Electronic Journal of Differential Equations, Vol. 2024 (2024), No. 22, pp. 1–27.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2024.22

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO NONCLASSICAL DIFFUSION EQUATIONS WITH DEGENERATE MEMORY AND A TIME-DEPENDENT PERTURBED PARAMETER

JIANGWEI ZHANG, ZHE XIE, YONGQIN XIE

Abstract. This article concerns the asymptotic behavior of solutions for a class of nonclassical diffusion equation with time-dependent perturbation coefficient and degenerate memory. We prove the existence and uniqueness of time-dependent global attractors in the family of time-dependent product spaces, by applying the operator decomposition technique and the contractive function method. Then we study the asymptotic structure of time-dependent global attractors as $t \to \infty$. It is worth noting that the memory kernel function satisfies general assumption, and the nonlinearity f satisfies a polynomial growth of arbitrary order.

1. Introduction

In this article, we discuss the long-term behavior of solutions of the perturbed nonclassical diffusion equation with degenerate memory,

$$u_t - \varepsilon(t)\Delta u_t - \Delta u - \int_0^\infty k(s)\operatorname{div}\{a(x)\nabla u(t-s)\}ds + f(u) = g(x), \tag{1.1}$$

with boundary condition

$$u(x,t)|_{\partial\Omega} = 0, \tag{1.2}$$

and initial conditions

$$u(x,\tau) = u_\tau(x), \quad u(x,\tau - s) = u_\tau(x,\tau - s), \quad s \geq 0, \tag{1.3}$$

where $(x,t) \in \Omega \times (\tau,\infty)$, and $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\tau \in \mathbb{R}$ is the initial time, $g = g(x) \in L^2(\Omega)$ is the the external force, $a(x)$ satisfies conditions specified in (H3) below. The perturbation parameter $\varepsilon(t) \in C^1(\mathbb{R})$ is assumed to be a decreasing bounded function satisfying

$$\lim_{t\to+\infty} \varepsilon(t) = 0, \tag{1.4}$$

and there exists $L > 0$, such that

$$\sup_{t\in\mathbb{R}}(|\varepsilon(t)| + |\varepsilon'(t)|) \leq L. \tag{1.5}$$

We use the following hypotheses

2020 *Mathematics Subject Classification.* 35K57, 35B40, 35B41.
Key words and phrases. Nonclassical diffusion equation; time-dependent global attractor; polynomial growth; contractive function; asymptotic structure.

Submitted November 9, 2023. Published March 12, 2024.

(H1) $k(s)=\int_s^\infty \mu(r)dr>0$, where the integrand μ satisfies that for any interval $[0,T]$ with $T>0$,

$$\mu\in L^1(\mathbb{R}^+) \text{ is a decreasing piecewise absolutely continuous function,} \tag{1.6}$$

and that there exists $\delta>0$, such that

$$k(s)\le \delta\mu(s),\quad \forall s\in\mathbb{R}^+. \tag{1.7}$$

As described in [13], inequality (1.7) is equivalent to the requirement that

$$\mu(t+s)\le \mathfrak{K}e^{-\delta t}\mu(s), \tag{1.8}$$

for some $\mathfrak{K}\ge 1$, $\delta>0$, any $t\ge 0$, and almost every $s>0$. Moreover, if μ belongs to $C^1(\mathbb{R}^+)$, then $\mu'(s)+\delta\mu(s)\le 0$ can derive (1.8) when $\mathfrak{K}=1$, which shows that the same condition for $\mathfrak{K}\ge 1$ is more general.

For simplicity, we let

$$m_0:=k(0)=\int_0^\infty \mu(s)ds<\infty.$$

(H2) The nonlinearity f satisfies $f\in C^1$, $f(0)=0$, and the arbitrary order polynomial growth restriction

$$\alpha_1|s|^p-\beta_1\le f(s)s\le \alpha_2|s|^p+\beta_2,\quad \forall s\in\mathbb{R},\ p\ge 2, \tag{1.9}$$

and the dissipative condition

$$f'(s)\ge -l, \tag{1.10}$$

where α_i,β_i $(i=1,2)$ and l are positive constants.

Denoting $F(s)=\int_\tau^s f(\sigma)d\sigma$ we can confirm that there exist $\tilde\alpha_i,\tilde\beta_i>0$ $(i=1,2)$ from (1.9), such that

$$\tilde\alpha_1|s|^p-\tilde\beta_1\le F(s)\le \tilde\alpha_2|s|^p+\tilde\beta_2,,\quad \forall s\in\mathbb{R}. \tag{1.11}$$

(H3) $a(x)\in C^\infty(\Omega)\cap C^0(\overline{\Omega})$ is non-negative, and there exists a connected set $A\subset\subset\Omega$ such that

$$a(x)=0 \iff x\in A.$$

Following Dafermos [14] we introducing an additional variable η^t, which is the history of u, i.e.

$$\eta^t=\eta^t(x,s):=\int_0^s u(x,t-r)\,dr,\quad s\in\mathbb{R}^+. \tag{1.12}$$

Let $\eta_t^t=\frac{\partial}{\partial t}\eta^t$ and $\eta_s^t=\frac{\partial}{\partial s}\eta^t$, it follows that

$$\eta_t^t=-\eta_s^t+u. \tag{1.13}$$

By (H1), (1.12) and (1.13) yields

$$\int_0^\infty k(s)\operatorname{div}\{a(x)\nabla u(t-s)\}ds=\int_0^\infty \mu(s)\operatorname{div}\{a(x)\nabla\eta^t(s)\}ds. \tag{1.14}$$

Thus, system (1.1)-(1.3) can be rewritten as

$$\begin{gathered} u_t-\varepsilon(t)\Delta u_t-\Delta u-\int_0^\infty \mu(s)\operatorname{div}\{a(x)\nabla\eta^t(s)\}ds+f(u)=g(x),\\ \eta_t^t=-\eta_s^t+u. \end{gathered} \tag{1.15}$$

with the initial and boundary conditions

$$\begin{gathered} u(x,t)|_{\partial\Omega}=0,\quad \eta^t(x,s)|_{\partial\Omega\times\mathbb{R}^+}=0,\quad t\in(\tau,\infty),\\ u(x,\tau)=u_\tau(x),\quad \eta^\tau(x,s)=\int_0^s u_\tau(x,\tau-r)dr,\quad (x,s)\in\Omega\times\mathbb{R}^+. \end{gathered}\tag{1.16}$$

Throughout this article, unless otherwise stated, we assume that $z(t)=(u(t),\eta^t)$ is the solution of system (1.15)-(1.16) with initial value $z_\tau=(u_\tau,\eta^\tau)$, and let $\mathbb{R}^\tau=[\tau,\infty)$, $\mathbb{R}^+=[0,\infty)$.

