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EXISTENCE OF PERIODIC SOLUTIONS AND STABILITY FOR A NONLINEAR SYSTEM OF NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the existence and uniqueness of periodic solutions, and stability of the zero solution to the nonlinear neutral system

$$\frac{d}{dt}x(t) = A(t)h\big(x(t-\tau_1(t))\big) + \frac{d}{dt}Q\big(t, x(t-\tau_2(t))\big) + G\big(t, x(t), x(t-\tau_2(t))\big).$$

We use integrating factors to transform the neutral differential equation into an equivalent integral equation. Then we construct appropriate mappings and employ Krasnoselskii's fixed point theorem to show the existence of a periodic solution. We also use the contraction mapping principle to show the existence of a unique periodic solution and the asymptotic stability of the zero solution. Our results generalize the corresponding results in the existing literature. An example is given to illustrate our results.

1. Introduction

Functional differential equations are widely applied in fields, such as neural networks, population dynamics, control theory, and many other fields. Recently, the theory about these equations has been an object of active research, mostly because by understanding the properties of solutions we can see the trend of events in real world problems.

Investigators have given special attention to the study of equations in which the delay occurs in the derivative of the state variable as well as in the independent variables. These equations are called neutral differential equations, and describe actual problems more accurately than other differential equations. In particular, qualitative analysis, such as periodicity and stability of solutions of neutral differential equations, has been studied extensively by many authors. For a long time, the direct Lyapunov method or Lyapunov function was the main tool for determining stability in many differential equations without solving the equations explicitly. However, there are a lot of problems when using this method: Lyapunov's direct method requires pointwise conditions while many practical problems do not meet these conditions; a suitable Lyapunov function is not easy to construct; and there are problems with ascertaining limit sets when the equation becomes unbounded

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or the derivative is not finite. Fortunately, Burton and other authors have applied fixed point theory to investigate the stability of systems and obtained some applicable techniques. Moreover, the advantage of fixed point theory is that it can prove the existence, uniqueness, boundedness, and stability of the equation at the same time. For recent works on periodicity and stability of neutral equations, we refer the reader to the references in this article.

In 2007, Islam and Raffoul [9] studied the existence of periodic solutions of the system

$$\frac{d}{dt}x(t) = A(t)x(t) + \frac{d}{dt}Q\big(t, x(t-g(t))\big) + G\big(t, x(t), x(t-g(t))\big). \tag{1.1}$$

Using Krasnoselskii's fixed point theorem, they obtained the existence of a unique periodic solution of (1.1) . Existence and uniqueness of solutions has been also investigated using the contraction mapping principle; see [9] for linear systems, and [14] for nonlinear systems. Mesmouli, Ardjouni, and Djoudi [14] extended the results of [9] by studying a nonlinear system with two variable delays

$$\frac{d}{dt}x(t) = A(t)x(t-\tau(t)) + \frac{d}{dt}Q(t, x(t-g(t))) + G(t, x(t), x(t-g(t))).$$

Motivated by the works mentioned above, we study the following system where we replace the linear term $A(t)x(t-\tau(t))$ by the nonlinear term $A(t)h\big(x(t-\tau_1(t))\big)$, and has two variable delays:

$$\frac{d}{dt}x(t) = A(t)h\big(x(t-\tau_1(t))\big) + \frac{d}{dt}Q\big(t, x(t-\tau_2(t)\big) + G\big(t, x(t), x(t-\tau_2(t))\big), \tag{1.2}$$

where $A(\cdot)$ is a nonsingular $n\times\ n$ matrix with continuous real-valued functions as entries. The functions $h:\mathbb{R}\to\mathbb{R}^n$, $Q:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$, and $G:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ are continuous in their respective arguments, and τ_2 is continuously differentiable. In our analysis we use the fundamental matrix solution of $x'(t) = A(t)x(t)$ coupled with Floquet theory to transform (1.2) into an integral system. The integral system obtained is the sum of two mappings, one completely continuous, and the other a contraction.

The organization of this article is as follows. In section 2 we present some definitions and transform (1.2) into an integral system. In section 3 we study the existence and uniqueness of a periodic solutions. In section 4 study the stability of the zero solution. And in section 5 we present an example that illustrates our results.

2. Preliminaries

For $T > 0$, let P_T be the set of all n-vector valued functions $x(t)$, which are continuous and periodic in t of period T. Then $(P_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|x(\cdot)\| = \sup_{t\in\mathbb{R}}|x(t)| = \sup_{t\in[0,T]}|x(t)|,$$

where $|\cdot|$ denotes the norm for $x\in\mathbb{R}^n$. Also, if A is an $n\times n$ real matrix, then we define the norm $|A| = \max_{1\le i\le n}\sum_{j=1}^n |a_{ij}|$.

Definition 2.1. Let matrix $A(\cdot)$ be periodic of period T. The linear system

$$y'(t) = A(t)y(t), \tag{2.1}$$

is said to be noncritical with respect to T, if it has no periodic solution of period T except for the trivial solution $y = 0$.

In this article, we assume that

$$A(t+T) = A(t), \quad \tau_1(t+T) = \tau_1(t) \geq \tau_1^* > 0, \quad \tau_2(t+T) = \tau_2(t) \geq \tau_2^* > 0, \tag{2.2}$$

with τ_1 is continuously differentiable and τ_1^*, τ_2^* are constants. For $t \in \mathbb{R}$, $x, y \in \mathbb{R}^n$, the functions $Q(t,x)$ and $G(t,x,y)$ are periodic in t of period T. That is

$$Q(t+T,x) = Q(t,x), \quad G(t+T,x,y) = G(t,x,y). \tag{2.3}$$

The functions Q, G, h are also globally Lipschitz continuous. That is, for $x, y, z, w \in \mathbb{R}^n$, there are positive constants k_1, k_2, k_3, k_4 such that

$$|Q(t,x) - Q(t,y)| \leq k_1 \|x - y\|, \tag{2.4}$$

$$|G(t,x,y) - G(t,z,w)| \leq k_2 \|x - z\| + k_3 \|y - w\|, \tag{2.5}$$

$$|h(x) - h(y)| \leq k_4 \|x - y\|. \tag{2.6}$$

Throughout this article we assume system (2.1) is noncritical. Let $K(t)$ represent the fundamental matrix of (2.1) with $K(0) = I$, where I is the $n \times n$ identity matrix. Then:

(a) $\det K(t) \neq 0$.
(b) There exists a constant matrix B such that $K(t+T) = K(t)e^{TB}$, by Floquet theory.
(c) System (2.1) is noncritical if and only if $\det(I - K(T)) \neq 0$.

The following lemma is fundamental for our results.

Lemma 2.2. *Suppose* (2.2) *and* (2.3) *hold, then x is a solution of* (1.2) *if and only if*

$$\begin{aligned}
&x(t) \\
&= Q(t, x(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(x(s))ds \\
&\quad + K(t)U(T) \int_t^{t+T} K^{-1}(s)A(s)\Big[Q(s, x(s-\tau_2(s))) - (x(s) - h(x(s))) \\
&\quad - \int_{s-\tau_1(s)}^{s} A(u)h(x(u))du\Big]ds + K(t)U(T) \int_t^{t+T} K^{-1}(s)[F(s)h(x(s-\tau_1(s))) \\
&\quad + G(s, x(s), x(s-\tau_2(s)))]ds,
\end{aligned} \tag{2.7}$$

where

$$\begin{gathered}
U(T) = (K^{-1}(T) - I)^{-1}, \\
F(t) = A(t) - (1 - \tau_1'(t))A(t - \tau_1(t)),
\end{gathered} \tag{2.8}$$

where $K(\cdot)$ is the fundamental matrix solution of (2.1).

