u_T(x) = \pm T$ if $\pm u(x) \ge T$, $u_T(x) = u(x)$ if $|u(x)| \le T$. By $\langle D\Gamma_{\varepsilon,\nu}(u), \phi\rangle = 0$, it is easy to obtain the inequality

$$\begin{aligned} \int_{\mathbb{R}^N} (\nabla u \nabla \phi + E(\varepsilon x) u\phi)\, dx &\le c_\nu \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^r |u(x)|^{r-1}\phi(x)}{|x-y|^\alpha}\, dx\, dy \\ &\quad + \tau \int_{\mathbb{R}^N} u\phi\, dx + c_\tau \int_{\mathbb{R}^N} |u|^{\frac{2Nr}{2N-\alpha}-2} u\phi\, dx, \end{aligned} \tag{4.1}$$

where $\tau \le \inf_{x\in\mathbb{R}^N} E(x)$. Hence

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla u \nabla \phi\, dx \\ &\le c_\nu \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} \frac{|u(y)|^r}{|x-y|^\alpha}\, dy + |u|^{\frac{\alpha r}{2N-\alpha}}\Big) |u|^{r-2} u\phi\, dx \\ &\le c_\nu \Big(\int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} \frac{|u(y)|^r}{|x-y|^\alpha}\, dy + |u|^{\frac{\alpha r}{2N-\alpha}}\Big)^{2N/\alpha} dx\Big)^{\frac{\alpha}{2N}} \\ &\quad \times \Big(\int_{\mathbb{R}^N} (|u|^r |u_T|^{2k-2})^{\frac{2N}{2N-\alpha}}\, dx\Big)^{\frac{2N-\alpha}{2N}} \\ &\le c_\nu \Big(\int_{\mathbb{R}^N} (|u||u_T|^{k-1})^{\frac{2Nr}{2N-\alpha}}\, dx\Big)^{\frac{2N-\alpha}{Nr}} \Big(\int_{\mathbb{R}^N} |u|^{\frac{2Nr}{2N-\alpha}}\, dx\Big)^{\frac{(r-2)(2N-\alpha)}{2Nr}} \\ &\le c_\nu \Big(\int_{\mathbb{R}^N} (|u||u_T|^{k-1})^{\frac{2Nr}{2N-\alpha}}\, dx\Big)^{\frac{2N-\alpha}{Nr}}. \end{aligned} \tag{4.2}$$

The left-hand side of (4.1) satisfies

$$\begin{aligned} \text{LHS} &\geq \int_{\mathbb{R}^N} |\nabla u|^2 |u_T|^{2k-2}\,dx \\ &\geq \frac{c}{k^2}\int_{\mathbb{R}^N} |\nabla(|u||u_T|^{k-1})|^2\,dx \\ &\geq \frac{c}{k^2}\Big(\int_{\mathbb{R}^N} (|u||u_T|^{k-1})^{2^*}\,dx\Big)^{2/2^*}. \end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3), we have

$$\Big(\int_{\mathbb{R}^N} (|u||u_T|^{k-1})^{2^*}\,dx\Big)^{2/2^*} \leq c_\nu k^2 \Big(\int_{\mathbb{R}^N} (|u||u_T|^{k-1})^{\frac{2Nr}{2N-\alpha}}\,dx Big)^{\frac{2N-\alpha}{Nr}}. \tag{4.4}$$

Letting $T\to\infty$ in (4.4) we obtain

$$\Big(\int_{\mathbb{R}^N} |u|^{2^*k}\,dx\Big)^{2/2^*} \leq c_\nu k^2 \Big(\int_{\mathbb{R}^N} |u|^{\frac{2Nrk}{2N-\alpha}}\,dx\Big)^{\frac{2N-\alpha}{Nr}}. \tag{4.5}$$

We denote $\chi=\frac{2N-\alpha}{r(N-2)}>1$, $k_1=\chi$, by iterations, we obtain

$$\Big(\int_{\mathbb{R}^N} |u|^{2^*\chi^n}\,dx\Big)^{\frac{1}{2^*\chi^n}} \leq \big(C_\nu \chi^{2n}\big)^{\frac{1}{2\chi^n}} \Big(\int_{\mathbb{R}^N} |u|^{2^*\chi^{n-1}}\,dx\Big)^{\frac{1}{2^*\chi^{n-1}}} \quad n=1,2,\dots. \tag{4.6}$$

Hence

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq c_\nu \|u\|_{L^{2^*}(\mathbb{R}^N)} \leq c_{\nu,L}. \tag{4.7}$$

(2) In view of $1<\alpha<N-1$, for $x\in\mathbb{R}^N$ we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{F_\nu(u(y))}{|x-y|^\alpha}\,dy &\leq b_\nu\Big(\int_{|x-y|\geq 1} \frac{|u(y)|^r}{|x-y|^\alpha}\,dx + \int_{|x-y|<1} \frac{|u(y)|^r}{|x-y|^\alpha}\,dx\Big) \\ &\leq b_\nu\Big(\|u\|^r_{L^r(\mathbb{R}^N)} + \int_{|x-y|<1} \frac{1}{|x-y|^\alpha}\,dx \|u\|^r_{L^\infty(\mathbb{R}^N)}\Big) \\ &\leq b_\nu(\|u\|^r_{L^r(\mathbb{R}^N)} + \|u\|^r_{L^\infty(\mathbb{R}^N)}) \\ &\leq b_{\nu,L}. \end{aligned} \tag{4.8}$$

(3) For $y\in\mathbb{R}^N$, $0<\rho<R\leq 1$. We choose $\eta\in C_0^\infty(\mathbb{R}^N,[0,1])$ such that $\eta(x)=0$ for $x\notin B(y,R)$; $\eta(x)=1$ for $x\in B(y,\rho)$ and $|\nabla\eta|\leq\frac{c}{R-\rho}$. Setting $\varphi=u|u|^{2k-2}\eta^m$, $k\geq 1$ as test function in $\langle D\Gamma_{\varepsilon,\nu}(u),\varphi\rangle=0$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla u\nabla\varphi\,dx \\ &\leq c\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\nu(u(y))|u(x)|^{2^*_\alpha-1}|\varphi(x)|}{|x-y|^\alpha}\,dx\,dy + c\int_{\mathbb{R}^N} |u|^{q-2}u\varphi\,dx. \end{aligned} \tag{4.9}$$

The left-hand side of (4.9) satisfies

$$\begin{aligned}\mathrm{LHS} &\geq \int_{\mathbb{R}^N} |\nabla u|^2 |u|^{2k-2}\eta^2\,dx - c\int_{\mathbb{R}^N} |\nabla u||\nabla\eta||u|^{2k-1}\eta\,dx\\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2|u|^{2k-2}\eta^2\,dx - c\int_{\mathbb{R}^N}|u|^{2k}|\nabla\eta|^2\eta^2\,dx\\ &\geq \frac{c}{k^2}\int_{\mathbb{R}^N}|\nabla(|u|^k\eta)|^2\,dx - c\int_{\mathbb{R}^N}|u|^{2k}|\nabla\eta|^2\,dx\\ &\geq \frac{c}{k^2}(\int_{B(y,\rho)}|u|^{2^*k}\,dx)^{2/2^*} - \frac{c}{(R-\rho)^2}\int_{B(y,R)}|u|^{2k}\,dx,\end{aligned} \tag{4.10}$$

By (4.7) and (4.8) we obtain

$$\begin{aligned}\mathrm{RHS} &\leq c_{\nu,L}\int_{\mathbb{R}^N}|u|^{2^*_\alpha-2}|u|^{2k}\eta^2\,dx + c\int_{\mathbb{R}^N}|u|^{q-2}|u|^{2k}\eta^2\,dx\\ &\leq c_{\nu,L}\int_{B(y,R)}|u|^{2k}\,dx.\end{aligned} \tag{4.11}$$

With the above estimations, we have

$$\Big(\int_{B(y,\rho)}|u|^{2^*k}\,dx\Big)^{2/2^*} \leq \frac{c_{\nu,L}k^2}{(R-\rho)^2}\int_{B(y,R)}|u|^{2k}\,dx, \quad \text{for } k\geq 1.$$

By iteration again, we obtain

$$\|u\|_{L^\infty(B(y,\frac{R}{2}))} \leq c_{\nu,L}\|u\|_{L^2(B(y,R))}.$$

Since

$$\int_{\mathbb{R}^N\setminus(\mathcal{M}_\varepsilon)^\delta} u^2\,dx \leq c_\delta\varepsilon^6,$$

it follows that $|u(x)|\leq c_{\delta,\nu,L}\varepsilon^3$ for all $x\in\mathbb{R}^N\setminus(\mathcal{M}_\varepsilon)^\delta$. □

For ν fixed, let $\varepsilon_n\to 0$, and assume $u_n\in H^1(\mathbb{R}^N)$ is such that $D\Gamma_{\varepsilon_n,\nu}(u_n)=0$ and $\Gamma_{\varepsilon_n,\nu}(u_n)\leq L$. It is easy to show that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, hence we have the following profile decomposition [38],

$$u_n = \sum_{k\in\Lambda} U_k(\cdot - y_{n,k}) + r_n, \tag{4.12}$$

where Λ is an index set, $y_{n,k}\in\mathbb{R}^N$. Moreover,

1. $u_n(\cdot + y_{n,k}) \rightharpoonup U_k$ in $H^1(\mathbb{R}^N)$ as $n\to\infty$.
2. $|y_{n,k}-y_{n,l}|\to\infty$ as $n\to\infty$ for $k\neq l$.
3. $\|u_n\|^2_{H^1(\mathbb{R}^N)} = \sum_{k\in\Lambda}\|U_k\|^2_{H^1(\mathbb{R}^N)} + \|r_n\|^2_{H^1(\mathbb{R}^N)} + o(1)$ as $n\to\infty$.
4. $\|r_n\|_{L^s(\mathbb{R}^N)}\to 0$ as $n\to\infty$, $2<s<2^*$,
$$\|u_n\|^s_{L^s(\mathbb{R}^N)} = \sum_{k\in\Lambda}\|U_k\|^s_{L^s(\mathbb{R}^N)} + o(1)$$
as $n\to\infty$.

By Lemma 4.2 (3) we have

$$\lim_{n\to\infty}\operatorname{dist}(y_{n,k},\mathcal{M}_{\varepsilon_n}) < +\infty.$$

We denote $y_k^* = \lim_{n\to\infty}\varepsilon_n y_{n,k}$. Since $\operatorname{dist}(y_{n,k},\mathcal{M}_{\varepsilon_n}) = \varepsilon_n^{-1}\operatorname{dist}(\varepsilon_n y_{n,k},\mathcal{M})$, we have

$$\operatorname{dist}(y_k^*,\overline{\mathcal{M}})=0, \quad \text{i.e. } y_k^*\in\overline{\mathcal{M}}. \tag{4.13}$$

Moreover, we obtain the following properties of $\{u_n\}$.