Equation (1.1), as a nonclassical diffusion equation, is well known for its mathematical and physical significance. For instance, it is usually utilized in the various fields, including fluid mechanics, solid mechanics, and heat conduction theory, see [8, 2, 21]. In contrast to the classical reaction-diffusion equation, it mainly considers viscous factor and historical influence of u and this historical influence (i.e., memory term $\int_0^\infty \mu(s)\operatorname{div}\{a(x)\nabla\eta^t(s)\}ds$) is degenerate. Specifically, the degeneracy is reflected in the sense that the function $a(x)\geq 0$ in $\int_0^\infty \mu(s)\operatorname{div}\{a(x)\nabla\eta^t(s)\}ds$ is allowed to vanish in some positive measure subset ω_0 of $\overline{\Omega}$.

When $\varepsilon(t)$ is zero (or a positive constant) and memory term is non-degenerate, it is easy to show that the equation (1.1) becomes the usual reaction-diffusion equation (or nonclassical diffusion equation) with memory, under these circumstances, the asymptotic behavior of solutions has been researched by many scholars in recent years (see [9, 17, 18, 19, 31, 35, 36, 37]). Especially to deserve to be mentioned, more recently, the authors considered the existence, regularity and upper semicontinuity of global and uniform attractors for autonomous and non-autonomous nonclassical diffusion equation lacking instantaneous damping $-\Delta u$ in bounded and unbounded domain when the nonlinearity satisfies critical exponential growth and polynomial growth of arbitrary order respectively, see [10, 29, 32, 33, 36, 38, 39].

Nevertheless, for equation (1.1) with time-dependent parameter, the current studies focus on the nonclassical diffusion equation without memory (i.e., $k(s)=0$ in (1.1)), see [23, 24, 30, 40] and the references therein. In [23, 24, 40], the authors proved the existence of time-dependent global attractors in $\mathcal{H}_t$ when the nonlinearity f satisfies the subcritical exponential growth, critical exponential growth, and polynomial growth of arbitrary $p-1$ ($p\geq 2$) order respectively. Particularly, when f meets polynomial growth of arbitrary order, the authors of [30] proved the existence, regularity and the asymptotic structure of the time-dependent global attractors for the equation

$$u_t-\varepsilon(t)\Delta u_t-\Delta u+\lambda u+f(u)=g(x).\tag{1.17}$$

To sum up, we try to consider the long-term behavior of equation (1.17) with memory (i.e., the case of non-degenerate) and without linear damping. At this point, if the memory term satisfies classical assumption $\mu(s)+\delta\mu'(s)\leq 0$ (see [31, 35]), then the results we obtain are perfectly predictable. This allows us to think of the asymptotic behavior of problem (1.1) under the premises that the (degenerate) memory term satisfies the weaker assumption (1.7) and the nonlinearity fulfills polynomial growth of arbitrary order?

So why would we consider the system (1.1) with the degenerated memory? In [4], the authors completed a pioneering work, namely, the uniform decay of solutions was obtained for degenerate problem

$$u_t-(a(x)u_x)_x+b(x)u=0,$$

where $a(x) \in C^1([0,1])$ satisfies $a(x) > 0$ and $a(x)|_{x=0,1} = 0$, and $b(x) \in C([0,1])$ meets $b(x) \geq 0$. Whereafter, some authors investigated the asymptotic behavior and stability of solutions for above problem under suitable assumptions, see [16, 1] and the references therein. In addition, many scholars considered wave equation with degenerate memory, see [5, 6, 7, 27] and the references therein. In should be emphasized that the authors in [27] obtained regularity of global attractors for the following wave equation with degenerate memory

$$u_{tt} - \Delta u + \int_0^\infty g(s) div[a(x)\nabla u(t-s)]ds + b(x)u_t + f(u) = g(x).$$

Furthermore, Faria et al. showed the existence of global attractors for the following heat equation with degenerate memory

$$\theta_t - k_0\Delta\theta - \int_{-\infty}^t k(t-s) div[a(x)\nabla\theta(s)]ds + f(\theta) = g$$

in recent a study [15] when the nonlinear term $f(\theta)$ fulfills critical exponential growth and the memory kernel function satisfies the weaker conditions (see (H1)).

The ideas in [15, 39] inspires us to take into account the existence and uniqueness of time-dependent global attractors for equation (1.1) when the assumptions (H1) and (H2) hold, which answers the question we posed earlier. Moreover, in this article, we also incidentally consider asymptotic structure of time-dependent global attractors based on existing studies in [11, 12, 30]. Thus, it is a comprehensive and innovative problem for us to think about the existence, uniqueness and the asymptotic structure of time-dependent global attractors for the equation (1.1), and this article improves the existing work in [15, 32, 33, 37].

Of course, we need to overcome the following two difficulties for solving foregoing problem:

(i) On the one hand, because the nonlinearity f has polynomial growth of arbitrary order and equation (1.1) includes the memory term, we cannot use Sobolev compact embedding to verify asymptotic compactness of the solution process generated by equation (1.1) as [15].
(ii) On the other hand, since the memory term is degenerate and the memory kernel function $k(s)$ satisfies the weaker condition (H1), which does not allow us to the classical estimation methods from [18, 31, 35, 37] in our problem.

To solve the above difficulties, we use some ingenious analytical techniques, and the operator decomposition method is adopted to obtain constructive function. Then we can verify that the process $\{U(t,\tau)\}_{t\geq\tau}$ generated by equation (1.1) is pullback asymptotically compact. Meanwhile, the asymptotic regularity of solutions for equation (1.1) is also obtained, and it follows that we can construct the contractive function and further show pullback asymptotical compactness of the corresponding process $\{U(t,\tau)\}_{t\geq\tau}$ associated with the equation (4.1). In addition, the study of asymptotic structures shows that the limit relation between the time-dependent attractors for the problem (1.1) and the global attractor for the reaction-diffusion equation with degenerate memory of [15] with the same conditions by using the method in [11].

This article is organized as follows: In Section 2, we recall some basic concepts with respect to the time-dependent global attractors and other some useful results that will be used later. In Section 3, we first prove pullback asymptotic compactness of the process $\{U(t,\tau)\}_{t\geq\tau}$ generated by problem (1.15) by establishing contractive function, and then the existence and uniqueness of time-dependent global attractors are attained for system (1.15)-(1.16). In Section 4, we obtain the limit relation between the time-dependent attractors for the equation (1.1) and the global attractor for the equation (1.1) in [15] satisfying the same conditions.

2. Preliminaries

In this section, we first give some notation used later, and then describe some basic concepts and theories of the existence of time-dependent global attractors, for details see [12, 11].