Proof. Let $x(t) \in P_T$ be a solution of (1.2). We rewrite (1.2) as

$$\begin{aligned}
\frac{d}{dt}x(t) &= A(t-\tau_1(t))h(x(t-\tau_1(t)))(1-\tau_1'(t)) \\
&\quad - A(t-\tau_1(t))h(x(t-\tau_1(t)))(1-\tau_1'(t)) + A(t)h(x(t)) - A(t)h(x(t)) \\
&\quad + A(t)h(x(t-\tau_1(t))) + \frac{d}{dt}Q(t, x(t-\tau_2(t))) + G(t, x(t), x(t-\tau_2(t)))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{dt} Q(t, x(t-\tau_2(t))) - \frac{d}{dt} \int_{t-\tau_1(t)}^t A(s)h(x(s))ds + A(t)h(x(t)) \\
&\quad + h(x(t-\tau_1(t)))[A(t) - A(t-\tau_1(t))(1-\tau_1'(t))] \\
&\quad + G(t, x(t), x(t-\tau_2(t))).
\end{aligned}
$$

Putting $F(t) = A(t) - (1-\tau_1'(t))A(t-\tau_1(t))$ we have

$$
\begin{aligned}
&\frac{d}{dt}\Big[x(t) + \int_{t-\tau_1(t)}^t A(s)h(x(s))ds - Q(t, x(t-\tau_2(t)))\Big] \\
&= A(t)\Big[x(t) + \int_{t-\tau_1(t)}^t A(s)h(x(s))ds - Q(t, x(t-\tau_2(t)))\Big] \\
&\quad + F(t)h(x(t-\tau_1(t))) - A(t)[x(t) - h(x(t))] + A(t)Q(t, x(t-\tau_2(t))) \\
&\quad - A(t)\int_{t-\tau_1(t)}^t A(s)h(x(s))ds + G(t, x(t), x(t-\tau_2(t))).
\end{aligned}
$$

Since $K(t)K^{-1}(t) = I$, it follows that

$$
\begin{aligned}
0 &= \frac{d}{dt}\left[K(t)K^{-1}(t)\right] \\
&= A(t)K(t)K^{-1}(t) + K(t)\frac{d}{dt}K^{-1}(t) \\
&= A(t) + K(t)\frac{d}{dt}K^{-1}(t).
\end{aligned}
$$

This implies

$$
\frac{d}{dt}K^{-1}(t) = -K^{-1}(t)A(t).
$$

If $x(t)$ is a solution of (1.2) with $x(0) = x_0$, then

$$
\begin{aligned}
&\frac{d}{dt}\Big[K^{-1}(t)\Big(x(t) + \int_{t-\tau_1(t)}^t A(s)h(x(s))ds - Q(t, x(t-\tau_2(t)))\Big)\Big] \\
&= \frac{d}{dt}K^{-1}(t)\Big[x(t) + \int_{t-\tau_1(t)}^t A(s)h(x(s))ds - Q(t, x(t-\tau_2(t)))\Big] \\
&\quad + K^{-1}(t)\frac{d}{dt}\Big[x(t) + \int_{t-\tau_1(t)}^t A(s)h(x(s))ds - Q(t, x(t-\tau_2(t)))\Big] \\
&= K^{-1}(t)A(t)\Big[Q(t, x(t-\tau_2(t))) - (x(t) - h(x(t))) - \int_{t-\tau_1(t)}^t A(s)h(x(s))ds\Big] \\
&\quad + K^{-1}(t)[F(t)h(x(t-\tau_1(t))) + G(t, x(t), x(t-\tau_2(t)))].
\end{aligned}
$$

Integrating of the above equation from 0 to t yields

$$
\begin{aligned}
x(t) = Q(t, x(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(x(s))ds + K(t)\Big(x(0) \\
+ \int_{-\tau_1(0)}^{0} A(s)h(x(s))ds - Q(0, -\tau_2(0))\Big) \\
+ K(t)\int_0^t K^{-1}(s)A(s)\Big[Q(s, x(s-\tau_2(s))) - (x(s) - h(x(s))) \\
- \int_{s-\tau_1(s)}^{s} A(u)h(x(u))du\Big]ds \\
+ K(t)\int_0^t K^{-1}(s)[F(s)h(x(s-\tau_1(s))) + G(s, x(s), x(s-\tau_2(s)))]ds.
\end{aligned}
\tag{2.9}
$$

Since $x(T) = x_0 = x(0)$ and

$$
(I - K(T))^{-1} = \left(K(T)\left(K^{-1}(T) - I\right)\right)^{-1} = \left(K^{-1}(T) - I\right)^{-1} K^{-1}(T),
$$

using (2.9) we obtain

$$
\begin{aligned}
& x(0) + \int_{-\tau_1(0)}^{0} A(s)h(x(s))ds - Q(0, -\tau_2(0)) \\
&= \left(K^{-1}(T) - I\right)^{-1} \int_0^T K^{-1}(s)A(s)\Big[Q(s, x(s-\tau_2(s))) - (x(s) - h(x(s))) \\
&\quad - \int_{s-\tau_1(s)}^{s} A(u)h(x(u))du\Big]ds \\
&\quad + \left(K^{-1}(T) - I\right)^{-1} \int_0^T K^{-1}(s)\big[F(s)h(x(s-\tau_1(s))) \\
&\quad + G(s, x(s), x(s-\tau_2(s)))\big]ds.
\end{aligned}
\tag{2.10}
$$

Substituting (2.10) into (2.9) yields

$$
\begin{aligned}
& x(t) \\
&= Q(t, x(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(x(s))ds \\
&\quad + K(t)\left(K^{-1}(T) - I\right)^{-1} \int_0^T K^{-1}(s)A(s)\Big[Q(s, x(s-\tau_2(s))) - (x(s) - h(x(s))) \\
&\quad - \int_{s-\tau_1(s)}^{s} A(u)h(x(u))du\Big]ds \\
&\quad + K(t)\left(K^{-1}(T) - I\right)^{-1} \int_0^T K^{-1}(s)\big[F(s)h(x(s-\tau_1(s))) \\
&\quad + G(s, x(s), x(s-\tau_2(s)))\big]ds + K(t)\int_0^t K^{-1}(s)A(s)\Big[Q(s, x(s-\tau_2(s))) \\
&\quad - (x(s) - h(x(s))) - \int_{s-\tau_1(s)}^{s} A(u)h(x(u))du\Big]ds \\
&\quad + K(t)\int_0^t K^{-1}(s)\big[F(s)h(x(s-\tau_1(s))) + G(s, x(s), x(s-\tau_2(s)))\big]ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
x(t) &= Q(t, x(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(x(s))ds \\
&\quad + K(t)\left(K^{-1}(T)-I\right)^{-1}\Big\{\int_0^T K^{-1}(s)A(s)\Big[Q(s,x(s-\tau_2(s))) \\
&\quad - (x(s)-h(x(s))) - \int_{s-\tau_1(s)}^{s} A(u)h(x(u))du\Big]ds \\
&\quad + \int_0^T K^{-1}(s)\left[F(s)h(x(s-\tau_1(s))) + G(s,x(s),x(s-\tau_2(s)))\right]ds \\
&\quad + \left(K^{-1}(T)-I\right)\int_0^t K^{-1}(s)A(s)\Big[Q(s,x(s-\tau_2(s))) - (x(s)-h(x(s))) \\
&\quad - \int_{s-\tau_1(s)}^{s} A(u)h(x(u))du\Big]ds \\
&\quad + \left(K^{-1}(T)-I\right)\int_0^t K^{-1}(s)\big[F(s)h(x(s-\tau_1(s))) \\
&\quad + G(s,x(s),x(s-\tau_2(s)))\big]ds\Big\} \\
&= Q(t, x(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(x(s))ds \\
&\quad + K(t)\left(K^{-1}(T)-I\right)^{-1}\Big\{\int_t^T K^{-1}(s)A(s)\Big[Q(s,x(s-\tau_2(s))) \\
&\quad - (x(s)-h(x(s))) - \int_{s-\tau_1(s)}^{s} A(u)h(x(u))du\Big]ds \\
&\quad + \int_t^T K^{-1}(s)\left[F(s)h(x(s-\tau_1(s))) + G(s,x(s),x(s-\tau_2(s)))\right]ds \\
&\quad + \int_0^t K^{-1}(T)K^{-1}(s)A(s)\Big[Q(s,x(s-\tau_2(s))) - (x(s)-h(x(s))) \\
&\quad - \int_{s-\tau_1(s)}^{s} A(u)h(x(u))du\Big]ds \\
&\quad + \int_0^t K^{-1}(T)K^{-1}(s)\left[F(s)h(x(s-\tau_1(s))) + G(s,x(s),x(s-\tau_2(s)))\right]ds\Big\}.
\end{aligned}
$$