Lemma 4.3. *If* $\tilde u_n = u_n(\cdot + y_n) \rightharpoonup U$ *in* $H^1(\mathbb{R}^N)$ *for* $y_n \in \mathbb{R}^N$, *and* $\lim_{n\to\infty} \varepsilon_n y_n = y^*$, *then* $Z = |U|$ *satisfies*

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \nabla Z \nabla \varphi\,dx + \int_{\mathbb{R}^N} Z\varphi\,dx \\
&\le C_\nu \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|Z(y)|^r |Z(x)|^{r-1}\varphi(x)}{|x-y|^\alpha}\,dx\,dy + c\int_{\mathbb{R}^N} Z^{q-1}\varphi\,dx,
\end{aligned}
\tag{4.14}
$$

for $\varphi \in H^1(\mathbb{R}^N)$ *and* $\varphi \ge 0$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, $R > 0$, such that $\varphi(x) = 1$ for $|x| \le R$; $\varphi(x) = 0$ for $|x| \ge 2R$ and $|\nabla\varphi| \le \frac{c}{R}$. Choosing $\varphi_n = \varphi(\cdot - y_n)$ as the test function in $\langle D\Gamma_{\varepsilon_n,\nu}(u_n), \varphi_n\rangle = 0$, we deduce

$$
\begin{aligned}
&\int_{\mathbb{R}^N} (\nabla \tilde u_n \nabla\varphi\,dx + E(\varepsilon_n(x+y_n))\tilde u_n\varphi)\,dx + \sigma\int_{R^N} k_{\varepsilon_n}(x+y_n,\tilde u_n)\varphi\,dx \\
&+ \Big(\int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x)u_n^2\,dx - 1\Big)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_{\varepsilon_n}(x+y_n)\tilde u_n\varphi\,dx \\
&= 2_\alpha^*\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F_\nu(\tilde u_n(y))f_\nu(\tilde u_n(x))\varphi(x)}{|x-y|^\alpha}\,dx\,dy + \vartheta\int_{\mathbb{R}^N}|\tilde u_n|^{q-2}\tilde u_n\varphi\,dx.
\end{aligned}
\tag{4.15}
$$

By Rellich's imbedding theorem, we have $\tilde u_n \to U$ in $L^s_{loc}(\mathbb{R}^N)(1 \le s < 2^*)$. By Lemma 2.1, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}(|\nabla(\tilde u_k - \tilde u_l)|^2\varphi\,dx \\
&= -\int_{\mathbb{R}^N}(\nabla(\tilde u_k - \tilde u_l), \nabla\varphi)(\tilde u_k - \tilde u_l)\,dx \\
&\quad - \int_{\mathbb{R}^N}(E(\varepsilon_k(x+y_k))\tilde u_k - E(\varepsilon_l(x+y_l))\tilde u_l)(\tilde u_k - \tilde u_l)\varphi\,dx \\
&\quad - \sigma\int_{\mathbb{R}^N}(k_{\varepsilon_k}(x+y_k,\tilde u_k) - k_{\varepsilon_l}(x+y_l,\tilde u_l))(\tilde u_k - \tilde u_l)\varphi\,dx \\
&\quad - \int_{\mathbb{R}^N}\big(\psi_{\varepsilon_k}(u_k)\chi_{\varepsilon_k}(x+y_k)\tilde u_k - \psi_{\varepsilon_l}(u_l)\chi_{\varepsilon_l}(x+y_l)\tilde u_l\big)(\tilde u_k - \tilde u_l)\varphi\,dx \\
&\quad + 2_\alpha^*\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F_\nu(\tilde u_k(y))f_\nu(\tilde u_k(x))(\tilde u_k(x) - \tilde u_l(x))\varphi(x)}{|x-y|^\alpha}\,dx\,dy \\
&\quad - 2_\alpha^*\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F_\nu(\tilde u_k(y))f_\nu(\tilde u_k(x))(\tilde u_k(x) - \tilde u_l(x))\varphi(x)}{|x-y|^\alpha}\,dx\,dy \\
&\quad + \vartheta\int_{\mathbb{R}^N}(|\tilde u_k|^{q-2}\tilde u_k - |\tilde u_l|^{q-2}\tilde u_l)(\tilde u_k - \tilde u_k)\varphi\,dx \\
&\le c\|\tilde u_k - \tilde u_l\|_{L^2(B(0,2R))} + c\|\tilde u_k - \tilde u_l\|_{L^m(B(0,2R))} + c\|\tilde u_k - \tilde u_l\|_{L^q(B(0,2R))} \\
&\quad + c\|\tilde u_k - \tilde u_l\|_{L^{\frac{2Nr}{2N-\alpha}}(B(0,2R))} \to 0, \quad \text{as } k,l\to\infty.
\end{aligned}
$$

Since $\varphi = 1$ in $B(0,R)$ and $\varphi \ge 0$, $\tilde u_n \to U$ in $H^1_{loc}(\mathbb{R}^N)$. Let $z_n = |\tilde u_n|$, $w_{n,\delta} = (\tilde u_n^2 + \delta^2)^{1/2} - \delta$. Then from Lebesgue's controlled convergence theorem it follows that $w_{n,\delta} \in H^1(\mathbb{R}^N)$, and $w_{n,\delta} \to z_n$ in $H^1(\mathbb{R}^N)$ as $\delta \to 0$. Now for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$\varphi \geq 0$, we have $\varphi_\delta = \varphi \tilde{u}_n (\tilde{u}_n^2 + \delta^2)^{-1/2} \in H^1_{loc}(\mathbb{R}^N)$, and

$$
\begin{aligned}
&\int_{\mathbb{R}^N} (\nabla w_{n,\delta} \nabla \varphi + E(\varepsilon_n(x+y_n)) w_{n,\delta} \varphi)\, dx \\
&= \int_{\mathbb{R}^N} (\tilde{u}_n \nabla \tilde{u}_n \nabla \varphi (\tilde{u}_n^2 + \delta^2)^{-1/2} + E(\varepsilon_n(x+y_n))((\tilde{u}_n^2+\delta^2)^{1/2} - \delta)\varphi)\, dx \\
&= \int_{\mathbb{R}^N} (\nabla \tilde{u}_n \nabla \varphi_\delta - |\nabla \tilde{u}_n|^p \varphi (\tilde{u}_n^2 + \delta^2)^{-\frac{3}{2}} \delta^2 + E(\varepsilon_n(x+y_n))((\tilde{u}_n^2+\delta^2)^{1/2} - \delta)\varphi)\, dx \\
&\leq \int_{\mathbb{R}^N} (\nabla \tilde{u}_n \nabla \varphi_\delta + E(\varepsilon_n(x+y_n)) \tilde{u}_n \varphi_\delta)\, dx \\
&\leq c \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|z_n(y)|^r |z_n(x)|^{r-1} |\varphi_\delta(x)|}{|x-y|^\alpha}\, dx\, dy + c \int_{\mathbb{R}^N} |z_n|^{q-1} |\varphi_\delta|\, dx.
\end{aligned}
$$

Letting $\delta \to 0$ in the above inequality we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} (\nabla z_n \nabla \varphi + z_n \varphi)\, dx \\
&\leq C_\nu \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|z_n(y)|^q |z_n(x)|^{q-1} \varphi(x)}{|x-y|^\alpha}\, dx\, dy + c \int_{\mathbb{R}^N} |z_n|^{q-1} \varphi\, dx.
\end{aligned} \tag{4.16}
$$

for $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$. By $\tilde{u}_n \to U$ in $H^1_{loc}(\mathbb{R}^N)$ as $n \to \infty$, we have $z_n \to Z$ in $W^{1,2}_{loc}(\mathbb{R}^N)$ as $n \to \infty$. Finally, by a density argument we complete the proof. ☐

Corollary 4.4. *Λ is a finite set.*

Proof. $Z_k = |U_k|$ satisfies (4.14) and taking $\varphi = Z_k$ in (4.14), we have

$$
\begin{aligned}
\|Z_k\|^2_{H^1(\mathbb{R}^N)} &\leq C_\nu \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|Z_k(y)|^r |Z_k(x)|^r}{|x-y|^\alpha}\, dx\, dy + c \int_{\mathbb{R}^N} |Z_k|^q\, dx \\
&\leq C_\nu \|Z_k\|^{2r}_{H^1(\mathbb{R}^N)} + c \|Z_k\|^q_{L^q(\mathbb{R}^N)}.
\end{aligned} \tag{4.17}
$$

So there exists $m > 0$ such that $\|U_k\|_{W^{1,p}(\mathbb{R}^N)} \geq m$. By property (3) of the profile decomposition (4.12), we know that Λ is a finite set. ☐

Assume that the sequence $\{u_n\}$ has the profile decomposition (4.12). By Corollary 4.4, we can assume that $\Lambda = \{1, \dots, k\}$. Meanwhile, we denote

$$
\Omega_R^{(n)} = \mathbb{R}^N \backslash \{\cup_{k\in\Lambda} B(y_{n,k}, R) \cup B(0,R)\},
$$

for the above $\{u_n\}$, we have the following statements.

Lemma 4.5. *Assume $\Gamma_{\varepsilon_n,\nu}(u_n) \leq L$, $D\Gamma_{\varepsilon_n,\nu}(u_n) = 0$, then there exist c, μ, independent of n, such that*

$$
\int_{\Omega_R^{(n)}} G_{\varepsilon_n}(x, u_n, \nabla u_n)\, dx \leq c \exp\{-\mu R\},
$$

where

$$
\begin{aligned}
&G_{\varepsilon_n}(x, u_n, \nabla u_n) \\
&= |\nabla u_n|^2 + E(\varepsilon_n x) u_n^2 + \sigma k_{\varepsilon_n}(x, u_n) u_n + \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x) u_n^2\, dx - 1 \Big)_+^{\beta-1} \chi_\varepsilon(x) u_n^2.
\end{aligned}
$$

Moreover,

$$
|u_n(x)| \leq c \exp\{-\mu R\}, \quad x \in \Omega_R^{(n)}.
$$

Proof. By the decomposition (4.12) we have

$$\|u_n\|_{L^s(\Omega_R^{(n)})} = o_R(1), \quad 2 < s < 2^*$$

where $o_R(1) \to 0$ as $R \to +\infty$. By Moser's iteration we have

$$\|u_n\|_{L^\infty(\Omega_R^{(n)})} = o_R(1). \tag{4.18}$$

Let $\eta \in C^\infty(\mathbb{R}^N)$ such that $\eta(x) = 0$ for $x \notin \Omega_R^{(n)}$, and $\eta(x) = 1$ for $x \in \Omega_{R+1}^{(n)}$, $|\nabla\eta| \le 2$. Take $\varphi_n = u_n\eta^2$ as a test function in $\langle D\Gamma_{\varepsilon_n}(u_n), \varphi\rangle = 0$, we have

$$\begin{aligned} &\int_{\Omega_R^{(n)}} (|\nabla u_n|^2 + E(\varepsilon_n x)|u_n|^2)\eta^2\,dx + \sigma \int_{\Omega_R^{(n)}} k_{\varepsilon_n}(x, u_n)u_n\eta^2\,dx \\ &+ \Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)u_n^2\,dx - 1\Big)_+^{\beta-1} \int_{\Omega_R^{(n)}} \chi_\varepsilon(x)u_n^2\eta^2\,dx \\ &= \frac{2_\alpha^*}{2}\int_{\Omega_R^{(n)}}\int_{\mathbb{R}^N} \frac{F(u_n(y))f(u_n(x))u_n(x)\eta^2(x)}{|x-y|^\alpha}\,dx\,dy + \vartheta\int_{\Omega_R^{(n)}} |u_n|^q\eta^2\,dx \\ &\quad - 2\int_{\Omega_R^{(n)}\setminus\Omega_{R+1}^{(n)}} \nabla u_n\nabla\eta u_n\eta\,dx. \end{aligned} \tag{4.19}$$