Basic concepts and notation. Hereafter let $|u|$ be the modular (or absolute value) of u and $|\cdot|_p$ be the norm of $L^p(\Omega)(p\geq 1)$, and $(\cdot,\cdot)$ be the inner product of $L^2(\Omega)$. Let $(\nabla\cdot,\nabla\cdot)$, $(\Delta\cdot,\Delta\cdot)$ and $|\nabla\cdot|_2^2$, $|\Delta\cdot|_2^2$ be the inner products and the equivalent norms of $H_0^1(\Omega)$ and $D(A)=H_0^1(\Omega)\cap H^2(\Omega)$ respectively. The time-dependent spaces $\mathcal{H}_t:=H_0^1(\Omega)$ and $\mathcal{H}_t^1:=D(A)$ are endowed with the corresponding norms

$$\|\cdot\|_{\mathcal{H}_t}^2=|\cdot|_2^2+\varepsilon(t)|\nabla\cdot|_2^2,\quad \|\cdot\|_{\mathcal{H}_t^1}^2=|\nabla\cdot|_2^2+\varepsilon(t)|\Delta\cdot|_2^2.$$

As in [7, 6], we assume that there is a Hilbert space

$$H_a^1(\Omega)=\big\{u\in L^2(\Omega):|\sqrt{a}\nabla u|\in L^2(\Omega),\, u|_{\partial\Omega}=0\big\}$$

with inner-product

$$(u,v)_{H_a^1}=(u,v)+(\sqrt{a}\nabla u,\sqrt{a}\nabla v).$$

We define the weight space

$$\mathcal{V}_a:=L_\mu^2(\mathbb{R}^+;H_a^1)=\big\{\eta^t:\mathbb{R}^+\to H_a^1,\int_0^\infty\mu(s)\|\eta^t\|_{H_a^1}^2<\infty\big\}$$

with inner product and norm

$$\langle\psi,\eta\rangle_{\mu,1}=\int_0^\infty\mu(s)(\psi,\eta)_{H_a^1}ds,\quad \|\eta^t\|_{\mu,1}^2=\int_0^\infty\mu(s)\|\eta^t\|_{H_a^1}^2ds.$$

As in [27] we define the regular Hilbert space

$$H_a^2(\Omega)=\big\{u\in H_0^1(\Omega),\sqrt{a}\Delta u\in L^2(\Omega)\big\},$$

with inner-product

$$(u,v)_{H_a^2}=(\nabla u,\nabla v)+(\sqrt{a}\Delta u,\sqrt{a}\Delta v).$$

We define the weight space

$$\mathcal{V}_a^1:=L_\mu^2(\mathbb{R}^+;H_a^2)=\big\{\eta^t:\mathbb{R}^+\to H_a^2,\int_0^\infty\mu(s)\|\eta^t\|_{H_a^2}^2<\infty\big\}$$

with inner product and norm

$$\langle\psi,\eta\rangle_{\mu,2}=\int_0^\infty\mu(s)(\psi,\eta)_{H_a^2}ds,\quad \|\eta^t\|_{\mu,2}=\int_0^\infty\mu(s)\|\eta^t\|_{H_a^2}^2ds.$$

Additionally, we denote $\mathcal{V}_0=L_\mu^2(\mathbb{R}^+;L^2(\Omega))$ and its inner product and norm

$$\langle\psi,\eta\rangle_{\mu,0}=\int_0^\infty\mu(s)(\psi,\eta)ds,\quad \|\eta^t\|_{\mu,0}^2=\int_0^\infty\mu(s)|\eta^t|_2^2ds.$$

Then phase space of equation (1.1) is

$$\mathcal{M}_t^r=\mathcal{H}_t^r\times\mathcal{V}_a^r\,(r=0,1).$$

and $\|\cdot\|^2_{\mathcal{M}_t^r} = \|\cdot\|^2_{\mathcal{H}_t^r} + \|\cdot\|^2_{\mu,r+1}$, where $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ represents a family of time-dependent normed spaces, and it should be noted that the superscript is omitted when $r = 0$.

Remark 2.1. As stated in [7, 27, 5], there exists reasonable continuous embedding $H_a^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow H_a^1(\Omega) \hookrightarrow L^2(\Omega)$; in particular, $H_a^2(\Omega) \hookrightarrow H_a^1(\Omega)$ is compact, but $D(A) \hookrightarrow H_a^2(\Omega)$ does not hold.

Next, we introduce some common notation based on processes of time-dependent space (see [23, 12, 11]).

Let $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ be a family of normed spaces. Note that the ball of radius R in $\mathcal{M}_t$ is

$$\mathscr{B}_t(R) = \{w \in \mathcal{M}_t : \|w\|_{\mathcal{M}_t} \le R\}.$$

For any given $\varepsilon > 0$, we define the ε neighborhood of set $B \subset \mathcal{M}_t$ as

$$\mathcal{O}_t^\varepsilon(B) = \cup_{x\in B}\{y \in \mathcal{M}_t : \|x-y\|_{\mathcal{M}_t} < \varepsilon\} = \cup_{x\in B}\{x + \mathscr{B}_t(\varepsilon)\}.$$

In particular, the Hausdorff semidistance of between two nonempty sets $A, B \subset \mathcal{M}_t$ is defined as

$$\operatorname{dist}_{\mathcal{M}_t}(A,B) = \sup_{x\in A}\inf_{y\in B}\|x-y\|_{\mathcal{M}_t}.$$

Definition 2.2. Let $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ be a family of normed spaces. A two-parameter family of operators $U(t,\tau) : \mathcal{M}_\tau \to \mathcal{M}_t$ is called a process if it satisfies the following properties:

(i) $U(\tau,\tau) = Id$ for $\tau \in \mathbb{R}$ (Identity operator);
(ii) $U(t,s)U(s,\tau) = U(t,\tau)$ for $t \ge s \ge \tau \in \mathbb{R}$.

Definition 2.3. A family of sets $\tilde{C} = \{C_t \subset \mathcal{M}_t : C_t \text{ is bounded}\}_{t\in\mathbb{R}}$ is called uniformly bounded if there exists a constant $R > 0$, such that $C_t \subset \mathscr{B}_t(R)$ for all $t \in \mathbb{R}$.

Definition 2.4. A family of sets $\tilde{B} = \{B_t\}_{t\in\mathbb{R}}$ is called pullback absorbing if $\tilde{B} = \{B_t\}_{t\in\mathbb{R}}$ is uniformly bounded and for all $R > 0$, there exists a constant $t_0 = t_0(t,R) \le t$ such that $U(t,\tau)\mathscr{B}_\tau(R) \subset B_t$ for any $\tau \le t_0$.