By letting $s = v - T$ and $U(T) = (K^{-1}(T) - I)^{-1}$, the above expression yields

$$
\begin{aligned}
&x(t) \\
&= Q(t, x(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(x(s))ds \\
&\quad + K(t)U(T)\int_t^T K^{-1}(s)A(s)\Big[Q(s,x(s-\tau_2(s))) - (x(s)-h(x(s))) \\
&\quad - \int_{s-\tau_1(s)}^{s} A(u)h(x(u))du\Big]ds
\end{aligned}
$$

$$
\begin{aligned}
&+ K(t)U(T)\int_t^T K^{-1}(s)\left[F(s)h(x(s-\tau_1(s))) + G(s,x(s),x(s-\tau_2(s)))\right] ds \\
&+ K(t)U(T)\int_T^{t+T} K^{-1}(T)K^{-1}(v-T)A(v-T)\Big[Q(v-T,x(v-T-\tau_2(v-T))) \\
&- (x(v-T)-h(x(v-T))) - \int_{v-T-\tau_1(v-T)}^{v-T} A(u)h(x(u))du\Big]dv \\
&+ K(t)U(T)\int_T^{t+T} K^{-1}(T)K^{-1}(v-T)\Big[F(v-T)h(x(v-T-\tau_1(v-T))) \\
&+ G(v-T,x(v-T),x(v-T-\tau_2(v-T)))\Big]ds. \qquad (2.11)
\end{aligned}
$$

By assumption (b) we have $K(t-T) = K(t)e^{-TB}$ and $K(T) = e^{TB}$. Hence

$$K^{-1}(T)K^{-1}(v-T) = K^{-1}(v).$$

Consequently, since (2.2) and (2.3) hold, equation (2.11) becomes

$$
\begin{aligned}
&x(t) \\
&= Q(t,x(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(x(s))ds \\
&\quad + K(t)U(T)\int_t^T K^{-1}(s)A(s)\Big[Q(s,x(s-\tau_2(s))) - (x(s)-h(x(s))) \\
&\quad - \int_{s-\tau_1(s)}^{s} A(u)h(x(u))du\Big]ds \\
&\quad + K(t)U(T)\int_t^T K^{-1}(s)\left[F(s)h(x(s-\tau_1(s))) + G(s,x(s),x(s-\tau_2(s)))\right] ds \\
&\quad + K(t)U(T)\int_T^{t+T} K^{-1}(s)A(s)\Big[Q(s,x(s-\tau_2(s))) - (x(s)-h(x(s))) \\
&\quad - \int_{s-\tau_1(s)}^{s} A(u)h(x(u))du\Big]ds \\
&\quad + K(t)U(T)\int_T^{t+T} K^{-1}(s)\left[F(s)h(x(s-\tau_1(s))) + G(s,x(s),x(s-\tau_2(s)))\right] ds. \qquad (2.12)
\end{aligned}
$$

By combining the two integrals of (2.12), we can easily obtain (2.7). By finding the derivative of (2.7), we can obtain (1.2). The converse implication is easily obtained and the proof is complete. □

We end this section by stating some theorems which will be useful for obtaining our main results. Next we state Krasnoselskii's fixed point theorem whose proof can be found in [19].

Theorem 2.3. *Let M be a closed convex non-empty subset of a Banach space $(X, \|\cdot\|)$. Suppose that A and B map M into X such that the following conditions hold*

(i) $Ax + By \in M$ *for all* $x, y \in M$;
(ii) A *is continuous, and* AM *is contained in a compact set;*
(iii) B *is a contraction.*

Then there is a $z \in M$, *with* $z = Az + Bz$.

Theorem 2.4 (Contraction mapping principle). *Let* (X, ρ) *a complete metric space and let* $P : X \to X$. *If there exists a constant* $\alpha < 1$, *such that for* $x, y \in X$ *we have*

$$\rho(Px, Py) \le \alpha\rho(x, y),$$

then there exists a unique point $z \in X$ *with* $Pz = z$.

3. Existence and uniqueness of periodic solutions

In this section, we discuss the existence and uniqueness of periodic solution of system (1.2) by using Krasnoselskii's fixed point theorem and the contraction mapping principle. To apply Theorem 2.3, we need to define a Banach space B, a closed bounded convex subset M of B and construct two mappings; one completely continuous and the other a contraction. So we let $(B, \|\cdot\|) = (P_T, \|\cdot\|)$ and

$$M = \{\varphi \in P_T : \|\varphi\| \le L\}, \tag{3.1}$$

where L is a positive constant.

Define the mapping $P : P_T \to P_T$ by

$$\begin{aligned}
&(P\varphi)(t)\\
&= Q(t, \varphi(t - \tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(\varphi(s))ds\\
&\quad + K(t)U(T)\int_t^{t+T} K^{-1}(s)A(s)\Big[Q(s, \varphi(s - \tau_2(s))) - (\varphi(s) - h(\varphi(s)))\\
&\quad - \int_{s-\tau_1(s)}^{s} A(u)h(\varphi(u))du\Big]ds\\
&\quad + K(t)U(T)\int_t^{t+T} K^{-1}(s)[F(s)h(\varphi(s - \tau_1(s)))\\
&\quad + G(s, \varphi(s), \varphi(s - \tau_2(s)))]ds.
\end{aligned} \tag{3.2}$$

Therefore, we express the above equation as

$$(P\varphi)(t) = (R\varphi)(t) + (B\varphi)(t), \tag{3.3}$$

where $R, B : P_T \to P_T$ are given by

$$\begin{aligned}
(R\varphi)(t) = Q(t, \varphi(t - \tau_2(t))) &- \int_{t-\tau_1(t)}^{t} A(s)h(\varphi(s))ds\\
&+ K(t)U(T)\int_t^{t+T} K^{-1}(s)A(s)\Big[Q(s, \varphi(s - \tau_2(s)))\\
&- (\varphi(s) - h(\varphi(s))) - \int_{s-\tau_1(s)}^{s} A(u)h(\varphi(u))du\Big]ds,
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
(B\varphi)(t) = K(t)U(T)\int_t^{t+T} &K^{-1}(s)[F(s)h(\varphi(s - \tau_1(s)))\\
&+ G(s, \varphi(s), \varphi(s - \tau_2(s)))]ds.
\end{aligned} \tag{3.5}$$

Lemma 3.1. *Let* R *be defined by* (3.4), *and assume that* (2.2)-(2.6) *hold. Then* R *is continuous and* RM *is contained in a compact set.*

Proof. Firstly, by (2.4)-(2.6), we obtain

$$|Q(t,x)| \le |Q(t,x) - Q(t,0) + Q(t,0)| \le k_1\|x\| + |Q(t,0)|\,, \tag{3.6}$$

$$\begin{aligned} |G(t,x,y)| &\le |G(t,x,y) - G(t,0,0) + G(t,0,0)| \\ &\le k_2\|x\| + k_3\|y\| + |G(t,0,0)|\,, \end{aligned} \tag{3.7}$$

$$|h(x)| \le k_4\|x\| + |h(0)|\,. \tag{3.8}$$

Let R defined by (3.4). For $\varphi \in M$, we have

$$\begin{aligned} (R\varphi)(t) &\le |Q(t,\varphi(t-\tau_2(t)))| + \int_{t-\tau_1(t)}^{t} |A||h(\varphi(s))|ds \\ &\quad + \int_t^{t+T} \big|\big[K(s)U^{-1}(T)K^{-1}(t)\big]^{-1}\big|\,|A|\Big[|Q(s,\varphi(s-\tau_2(s)))| \\ &\quad + |\varphi(s)| + |h(\varphi(s))| + \int_{s-\tau_1(s)}^{s} |A||h(\varphi(u))|du\Big]ds \\ &\le k_1L + \beta + \alpha|A|(k_4L+\eta) + cT|A|[k_1L + \beta + L + k_4L + \eta \\ &\quad + \alpha|A|(k_4L+\eta)] = E, \end{aligned} \tag{3.9}$$

where E is a constant, and

$$\begin{gathered} \alpha = \sup_{t\in[0,T]} |\tau_1(t)|, \quad \beta = \sup_{t\in[0,T]} |Q(t,0)|, \quad \gamma = \sup_{t\in[0,T]} |G(t,0,0)|, \\ \eta = |h(0)|, \quad c = \sup_{t\in[0,T]} \big(\sup_{s\in[t,t+T]} \big|[K(s)U^{-1}(T)K^{-1}(t)]^{-1}\big|\big). \end{gathered}$$

Next, we show that R is continuous in the supremum norm. Let $\varphi_1, \varphi_2 \in M$, for $\epsilon > 0$, choose $\eta = \epsilon/\Delta$, where

$$\Delta = k_1 + \alpha|A|k_4 + cT|A|k_1 + cT|A| + cT|A|k_4 + cT|A|(\alpha|A|k_4) + cT|F|k_4.$$