By Lemma 4.2(2) and (4.18), for n large enough, we have

$$\begin{aligned} &\frac{2_\alpha^*}{2}\int_{\Omega_R^{(n)}}\int_{\mathbb{R}^N} \frac{F(u_n(y))f(u_n(x))u_n(x)\eta^2(x)}{|x-y|^\alpha}\,dx\,dy + \vartheta\int_{\Omega_R^{(n)}} |u_n|^q\eta^2\,dx \\ &\le \frac{1}{2}\int_{\Omega_R^{(n)}} E(\varepsilon_n x)|u_n|^p\eta^p\,dx. \end{aligned} \tag{4.20}$$

Also

$$2\Big|\int_{\Omega_R^{(n)}\setminus\Omega_{R+1}^{(n)}} \nabla u_n\nabla\eta u_n\eta\,dx\Big| \le c\int_{\Omega_R^{(n)}\setminus\Omega_{R+1}^{(n)}} (|\nabla u_n|^2 + |u_n|^2)\,dx. \tag{4.21}$$

By (4.19)-(4.21), we have

$$\int_{\Omega_{R+1}^{(n)}} G_{\varepsilon_n}(x, u_n, \nabla u_n)\,dx \le c\int_{\Omega_R^{(n)}\setminus\Omega_{R+1}^{(n)}} G_{\varepsilon_n}(x, u_n, \nabla u_n)\,dx.$$

Consequently,

$$\int_{\Omega_{R+1}^{(n)}} G_{\varepsilon_n}(x, u_n, \nabla u_n)\,dx \le \theta\int_{\Omega_R^{(n)}} G_{\varepsilon_n}(x, u_n, \nabla u_n)\,dx,$$

where $\theta = \frac{c}{c+1} < 1$. Finally

$$\int_{\Omega_R^{(n)}} G_{\varepsilon_n}(x, u_n, \nabla u_n)\,dx \le c\exp\{-\mu R\},$$

where $\mu = -\ln\theta > 0$. And by Moser's iteration, we have

$$|u_n(x)| \le c\exp\{-\mu R\}, \ x \in \Omega_R^{(n)}. \qquad \square$$

Lemma 4.6. *For every $k \in \Lambda$ it holds $k^* = \lim_{n\to\infty} \varepsilon_n y_n^k \in \bar{\mathcal{A}}$.*

Proof. If the lemma does not hold, we assume that there exist $k \in \Lambda$ and $\varepsilon_n > 0$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $\operatorname{dist}(y_k^*, \mathcal{A}) > 0$. Let $t_k = \nabla V(y_k^*) \neq 0$, by (A2) we deduce that there exists $\delta_1 > 0$ such that

$$(t_k, \nabla V(x)) \geq \frac{1}{2}|t_k|^2 > 0, \quad (t_k, \nabla \operatorname{dist}(x, \mathcal{M})) \geq 0 \quad \text{for } x \in B_{\delta_1}(y_k^*). \tag{4.22}$$

Let

$$\delta_2 = \min\{|y_k^* - y_l^*| \Big| y_k^* \neq y_l^*, k, l = 1, 2, \dots, k\}, \quad 0 < \delta < \min\{\frac{1}{2}\delta_1, \frac{1}{100}\delta_2\}.$$

From $\langle D\Gamma_{\varepsilon_n,\nu}(u_n), \varphi\rangle = 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi + E(\varepsilon_n x) u_n \varphi)\, dx + \sigma \int_{\mathbb{R}^N} k_{\varepsilon_n}(x, u_n)\varphi\, dx \\
&+ \Big(\int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x) u_n^2\, dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x) u_n \varphi\, dx \\
&= 2_\alpha^* \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_\nu(u_n(y)) f_\nu(u_n(x))\varphi(x)}{|x-y|^\alpha}\, dx\, dy + \vartheta \int_{\mathbb{R}^N} |u_n|^{q-2} u_n \varphi\, dx,
\end{aligned} \tag{4.23}$$

for all $\varphi \in H^1(\mathbb{R}^N)$. Let $\eta \in C_0^\infty(\mathbb{R}^N)$ be such that $\eta(x) = 0$ if $|x - y_{n,k}| \geq 2\delta\varepsilon_n^{-1}$; $\eta(x) = 1$ if $|x - y_{n,k}| \leq \delta\varepsilon_n^{-1}$ and $|\nabla\eta| \leq \frac{2}{\delta}\varepsilon_n (\leq 1)$. Choosing $\varphi = (t_k, \nabla u_n)\eta$ as test function in (4.23), we obtain the local Pohožaev identity

$$\begin{aligned}
&\frac{\varepsilon_n}{2}\int_{\mathbb{R}^N} (t_k, \nabla E(\varepsilon_n x)) u_n^2 \eta\, dx + \sigma \int_{\mathbb{R}^N} (t_k, \nabla_x K_{\varepsilon_n}(x, u_n))\eta\, dx \\
&+ \frac{1}{2}\Big(\int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x) u_n^2\, dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} (\nabla \chi_{\varepsilon_n}(x), t_k) u_n^2 \eta\, dx \\
&= \int_{\mathbb{R}^N} (\nabla u_n, \nabla\eta)(t_k, \nabla\eta)\, dx - \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u_n|^2 + E(\varepsilon_n x)) u_n^2)(t_k, \nabla\eta)\, dx \\
&\quad - \frac{1}{2}\Big(\int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x) u_n^2\, dx - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_{\varepsilon_n}(x) u_n^2 (t_k, \nabla\eta)\, dx \\
&\quad - \alpha 2_\alpha^* \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (t_k, x-y)\frac{F_\nu(u_n(y)) F_\nu(u_n(x))\eta(x)}{|x-y|^{\alpha+2}}\, dx\, dy \\
&\quad + 2_\alpha^* \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (t_k, \nabla\eta)\frac{F_\nu(u_n(y)) F_\nu(u_n(x))}{|x-y|^\alpha}\, dx\, dy \\
&\quad + \sigma \int_{\mathbb{R}^N} K_{\varepsilon_n}(x, u_n)(t_k, \nabla\eta)\, dx + \frac{\vartheta}{q}\int_{\mathbb{R}^N} |u_n|^q (t_k, \nabla\eta)\, dx.
\end{aligned} \tag{4.24}$$

We denote

$$\begin{gathered}
B_n = B(y_{n,k}, 2\delta\varepsilon_n^{-1}), \quad T_n = B(y_{n,k}, 2\delta\varepsilon_n^{-1}) \backslash B(y_{n,k}, \delta\varepsilon_n^{-1}), \\
\tilde{T_n} = B(y_{n,k}, 3\delta\varepsilon_n^{-1}) \backslash B(y_{n,k}, \delta\varepsilon_n^{-1}).
\end{gathered}$$

Next, we estimate equation (4.24). By (4.22), we have

$$\begin{gathered}
\varepsilon_n \int_{B_n} (t_k, \nabla E(\varepsilon_n x)) u_n^2 \eta\, dx \geq c\varepsilon_n, \\
(t_k, \nabla_x K_{\varepsilon_n}(x, u_n)) = c(t_k, \nabla \operatorname{dist}(\varepsilon_n x, \mathcal{M})) \geq 0, \ \forall x \in B_n, \\
\Big(\int_{R^N} \chi_{\varepsilon_n}(x) u_n^2\, dx - 1\Big)_+^{\beta-1} \int_{B_n} (\nabla \chi_{\varepsilon_n}(x), t_k) u_n^2 \eta\, dx \geq 0.
\end{gathered}$$

Hence the left-hand side of (4.24) satisfies

$$\text{LHS} \geq c\varepsilon_n. \tag{4.25}$$

We estimate the right-hand side of (4.24). Since

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (t_k, x-y)\frac{F_\nu(u_n(y))F_\nu(u_n(x))\eta(x)\eta(y)}{|x-y|^{\alpha+2}}\,dx\,dy = 0,$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (t_k, x-y)\frac{F_\nu(u_n(y))F_\nu(u_n(x))\eta(x)}{|x-y|^{\alpha+2}}\,dx\,dy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (t_k, x-y)\frac{F_\nu(u_n(y))F_\nu(u_n(x))(1-\eta(y))}{|x-y|^{\alpha+2}}\,dx\,dy \\
&\leq c\int_{B_n}\int_{(\mathbb{R}^N\setminus B_n)\cup T_n} \frac{|u_n(y)|^r|u_n(x)|^r}{|x-y|^{\alpha+1}}\,dx\,dy \\
&\leq c\int_{B_n}\int_{\tilde{T_n}} \frac{|u_n(y)|^r|u_n(x)|^r}{|x-y|^{\alpha+1}}\,dx\,dy + c\int_{B_n}\int_{\mathbb{R}^N\setminus(\tilde{T_n}\cup B_n)} \frac{|u_n(y)|^r|u_n(x)|^r}{|x-y|^{\alpha+1}}\,dx\,dy \\
&=: \text{I} + \text{II}.
\end{aligned}$$

Then

$$\text{II} \leq c\int_{B_n}\int_{\mathbb{R}^N\setminus(\tilde{T_n}\cup B_n)} |u_n(y)|^r|u_n(x)|^r\frac{1}{\delta^{\alpha+1}}\varepsilon_n^{\alpha+1}\,dx\,dy \leq c\varepsilon_n^{\alpha+1}\,.$$

By Lemma 4.5, we have

$$|u_n(y)| \leq c\exp\{-\mu\delta\varepsilon_n^{-1}\}, \quad \forall y \in \tilde{T_n}.$$

Consequently, for n large enough, we have

$$\begin{aligned}
\text{I} &\leq c\exp\{-r\mu\delta\varepsilon_n^{-1}\}\int_{B_n}\int_{\tilde{T_n}} \frac{|u_n(x)|^r}{|x-y|^{\alpha+1}}\,dx\,dy \\
&\leq c\exp\{-r\mu\delta\varepsilon_n^{-1}\}\int_{B_n}|u_n(x)|^r\,dx\int_{|x-y|\leq 5\delta\varepsilon_n^{-1}} \frac{1}{|x-y|^{\alpha+1}}\,dy \\
&\leq c\exp\{-r\mu\delta\varepsilon_n^{-1}\}\varepsilon_n^{-N+\alpha+1} \leq c\varepsilon_n^{\alpha+1}\,.
\end{aligned}$$

By Lemma 4.5, for n large enough, the right hand side of (4.24) satisfies

$$\begin{aligned}
\text{RHS} &\leq c\int_{T_n} G_{\varepsilon_n}(x, u_n, \nabla u_n)\,dx + c\exp\{-\frac{\mu\delta}{2}\varepsilon_n^{-1}\} + c\exp\{-q\mu\delta\varepsilon_n^{-1}\}\varepsilon_n^{-N} + c\varepsilon_n^{\alpha+1} \\
&\leq c\exp\{-\mu\delta\varepsilon_n^{-1}\} + c\exp\{-\frac{\mu\delta}{2}\varepsilon_n^{-1}\} + c\exp\{-q\mu\delta\varepsilon_n^{-1}\}\varepsilon_n^{-N} + c\varepsilon_n^{\alpha+1} \leq c\varepsilon_n^{\alpha+1}.
\end{aligned}$$