The process $\{U(t,\tau)\}_{t\ge\tau}$ is called dissipative whenever it enters a pullback absorbing family $\tilde{B}_0 = \{B_t^0\}_{t\in\mathbb{R}}$.

Definition 2.5. A time-dependent absorbing set for the process $U(t,\tau)$ is a uniformly bounded family $\tilde{B} = \{B_t\}_{t\in\mathbb{R}}$ with the following characteristic: for any $R > 0$, there exists $t_0 = t_0(t,R) \ge 0$, such that

$$U(t,\tau)\mathscr{B}_\tau(R) \subset B_t \quad \text{for all } \tau \le t - t_0.$$

Definition 2.6. The process $U(t,\tau)$ is called pullback asymptotic compact if for any $t \in \mathbb{R}$, any bounded sequence $\{z_n\}_{n=1}^\infty \subset \mathcal{M}_{\tau_n}$ and $\{\tau_n\}_{n=1}^\infty \subset (-\infty, t]$ with $\tau_n \to -\infty$ as $n \to \infty$, the sequence $\{U(t,\tau_n)z_n\}_{n=1}^\infty$ has a convergent subsequence in $\mathcal{M}_t$.

Definition 2.7. A time-dependent global attractor of the process $U(t,\tau)$ is the smallest family $\tilde{\mathscr{A}} = \{\mathcal{A}_t\}_{t\in\mathbb{R}}$ such that

(i) for every $t \in \mathbb{R}$, $\mathscr{A}_t$ is compact in $\mathcal{M}_t$;

(ii) $\tilde{\mathscr{A}}$ is pullback attracting, namely, $\tilde{\mathscr{A}}$ is uniformly bounded and

$$\lim_{\tau\to-\infty} \operatorname{dist}_{\mathcal{M}_t}(U(t,\tau)C_\tau, \mathcal{A}_t) = 0$$

holds for all uniformly bounded family $\tilde{C} = \{C_\tau\}_{\tau\in\mathbb{R}}$ and every fixed $t \in \mathbb{R}$ and $\tau \le t$.

Remark 2.8. The pullback attracting essence can be equivalently described in the light of pullback absorbing: a (uniformly bounded) family $\mathcal{K} = \{K_t\}_{t\in\mathbb{R}}$ is said to be pullback attracting if for all $\varepsilon > 0$ the family $\{\mathcal{O}_t^\varepsilon(K_t)\}_{t\in\mathbb{R}}$ is pullback absorbing.

Theorem 2.9. *A time-dependent global attractor $\tilde{\mathscr{A}}$ exists and it is unique if and only if the process $U(t,\tau)$ is asymptotically compact, i.e., the set*

$$\mathbb{K} = \{\mathcal{K} = \{K_t\}_{t\in\mathbb{R}} : K_t \subset \mathcal{M}_t \text{ is compact, and } \mathbb{K} \text{ is pullback attracting}\}$$

is non-empty.

It can be seen from Definition 2.7 that the time-dependent global attractor does not have to be invariant, which is because the process does not require to meet some continuity. If the process $U(t,\tau)$ satisfies appropriate continuity, then the invariance of time-dependent global attractor $\tilde{\mathscr{A}}$ can be obtained.

Definition 2.10. The time-dependent global attractor $\tilde{\mathscr{A}} = \{\mathcal{A}_t\}_{t\in\mathbb{R}}$ is said to be invariant if

$$U(t,\tau)\mathcal{A}_\tau = \mathcal{A}_t, \quad t \ge \tau \in \mathbb{R}.$$

Lemma 2.11. *If the time-dependent global attractor $\tilde{\mathscr{A}}$ exists and the process $U(t,\tau)$ is a strongly continuous process, then $\tilde{\mathscr{A}}$ is invariant.*

Next, we will state the definitions of contractive function and $\mathcal{M}_t$-contractive process, which will be utilized to prove asymptotic compactness of a family of process $\{U(t,\tau)\}_{t\ge\tau}$ (see [22, 25, 28, 34, 35]).

Definition 2.12. Let $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ be a family of Banach spaces and $\tilde{B} = \{B_t \subset \mathcal{M}_t\}_{t\in\mathbb{R}}$ be a family of uniformly bounded subset. We call function $\varphi(\cdot,\cdot)$, defined on $\mathcal{M}_\tau\times\mathcal{M}_\tau$, to be a contractive function on $B_\tau\times B_\tau$ if for any sequence $\{z_n\}_{n=1}^\infty \subset B_\tau$, there exists a subsequence $\{z_{n_k}\}_{k=1}^\infty \subset \{z_n\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty} \varphi_\tau^t(z_{n_k}, z_{n_l}) = 0, \quad \forall\, t \ge \tau \in \mathbb{R}.$$

We use $\mathfrak{C}(B_\tau)$ to denote the set all contractive function on $B_\tau \times B_\tau$.

Definition 2.13. Assume that $U(t,\tau)$ is a process on $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ and it has a pullback bounded absorbing set $\tilde{B} = \{B_t\}_{t\in\mathbb{R}}$. $U(t,\tau)$ is called $\mathcal{M}_t$-contractive process if for any given $\varepsilon > 0$, there exist $T = T(\varepsilon)$ and $\varphi_T^t(\cdot,\cdot) \in \mathfrak{C}(B_T)$ such that

$$\|U(t,T)z_1 - U(t,T)z_2\|_{\mathcal{M}_t} \le \varepsilon + \varphi_T^t(z_1,z_2), \quad \forall\, z_i \in B_T (i = 1,2).$$

where φ_T^t depends on T.

Next, we give the method to prove the existence of time-dependent global attractors for evolution equations, which will be used in the later discussion.

Theorem 2.14 ([25])**.** *Let $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ be a family of Banach spaces, then $U(t,\tau)$ has a time-dependent global attractor in $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$, if the following conditions hold*

(i) *$U(t,\tau)$ has a pullback absorbing set $\tilde{B} = \{B_t\}_{t\in\mathbb{R}}$ in $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$;*

(ii) *$U(t,\tau)$ is a $\mathcal{M}_t$-contractive process.*

In what follows, we give two Lemmas, which shall be used to prove the compactness of solution process.