For $\|\varphi_1 - \varphi_2\| < \eta$, we obtain

$$\begin{aligned} &|(R\varphi_1)(t) - (R\varphi_2)(t)| \\ &\le |Q(t,\varphi_1(t-\tau_2(t))) - Q(t,\varphi_2(t-\tau_2(t)))| + \int_{t-\tau_1(t)}^{t} |A|\,|h(\varphi_1(s)) - h(\varphi_2(s))|\,ds \\ &\quad + \int_t^{t+T} \big|[K(s)U^{-1}(T)K^{-1}(t)]^{-1}\big|\,|A|\,|Q(s,\varphi_1(s-\tau_2(s))) - Q(s,\varphi_2(s-\tau_2(s)))|\,ds \\ &\quad + \int_t^{t+T} \big|[K(s)U^{-1}(T)K^{-1}(t)]^{-1}\big|\,|A||\varphi_1(s) - \varphi_2(s)|ds \\ &\quad + \int_t^{t+T} \big|[K(s)U^{-1}(T)K^{-1}(t)]^{-1}\big|\,|A||h(\varphi_1(s)) - h(\varphi_2(s))|ds \\ &\quad + \int_t^{t+T} \big|[K(s)U^{-1}(T)K^{-1}(t)]^{-1}\big|\,|A| \int_{s-\tau_1(s)}^{s} |A|\,|h(\varphi_1(u)) - h(\varphi_2(u))|\;du\,ds \\ &\quad + \int_t^{t+T} \big|[K(s)U^{-1}(T)K^{-1}(t)]^{-1}\big|\,|F|\,|h(\varphi_1(s-\tau_1(s))) - h(\varphi_2(s-\tau_1(s)))|\,ds \\ &\le k_1\|\varphi_1-\varphi_2\| + \alpha|A|k_4\|\varphi_1-\varphi_2\| + cT|A|k_1\|\varphi_1-\varphi_2\| + cT|A|\|\varphi_1-\varphi_2\| \\ &\quad + cT|A|k_4\|\varphi_1-\varphi_2\| + cT|A|\,(\alpha|A|k_4\|\varphi_1-\varphi_2\|) + cT|F|k_4\|\varphi_1-\varphi_2\| \\ &= \big[k_1 + \alpha|A|k_4 + cT|A|k_1 + cT|A| + cT|A|k_4 + cT|A|(\alpha|A|k_4) \end{aligned}$$

$$+ cT|F|k_4\big]\|\varphi_1 - \varphi_2\| < \epsilon,$$

which shows the continuity of R.

Finally, we show that RM is contained in a compact set. Let $\varphi \in M$, then by (3.9), we see that $R\varphi$ is uniformly bounded. Then, let $\varphi \in M$, without loss of generality, we can pick $\omega < t$ such that $t - \omega < T$. Then we have

$$\begin{aligned}
&|(R\varphi)(t) - (R\varphi)(\omega)| \\
&\le |Q(t, \varphi(t - \tau_2(t))) - Q(\omega, \varphi(\omega - \tau_2(\omega)))| + \Big| \int_{t-\tau_1(t)}^{t} A(s)h(\varphi(s))ds \\
&\quad - \int_{\omega-\tau_1(\omega)}^{\omega} A(s)h(\varphi(s))ds\Big| \\
&\quad + \Big| \int_{t}^{t+T} [K(s)U^{-1}(T)K^{-1}(t)]^{-1} A(s)Q(s, \varphi(s - \tau_2(s)))ds \\
&\quad - \int_{\omega}^{\omega+T} [K(s)U^{-1}(T)K^{-1}(\omega)]^{-1} A(s)Q(s, \varphi(s - \tau_2(s)))ds\Big| \\
&\quad + \Big| \int_{t}^{t+T} [K(s)U^{-1}(T)K^{-1}(t)]^{-1} A(s)\varphi(s)ds \\
&\quad - \int_{\omega}^{\omega+T} [K(s)U^{-1}(T)K^{-1}(\omega)]^{-1} A(s)\varphi(s)ds\Big| \\
&\quad + \Big| \int_{t}^{t+T} [K(s)U^{-1}(T)K^{-1}(t)]^{-1} A(s)h(\varphi(s))ds \\
&\quad - \int_{\omega}^{\omega+T} [K(s)U^{-1}(T)K^{-1}(\omega)]^{-1} A(s)h(\varphi(s))ds\Big| \\
&\quad + \Big| \int_{t}^{t+T} [K(s)U^{-1}(T)K^{-1}(t)]^{-1} A(s) \int_{s-\tau_1(s)}^{s} A(u)h(\varphi(u))\,du\,ds \\
&\quad - \int_{\omega}^{\omega+T} [K(s)U^{-1}(T)K^{-1}(\omega)]^{-1} A(s) \int_{s-\tau_1(s)}^{s} A(u)h(\varphi(u))\,du\,ds\Big|.
\end{aligned}$$

Since (3.6)-(3.8) hold, we have

$$\begin{aligned}
&\Big| \int_{t-\tau_1(t)}^{t} A(s)h(\varphi(s))ds - \int_{\omega-\tau_1(\omega)}^{\omega} A(s)h(\varphi(s))ds\Big| \\
&\le (k_4L + \eta)\Big(\int_{\omega}^{t} |A|ds + \int_{\omega-\tau_1(\omega)}^{t-\tau_1(t)} |A|ds\Big),
\end{aligned}$$

and

$$\begin{aligned}
&\Big| \int_{t}^{t+T} [K(s)U^{-1}(T)K^{-1}(t)]^{-1} A(s)Q(s, \varphi(s - \tau_2(s)))ds \\
&- \int_{\omega}^{\omega+T} [K(s)U^{-1}(T)K^{-1}(\omega)]^{-1} A(s)Q(s, \varphi(s - \tau_2(s)))ds\Big| \\
&= \Big| \int_{t}^{\omega} [K(s)U^{-1}(T)K^{-1}(t)]^{-1} A(s)Q(s, \varphi(s - \tau_2(s)))ds \\
&\quad + \int_{\omega+T}^{t+T} [K(s)U^{-1}(T)K^{-1}(t)]^{-1} A(s)Q(s, \varphi(s - \tau_2(s)))ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\omega}^{\omega+T}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}-[K(s)U^{-1}(T)K^{-1}(\omega)]^{-1}\\
&\times A(s)Q(s,\varphi(s-\tau_2(s)))ds\Big|\\
&\le |K(T)-I|\Big|\int_{\omega}^{t}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}A(s)Q(s,\varphi(s-\tau_2(s)))ds\Big|\\
&+\Big|\int_{\omega}^{\omega+T}\big([K(s)U^{-1}(T)K^{-1}(t)]^{-1}-[K(s)U^{-1}(T)K^{-1}(\omega)]^{-1}\big)\\
&\times A(s)Q(s,\varphi(s-\tau_2(s)))ds\Big|\\
&\le c|K(T)-I|(k_1L+\beta)\int_{\omega}^{t}|A|ds+(k_1L+\beta)\\
&\times\int_0^T|K(t)-K(\omega)|\left|U(T)K(T)K^{-1}(s)\right||A|ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_t^{t+T}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}A(s)\varphi(s)ds\\
&\quad-\int_{\omega}^{\omega+T}[K(s)U^{-1}(T)K^{-1}(\omega)]^{-1}A(s)\varphi(s)ds\Big|\\
&=\Big|\int_t^{\omega}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}A(s)\varphi(s)ds\\
&\quad+\int_{\omega+T}^{t+T}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}A(s)\varphi(s)ds\\
&\quad+\int_{\omega}^{\omega+T}\left|[K(s)U^{-1}(T)K^{-1}(t)]^{-1}-[K(s)U^{-1}(T)K^{-1}(\omega)]^{-1}\right|A(s)\varphi(s)ds\Big|\\
&\le|K(T)-I|\Big|\int_{\omega}^{t}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}A(s)\varphi(s)ds\Big|\\
&\quad+\Big|\int_{\omega}^{\omega+T}\left([K(s)U^{-1}(T)K^{-1}(t)]^{-1}-[K(s)U^{-1}(T)K^{-1}(\omega)]^{-1}\right)A(s)\varphi(s)ds\Big|\\
&\le c\,|K(T)-I|\,L\int_{\omega}^{t}|A|ds+L\int_0^T|K(t)-K(\omega)|\left|U(T)K(T)K^{-1}(s)\right||A|ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_t^{t+T}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}A(s)h(\varphi(s))ds\\
&\quad-\int_{\omega}^{\omega+T}[K(s)U^{-1}(T)K^{-1}(\omega)]^{-1}A(s)h(\varphi(s))ds\Big|\\
&=\Big|\int_t^{\omega}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}A(s)h(\varphi(s))ds\\
&\quad+\int_{\omega+T}^{t+T}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}A(s)h(\varphi(s))ds\\
&\quad+\int_{\omega}^{\omega+T}\left|[K(s)U^{-1}(T)K^{-1}(t)]^{-1}-[K(s)U^{-1}(T)K^{-1}(\omega)]^{-1}\right|A(s)h(\varphi(s))ds\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\le |K(T)-I|\Big|\int_\omega^t [K(s)U^{-1}(T)K^{-1}(t)]^{-1}A(s)h(\varphi(s))ds\Big| \\
&\quad + \Big|\int_\omega^{\omega+T} \big([K(s)U^{-1}(T)K^{-1}(t)]^{-1}-[K(s)U^{-1}(T)K^{-1}(\omega)]^{-1}\big)\,A(s)h(\varphi(s))ds\Big| \\
&\le c\Big|K(T)-I\Big|(k_4L+\eta)\int_\omega^t |A|ds+(k_4L+\eta) \\
&\quad \times \int_0^T |K(t)-K(\omega)||U(T)K(T)K^{-1}(s)||A|ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_t^{t+T}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}A(s)\int_{s-\tau_1(s)}^s A(u)h(\varphi(u))\,du\,ds \\
&-\int_\omega^{\omega+T}[K(s)U^{-1}(T)K^{-1}(\omega)]^{-1}A(s)\int_{s-\tau_1(s)}^s A(u)h(\varphi(u))\,du\,ds\Big| \\
&=\Big|\int_t^{\omega}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}A(s)\int_{s-\tau_1(s)}^s A(u)h(\varphi(u))\,du\,ds \\
&\quad +\int_{\omega+T}^{t+T}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}A(s)\int_{s-\tau_1(s)}^s A(u)h(\varphi(u))\,du\,ds \\
&\quad +\int_\omega^{\omega+T}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}-[K(s)U^{-1}(T)K^{-1}(\omega)]^{-1} \\
&\quad \times A(s)\int_{s-\tau_1(s)}^s A(u)h(\varphi(u))\,du\,ds\Big| \\
&\le |K(T)-I|\Big|\int_\omega^{t}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}A(s)\int_{s-\tau_1(s)}^s A(u)h(\varphi(u))\,du\,ds\Big| \\
&\quad +\Big|\int_\omega^{\omega+T}\big([K(s)U^{-1}(T)K^{-1}(t)]^{-1}-[K(s)U^{-1}(T)K^{-1}(\omega)]^{-1}\big) \\
&\quad \times A(s)\int_{s-\tau_1(s)}^s A(u)h(\varphi(u))\,du\,ds\Big| \\
&\le |K(T)-I|\alpha|A|(k_4L+\eta)\int_\omega^t|A|ds+\alpha|A|(k_4L+\eta) \\
&\quad \times\int_0^T|K(t)-K(\omega)|\,|U(T)K(T)K^{-1}(s)|\,|A|ds,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&|(R\varphi)(t)-(R\varphi)(\omega)| \\
&\le |Q(t,\varphi(t-\tau_2(t)))-Q(\omega,\varphi(\omega-\tau_2(\omega)))|+(k_4L+\eta)\Big(\int_\omega^t|A|ds+\int_{\omega-\tau_1(\omega)}^{t-\tau_1(t)}|A|ds\Big) \\
&\quad +c|K(T)-I|(k_1L+\beta)\int_\omega^t|A|ds+(k_1L+\beta) \\
&\quad \times\int_0^T|K(t)-K(\omega)|\,\big|U(T)K(T)K^{-1}(s)\big|\,|A|ds
\end{aligned}
$$