Hence

$$\varepsilon_n \leq c\varepsilon_n^{\alpha+1}.$$

Since $0 < \alpha < \min\{N-1, 4\}$, we arrive at a contradiction as $n \to \infty$ and complete the proof. □

Proof of Theorem 4.1 part 2. By Lemma 4.5,

$$|u_n(x)| \leq c\exp\{-\mu R\} \quad \text{for } x \in \Omega_R^{(n)}.$$

Let $R_n(x) = \min\{|x - y_{n,k}|\Big| k \in \Lambda\}$, then

$$|u_n(x)| \leq c\exp\{-\mu R_n(x)\} \quad \text{for } x \in \Omega_{R_n}^{(n)}.$$

Since $\varepsilon_n y_{n,k} \to y_k^* \in \mathcal{A}$, for any $\delta > 0$, there exists $\varepsilon(\delta)$ such that for $\varepsilon_n \le \varepsilon(\delta)$, $\varepsilon_n y_{n,k} \in \mathcal{A}^\delta$, hence

$$|u_n(x)| \le c\exp\{-\mu R_n\} \le c\exp\{-\mu\,\mathrm{dist}(x,(\mathcal{A}^\delta)_{\varepsilon_n})\}, \quad x \in \mathbb{R}^N. \qquad \square$$

In the following , we assume $u_n \in H^1(\mathbb{R}^N)$, $L > 0$, $I_{\varepsilon_n,\nu_n}(u_n) \le L$, $DI_{\varepsilon_n,\nu_n}(u_n) = 0$, $\nu_n \to 0$,and $\varepsilon_n \to \varepsilon^* \in (0,1)$. The case $\nu_n \to \nu^* \in (0,1)$ is easier, since in that case we need only to deal with subcritical problems. It is easy to show that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, we have the following profile decomposition [38].

$$u_n = \sum_{k\in\Lambda_1} U_k(\cdot - y_{n,k}) + \sum_{k\in\Lambda_\infty} \sigma_{n,k}^{\frac{N-2}{2}} U_k(\sigma_{n,k}(\cdot - y_{n,k})) + r_n, \tag{4.26}$$

where $y_{n,k} \in \mathbb{R}^N$, $\sigma_{n,k} \in \mathbb{R}^+$, Λ is an index set, $U_k \in H^1(\mathbb{R}^N)$ for $k \in \Lambda_1$, $U_k \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ for $k \in \Lambda_\infty$, $r_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ satisfying

(1) For $k \in \Lambda_1$, $u_n(\cdot + y_{n,k}) \rightharpoonup U_k$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. For $k \in \Lambda_\infty$, $\sigma_{n,k}^{-\frac{N-2}{2}} u_n(\sigma_{n,k}^{-1}(\cdot + y_{n,k})) \rightharpoonup U_k$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ as $n \to \infty$.
(2) For $k \in \Lambda_1$, $k \ne l$, $|y_{n,k} - y_{n,l}| \to \infty$ as $n \to \infty$. For $k \in \Lambda_\infty$, $k \ne l$, $\frac{\sigma_{n,k}}{\sigma_{n,l}} + \frac{\sigma_{n,l}}{\sigma_{n,k}} + \sigma_{n,k}\sigma_{n,l}|y_{n,k} - y_{n,l}|^2 \to \infty$ as $n \to \infty$.
(3) $\|u_n\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \sum_{k\in\Lambda} \|U_k\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} + \|r_n\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} + o(1)$ as $n \to \infty$, $\Lambda = \Lambda_1 \cup \Lambda\infty$. $\|u_n\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} = \sum_{k\in\Lambda} \|U_k\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} + o(1)$ as $n \to \infty$, $\Lambda = \Lambda_1 \cup \Lambda\infty$.
(4) $r_n = u_n - \sum_{k\in\Lambda_1} U_k(\cdot - y_{n,k}) - \sum_{k\in\Lambda_\infty} \sigma_{n,k}^{\frac{N-2}{2}} U_k(\sigma_{n,k}(\cdot - y_{n,k})) \to 0$ in $L^{2^*}(\mathbb{R}^N)$ as $n \to \infty$.

Similar to Lemma 4.3 and Corollary 4.4, we have the following lemma.

Lemma 4.7. *(1) Assume $y_n \in \mathbb{R}^N$ and set $\tilde{u}_n = u_n(\cdot + y_n) \rightharpoonup U$ in $H^1(\mathbb{R}^N)$. Then $Z = |U|$ satisfies*

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla Z \nabla \varphi\,dx + \int_{\mathbb{R}^N} Z\varphi\,dx \\ &\le c\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|Z(y)|^{2_\alpha^*}|Z(x)|^{2_\alpha^*-1}\varphi(x)}{|x-y|^\alpha}\,dx\,dy + c\int_{\mathbb{R}^N} Z^{q-1}\varphi\,dx, \end{aligned} \tag{4.27}$$

for $\varphi \in H^1(\mathbb{R}^N)$, $\varphi \ge 0$.

(2) The index sets Λ_1, Λ_∞ in the profile decomposition (4.26) *are infinite.*

Lemma 4.8. *Assume $y_n \in \mathbb{R}^N$, $\sigma_n \to \infty$. Set $\tilde{u}_n = \sigma_n^{-\frac{N-2}{2}} u_n(\sigma_n^{-1} \cdot + y_n) \rightharpoonup U$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Then $Z = |U|$ satisfies*

$$\int_{\mathbb{R}^N} \nabla Z \nabla \varphi\,dx \le c\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|Z(y)|^{2_\alpha^*}|Z(x)|^{2_\alpha^*-1}\varphi(x)}{|x-y|^\alpha}\,dx\,dy, \tag{4.28}$$

for $\varphi \in H^1(\mathbb{R}^N)$, $\varphi \ge 0$.

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, $R > 0$, such that $\varphi(x) = 1$ for $|x| \le R$; $\varphi(x) = 0$ for $|x| \ge 2R$, and $|\nabla\varphi| \le \frac{c}{R}$. Select $\varphi_n = \sigma_n^{\frac{N-2}{2}} \varphi(\sigma_n(\cdot - y_n))$ as the test function in

$\langle DI_{\varepsilon_n,\nu_n}(u_n),\varphi_n\rangle=0$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N}(\nabla\tilde u_n\nabla\varphi\,dx+\sigma_n^{-2}V(\varepsilon_n(x+y_n))\tilde u_n\varphi)\,dx\\ &=2_\alpha^*\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\tilde F_{\nu_n}(\tilde u_n(y))\tilde f_{\nu_n}(\tilde u_n(x))\varphi(x)}{|x-y|^\alpha}\,dx\,dy\\ &\quad+\vartheta\sigma_n^{q\frac{N-2}{2}-N}\int_{\mathbb{R}^N}|\tilde u_n|^{q-2}\tilde u_n\varphi\,dx, \end{aligned}\tag{4.29}$$

where $\tilde F_{\nu_n}(t)=\sigma_n^{\frac{\alpha}{2}-N}F_{\nu_n}(\sigma_n^{\frac{N-2}{N}}t)$, $\tilde f_{\nu_n}(t)=\sigma_n^{\frac{\alpha-N}{2}-1}f_{\nu_n}(\sigma_n^{\frac{N-2}{N}}t)$. By Rellich's imbedding theorem, we have $\tilde u_n\to U$ in $L^s_{loc}(\mathbb{R}^N)(1\le s<2^*)$. By (4.29) and Lemma 2.1, we have

$$\begin{aligned} &\int_{\mathbb{R}^N}(|\nabla(\tilde u_k-\tilde u_l)|^2\varphi\,dx\\ &=-\int_{\mathbb{R}^N}(\nabla(\tilde u_k-\tilde u_l),\nabla\varphi)(\tilde u_k-\tilde u_l)\,dx\\ &\quad-\int_{\mathbb{R}^N}(V(\varepsilon_k(x+y_k))\tilde u_k-V(\varepsilon_l(x+y_l))\tilde u_l)(\tilde u_k-\tilde u_l)\varphi\,dx\\ &\quad+2_\alpha^*\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F_{\nu_n}(\tilde u_k(y))f_{\nu_n}(\tilde u_k(x))(\tilde u_k(x)-\tilde u_l(x))\varphi(x)}{|x-y|^\alpha}\,dx\,dy\\ &\quad-2_\alpha^*\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F_{\nu_n}(\tilde u_k(y))f_{\nu_n}(\tilde u_k(x))(\tilde u_k(x)-\tilde u_l(x))\varphi(x)}{|x-y|^\alpha}\,dx\,dy\\ &\quad+\vartheta\int_{\mathbb{R}^N}(|\tilde u_k|^{q-2}\tilde u_k-|\tilde u_l|^{q-2}\tilde u_l)(\tilde u_k-\tilde u_k)\varphi\,dx\\ &\le c\|\tilde u_k-\tilde u_l\|_{L^2(B(0,2R))}+c\|\tilde u_k-\tilde u_l\|_{L^q(B(0,2R))}+c\|\tilde u_k-\tilde u_l\|_{L^{\frac{2Nr}{2N-\alpha}}(B(0,2R))}\\ &\to0,\quad\text{as }k,l\to\infty. \end{aligned}$$

Since $\varphi=1$ in $B(0,R)$ and $\varphi\ge0$, $\tilde u_n\to U$ in $\mathcal{D}^{1,2}_{loc}(\mathbb{R}^N)$. Let $\sigma_n\to\infty$ in (4.29), we have

$$\int_{\mathbb{R}^N}\nabla\tilde u_n\nabla\varphi\,dx=2_\alpha^*\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\tilde F_{\nu_n}(\tilde u_n(y))\tilde f_{\nu_n}(\tilde u_n(x))\varphi(x)}{|x-y|^\alpha}\,dx\,dy\tag{4.30}$$

Let $z_n=|\tilde u_n|$, $w_{n,\delta}=(\tilde u_n^2+\delta^2)^{1/2}-\delta$, then it follows from Lebesgue's controlled convergence theorem that $w_{n,\delta}\in\mathcal{D}^{1,2}_{loc}(\mathbb{R}^N)$, and $w_{n,\delta}\to z_n$ in $\mathcal{D}^{1,2}_{loc}(\mathbb{R}^N)$ as $\delta\to0$. Now for any $\varphi\in C_0^\infty(\mathbb{R}^N)$, $\varphi\ge0$, we have $\varphi_\delta=\varphi\tilde u_n(\tilde u_n^2+\delta^2)^{-1/2}\in\mathcal{D}^{1,2}_{loc}(\mathbb{R}^N)$, and

$$\begin{aligned} \int_{\mathbb{R}^N}\nabla w_{n,\delta}\nabla\varphi\,dx&=\int_{\mathbb{R}^N}\tilde u_n\nabla\tilde u_n\nabla\varphi(\tilde u_n^2+\delta^2)^{-1/2}\,dx\\ &=\int_{\mathbb{R}^N}\big(\nabla\tilde u_n\nabla\varphi_\delta-|\nabla\tilde u_n|^p\varphi(\tilde u_n^2+\delta^2)^{-\frac32}\delta^2\big)\,dx\\ &\le\int_{\mathbb{R}^N}\nabla\tilde u_n\nabla\varphi_\delta\,dx\\ &\le c\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|z_n(y)|^{2_\alpha^*}|z_n(x)|^{2_\alpha^*-1}|\varphi_\delta(x)|}{|x-y|^\alpha}\,dx\,dy. \end{aligned}\tag{4.31}$$