Lemma 2.15 ([26]). *Let $X \subset\subset H \subset Y$ be Banach spaces, with X reflexive and T, τ be two constants with $\tau \leq T$. Suppose that u_n is a sequence that is uniformly bounded in $L^2(\tau, T; X)$ and du_n/dt is uniformly bounded in $L^p(\tau, T; Y)$, for some $p > 1$. Then there is a subsequence of u_n that converges strongly in $L^2(\tau, T; H)$.*

Lemma 2.16 ([3]). *Suppose that the nonnegative function $\mu \in L^1(\mathbb{R}^+)$ is decreasing piecewise absolutely continuous, and it satisfies that if there exists $s_0 \in \mathbb{R}^+$ such that $\mu(s_0) = 0$, then $\mu(s) = 0$ is true for any $s \geq s_0$. Furthermore, let $B_0 \hookrightarrow\hookrightarrow B_1 \hookrightarrow B_2$ be Banach spaces, where B_0, B_1 are reflexive. If $\mathscr{C} \subset L^2_\mu(\mathbb{R}^+, B_1)$ and satisfies*

(i) *$\mathscr{C}$ in $L^2_\mu(\mathbb{R}^+, B_0) \cap H^1_\mu(\mathbb{R}^+, B_2)$;*
(ii) *$\sup_{\vartheta \in \mathscr{C}} \|\vartheta\|^2_{B_1} \leq h(s)$, for all $s \in \mathbb{R}^+$, $h(s) \in L^2_\mu(\mathbb{R}^+)$; then $\mathscr{C}$ is relatively compact in $L^2_\mu(\mathbb{R}^+, B_1)$, where $H^1_\mu(\mathbb{R}^+, B_2) = \{f : f(s), \partial_s f(s) \in L^2_\mu(\mathbb{R}^+, B_2)\}$.*

3. Existence and uniqueness of a time-dependent global attractor

It is easy to know that the key for existence and uniqueness is to verify the pullback asymptotic compactness of process generated by (1.1). To do this, we first prove the asymptotic regularity of solutions, then the contractive function can be constructed by it, which can ensure the pullback asymptotic compactness of corresponding process.

3.1. Well-posedness.

We now describe the well-posedness for the equation (1.1), which can be obtained by using standard Faedo-Galerkin method (see e.g., [30, 26, 29]). For simplicity, we only give the final conclusion.

Lemma 3.1. *Let Ω be a bounded domain of $\mathbb{R}^n$ ($n \geq 3$) with smooth boundary $\partial\Omega$, $z_\tau = (u_\tau, \eta^\tau) \in \mathcal{M}_\tau$, and (H1)–(H3) be satisfied. Then for any $T > \tau$, the system (1.15)-(1.16) possesses a unique weak solution $z(t) = (u(t), \eta^t)$ satisfying*

$$u \in C(\tau, T; \mathcal{H}_t) \cap L^p(\tau, T; L^p(\Omega)), \quad \eta^t \in C(\tau, T; \mathcal{V}_a). \tag{3.1}$$

In addition, if $z_i = U(t,\tau)z^i_\tau = (u_i(t), \eta^t_i)$ ($i = 1, 2$) are two weak solutions of (1.15)-(1.16), then for any $t \geq \tau$, it is easy to obtain a positive constant $C_\Re$ independent of t, such that

$$\|U(t,\tau)z^1_\tau - U(t,\tau)z^2_\tau\|_{\mathcal{M}_t} \leq C_\Re \|z^1_\tau - z^2_\tau\|_{\mathcal{M}_\tau}. \tag{3.2}$$

Remark 3.2. By Lemma 3.1, the following solution process can be defined on the family of time-dependent spaces $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$

$$U(t,\tau) : \mathcal{M}_\tau \to \mathcal{M}_t, \quad U(t,\tau)z_\tau = z(t), \quad \forall t \geq \tau. \tag{3.3}$$

In particular, from (3.2), it is easy to find the process $U(t,\tau)$ is Lipschitz continuous. That is to say, $U(t,\tau)$ is a strongly continuous process over the family of time-dependent phase space $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$.

3.2. **Time-dependent absorbing sets.** In this subsection, we shall study the dissipative feature for the process $\{U(t,\tau)\}_{t\geq\tau}$. To this end, we need a series of prior estimates. Throughout this subsection and subsequent sections, we always assume that $\Omega\subset\mathbb{R}^n(n\geq 3)$ is a bounded domain with smooth boundary, the initial value $z_\tau=(u_\tau,\eta^\tau)\in\mathscr{B}(R)\subset\mathcal{M}_\tau$, $\varepsilon(t)$ satisfies (1.4)-(1.5), $g\in L^2(\Omega)$ and the assumptions (H1)–(H3) hold.

Lemma 3.3. *Assume that $z(t)=(u(t),\eta^t)$ is sufficiently regular solution of* (1.15)-(1.16). *Let*

$$L(t)=\int_0^\infty k(s)|\sqrt{a}\nabla\eta^t(s)|_2^2ds,$$

then $L(t)$ satisfies the differential inequality

$$\frac{d}{dt}L(t)\leq 2\delta^2m_0|a|_\infty|\nabla u|_2^2-\frac{1}{2}\int_0^\infty\mu(s)|\sqrt{a}\nabla\eta^t(s)|_2^2ds, \tag{3.4}$$

and

$$|L(t)|\leq\delta E_1(t), \tag{3.5}$$

where $E_1(t)=\|u\|_{\mathcal{H}_t}^2+\int_0^\infty\mu(s)|\sqrt{a}\nabla\eta^t(s)|_2^2ds$.

Proof. Combining with Hölder inequality, Young inequality and assumption (H3), we have

$$\begin{aligned}
\frac{d}{dt}L(t)&=\int_0^\infty k(s)\frac{d}{dt}(\sqrt{a}\nabla\eta^t,\sqrt{a}\nabla\eta^t)ds\\
&=2\int_0^\infty k(s)(\sqrt{a}\nabla\eta_t^t,\sqrt{a}\nabla\eta^t)ds\\
&\leq 2\int_0^\infty k(s)(\sqrt{a}\nabla u,\sqrt{a}\nabla\eta^t)ds-2\int_0^\infty k(s)(\sqrt{a}\nabla\eta_s^t,\sqrt{a}\nabla\eta^t)ds\\
&\leq 2\Big(\int_0^\infty k(s)|\sqrt{a}\nabla u|_2^2ds\Big)^{1/2}\Big(\int_0^\infty k(s)|\sqrt{a}\nabla\eta^t|_2^2ds\Big)^{1/2}\\
&\quad-\int_0^\infty k(s)\frac{d}{ds}(\sqrt{a}\nabla\eta^t,\sqrt{a}\nabla\eta^t)ds\\
&\leq 2\delta m_0^{1/2}|\sqrt{a}\nabla u|_2\Big(\int_0^\infty k(s)|\sqrt{a}\nabla\eta^t|_2^2ds\Big)^{1/2}\\
&\quad-\int_0^\infty\mu(s)|\sqrt{a}\nabla\eta^t(s)|_2^2ds\\
&\leq 2\delta^2m_0|\sqrt{a}\nabla u|_2^2-\frac{1}{2}\int_0^\infty\mu(s)|\sqrt{a}\nabla\eta^t(s)|_2^2ds.
\end{aligned}\tag{3.6}$$