$$
\begin{aligned}
&+ c\,|K(T)-I|\,L\int_{\omega}^{t}|A|ds + L\int_{0}^{T}|K(t)-K(\omega)|\left|U(T)K(T)K^{-1}(s)\right||A|ds\\
&+ c\,|K(T)-I|\,(k_4L+\eta)\int_{\omega}^{t}|A|ds + (k_4L+\eta)\\
&\times\int_{0}^{T}|K(t)-K(\omega)|\left|U(T)K(T)K^{-1}(s)\right||A|ds\\
&+ |K(T)-I|\,\alpha|A|(k_4L+\eta)\int_{\omega}^{t}|A|ds + \alpha|A|(k_4L+\eta)\\
&\times\int_{0}^{T}|K(t)-K(\omega)|\left|U(T)K(T)K^{-1}(s)\right||A|ds.
\end{aligned}
$$

Then by the dominated convergence theorem $|(R\varphi)(t)-(R\varphi)(\omega)|\to 0$ as $t-\omega\to 0$ independently of $\varphi\in M$. Thus (Rφ) is equicontinuous. Hence by Ascoli-Arzela's theorem, RM is contained in a compact set. □

Lemma 3.2. *Suppose* (2.2)-(2.6) *hold and*

$$
cT\left(k_4|F|+k_2+k_3\right)<1. \tag{3.10}
$$

If B is defined by (3.5)*, and F by* (2.8)*, then B is a contraction.*

Proof. Let B defined by (3.5), and $\varphi,\psi\in M$. By (2.4)-(2.6), we have

$$
\begin{aligned}
&|(B\varphi)(t)-(B\psi)(t)|\\
&=\Big|\int_{t}^{t+T}[K(s)U^{-1}(T)K^{-1}(t)]^{-1}\Big[F(s)\left(h(\varphi(s-\tau_1(s)))-h(\psi(s-\tau_1(s)))\right)\\
&\quad+\left(G(s,\varphi(s),\varphi(s-\tau_2(s)))-G(s,\psi(s),\psi(s-\tau_2(s)))\right)\Big]ds\Big|\\
&\le cT\left(k_4|F|+k_2+k_3\right)\|\varphi-\psi\|.
\end{aligned}
$$

The proof is complete. □

Theorem 3.3. *Let the hypothesis of Lemmas 3.1 and 3.2 hold, M defined by* (3.1)*. If there exist a constant $L>0$ such that*

$$
\begin{aligned}
&k_1L+\beta+\alpha|A|(k_4L+\eta)+cT|A|[k_1L+\beta+L+(k_4L+\eta)+\alpha|A|(k_4L+\eta)]\\
&+cT[|F|(k_4L+\eta)+(k_2+k_3)L+\gamma]\le L,
\end{aligned}
$$

then (1.2) *has a T-periodic solution in M.*

Proof. By Lemma 3.1, R is continuous and RM is contained in a compact set, by Lemma 3.2, B is a contraction. Next, we will show that if $\varphi,\psi\in M$, we have $R\varphi+B\psi\in M$. Let $\varphi,\psi\in M$, we have

$$
\begin{aligned}
\|R\varphi+B\psi\|&\le k_1L+\beta+\alpha|A|(k_4L+\eta)+cT|A|[k_1L+\beta+L+(k_4L+\eta)\\
&\quad+\alpha|A|(k_4L+\eta)]+cT[|F|(k_4L+\eta)+(k_2+k_3)L+\gamma]\le L.
\end{aligned}
$$

Clearly, all the hypotheses of Krasnoselskii's fixed point theorem are satisfied. Thus there exists a fixed point $z\in M$ such that $z=Az+Bz$. By Lemma 2.2, this fixed point is a solution of (1.2). Hence (1.2) has a T-periodic solution in M. □

Theorem 3.4. *Suppose* (2.4)-(2.6) *hold. If*

$$
\begin{aligned}
&k_1+\alpha|A|k_4+cT|A|k_1+cT|A|+cT|A|k_4+cT|A|(\alpha|A|k_4)\\
&+cT|F|k_4+cT(k_2+k_3)<1,
\end{aligned} \tag{3.11}
$$

then (1.2) *has a unique T-periodic solution in P_T.*

Proof. Let the mapping P given by (3.2). For $\varphi_1, \varphi_2 \in P_T$, we have