Letting $\delta \to 0$ in the above inequality, we obtain

$$\int_{\mathbb{R}^N} \nabla z_n \nabla \varphi \, dx \le c \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|z_n(y)|^{2^*_\alpha} |z_n(x)|^{2^*_\alpha - 1} \varphi(x)}{|x-y|^\alpha} \, dx \, dy. \tag{4.32}$$

for $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \ge 0$. By $\tilde{u}_n \to U$ in $\mathcal{D}^{1,2}_{loc}(\mathbb{R}^N)$ as $n \to \infty$, we have $z_n \to Z$ in $\mathcal{D}^{1,2}_{loc}(\mathbb{R}^N)$ as $n \to \infty$. Hence by a density argument we complete the proof. □

Lemma 4.9 ([49])**.** *Let $w \ge 0$, $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a solution of $-\Delta w \le av$, where $a \in L^{N/2}(\mathbb{R}^N)$, $v \ge 0$, $v \in L^{2^*}(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$, $s > \frac{2^*}{2}$. Then*

$$\|w\|_{L^s(\mathbb{R}^N)} \le c \|a\|_{L^{N/2}(\mathbb{R}^N)} \|v\|_{L^s(\mathbb{R}^N)}.$$

Lemma 4.10. *Let $w \ge 0$, $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a solution of $-\Delta w \le a w^{2^*_\alpha - 1}$, where $a(x) = \int_{\mathbb{R}^N} \frac{w^{2^*_\alpha}(y)}{|x-y|^\alpha} \, dy + w^{2^* - 2^*_\alpha}(x)$, $a \in L^{2N/\alpha}(\mathbb{R}^N)$, and $\|a\|_{L^{2N/\alpha}(\mathbb{R}^N)} \le d_1$. Assume $\int_{B(y,2\rho)} w^{2^*} \, dx \le d_2 := (\frac{S}{2^* d_1})^{\frac{2^*\kappa}{2^* - 2\kappa}}$, ll $y \in \mathbb{R}^N$ and $0 < \rho < 1$, where $\kappa = \frac{2N}{2N-\alpha}$, S is optimal constant of the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.*

$$S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*}},$$

then there exist constants c, c^*, depending on $\|w\|_{L^{2^*}(\mathbb{R}^N)}$ and ρ such that

$$\begin{gathered} \|w\|_{L^\infty(B(y,\frac{\rho}{2}))} \le c \|w\|_{L^{2^*}(B(y,\rho))} \le c^*, \\ \|w\|_{L^\infty(B(y,\frac{\rho}{2}))} \le c \|w\|_{L^\tau(B(y,\rho))}, \quad \forall \tau \in (0, 2^*]. \end{gathered}$$

Proof. For every $\varphi \in H^1(\mathbb{R}^N)$ with $\varphi \ge 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla w \nabla \varphi \, dx &\le \|a\|_{L^{2N/\alpha}(\mathbb{R}^N)} \Big(\int_{\mathbb{R}^N} w^{2^* - \kappa} \varphi^\kappa \, dx \Big)^{1/\kappa} \\ &\le d_1 \Big(\int_{\mathbb{R}^N} w^{2^* - \kappa} \varphi^\kappa \, dx \Big)^{1/\kappa}. \end{aligned} \tag{4.33}$$

Let $p > 1$, $\psi \in C_0^\infty(\mathbb{R}^N)$, $\psi(x) = 1$ if $x \in B(y,\rho)$; $\psi(x) = 0$ if $x \notin B(y, 2\rho)$, and $|\nabla \psi| \le \frac{2}{\rho}$. By choosing the test function $\varphi = w^{2p-1}\psi^2$ in (4.33), we deduce that

$$\int_{\mathbb{R}^N} |\nabla (w^p \psi)|^2 \, dx \le p d_1 \Big(\int_{\mathbb{R}^N} w^{2^* - 2\kappa} (w^{2p} \psi^2)^\kappa \, dx \Big)^{1/\kappa} + \int_{\mathbb{R}^N} w^{2p} |\nabla \psi|^2 \, dx. \tag{4.34}$$

Taking $p = 1 + \delta$, and $\delta > 0$ small, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla (w^{1+\delta} \psi)|^2 \, dx \\ &\le p d_1 \Big(\int_{\mathbb{R}^N} w^{2^* - 2\kappa} (w^{1+\delta} \psi)^{2\kappa} \, dx \Big)^{1/\kappa} + \int_{\mathbb{R}^N} w^{2(1+\delta)} |\nabla \psi|^2 \, dx. \end{aligned} \tag{4.35}$$

The left-hand side of (4.35) satisfies

$$\text{LHS} \ge S \Big(\int_{\mathbb{R}^N} (w^{1+\delta} \psi)^{2^*} \, dx \Big)^{2/2^*}, \tag{4.36}$$

where S is the Sobolev constant. The first term of right-hand side of (4.35),

$$\begin{aligned} & pd_1\Big(\int_{\mathbb{R}^N} w^{2^*-2\kappa}(u^{1+\delta}\psi)^{2\kappa}\,dx\Big)^{1/\kappa} \\ & \le pd_1\Big(\int_{B(y,2\rho)} w^{2^*}\,dx\Big)^{\frac{2^*-2\kappa}{2^*\cdot\kappa}}\Big(\int_{\mathbb{R}^N}(w^{1+\delta}\psi)^{2^*}\,dx\Big)^{2/2^*}. \end{aligned} \tag{4.37}$$

From (4.35)-(4.37), we have

$$\begin{aligned} & S\Big(\int_{\mathbb{R}^N}(w^{1+\delta}\psi)^{2^*}\,dx\Big)^{2/2^*} \\ & \le pd_1\Big(\int_{B(y,2\rho)} w^{2^*}\,dx\Big)^{\frac{2^*-2r}{2^*r}}\Big(\int_{\mathbb{R}^N}(w^{1+\delta}\psi)^{2^*}\,dx\Big)^{2/2^*} + \int_{\mathbb{R}^N} w^{2(1+\delta)}|\nabla\psi|^2\,dx. \end{aligned} \tag{4.38}$$

Taking $1+\delta = 2^*/2$, $q=(1+\delta)2^* > 2^*$, and denoting $\int_{B(y,2\rho)} w^{2^*}\,dx \le d_2$, we have

$$\frac{S}{2}\Big(\int_{B(y,\rho)} w^q\,dx\Big)^{2/2^*} \le \int_{\mathbb{R}^N} w^{2^*}|\nabla\psi|^2\,dx \le \frac{4}{\rho^2}\int_{B(y,2\rho)} w^{2^*}\,dx \le \frac{4d_2}{\rho^2}.$$

hence

$$\int_{B(y,\rho)} w^q\,dx \le d_3 := \big(\frac{8d_2}{S\rho^2}\big)^{\frac{2^*}{2}}.$$

Let $0<r<R<\rho<1$, $\psi\in C_0^\infty(\mathbb{R}^N)$, $\psi(x)=1$ if $x\in B(y,r)$; $\psi(x)=0$ if $x\notin B(y,R)$, and $|\nabla\psi|\le\frac{2}{R-r}$. Then the left-hand side of (4.34) satisfies

$$\text{LHS} \ge S\Big(\int_{\mathbb{R}^N}(w^p\psi)^{2^*}\,dx\Big)^{2/2^*} \ge S\Big(\int_{B(y,r)}(w^{2^*\cdot p}\,dx\Big)^{2/2^*}. \tag{4.39}$$

The right-hand side of (4.34) satisfies

$$\begin{aligned} \text{RHS} & \le \Big(\int_{B(y,\rho)} w^q\,dx\Big)^{\frac{2^*-2\kappa}{q\kappa}}\Big(\int_{\mathbb{R}^N}(w^{2p}\psi^2)^{\frac{q\kappa}{q-2^*+2\kappa}}\,dx\Big)^{\frac{q-2^*+2\kappa}{q\kappa}} \\ & \quad + \frac{c}{(R-r)^2}\Big(\int_{B(y,R)} w^{\frac{2^*\cdot p}{d}}\,dx\Big)^{2d/2^*} \\ & \le \big(c+\frac{c}{(R-r)^2}\big)\Big(\int_{B(y,R)} w^{\frac{2^*\cdot p}{d}}\,dx\Big)^{2d/2^*}, \end{aligned} \tag{4.40}$$

where $d=\frac{2^*(q-2^*+2\kappa)}{2q\kappa}$. By (4.34), (4.39) and (4.40), we have

$$\Big(\int_{B(y,r)} w^{2^*\cdot p}\,dx\Big)^{\frac{1}{2^*\cdot p}} \le \big(\frac{c}{R-r}\big)^{\frac1p}\Big(\int_{B(y,R)} w^{\frac{2^*\cdot p}{d}}\,dx\Big)^{\frac{d}{2^*\cdot p}}, \tag{4.41}$$

Taking $p=p_k=d^k$, $R=r_k=\frac{\rho}{2}+\frac{\rho}{2^k}$, $r=r_{k+1}$, $k=1,2,\dots$. By (4.41), we have

$$\Big(\int_{B(y,r_{k+1})} w^{2^*d^k}\,dx\Big)^{\frac{1}{2^*d^k}} \le \big(\frac{c2^{k+1}}{\rho}\big)^{1/d^k}\Big(\int_{B(y,r_k)} w^{2^*d^{k-1}}\,dx\Big)^{\frac{1}{2^*d^{k-1}}},$$

Letting $k\to\infty$, we obtain

$$\|w\|_{L^\infty(B(y,\frac{\rho}{2}))} \le c\|w\|_{L^{2^*}(B(y,\rho))} \le c^*.$$

Taking $p = p_k = d^k$, $r_k = r + \frac{R-r}{2^{k-1}}$, $k = 1, 2, \dots$. By (4.41), we obtain

$$\Big(\int_{B(y,r_{k+1})} w^{2^* d^k}\,dx\Big)^{\frac{1}{2^* d^k}} \le \big(\frac{c2^k}{R-r}\big)^{1/d^k}\Big(\int_{B(y,r_k)} w^{2^* d^{k-1}}\,dx\Big)^{\frac{1}{2^* d^{k-1}}},$$

Letting $k \to \infty$, we have

$$\|w\|_{L^\infty(B(y,r))} \le c\big(\frac{1}{R-r}\big)^{\frac{1}{d-1}}\|w\|_{L^{2^*}(B(y,R))}.$$

hence

$$\begin{aligned} \|w\|_{L^\infty(B(y,r))} &\le c\big(\frac{1}{R-r}\big)^{\frac{1}{d-1}}\|w\|_{L^{2^*}(B(y,R))} \\ &\le c\big(\frac{1}{R-r}\big)^{\frac{1}{d-1}}\|w\|^{\frac{2^*-\tau}{2^*}}_{L^\infty(B(y,R))}\|w\|^{\frac{\tau}{2^*}}_{L^\tau(B(y,R))} \\ &\le \frac{1}{2}\|w\|_{L^\infty(B(y,R))} + c\big(\frac{1}{R-r}\big)^{\frac{2^*}{\tau(d-1)}}\|w\|_{L^\tau(B(y,R))}. \end{aligned} \tag{4.42}$$

By iteration, we have

$$\|w\|_{L^\infty(B(y,\frac{\rho}{2}))} \le c\|w\|_{L^\tau(B(y,\rho))}. \qquad \square$$

from the above lemma, we have the following Lemma.