In addition,

$$|\sqrt{a}\nabla u|_2^2=\int_\Omega|a||\nabla u|^2dx\leq|a|_\infty|\nabla u|_2^2. \tag{3.7}$$

By (3.6) and (3.7), one can obtain (3.4). Then it is easy to obtain

$$|L(t)|\leq\int_0^\infty k(s)|\sqrt{a}\nabla\eta^t(s)|_2^2ds\leq\delta\int_0^\infty\mu(s)|\sqrt{a}\nabla\eta^t(s)|_2^2ds\leq\delta E_1(t). \tag{3.8}$$

The proof is complete. □

Lemma 3.4. *For each positive constant R, let $z_\tau = (u_\tau, \eta^\tau) \in \mathscr{B}_\tau(R) \subset \mathcal{M}_\tau$. Then there exist $\kappa_0 > 0$ and $T_0 = T_0(R) \geq 0$, such that*

$$\|u\|^2_{\mathcal{H}_t} + \int_0^\infty \mu(s)|\sqrt{a}\nabla\eta^t(s)|_2^2 ds \leq \kappa_0, \quad \forall t \geq \tau + T_0.$$

Proof. Multiplying (1.15) by u, and integrating over Ω yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u\|^2_{\mathcal{H}_t} - \frac{1}{2}\varepsilon'(t)|\nabla u|_2^2 + |\nabla u|_2^2 + \int_0^\infty \mu(s)(a(x)\nabla\eta^t, \nabla u)ds + \langle f(u), u\rangle \\ &= \langle g, u\rangle.\end{aligned} \tag{3.9}$$

In addition, we have

$$\begin{aligned}&\int_0^\infty \mu(s)(a(x)\nabla\eta^t, \nabla u)ds \\ &= \int_0^\infty \mu(s)(\sqrt{a}\nabla\eta^t, \sqrt{a}\nabla\eta^t_t)ds + \int_0^\infty \mu(s)(\sqrt{a}\nabla\eta^t, \sqrt{a}\nabla\eta^t_s)ds \\ &= \frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(s)|\sqrt{a}\nabla\eta^t|_2^2 ds - \int_0^\infty \mu'(s)|\sqrt{a}\nabla\eta^t|_2^2 ds\end{aligned} \tag{3.10}$$

and

$$\langle g, u\rangle \leq \frac{2}{\lambda_1}|g|_2^2 + \frac{\lambda_1}{8}|u|_2^2. \tag{3.11}$$

Combining this, (3.9)-(3.10), the Poincaré inequality, (1.4) and (H2), we obtain

$$\frac{1}{2}\frac{d}{dt}E_1(t) + \frac{3}{4}|\nabla u|_2^2 + \frac{\lambda_1}{8}|u|_2^2 + \alpha_1|u|_p^p \leq \beta_1|\Omega| + \frac{2}{\lambda_1}|g|_2^2, \tag{3.12}$$

where $|\Omega|$ denotes Lebesgue measure of domain Ω.

Next, we define functional

$$S(t) = E_1(t) + 2\gamma L(t).$$

Then by (3.5), it is easy to obtain

$$(1 - 2\gamma\delta)E_1(t) \leq S(t) \leq (1 + 2\gamma\delta)E_1(t), \tag{3.13}$$

where γ is small enough to guarantee $1 - 2\gamma\delta > 0$.

Furthermore, from (3.4) and (3.12), it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}S(t) + \frac{\varepsilon(t)}{2L}|\nabla u|_2^2 + (\frac{1}{4} - 2\gamma\delta^2 m_0|a|_\infty)|\nabla u|_2^2 \\ &+ \gamma\int_0^\infty \mu(s)|\sqrt{a}\nabla\eta^t(s)|_2^2 ds + \frac{\lambda_1}{8}|u|_2^2 + \alpha_1|u|_p^p \\ &\leq \beta_1|\Omega| + \frac{2}{\lambda_1}|g|_2^2,\end{aligned} \tag{3.14}$$

similarly, we can choose suitable γ such that $\frac{1}{4} - 2\gamma\delta^2 m_0|a|_\infty \geq 0$.

In conclusion, letting $\gamma = \min\{\frac{1}{2\delta}, \frac{1}{8\delta^2 m_0|a|_\infty}\} > 0$, and $c_1 = \min\{\frac{1}{2L}, \gamma, \frac{\lambda_1}{8}\}$, inequality (3.14) becomes

$$\frac{d}{dt}S(t) + 2c_1E_1(t) + 2\alpha_1|u|_p^p \leq 2\beta_1|\Omega| + \frac{4}{\lambda_1}|g|_2^2, \tag{3.15}$$

Combining this, (3.13), and (3.15), we have

$$\frac{d}{dt}S(t) + c_2S(t) \leq 2\beta_1|\Omega| + \frac{4}{\lambda_1}|g|_2^2, \tag{3.16}$$

where $c_2 = \frac{2c_1}{1+2\gamma\delta}$. Then Gronwall's inequality yields

$$S(t) \le e^{-c_2(t-\tau)}S(\tau) + \frac{1}{c_2}(2\beta_1|\Omega| + \frac{4}{\lambda_1}|g|_2^2). \tag{3.17}$$

From (3.13) and (3.17) we have

$$\begin{aligned} E_1(t) &\le \frac{1+2\gamma\delta}{1-2\gamma\delta}e^{-c_2(t-\tau)}E_1(\tau) + \frac{1}{c_2(1-2\gamma\delta)}(2\beta_1|\Omega| + \frac{4}{\lambda_1}|g|_2^2) \\ &\le \frac{1+2\gamma\delta}{1-2\gamma\delta}Re^{-c_2(t-\tau)} + \frac{1}{c_2(1-2\gamma\delta)}(2\beta_1|\Omega| + \frac{4}{\lambda_1}|g|_2^2). \end{aligned} \tag{3.18}$$

Thus, in light of (3.18), there exists $T_0 = T_0(R) > 0$, such that

$$\|u\|_{\mathcal{H}_t}^2 + \int_0^\infty \mu(s)|\sqrt{a}\nabla\eta^t(s)|_2^2 ds \le \kappa_0$$

for all $t - \tau \ge T_0$, where $\kappa_0 = \frac{2}{c_2(1-2\gamma\delta)}(2\beta_1|\Omega| + \frac{4}{\lambda_1}|g|_2^2)$. This completes the proof. □

From (3.13) and (3.17) (or just use (3.18)), we have the following result.