$$
\begin{aligned}
&|(P\varphi_1)(t) - (P\varphi_2)(t)| \\
&\le |Q(t, \varphi_1(t-\tau_2(t))) - Q(t, \varphi_2(t-\tau_2(t)))| + \int_{t-\tau_1(t)}^{t} |A|\,|h(\varphi_1(s)) - h(\varphi_2(s))|\,ds \\
&\quad + \int_t^{t+T} \big|[K(s)U^{-1}(T)K^{-1}(t)]^{-1}\big|\,|A|\,|Q(s, \varphi_1(s-\tau_2(s))) - Q(s, \varphi_2(s-\tau_2(s)))|\,ds \\
&\quad + \int_t^{t+T} \big|[K(s)U^{-1}(T)K^{-1}(t)]^{-1}\big|\,|A||\varphi_1(s) - \varphi_2(s)|ds \\
&\quad + \int_t^{t+T} \big|[K(s)U^{-1}(T)K^{-1}(t)]^{-1}\big|\,|A||h(\varphi_1(s)) - h(\varphi_2(s))|ds \\
&\quad + \int_t^{t+T} \big|[K(s)U^{-1}(T)K^{-1}(t)]^{-1}\big|\,|A| \int_{s-\tau_1(s)}^{s} |A|\,|h(\varphi_1(u)) - h(\varphi_2(u))|\;du\,ds \\
&\quad + \int_t^{t+T} \big|[K(s)U^{-1}(T)K^{-1}(t)]^{-1}\big|\,|F|\,|h(\varphi_1(s-\tau_1(s))) - h(\varphi_2(s-\tau_1(s)))|\,ds \\
&\quad + \int_t^{t+T} \big|[K(s)U^{-1}(T)K^{-1}(t)]^{-1}\big|\,\big|G(s, \varphi_1(s), \varphi_1(s-\tau_2(s))) \\
&\quad - G(s, \varphi_2(s), \varphi_2(s-\tau_2(s)))\big|ds \\
&\le k_1\|\varphi_1 - \varphi_2\| + \alpha|A|k_4\|\varphi_1 - \varphi_2\| + cT|A|k_1\|\varphi_1 - \varphi_2\| + cT|A|\|\varphi_1 - \varphi_2\| \\
&\quad + cT|A|k_4\|\varphi_1 - \varphi_2\| + cT|A|\,(\alpha|A|k_4\|\varphi_1 - \varphi_2\|) + cT|F|k_4\|\varphi_1 - \varphi_2\| \\
&\quad + cT(k_2 + k_3)\|\varphi_1 - \varphi_2\| \\
&= [k_1 + \alpha|A|k_4 + cT|A|k_1 + cT|A| + cT|A|k_4 + cT|A|(\alpha|A|k_4) \\
&\quad + cT|F|k_4 + cT(k_2 + k_3)]\|\varphi_1 - \varphi_2\|.
\end{aligned}
$$

Since (3.11) holds, with the contraction mapping principle we complete the proof. $\square$

Note that, when $h(x) \equiv x$, Theorem 3.3 and 3.4 reduce to [14, Theorems 3.1 and 3.2].

4. Asymptotic stability of the zero solution

In this section, we study the asymptotic stability of the zero solution of the nonlinear system

$$
\frac{d}{dt}x(t) = A(t)h(x(t-\tau_1(t))) + \frac{d}{dt}Q(t, x(t-\tau_2(t))) + G(t, x(t), x(t-\tau_2(t))), \tag{4.1}
$$

with the initial function

$$
x(t) = \psi(t), \quad t \in [m(t_0), t_0], \tag{4.2}
$$

where $\psi \in C([m(t_0), t_0], \mathbb{R}^n)$. $A, h, Q, G, \tau_1, \tau_2$ are defined as in the previous section, and for $t \ge t_0$,

$$
m_j(t_0) = \inf\{t - r_j(t), t \ge t_0\}, \quad m(t_0) = \inf\{m_j(t_0), j = 1, 2\}.
$$

We assume that

$$
Q(t, 0) = G(t, 0, 0) = h(0) = 0. \tag{4.3}
$$

Now we obtain sufficient conditions for the asymptotic stability of the zero solution for system (4.1) by using contraction mapping principle. We define the space

$$\begin{aligned}S_\psi = \big\{\varphi \in C(\mathbb{R}, \mathbb{R}^n) : \varphi(t) = \psi(t) \text{ if } m(t_0) \le t \le t_0,\\ \varphi(t) \to 0 \text{ as } t \to \infty,\ \varphi \text{ is bounded}\big\}.\end{aligned}$$

Then, $(S_\psi, \|\cdot\|)$ is a complete metric space with the supremum norm $\|\cdot\|$.

Definition 4.1. For each initial value $(t_0, \psi) \in (0, \infty) \times S_\psi$, a function x is called a solution of (4.1) associated with (t_0, ψ) if $x \in C([m(t_0), \infty), \mathbb{R}^n)$ satisfies (4.1) for almost all $t \ge t_0$ and $x = \psi$ for $t \le t_0$. Such a solution is denoted by $x(t) = x(t, t_0, \psi)$.

Definition 4.2. If $\Phi(t)$ is a fundamental matrix solution for system (2.1), then $\Phi(t, r) := \Phi(t)\Phi^{-1}(r)$ is the state transition matrix. Also, the state transition matrix satisfies the Chapman-Kolmogorov identities

$$\begin{gathered}\Phi(r, r) = I, \quad \Phi(t, s) \quad \Phi(s, r) = \Phi(t, r),\\ \Phi^{-1}(t, s) = \Phi(s, t), \quad \frac{\partial \Phi(t, s)}{\partial s} = -\Phi(t, s)A(s).\end{gathered}$$

In our analysis we use the fundamental matrix solution of (2.1) to transform (4.1) into an integral equation. Then we employ the contraction mapping principle to show the asymptotic stability of the zero solution of (4.1).

Lemma 4.3. *x is a solution of the* (4.1) *if and only if*

$$\begin{aligned}x(t) &= Q(t, x(t-\tau_2(t))) - \int_{t-\tau_1(t)}^t A(s)h(x(s))ds\\ &\quad + \Phi(t, t_0)\Big[x(t_0) + \int_{t_0-\tau_1(t_0)}^{t_0} A(s)h(x(s))ds - Q(t_0, x(t_0 - \tau_2(t_0)))\Big]\\ &\quad + \int_{t_0}^t \Phi(t, s)\Big\{F(s)h(x(s)) - A(s)[x(s) - h(x(s))] + A(s)Q(s, x(s-\tau_2(s)))\\ &\quad - A(s)\int_{s-\tau_1(s)}^t A(u)h(x(u))du + G(s, x(s), x(s-\tau_2(s)))\Big\}ds,\end{aligned} \tag{4.4}$$

where

$$F(t) = A(t) - (1 - \tau_1'(t))A(t - \tau_1(t)).$$

Proof. Let x be a solution of (4.1) and $\Phi(t)$ is a fundamental matrix solution of (2.1). We rewrite (4.1) as

$$\begin{aligned}&\frac{d}{dt}\Big[x(t) + \int_{t-\tau_1(t)}^t A(s)h(x(s))ds - Q(t, x(t-\tau_2(t)))\Big]\\ &= A(t)\Big[x(t) + \int_{t-\tau_1(t)}^t A(s)h(x(s))ds - Q(t, x(t-\tau_2(t)))\Big]\\ &\quad + F(t)h(x(t-\tau_1(t))) - A(t)[x(t) - h(x(t))] + A(t)Q(t, x(t-\tau_2(t)))\\ &\quad - A(t)\int_{t-\tau_1(t)}^t A(s)h(x(s))ds + G(t, x(t), x(t-\tau_2(t))),\end{aligned}$$

where $F(t) = A(t) - (1 - \tau_1'(t))A(t - \tau_1(t))$.

Defining a new function

$$z(t)=\Phi^{-1}(t)\Big[x(t)+\int_{t-\tau_1(t)}^{t}A(s)h(x(s))ds-Q(t,x(t-\tau_2(t)))\Big],$$

we have

$$\begin{aligned}\frac{d}{dt}z(t)&=\frac{d}{dt}\Phi^{-1}(t)\Big[x(t)+\int_{t-\tau_1(t)}^{t}A(s)h(x(s))ds-Q(t,x(t-\tau_2(t)))\Big]\\&\quad+\Phi^{-1}(t)\frac{d}{dt}\Big[x(t)+\int_{t-\tau_1(t)}^{t}A(s)h(x(s))ds-Q(t,x(t-\tau_2(t)))\Big].\end{aligned}$$

From Definition 4.2, it follows that $\frac{d}{dt}\Phi^{-1}(t)=-\Phi^{-1}(t)A(t)$. Then

$$\begin{aligned}\frac{d}{dt}z(t)&=\frac{d}{dt}\Phi^{-1}(t)\Big[x(t)+\int_{t-\tau_1(t)}^{t}A(s)h(x(s))ds-Q(t,x(t-\tau_2(t)))\Big]\\&\quad+\Phi^{-1}(t)\frac{d}{dt}\Big[x(t)+\int_{t-\tau_1(t)}^{t}A(s)h(x(s))ds-Q(t,x(t-\tau_2(t)))\Big]\\&=-\Phi^{-1}(t)A(t)\Big[x(t)+\int_{t-\tau_1(t)}^{t}A(s)h(x(s))ds-Q(t,x(t-\tau_2(t)))\Big]\\&\quad+\Phi^{-1}(t)\frac{d}{dt}\Big[x(t)+\int_{t-\tau_1(t)}^{t}A(s)h(x(s))ds-Q(t,x(t-\tau_2(t)))\Big]\\&=\Phi^{-1}(t)\Big\{F(t)h(x(t-\tau_1(t)))-A(t)\left[x(t)-h(x(t))\right]+A(t)Q(t,x(t-\tau_2(t)))\\&\quad-A(t)\int_{t-\tau_1(t)}^{t}A(s)h(x(s))ds+G(t,x(t),x(t-\tau_2(t)))\Big\}.\end{aligned}$$