Lemma 4.11. *Let $w \ge 0$, $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a solution of*

$$-\Delta w \le a w^{2^*_\alpha - 1},$$

where $a(x) = \int_{\mathbb{R}^N} \frac{w^{2^*_\alpha}(y)}{|x-y|^\alpha}\,dy + w^{2^*-2^*_\alpha}(x)$, and $a \in L^{2N/\alpha}(\mathbb{R}^N)$. Then there exist $R > 0$ and a constant c depending on $\|w\|_{L^{2^*}(\mathbb{R}^N)}$, such that

$$|w(x)| \le c\Big(\int_{|x|\ge\frac{R}{2}} w^{2^*}\,dx\Big)^{1/2^*}, \quad x \in \mathbb{R}^N,\ |x| \ge R.$$

Lemma 4.12. *There exist positive constants c, μ such that*

$$\begin{gathered} |U_k(x)| \le c(1+|x|^2)^{\frac{2-N}{2}} \quad \text{for } k \in \Lambda_\infty, \\ |U_k(x)| \le c\exp\{-\mu|x|\} \quad \text{for } k \in \Lambda_1. \end{gathered}$$

Proof. Let $w \ge 0$, $w \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a solution of

$$-\Delta w \le a w^{2^*_\alpha - 1},$$

where $a \in L^{2N/\alpha}(\mathbb{R}^N)$. Using the Kelvin transformation $v(x) = |x|^{2-N} w(\frac{x}{|x|^2})$, we know that v satisfies

$$-\Delta v \le \tilde{a} v^{2^*_\alpha - 1}, \quad \forall |x| \le 1,$$

where $\tilde{a}(x) = |x|^{\alpha-N} a(\frac{x}{|x|^2})$. We also have $\int_{B_\rho(0)} |v|^{2^*}\,dx = o_\rho(1)$, and

$$\|\tilde{a}\|_{L^{2N/\alpha}(\mathbb{R}^N)} = \|a\|_{L^{2N/\alpha}(\mathbb{R}^N)} \le b.$$

Choose ρ small enough, By Lemma 4.10, we obtain $|v(x)| \le c$ in $B_{\frac{\rho}{2}(0)}$. This shows that

$$|w(x)| \le \frac{c}{|x|^{N-2}}, \quad |x| \ge 2\rho.$$

Similarly, we obtain that $|w(x)| \le c$, $|x| \le 2\rho$. So we have

$$w(x) \le c(1+|x|^2)^{\frac{2-N}{2}}.$$

We use the hole-filling technique. For

$$-\Delta w + w \le a w^{2_\alpha^*-1},$$

we also have

$$w(x) \le c(1+|x|^2)^{\frac{2-N}{2}}.$$

So there exists $R_0 > 0$ such that

$$w(x)^{2^*-2} \le \frac{1}{2}, \quad \text{for } x \in \mathbb{R}^N, \quad |x| \ge R_0.$$

Without loss of generality, we assume that

$$-\Delta w + w \le 0, \quad \text{for } x \in \mathbb{R}^N, \quad |x| \ge R_0.$$

For $R > R_0$, we choose $\psi \in C^\infty(\mathbb{R}^N)$ such that $\psi(x) = 1$, $|x| \ge R+1$; $\psi(x) = 0$, $|x| \le R$, $|\nabla\psi| \le 2$. Taking $\varphi = w\psi^2$ as test function we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w|^2\psi^2\,dx + \int_{\mathbb{R}^N} w^2\psi^2\,dx &\le 2\int_{\mathbb{R}^N} \nabla w\psi w\nabla\psi\,dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2\psi^2\,dx + c\int_{\mathbb{R}^N} w^2|\nabla\psi|^2\,dx. \end{aligned} \tag{4.43}$$

Therefore,

$$\int_{|x|\ge R+1} w^2\,dx \le \int_{\mathbb{R}^N} w^2\psi^2\,dx \le c\int_{\mathbb{R}^N} w^2|\nabla\psi|^2\,dx \le \int_{R\le|x|\le R+1} w^2\,dx,$$

$$\int_{|x|\ge R+1} w^2\,dx \le \frac{c}{c+1}\int_{|x|\ge R} w^2\,dx.$$

It follows that there exist c, μ such that

$$\int_{|x|\ge \frac{R}{2}} w^2\,dx \le c\exp\{-\mu R\}.$$

By Lemma 4.11, there exist c', μ' such that

$$w(x) \le c\Big(\int_{|x|\ge\frac{R}{2}} w^2\,dx\Big)^{1/2^*} \le c'\exp\{-\mu' R\}, \ x \in \mathbb{R}^N, \quad |x| \ge R.$$

□

Suppose $1 \le p_2 < 2^* < p_1$, $\sigma \ge 1$ and $m > 0$. Consider the system of inequalities

$$\begin{gathered} \|u_1\|_{L^{p_1}(\mathbb{R}^N)} \le m, \\ \|u_2\|_{L^{p_2}(\mathbb{R}^N)} \le m\sigma^{\frac{N}{2^*}-\frac{N}{p_2}}. \end{gathered} \tag{4.44}$$

We define the norm

$$\|u\|_{p_1,p_2,\sigma} = \inf\{m : \text{there exists } u_1, u_2 \text{ satisfying } |u| \le u_1+u_2 \text{ and (4.44) holds}\}.$$

Lemma 4.13. *Assume the profile decomposition* (4.26) *holds. Without loss of generality assume* $k_\infty \in \Lambda_\infty$, $\sigma_n = \sigma_{n,k_\infty} = \min\{\sigma_{n,k} \big| k \in \Lambda_\infty\}$. $\frac{N}{N-2} < p_2 < 2^* < p_1$*, then there exists a constant c, depending on* p_1, p_2 *such that* $\|u\|_{p_1,p_2,\sigma} \le c$.

Proof. We divide u_n into three parts

$$u_n = z_n + w_n + r_n,$$

where

$$z_n = \sum_{k\in\Lambda_1} U_k(\cdot - y_{n,k}), \quad w_n = \sum_{k\in\Lambda_\infty} \sigma_{n,k}^{\frac{N-2}{2}} U_k(\sigma_{n,k}(\cdot - y_{n,k})).$$

By Lemma 4.12, for $k \in \Lambda_1$, U_k decays exponentially. Hence $|z_n|_{p_1} \leq c$ for all $p_1 \geq 1$. For $k \in \Lambda_\infty$, U_k decays polynomially. Hence for $\frac{N}{N-2} < p_2 < 2^*$,

$$
\begin{aligned}
\|\sigma_{n,k}^{\frac{N-2}{2}} U_k(\sigma_{n,k}(\cdot - y_{n,k}))\|_{L^{p_2}(\mathbb{R}^N)} &= \sigma_{n,k}^{\frac{N-2}{2}-\frac{N}{p_2}} \|U_k\|_{L^{p_2}(\mathbb{R}^N)} \\
&\leq c\sigma_{n,k}^{\frac{N}{2^*}-\frac{N}{p_2}} \Big(\int_{\mathbb{R}^N} (1+|x|^2)^{\frac{2-N}{2}p_2}\, dx \Big)^{\frac{1}{p_2}} \\
&\leq c\sigma_{n,k}^{\frac{N}{2^*}-\frac{N}{p_2}} \leq c\sigma_n^{\frac{N}{2^*}-\frac{N}{p_2}}
\end{aligned} \tag{4.45}
$$

and $\|w_n\|_{L^{p_2}(\mathbb{R}^N)} \leq c\sigma_n^{\frac{N}{2^*}-\frac{N}{p_2}}$. We define $Z_n, W_n, R_n \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ as follows:

$$
\begin{gathered}
-\Delta Z_n = a_n(x)|z_n|^{2_\alpha^*-1}, \quad \text{in } \mathbb{R}^N, \\
-\Delta W_n = a_n(x)|w_n|^{2_\alpha^*-1}, \quad \text{in } \mathbb{R}^N, \\
-\Delta R_n = a_n(x)|r_n|^{2_\alpha^*-1}, \quad \text{in } \mathbb{R}^N.
\end{gathered}
$$

Here $a_n(x) = \int_{\mathbb{R}^N} \frac{|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha}\, dy + |u_n(x)|^{\frac{\alpha}{N-2}}$. By Lemma 4.9, we have

$$
\begin{aligned}
\|Z_n\|_{L^{p_1}(\mathbb{R}^N)} &\leq c\|a_n|z_n|^{2_\alpha^*-2}\|_{L^{N/2}(\mathbb{R}^N)} \|z_n\|_{L^{p_1}(\mathbb{R}^N)} \\
&\leq c\|a_n\|_{L^{2N/\alpha}(\mathbb{R}^N)} \||z_n|^{2_\alpha^*-2}\|_{L^{\frac{2N}{4-\alpha}}(\mathbb{R}^N)} \|z_n\|_{L^{p_1}(\mathbb{R}^N)} \\
&\leq c\|z_n\|_{L^{p_1}(\mathbb{R}^N)} \leq c.
\end{aligned} \tag{4.46}
$$

Similar to (4.46), we have

$$
\begin{gathered}
\|W_n\|_{L^{p_2}(\mathbb{R}^N)} \leq c\|w_n\|_{L^{p_2}(\mathbb{R}^N)} \leq c\sigma_n^{\frac{N}{2^*}-\frac{N}{p_2}}, \\
\|R_n\|_{p_1,p_2,\sigma_n} \leq o(1)\|r_n\|_{p_1,p_2,\sigma_n}.
\end{gathered}
$$

Let u_n satisfy the equation

$$
-\Delta u_n + V(\varepsilon_n x)u_n = 2_\alpha^* \Big(\int_{\mathbb{R}^N} \frac{F_{\nu_n}(u_n(y))}{|x-y|^\alpha}\, dy \Big) f_{\nu_n}(u_n) + \vartheta |u_n|^{q-2}u_n, \tag{4.47}
$$

where $x \in \mathbb{R}^N$.