Corollary 3.5. *Let $z_\tau = (u_\tau, \eta^\tau) \in \mathscr{B}_\tau(R) \subset \mathcal{M}_\tau$. Then there exists $\mathscr{K}_0 = \mathscr{K}_0(R)$ such that for all $t \ge \tau$,*

$$\|u\|_{\mathcal{H}_t}^2 + \int_0^\infty \mu(s)|\sqrt{a}\nabla\eta^t(s)|_2^2 ds \le \mathscr{K}_0\,.$$

Proof. By (3.18) by letting

$$\mathscr{K}_0 = \frac{1+2\gamma\delta}{1-2\gamma\delta}R + \frac{1}{c_2(1-2\gamma\delta)}(2\beta_1|\Omega| + \frac{4}{\lambda_1}|g|_2^2)$$

the conclusion holds. □

The following Lemma can be obtained from Lemma 3.3. we omit its proof.

Lemma 3.6. *Under the assumption of Lemma 3.3. Let*

$$B(t) = \int_0^\infty k(s)|\eta^t(s)|_2^2 ds.$$

Then $B(t)$ satisfies the differential inequality

$$\frac{d}{dt}B(t) \le 2\delta^2 m_0|u|_2^2 - \frac{1}{2}\int_0^\infty \mu(s)|\eta^t(s)|_2^2 ds, \tag{3.19}$$

and the estimate

$$|B(t)| \le \delta E_2(t), \tag{3.20}$$

where $E_2(t) = \int_0^\infty \mu(s)|\eta^t(s)|_2^2 ds$.

To obtain the bounded absorbing set of $\mathcal{M}_t$, we need the following result.

Lemma 3.7. *Under the assumption of Lemma 3.4, for each $t - \tau \ge T_1 = T(R)$, there exists $\kappa_1 > 0$, such that*

$$\int_0^\infty \mu(s)|\eta^t(s)|_2^2 ds \le \kappa_1\,.$$

Proof. Taking the inner product of η^t and the second equation of (1.15), we have

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(s)|\eta^t|_2^2 ds - \int_0^\infty \mu'(s)|\eta^t|_2^2 ds = \int_0^\infty \mu(s)(u,\eta^t)ds. \tag{3.21}$$

For the right-hand side we have

$$\int_0^\infty \mu(s)(u,\eta^t)ds \le \frac{m_0}{\gamma_1}|u|_2^2 + \frac{\gamma_1}{4}\int_0^\infty \mu(s)|\eta^t|_2^2 ds. \tag{3.22}$$

From (3.21) and (3.22) it follows that

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(s)|\eta^t|_2^2 ds \le \frac{m_0}{\gamma_1}|u|_2^2 + \frac{\gamma_1}{4}\int_0^\infty \mu(s)|\eta^t|_2^2 ds. \tag{3.23}$$

Furthermore, letting

$$S_1(t) = E_2(t) + 2\gamma_1 B(t),$$

by (3.17), we have

$$(1-2\gamma_1\delta)E_2(t) \le S_1(t) \le (1+2\gamma_1\delta)E_2(t), \tag{3.24}$$

where γ_1 is appropriately small to ensure $1-2\gamma_1\delta > 0$.

By combining with (3.19) and (3.23), we obtain

$$\frac{d}{dt}S_1(t) + \frac{\gamma}{2}E_2(t) \le 4\delta^2 m_0\gamma|u|_2^2 + \frac{2m_0}{\gamma_1}|u|_2^2. \tag{3.25}$$

Then by Corollary 3.5 and (3.24)-(3.25), we have

$$\frac{d}{dt}S_1(t) + c_3S_1(t) \le (4\delta^2 m_0\gamma_1 + \frac{2m_0}{\gamma_1})\mathscr{K}_0, \tag{3.26}$$

where $c_3 = \frac{\gamma_1}{2(1+2\gamma_1\delta)}$.

Applying Gronwall's Lemma to (3.26), we have

$$S_1(t) \le e^{c_3(t-\tau)}S_1(\tau) + \frac{2\mathscr{K}_0}{c_3}(2\delta^2 m_0\gamma_1 + \frac{m_0}{\gamma_1}). \tag{3.27}$$

From this, (3.24), and (3.27), there exists $T_1 = T_1(R)(>0)$, such that

$$\int_0^\infty \mu(s)|\eta^t(s)|_2^2 ds \le \kappa_1,$$

where $\kappa_1 = \frac{4\mathscr{K}_0}{c_3}(2\delta^2 m_0\gamma_1 + \frac{m_0}{\gamma_1})$. □

Next, we show the existence of a bounded absorbing set.

Theorem 3.8. *Let $z_\tau = (u_\tau, \eta^\tau) \in \mathscr{B}_\tau(R) \subset \mathcal{M}_\tau$. Then, for any given $R \in \mathbb{R}^+$, there exists $\rho_0 > 0$ such that the process $U(t,\tau)$ generated by (1.15) possesses a time-dependent bounded absorbing set $\bar{B}_0 = \{B_t^0\}_{t\in\mathbb{R}}$ $(:= \{\mathscr{B}_t(\rho_0)\}_{t\in\mathbb{R}})$, i.e.,*

$$\bar{B}_0 = \left\{z = (u,\eta^t) \in \mathcal{M}_t : \|u\|_{\mathcal{H}_t}^2 + \|\eta^t\|_{\mu,1}^2 \le \rho_0, \forall\, t \in \mathbb{R}\right\}; \tag{3.28}$$

this is, there exists $\mathcal{T}_0 = \max\{T_0, T_1\} \ge 0$, such that

$$U(t,\tau)\mathscr{B}_\tau(R) \subset B_t^0, \quad \forall \tau \le t - \mathcal{T}_0.$$

Proof. In Lemmas 3.4 and 3.7, just take $\rho_0 = \kappa_0 + \kappa_1$ and $\mathcal{T}_0 = \max\{T_0, T_1\} \ge 0$. Then the above conclusion of the theorem follows. □

3.3. **Time-dependent global attractors.** Now we prove the existence of time-dependent global attractors in $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ for the process defined by (3.3). This will be Theorem 3.17, but we first give the following lemmas.