Also note that

$$z(t_0)=\Phi^{-1}(t_0)\Big[x(t_0)+\int_{t_0-\tau_1(t_0)}^{t_0}A(s)h(x(s))ds-Q(t_0,x(t_0-\tau_2(t_0)))\Big].$$

Integrating from t_0 to t, we have

$$\begin{aligned}&z(t)-z(t_0)\\&=\int_{t_0}^{t}\Phi^{-1}(s)\Big\{F(s)h(x(s-\tau_1(s)))-A(s)\left[x(s)-h(x(s))\right]+A(s)Q(s,x(s-\tau_2(s)))\\&\quad-A(s)\int_{s-\tau_1(s)}^{s}A(u)h(x(u))du+G(s,x(s),x(s-\tau_2(s)))\Big\}ds.\end{aligned}$$

This yields

$$\begin{aligned}&\Phi^{-1}(t)\Big[x(t)+\int_{t-\tau_1(t)}^{t}A(s)h(x(s))ds-Q(t,x(t-\tau_2(t)))\Big]\\&=\Phi^{-1}(t_0)\Big[x(t_0)+\int_{t_0-\tau_1(t_0)}^{t_0}A(s)h(x(s))ds-Q(t_0,x(t_0-\tau_2(t_0)))\Big]\\&\quad+\int_{t_0}^{t}\Phi^{-1}(s)\Big\{F(s)h(x(s-\tau_1(s)))-A(s)\left[x(s)-h(x(s))\right]\\&\quad+A(s)Q(s,x(s-\tau_2(s)))\end{aligned}$$

$$- A(s)\int_{s-\tau_1(s)}^{s} A(u)h(x(u))du + G(s,x(s),x(s-\tau_2(s)))\Big\}ds,$$

which yields

$$\begin{aligned}
x(t) &= Q(t,x(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(x(s))ds \\
&\quad + \Phi(t,t_0)\Big[x(t_0) + \int_{t_0-\tau_1(t_0)}^{t_0} A(s)h(x(s))ds - Q(t_0,x(t_0-\tau_2(t_0)))\Big] \\
&\quad + \int_{t_0}^{t} \Phi(t,s)\Big\{F(s)h(x(s-\tau_1(s))) - A(s)\left[x(s) - h(x(s))\right] \\
&\quad + A(s)Q(s,x(s-\tau_2(s))) \\
&\quad - A(s)\int_{s-\tau_1(s)}^{s} A(u)h(x(u))du + G(s,x(s),x(s-\tau_2(s)))\Big\}ds.
\end{aligned}$$

The converse implication is easily obtained. the proof is complete. □

To obtain a sufficient condition of the asymptotic stability of the zero solution of (4.1), we assume that

$$\Phi(t) \to 0, \quad t \to \infty, \tag{4.5}$$

$$t - \tau_1(t) \to \infty, \quad t \to \infty, \tag{4.6}$$

$$t - \tau_2(t) \to \infty, \quad t \to \infty, \tag{4.7}$$

and that there is $\beta > 0$, such that

$$\begin{aligned}
&k_1 + k_4\int_{t-\tau_1(t)}^{t} |A|ds + \int_{t_0}^{t} |\Phi(t,s)|\Big[k_4|F| + (1+k_1+k_4)|A| \\
&+ k_4|A|\int_{s-\tau_1(s)}^{s} |A|du + k_2 + k_3\Big]ds \le \beta < 1, \quad \text{for } t \ge t_0.
\end{aligned}\tag{4.8}$$

Theorem 4.4. *Let us assume that* (2.4)-(2.6) *and* (4.5)-(4.8) *hold. Then every solution* $x(t,t_0,\psi)$ *of* (4.1)*, with small continuous initial function* ψ*, is bounded and asymptotically stable.*

Proof. We define the mapping P based on Lemma 4.3:

$$\begin{aligned}
(P\varphi)(t) &= Q(t,\varphi(t-\tau_2(t))) - \int_{t-\tau_1(t)}^{t} A(s)h(\varphi(s))ds \\
&\quad + \Phi(t,t_0)\Big[\varphi(t_0) + \int_{t_0-\tau_1(t_0)}^{t_0} A(s)h(\varphi(s))ds - Q(t_0,\varphi(t_0-\tau_2(t_0)))\Big] \\
&\quad + \int_{t_0}^{t} \Phi(t,s)\Big\{F(s)h(\varphi(s)-\tau_1(s)) - A(s)[\varphi(s) - h(\varphi(s))] \\
&\quad + A(s)Q(s,\varphi(s-\tau_2(s))) \\
&\quad - A(s)\int_{s-\tau_1(s)}^{t} A(u)h(\varphi(u))du + G(s,\varphi(s),\varphi(s-\tau_2(s)))\Big\}ds.
\end{aligned}$$

Since Q, G, h are continuous, it is easy to show that P is continuous. Let ψ be a small given continuous initial function with $\|\psi\| < \delta(\delta > 0)$. Since $\varphi \in S_\psi$, there is a constant $L > 0$ such that $\|\varphi\| \le L$. By choosing a suitable δ, we have

$$\begin{aligned}
|(P\varphi)(t)| &\le |Q(t,\varphi(t-\tau_2(t)))| + \int_{t-\tau_1(t)}^{t} |A||h(\varphi(s))|ds \\
&\quad + |\Phi(t,t_0)|\Big[|\varphi(t_0)| + \int_{t_0-\tau_1(t_0)}^{t_0} |A||h(\varphi(s))|ds + |Q(t_0,\varphi(t_0-\tau_2(t_0)))|\Big] \\
&\quad + \int_{t_0}^{t} |\Phi(t,s)|\Big\{|F||h(\varphi(s)-\tau_2(s))| + |A|[|\varphi(s)| + |h(\varphi(s))|] \\
&\quad + |A||Q(s,\varphi(s-\tau_2(s)))| \\
&\quad + |A|\int_{s-\tau_1(s)}^{s} |A(u)||h(\varphi(u))|du + |G(s,\varphi(s),\varphi(s-\tau_2(s)))|\Big\}ds \\
&\le k_1 L + k_4 L \int_{t-\tau_1(t)}^{t} |A|ds + |\Phi|\delta(1 + k_4\int_{t_0-\tau_1(t_0)}^{t_0} |A|ds + k_1) \\
&\quad + L\int_{t_0}^{t} |\Phi(t,s)|\Big[k_4|F| + (1+k_1+k_4)|A| \\
&\quad + k_4|A|\int_{s-\tau_1(s)}^{s} |A|du + (k_2+k_3)\Big] \\
&\le \beta L + |\Phi|\delta(1 + k_4\int_{t_0-\tau_1(t_0)}^{t_0} |A|ds + k_1),
\end{aligned}$$

which implies that $P\varphi$ is bounded.

Then we show that $(P\varphi)(t) \to 0$ as $t \to \infty$. By (4.3) and (4.5)-(4.7), we can easily have that $Q(t,\varphi(t-\tau_2(t))) \to 0$, $\int_{t-\tau_1(t)}^{t} A(s)h(\varphi(s))ds \to 0$ and

$$\Phi(t,t_0)\Big[\varphi(t_0) + \int_{t_0-\tau_1(t_0)}^{t_0} A(s)h(\varphi(s))ds - Q(t_0,\varphi(t_0-\tau_2(t_0)))\Big] \to 0 \quad \text{as } t \to \infty.$$

Let $\epsilon > 0$ be given, then there exist a $t_1 > t_0$ such that for $t > t_1$, $|\varphi(t-\tau_1(t))| < \epsilon$. By (4.5), there exist a $t_2 > t_1$ such that for $t > t_2$ implies $|\Phi(t,t_2)| < \frac{\epsilon}{\beta L}$. Thus for $t > t_2$, we have

$$\begin{aligned}
&\int_{t_0}^{t} |\Phi(t,s)|\Big\{|F||h(\varphi(s)-\tau_2(s))| + |A|[|\varphi(s)| + |h(\varphi(s))|] + |A||Q(s,\varphi(s-\tau_2(s)))| \\
&\quad + |A|\int_{s-\tau_1(s)}^{s} |A||h(\varphi(u))|du + |G(s,\varphi(s),\varphi(s-\tau_2(s)))|\Big\}ds \\
&\le L\int_{t_0}^{t_1} |\Phi(t,s)|\Big[k_4|F| + (1+k_1+k_4)|A| + k_4|A|\int_{s-\tau_1(s)}^{s} |A|du + (k_2+k_3)\Big]ds \\
&\quad + \epsilon\int_{t_1}^{t} |\Phi(t,s)|\Big[k_4|F| + (1+k_1+k_4)|A| + k_4|A|\int_{s-\tau_1(s)}^{s} |A|du + (k_2+k_3)\Big]ds \\
&\le L|\Phi(t,t_2)|\int_{t_0}^{t_1} |\Phi(t_2,s)|\Big[k_4|F| + (1+k_1+k_4)|A| \\
&\quad + k_4|A|\int_{s-\tau_1(s)}^{s} |A|du + (k_2+k_3)\Big]ds + \beta\epsilon \\
&\le \beta L|\Phi(t,t_2)| + \beta\epsilon < \epsilon + \beta\epsilon.
\end{aligned}$$

Hence, $(P\varphi)(t) \to 0$ as $t \to \infty$.