By Lemma 2.4 and the inequality $|t| \leq \varepsilon|t| + c_\varepsilon|t|^{2^*}$, the function $v_n = |u_n|$ satisfies

$$
-\Delta v_n \leq ca_n(x)|u_n|^{2_\alpha^*-1} \leq -c\Delta(Z_n + W_n + R_n).
$$

By the maximum principle, we obtain

$$
|u_n| = v_n \leq c(Z_n + W_n + R_n).
$$

Now we have the estimate

$$
\begin{aligned}
\|u_n\|_{p_1,p_2,\sigma_n} &\leq c(\|Z_n\|_{p_1,p_2,\sigma_n} + \|W_n\|_{p_1,p_2,\sigma_n} + \|R_n\|_{p_1,p_2,\sigma_n}) \\
&\leq c + o(1)\|r_n\|_{p_1,p_2,\sigma_n} \\
&\leq c + o(1)(\|u_n\|_{p_1,p_2,\sigma_n} + \|z_n\|_{p_1,p_2,\sigma_n} + \|w_n\|_{p_1,p_2,\sigma_n}) \\
&\leq c + o(1)\|u_n\|_{p_1,p_2,\sigma_n},
\end{aligned} \tag{4.48}
$$

and $\|u_n\|_{p_1,p_2,\sigma_n} \leq c$. □

Assume the profile decomposition (4.26) holds. Let $k_\infty \in \Lambda_\infty$ be such that $\sigma_n = \sigma_{n,k_\infty} = \min\{\sigma_{n,k} \big| k \in \Lambda_\infty\}$. We denote $y_n = y_{n,k_\infty}$. Since the index set Λ_∞ is finite, we can find a constant $\bar{c} > 0$ such that the region

$$
\mathcal{A}_n^1 = B(y_n, 7\bar{c}\sigma_n^{-1/2}) \backslash B(y_n, \bar{c}\sigma_n^{-1/2})
$$

does not contain any concentration points, corresponding to the index set Λ_∞,

$$\mathcal{A}_n^1 \cap \{y_{n,k} \big| k \in \Lambda_\infty\} = \emptyset.$$

Lemma 4.14 ([20])**.** *Let $u \geq 0$, $u \in H^1(\mathbb{R}^N)$ and satisfy*

$$-\Delta u \leq f, \quad \text{in } \mathbb{R}^N,$$

where $f \geq 0$, $f \in L^1_{loc}(\mathbb{R}^N)$. Then for $\gamma \in (1, \frac{N}{N-1})$, there exists a positive constant $c = c(N, \gamma)$ such that for $x_0 \in \mathbb{R}^N$, $r \in (0,1)$

$$\Big(r^{-N} \int_{B(x_0,r)} |u|^\gamma \, dx\Big)^{1/\gamma} \leq c\Big(1 + \int_r^1 (t^{1-N} \int_{B(x_0,t)} f \, dx) dt\Big).$$

Lemma 4.15. *Assume the profile decomposition (4.26) holds. Then there exists a constant $c > 0$, independent of n, such that $|u_n(x)| \leq c$ for $x \in \mathcal{A}_n^2$, and*

$$\int_{\mathcal{A}_n^3} |\nabla u_n|^2 \, dx \leq c \sigma_n^{\frac{2-N}{2}},$$

where

$$\mathcal{A}_n^2 = B(y_n, 6\bar{c}\sigma_n^{-1/2}) \backslash B(y_n, 2\bar{c}\sigma_n^{-1/2}), \quad \mathcal{A}_n^3 = B(y_n, 5\bar{c}\sigma_n^{-1/2}) \backslash B(y_n, 3\bar{c}\sigma_n^{-1/2}).$$

Proof. It is easy to show that

$$-\Delta v_n \leq c w_n v_n^{2^*_\alpha - 1},$$

where

$$w_n(x) = \int_{\mathbb{R}^N} \frac{|v_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} \, dy + |v_n(x)|^{2^* - 2^*_\alpha}.$$

By Lemma 4.14, for all $y \in \mathbb{R}^N$ and $r \in (0,1)$, we have

$$\begin{aligned} \Big(r^{-N} \int_{B(y,r)} |u_n|^\gamma \, dx\Big)^{1/\gamma} &= \Big(r^{-N} \int_{B(y,r)} v_n^\gamma \, dx\Big)^{1/\gamma} \\ &\leq c\Big(1 + \int_r^1 (t^{1-N} \int_{B(y,t)} w_n v_n^{2^*_\alpha - 1} \, dx) dt\Big). \end{aligned} \tag{4.49}$$

By Lemma 4.13, $\|v_n\|_{p_1, p_2, \sigma_n} \leq c$ for $\frac{N}{N-2} < p_2 < 2^* < p_1$. There exist functions $v_n^{(1)}, v_n^{(2)}$ such that $v_n \leq v_n^{(1)} + v_n^{(2)}$, $\|v_n^{(1)}\|_{L^{p_1}(\mathbb{R}^N)} \leq c$, and

$$\|v_n^{(2)}\|_{L^{p_2}(\mathbb{R}^N)} \leq c \sigma_n^{\frac{N}{2^*} - \frac{N}{p_2}}.$$

We choose $p_1 = \frac{N(N+2)}{N-2}$ and $p_2 = \frac{2N(N+2-\alpha)}{(N-2)(2N-\alpha)}$. We have the estimate

$$\begin{aligned} \Big(r^{-N} \int_{B(y,r)} |u_n|^{\gamma \, dx}\Big)^{1/\gamma} &\leq c + c \int_r^1 \Big(t^{1-N} \int_{B(y,t)} w_n^{(1)} |v_n^{(1)}|^{2^*_\alpha - 1} \, dx\Big) dt \\ &\quad + c \int_r^1 \Big(t^{1-N} \int_{B(y,t)} w_n^{(2)} |v_n^{(2)}|^{2^*_\alpha - 1} \, dx\Big) dt, \end{aligned}$$

where

$$\begin{aligned} &\int_r^1 \Big(t^{1-N} \int_{B(y,t)} w_n^{(1)} |v_n^{(1)}|^{2^*_\alpha - 1} \, dx\Big) dt \\ &\leq \Big(\int_r^1 t^{1-N} dt\Big) \Big(\int_{B(y,t)} |w_n^{(1)}|^N |v_n^{(1)}|^{N(2^*_\alpha - 1)} \, dx\Big)^{1/N} \Big(\int_{B(y,t)} dt\Big)^{\frac{N-1}{N}} \end{aligned}$$

$$
\begin{aligned}
&\le c\Big(\int_{B(y,1)}|w_n^{(1)}|^N|v_n^{(1)}|^{N(2_\alpha^*-1)}\,dx\Big)^{1/N}\\
&\le c\Big(\int_{B(y,1)}|w_n^{(1)}|^{\frac{N(N+2)}{\alpha}}\,dx\Big)^{\frac{\alpha}{N+2}}\cdot\Big(\int_{B(y,1)}|v_n^{(1)}|^{N(2_\alpha^*-1)\frac{N+2}{N+2-\alpha}}\,dx\Big)^{\frac{N+2-\alpha}{N+2}}\\
&\le c\Big(\int_{B(y,1)}|v_n^{(1)}|^{\frac{N(N+2)}{N-2}}\,dx\Big)^{\frac{N+2-\alpha}{N+2}}\le c,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_r^1\Big(t^{1-N}\int_{B(y,t)}w_n^{(2)}|v_n^{(2)}|^{2_\alpha^*-1}\,dx\Big)dt\\
&\le\Big(\int_r^1 t^{1-N}dt\Big)\Big(\int_{B(y,t)}|w_n^{(2)}|^{2N/\alpha}\,dx\Big)^{\frac{\alpha}{2N}}\Big(\int_{B(y,t)}|v_n^{(2)}|^{(2_\alpha^*-1)\frac{2N}{2N-\alpha}}\,dx\Big)^{\frac{2N-\alpha}{2N}}\\
&\le c\Big(\int_r^1 t^{1-N}dt\Big)\Big(\int_{B(y,t)}|v_n^{(2)}|^{p_2}\,dx\Big)^{\frac{2_\alpha^*-1}{p_2}}\\
&\le c\Big(\int_r^1 t^{1-N}dt\Big)\sigma_n^{\frac{2-N}{2}}\\
&\le c(r\sigma_n^{1/2})^{2-N}\le c,\quad\text{provided } r\ge\frac{\overline{c}}{4}\sigma_n^{-1/2}.
\end{aligned}
$$

Therefore

$$
\Big(\sigma_n^{N/2}\int_{B(y,\frac{\overline{c}}{4}\sigma_n^{-1/2})}|u_n|^\gamma\,dx\Big)^{1/\gamma}\le c,\quad y\in\mathcal{A}_n^2.
$$

In (4.26), with $u_n=z_n+w_n+r_n$, $z_n\in L^\infty(\mathbb{R}^N)$, $|r_n|_{2^*}=o(1)$, by (4) of the profile decomposition (4.26) becomes $w_n=\sum_{k\in\Lambda_\infty}\sigma_{n,k}^{\frac{N-2}{2}}U_k(\sigma_{n,k}(\cdot-x_{n,k}))$. For $y\in\mathcal{A}_n^2$, $x\in B(y,\frac{\overline{c}}{2}\sigma_n^{-1/2})$, by our choice $\mathcal{A}_n^1$, $|y-y_{n,k}|\ge\overline{c}\sigma_n^{-1/2}$, $|x-y_{n,k}|\ge|y_{n,k}-y|-|x-y|\ge\frac{\overline{c}}{2}\sigma_n^{-1/2}\ge\frac{\overline{c}}{2}\sigma_{n,k}^{-1/2}$. Hence for n is large enough,

$$
\begin{aligned}
\int_{B(y,\frac{\overline{c}}{2}\sigma_n^{-1/2})}|u_n|^{2^*}\,dx&=\int_{B(y,\frac{\overline{c}}{2}\sigma_n^{-1/2})}|\sigma_{n,k}^{\frac{N-2}{2}}U_k(\sigma_{n,k}(\cdot-y_{n,k}))|^{2^*}\,dx+o(1)\\
&\le\int_{|x-x_{n,k}|\ge\frac{\overline{c}}{2}\sigma_{n,k}^{-1/2}}|\sigma_{n,k}^{\frac{N-2}{2}}U_k(\sigma_{n,k}(\cdot-y_{n,k}))|^{2^*}\,dx+o(1)\\
&=\int_{|x|\ge\frac{\overline{c}}{2}\sigma_{n,k}^{-1/2}}|U_k(x)|^{2^*}\,dx+o(1)=o(1),\quad\forall y\in\mathcal{A}_n^2.
\end{aligned}
$$

By Lemma 4.10, we have

$$
|u_n(x)|\le c\Big(\sigma_n^{N/2}\int_{B(y,\frac{\overline{c}}{4}\sigma_n^{-1/2})}|u_n|^\gamma\,dx\Big)^{1/\gamma}\le c,\quad\forall x\in B(y,\frac{\overline{c}}{8}\sigma_n^{-1/2}),\ y\in\mathcal{A}_n^2.
$$

Hence $|u_n(x)|\le c$ for $x\in\mathcal{A}_n^2$.

Let $\varphi\in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi(x)=1$ for $x\in\mathcal{A}_n^3$; $\varphi(x)=0$ for $x\notin\mathcal{A}_n^2$ and $|\nabla\varphi|\le 2\overline{c}^{-1}\sigma_n^{1/2}$. Taking $\phi=u_n\varphi^2$ as test function in $\langle DI_{\varepsilon_n,\nu_n}(u_n),\phi\rangle=0$, we obtain

$$
\int_{\mathbb{R}^N}(|\nabla u_n|^2\varphi^2+V(\varepsilon_n x)u_n^2\varphi^2)\,dx
$$

$$
\begin{aligned}
&= -2\int_{\mathbb{R}^N} u_n \nabla u_n \varphi \nabla\varphi\,dx + 2_\alpha^* \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{F_{\nu_n}(u_n(y))}{|x-y|^\alpha}\,dy f_{\nu_n}(u_n)u_n\varphi^2\,dx \\
&\quad + \vartheta\int_{\mathbb{R}^N} |u_n|^q\varphi^2\,dx \\
&\le \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2\varphi^2\,dx + c\int_{\mathbb{R}^N} |u_n|^2|\nabla\varphi|^2\,dx \\
&\quad + c\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha}\,dy|u_n|^{2_\alpha^*}\varphi^2\,dx + c\int_{\mathbb{R}^N} |u_n|^q\varphi^2\,dx.
\end{aligned}
$$

Since $|u_n(x)| \le c$ for $x \in \mathcal{A}_n^2$, we have

$$
\begin{aligned}
&\int_{\mathcal{A}_n^3} |\nabla u_n|^2\,dx \\
&\le \int_{\mathbb{R}^N} |\nabla u_n|^2\varphi^2\,dx \\
&\le c\sigma_n\int_{\mathcal{A}_n^2} |u_n|^2\,dx + c\int_{\mathcal{A}_n^2}\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha}\,dy|u_n|^{2_\alpha^*}\varphi^2\,dx + c\int_{\mathcal{A}_n^2}|u_n|^q\,dx \\
&\le c\Big(\int_{\mathcal{A}_n^2} \Big|\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2_\alpha^*}}{|x-y|^\alpha}\,dy\Big|^{2N/\alpha}\,dx\Big)^{\frac{\alpha}{2N}}\Big(\int_{\mathcal{A}_n^2} |u_n|^{2^*}\,dx\Big)^{\frac{2N-\alpha}{2N}} + c\sigma_n^{1-\frac{N}{2}} \\
&\le c\sigma_n^{\frac{2-N}{2}},
\end{aligned}
$$

for n large enough. □

Similar to [16, Lemma 4.3], we can prove that the index set Λ_∞ in the profile decomposition (4.26) is empty.

The proof part 3 of Theorem 4.1. Note that Λ_∞ is empty, the profile decomposition in (4.26) reduces to

$$
u_n = \sum_{k\in\Lambda_1} U_k(\cdot - y_{n,k}) + r_n, \tag{4.50}
$$

where $r_n \to 0$ in $L^{2^*}(\mathbb{R}^N)$ as $n \to \infty$. By Lemma 4.12, there exist c, μ, such that $|U_k(x)| \le c\exp\{-\mu|x|\}$ for $k \in \Lambda_1$. Hence, $\sum_{k\in\Lambda_1} U_k(\cdot - y_{n,k})$ are uniformly bounded. It follows that $\{u_n\}$ are uniformly bounded, that is there exists a constant M depending on L, but not on n, such that

$$
|u_n(x)| \le M, \quad \text{for } x \in \mathbb{R}^N,\ n = 1, 2, \dots .
$$

□

5. Proof of Theorem 1.1

From Theorem 4.1, we deduce the following corollary.

Corollary 5.1. *(1) Assume $\Gamma_{\varepsilon,\nu,\lambda}(u) \le L$ and $D\Gamma_{\varepsilon,\nu,\lambda}(u) = 0$. Then there exists $\overline{\lambda} = \overline{\lambda}(L)$ such that $\Gamma_{\varepsilon,\nu,\lambda}(u) = \Gamma_{\varepsilon,\nu}(u)$ and $D\Gamma_{\varepsilon,\nu}(u) = 0$ if $0 < \lambda \le \overline{\lambda}$.*

(2) Assume $\Gamma_{\varepsilon,\nu}(u) \le L$ and $D\Gamma_{\varepsilon,\nu}(u) = 0$. Then there exists $\overline{\varepsilon} = \overline{\varepsilon}(L)$ such that $\Gamma_{\varepsilon,\nu}(u) = I_{\varepsilon,\nu}(u)$ and $DI_{\varepsilon,\nu}(u) = 0$ if $0 < \varepsilon \le \overline{\varepsilon}$.

(3) Assume $I_{\varepsilon,\nu}(u) \le L$ and $DI_{\varepsilon,\nu}(u) = 0$. Then there exists $\overline{\nu} = \overline{\nu}(L)$ such that $I_{\varepsilon,\nu}(u) = I_\varepsilon(u)$ and $DI_\varepsilon(u) = 0$ if $0 < \nu \le \overline{\nu}$.

Proof. (1) By Theorem 4.1(1), if $0 < \lambda < \overline{\lambda}(L) = \frac{1}{CH^{2^*_\alpha}}$, then
$$\|u\|_{H^1(\mathbb{R}^N)} \leq (\frac{1}{C\lambda})^{\frac{1}{2^*_\alpha}}.$$
It follows that $\Gamma_{\varepsilon,\nu,\lambda}(u) = \Gamma_{\varepsilon,\nu}(u)$ and $D\Gamma_{\varepsilon,\nu}(u) = 0$.

(2) By Theorem 4.1(2), there exist constants μ, $c = c(L)$ such that, for every $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta) > 0$, for $0 < \varepsilon < \varepsilon(\delta)$ and $x \in \mathbb{R}^N$
$$|u(x)| \leq c \exp\{-\mu \text{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\}.$$
Let $\overline{\varepsilon}(L) \leq \min\{\mu, \frac{1}{c}\}$, then for $0 < \varepsilon \leq \overline{\varepsilon}$ and $x \in \mathbb{R}^N$, it holds
$$\begin{aligned}|u(x)| &\leq c \exp\{-\mu \text{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\} \\ &\leq \frac{1}{\varepsilon} \exp\{-\varepsilon \text{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\} \\ &\leq \frac{1}{\varepsilon} \exp\{-\text{dist}(\varepsilon x, \mathcal{M})\}.\end{aligned}$$
Hence $m_\varepsilon(x, u) = u$, for $x \in \mathbb{R}^N$. Moreover we denote $D = \max\{|y| \Big| y \in \overline{\mathcal{M}}\}$, $d = \text{dist}(\mathcal{A}^\delta, \partial\mathcal{M})$. We choose an integer $l > 1$ such that $ld \geq D$. Then for $x \notin \overline{\mathcal{M}}_\varepsilon$,
$$\begin{aligned}l\,\text{dist}(x, (\mathcal{A}^\delta)_\varepsilon) &\geq l\,\text{dist}((\mathcal{A}^\delta)_\varepsilon, \partial\mathcal{M}_\varepsilon) + \text{dist}(x, \partial\mathcal{M}_\varepsilon) \\ &\geq \frac{l}{\varepsilon} d + |x| - \frac{D}{\varepsilon} \geq |x|,\end{aligned}$$
and hence
$$|u(x)| \leq c \exp\{-c \text{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\} \leq c \exp\{-\frac{c}{l}|x|\}, \quad \forall x \notin \mathcal{M}_\varepsilon.$$
As a consequence, for $0 < \varepsilon < \varepsilon(\delta)$ sufficiently small, one has
$$\int_{\mathbb{R}^N} \chi_\varepsilon(x) u^2\, dx \leq c\varepsilon^{-6} \int_{|x| \geq c\varepsilon^{-1}} \exp\{-\frac{c}{l}|x|\}\, dx \leq c\varepsilon^{-N-5} \exp\{-\frac{c}{\varepsilon}\} < 1$$
and
$$\Big(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p\, dx - 1\Big)_+ = 0.$$
It follows that $\Gamma_\varepsilon(u) = I_\varepsilon(u)$ and $DI_\varepsilon(u) = 0$.

(3) By Theorem 4.1(3), if $0 < \nu < \overline{\nu}(L) = \frac{1}{M}$, then
$$|u(x)| \leq \frac{1}{\nu},\ \forall x \in \mathbb{R}^N.$$
Hence, $I_{\varepsilon,\nu}(u) = I_\varepsilon(u)$ and $DI_\varepsilon(u) = 0$. □

Proof of Theorem 1.1. Given a positive integer k, by Theorem 3.10, there exist $0 < \tilde{\varepsilon} < 1$, $0 < \tilde{\nu} < 1$, and $0 < \tilde{\lambda} < 1$, such that if $0 < \varepsilon < \tilde{\varepsilon}$, $0 < \nu < \tilde{\nu}$, $0 < \lambda < \tilde{\lambda}$, then the functional $\Gamma_{\varepsilon,\nu,\lambda}$ has k pairs of sign-changing critical points $\pm u_j$, $j = 1, \dots, k$, and the corresponding critical values satisfy
$$0 < c_1(\varepsilon, \nu, \lambda) \leq \cdots \leq c_k(\varepsilon, \nu, \lambda) \leq m_k.$$
By Corollary 5.1(3), there exists $\nu_k = \nu_k(m_k)$, such that if
$$0 < \nu < \tilde{\nu}_k = \min\{\nu_k, \tilde{\nu}\}, \quad I_{\varepsilon,\nu}(u) \leq m_k, \quad DI_{\varepsilon,\nu}(u) = 0,$$
then
$$I_{\varepsilon,\nu}(u) = I_\varepsilon(u), \quad DI_\varepsilon(u) = 0.$$

Fixed $\overline{\nu} \in (0, \nu_k)$. By Corollary 5.1(2), there exists $\varepsilon_k = \varepsilon_k(m_k)$, such that if

$$0 < \varepsilon < \tilde{\varepsilon}_k = \min\{\varepsilon_k, \tilde{\varepsilon}\}, \quad \Gamma_{\varepsilon,\overline{\nu}}(u) \le m_k, \quad D\Gamma_{\varepsilon,\overline{\nu}}(u) = 0,$$

then

$$\Gamma_{\varepsilon,\overline{\nu}}(u) = I_{\varepsilon,\overline{\nu}}(u), \quad DI_{\varepsilon,\overline{\nu}}(u) = 0.$$

We fix $\overline{\nu} \in (0, \nu_k)$ and $\overline{\varepsilon} \in (0, \varepsilon_k)$. By Corollary 5.1(1), there exists $\lambda_k = \lambda_k(m_k)$, such that if

$$0 < \lambda < \tilde{\lambda}_k = \min\{\lambda_k, \tilde{\lambda}\}, \quad \Gamma_{\overline{\varepsilon},\overline{\nu},\lambda}(u) \le m_k, \quad DI_{\overline{\varepsilon},\overline{\nu},\lambda}(u) = 0,$$

then

$$\Gamma_{\overline{\varepsilon},\overline{\nu},\lambda}(u) = \Gamma_{\overline{\varepsilon},\overline{\nu}}(u), \quad D\Gamma_{\overline{\varepsilon},\overline{\nu}}(u) = 0.$$

Now for $0 < \nu < \tilde{\nu}_k$, $0 < \varepsilon < \tilde{\varepsilon}_k$, $0 < \lambda < \tilde{\lambda}_k$, we have that $u_{j,\varepsilon} = u_j(\varepsilon, \nu, \lambda)$, $j = 1, \dots, k$ are critical points of the functional I_ε. Moreover, by Theorem 4.1, there exist constants $\mu > 0$, $c = c(m_k)$, such that for any $\delta > 0$, there exists $\overline{\varepsilon}_k(\delta)$ such that for $0 < \varepsilon < \overline{\varepsilon}_k(\delta)$ it holds

$$|u_{j,\varepsilon}| \le c \exp\{-\mu \operatorname{dist}(x, (\mathcal{A}^\delta)_\varepsilon)\}, \quad \text{for } x \in \mathbb{R}^N,$$

hence

$$|v_{j,\varepsilon}| \le c \exp\{-\frac{\mu}{\varepsilon} \operatorname{dist}(x, \mathcal{A}^\delta)\}, \quad \text{for } x \in \mathbb{R}^N. \qquad \square$$

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