Now we show that P is a contraction under the supremum norm. Let $\varphi_1, \varphi_2 \in S_\psi$, then

$$
\begin{aligned}
&|(P\varphi_1)(t) - (P\varphi_2)(t)| \\
&\le |Q(t, \varphi_1(t-\tau_2(t))) - Q(t, \varphi_2(t-\tau_2(t)))| + \int_{t-\tau_1(t)}^t |A|\,|h(\varphi_1(s)) - h(\varphi_2(s))|\,ds \\
&\quad + \int_{t_0}^t |\Phi(t,s)| \Big[k_4|F| + (1+k_1+k_4)|A| \\
&\quad + k_4|A| \int_{s-\tau_1(s)}^s |A| du + k_2 + k_3\Big] ds \|\varphi_1 - \varphi_2\| \\
&\le \beta\|\varphi_1 - \varphi_2\|.
\end{aligned}
$$

Since $\beta < 1$ as defined by (4.8), the contraction mapping principle implies that P has a unique fixed point in S_ψ which satisfies (4.1).

Lastly, we need to show that the zero solution of (4.1) is stable. Choose a δ such that $2\delta K + \beta\epsilon < \epsilon$, where $K = \sup_{t\in[t_0,\infty)} \Phi(t,t_0)$, and β is defined by (4.8). For $\|\psi\| \le \delta$, we claim that $|x(t)| < \epsilon$. Suppose that there exists a $t' > t_0$ such that $|x(t')| \ge \epsilon$, and $t^* = \inf\{t' : x(t') \ge \epsilon\}$. By the integral representation of $x(t)$, we have

$$
\begin{aligned}
&|x(t^*)| \\
&\le k_1\epsilon + k_4 L \int_{t-\tau_1(t)}^t |A| ds + |\Phi(t,t_0)|\delta(1 + k_4 \int_{t_0-\tau_1(t_0)}^{t_0} |A| h ds + k_1) \\
&\quad + \epsilon \int_{t_0}^t |\Phi(t,s)| \Big[k_4|F| + (1+k_1+k_4)|A| + k_4|A| \int_{s-\tau_1(s)}^s |A| du + (k_2+k_3)\Big] \\
&\le 2\delta K + \beta\epsilon < \epsilon,
\end{aligned}
$$

which contradicts the definition of t^*. Therefore, the zero solution of (4.1) is stable. Hence, the fixed point is bounded and asymptotically stable. ☐

5. An example

We consider the two-dimensional system

$$
\begin{aligned}
\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} &= \begin{pmatrix} p(t) & 1 \\ 0 & q(t) \end{pmatrix} \begin{pmatrix} h(x_1(t-\tau_1(t))) \\ h(x_2(t-\tau_1(t))) \end{pmatrix} + \frac{d}{dt} \begin{pmatrix} 0 \\ V(t, x_1(t-\tau_2(t))) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ W(t, x_1(t-\tau_2(t))) \end{pmatrix},
\end{aligned} \tag{5.1}
$$

where p and q are positive periodic continuous functions with periodic T. The functions $h : \mathbb{R} \to \mathbb{R}$, $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $W : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous in their respective arguments, $\tau_1(\cdot), \tau_2(\cdot)$ satisfy (2.2).

Functions $V(t,x)$ and $W(t,x,y)$ are periodic in t with period T. They are also globally Lipschitz continuous in x and y respectively. That is

$$
V(t+T, x) = V(t,x), \quad W(t+T, x, y) = W(t,x,y), \tag{5.2}
$$

and there are positive constants k_1, k_2, k_3 such that

$$
|V(t,x) - V(t,y)| \le k_1\|x - y\|, \tag{5.3}
$$

$$|W(t,x,y) - W(t,z,w)| \le k_2\|x-z\| + k_3\|y-w\|, \tag{5.4}$$

$$|h(x) - h(y)| \le k_4\|x-y\|. \tag{5.5}$$

Let $q(t) = p(t) = -1$, $h(x) = \frac{1}{3}\sin(x+\frac{\pi}{3})$, $\tau_1(t) = \frac{\pi}{50}\sin^2(\pi t)$, $\tau_2(\cdot)$ is a non-negative and continuous function with period of T, $V(t,x) = \frac{1}{5}sin(2\pi t)\sin(x+\frac{\pi}{6})$, $W(t,x,y) = \frac{1}{8}\cos(2\pi t)\sin(x) + \frac{1}{7}\sin(y+\frac{\pi}{3})$. Consider the Banach space $(C_1, \|\cdot\|)$

$$C_1 = \{\varphi \in C(\mathbb{R},\mathbb{R}),\ \varphi(t+1) = \varphi,\ t\in\mathbb{R}\},$$

and the closed bounded convex subset $M = \{\varphi \in C_1 : \|\varphi\| \le \pi\}$. Then for $x,y,z,w \in M$, we have

$$|V(t,x) - V(t,y)| \le \frac{1}{5}\|x-y\|, \quad |W(t,x,y) - W(t,z,w)| \le \frac{1}{8}\|x-z\| + \frac{1}{7}\|y-w\|,$$

$$|h(x)-h(y)| \le \frac{1}{3}\|x-y\|, \quad \alpha = \sup_{t\in[0,T]}|\tau_1(t)| = \frac{\pi}{50},$$

$$\beta = \sup_{t\in[0,T]}|V(t,0)| = \frac{1}{10}, \quad \gamma = \sup_{t\in[0,T]}|W(t,0,0)| = \frac{1}{7},$$

$$\eta = |h(0)| = \frac{\sqrt{3}}{6}, \quad |F| = |A(t) - (1-\tau_1'(t))A(1-\tau_1(t))| < 0.04, \quad c \le 0.34.$$

Consequently,

$$\begin{aligned}
&k_1L + \beta + \alpha|A|(k_4L+\eta) + cT|A|[k_1L + \beta + L + (k_4L+\eta) + \alpha|A|(k_4L+\eta)]\\
&+ cT[|F|(k_4L+\eta) + (k_2+k_3)L + \gamma]\\
&\le \frac{\pi}{5} + \frac{1}{10} + \frac{\pi}{50}(\frac{\pi}{3} + \frac{\sqrt{3}}{6}) + 0.34[\frac{\pi}{5} + \frac{1}{10} + \pi + \frac{\pi}{3} + \frac{\sqrt{3}}{6} + \frac{\pi}{50}(\frac{\pi}{3} + \frac{\sqrt{3}}{6})]\\
&\quad + 0.34[0.04(\frac{\pi}{3} + \frac{\sqrt{3}}{6}) + (\frac{1}{7} + \frac{1}{8})\pi + \frac{1}{7}] \le \pi.
\end{aligned}$$

Then (5.1) has a 1-periodic solution in M. Moreover,

$$\begin{aligned}
&k_1 + \alpha|A|k_4 + cT|A|k_1 + cT|A| + cT|A|k_4 + cT|A|(\alpha|A|k_4) + cT|F|k_4\\
&+ cT(k_2+k_3)\\
&\le \frac{1}{5} + \frac{\pi}{50}\frac{1}{3} + 0.34\frac{1}{5} + 0.34 + 0.34\frac{1}{3} + 0.34\frac{\pi}{50}(\frac{\pi}{50}\frac{1}{3}) + 0.34(0.04)\frac{1}{3} + 0.34\frac{15}{56}\\
&\le 1.
\end{aligned}$$

Then (5.1) has a unique 1-periodic solution in M.

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