Lemma 3.9. *Under the assumption of Lemma 3.4, there exists positive constant $\rho_1 = \rho_1(\rho_0)$ such that*

$$\int_t^{t+1} (|\nabla u(r)|_2^2 + |u(r)|_p^p)dr + \int_t^{t+1}\int_0^\infty \mu(s)|\sqrt{a}\eta^r(s)|\,ds\,dr \le \rho_1, \quad \forall t-\tau \ge \mathcal{T}_0\,.$$

Proof. According to (3.12), by letting $c_4 = \min\{\frac{3}{4}, 2\alpha_1\}$, we have

$$\frac{d}{dt}E_1(t) + c_4(|\nabla u|_2^2 + |u|_p^p) \le 2\beta_1|\Omega| + \frac{4}{\lambda_1}|g|_2^2. \tag{3.29}$$

Integrating (3.29) on $[t, t+1]$, and using Theorem 3.8, we obtain

$$\int_t^{t+1} (|\nabla u(r)|_2^2 + |u(r)|_p^p)dr \le \kappa_3, \quad \forall t-\tau \ge \mathcal{T}_0, \tag{3.30}$$

where $\kappa_3 = \frac{1}{c_4}(\rho_0 + 2\beta_1|\Omega| + \frac{4}{\lambda_1}|g|_2^2)$. Similarly, by (3.15), we have

$$\frac{d}{dt}S(t) + c_1\int_0^\infty \mu(s)|\sqrt{a}\nabla\eta^t(s)|_2^2 ds \le 2\beta_1|\Omega| + \frac{4}{\lambda_1}|g|_2^2. \tag{3.31}$$

Then integrating (3.31) from t to $t+1$ about t yields

$$c_1\int_t^{t+1}\int_0^\infty \mu(s)|\sqrt{a}\nabla\eta^r(s)|_2^2\,ds\,dr \le S(t+1) + 2\beta_1|\Omega| + \frac{4}{\lambda_1}|g|_2^2. \tag{3.32}$$

Combining with this, (3.13), and Theorem 3.8, we have

$$\int_t^{t+1}\int_0^\infty \mu(s)|\sqrt{a}\nabla\eta^r(s)|_2^2\,ds\,dr \le \kappa_4, \quad \forall t-\tau \ge \mathcal{T}_0, \tag{3.33}$$

where

$$\kappa_4 = \frac{1}{c_1}[(1+2\gamma\delta)\rho_0 + 2\beta_1|\Omega| + \frac{4}{\lambda_1}|g|_2^2].$$

Therefore, from (3.30) and (3.33), by letting $\rho_1 = \kappa_3 + \kappa_4$, it is easy to see that the aforementioned result is true. □

Corollary 3.10. *Under the assumptions of Lemma 3.4, there exists a positive constant $\mathscr{K}_1$, such that*

$$\int_t^{t+1} \big(|\nabla u(r)|_2^2 + |u(r)|_p^p\big)dr \le \mathscr{K}_1$$

for all $t \ge \tau$.

Proof. Integrating (3.29) on $[t, t+1]$, then using Corollary 3.5, we have

$$\int_t^{t+1} (|\nabla u(r)|_2^2 + |u(r)|_p^p)dr \le \mathscr{K}_1, \quad \forall t \ge \tau,$$

where $\mathscr{K}_1 = \frac{1}{c_4}(\mathscr{K}_0 + 2\beta_1|\Omega| + \frac{4}{\lambda_1}|g|_2^2)$. □

Lemma 3.11. *Under the assumptions of Lemma 3.4, there exists a positive constant $\rho_2 = \rho_2(R)$ such that*

$$\int_t^{t+1} \big(\varepsilon(s)|u_t(s)|_2^2 + \varepsilon^2(s)|\nabla u_t(s)|_0^2\big)ds \le \rho_2, \quad \forall t \ge \tau.$$

Proof. Using $\varepsilon(t)u_t$ to make inner product with the first equation of (1.16) in $L^2(\Omega)$, and combining with (1.4)-(1.5), we have

$$\begin{aligned} &\frac{d}{dt}\Big[\frac{\varepsilon(t)}{2}|\nabla u|_2^2+\varepsilon(t)\int_\Omega F(u)dx\Big]+\frac{\varepsilon(t)}{2}|u_t|_2^2+\varepsilon^2(t)|\nabla u_t|_2^2\\ &\le -\int_0^\infty \mu(s)(a(x)\nabla\eta^t,\varepsilon(t)\nabla u_t)ds+\frac{L}{2}|g|_2^2+L\widetilde{\beta}_1|\Omega|. \end{aligned}\tag{3.34}$$

Estimating the first term at the right-hand side hand of (3.34), we obtain

$$\begin{aligned} &\big|-\int_0^\infty \mu(s)(a(x)\nabla\eta^t,\varepsilon(t)\nabla u_t)ds\big|\\ &\le \frac{m_0}{2}|a|_\infty\int_0^\infty \mu(s)|\sqrt{a}\nabla\eta^t|_2^2ds+\frac{1}{2}\varepsilon^2(t)|\nabla u_t|_2^2. \end{aligned}\tag{3.35}$$

By (3.34)-(3.35) we have

$$\begin{aligned} &\frac{d}{dt}\Big[\varepsilon(t)|\nabla u|_2^2+2\varepsilon(t)\int_\Omega F(u)dx\Big]+\varepsilon(t)|u_t|_2^2+\varepsilon^2(t)|\nabla u_t|_2^2\\ &\le m_0|a|_\infty\int_0^\infty \mu(s)|\sqrt{a}\nabla\eta^t|_2^2ds+L|g|_2^2+2L\widetilde{\beta}_1|\Omega|. \end{aligned}\tag{3.36}$$

Integrating (3.36) about t from s to $t+1$, $(t\le s\le t+1)$; then by Corollary 3.5, we have

$$\varepsilon(t+1)\int_\Omega F(u(t+1))dx\le \frac{m_0}{2}|a|_\infty\mathscr{K}_0+\frac{L}{2}|g|_2^2+L\widetilde{\beta}_1|\Omega|+\varepsilon(s)\int_\Omega F(u(s))+L|\nabla u(s)|_2^2,$$

from (1.11), we know that the above inequality can be turned into

$$\begin{aligned} &\varepsilon(t+1)|u(t+1)|_p^p\\ &\le \frac{m_0}{2\widetilde{\alpha}_1}|a|_\infty\mathscr{K}_0+\frac{L}{2\widetilde{\alpha}_1}|g|_2^2+\frac{L}{\widetilde{\alpha}_1}|\Omega|(2\widetilde{\beta}_1+\widetilde{\beta}_2)+\frac{L}{\widetilde{\alpha}_1}(\widetilde{\alpha}_2|u(s)|_p^p+|\nabla u(s)|_2^2). \end{aligned}\tag{3.37}$$

Then integrating (3.37) over $[t,t+1]$, and combining with Corollary 3.10, we have

$$\varepsilon(t+1)|u(t+1)|_p^p\le c_5,\quad \forall t\ge\tau,\tag{3.38}$$

where $c_5=\frac{m_0}{2\widetilde{\alpha}_1}|a|_\infty\mathscr{K}_0+\frac{L}{2\widetilde{\alpha}_1}|g|_2^2+\frac{L}{\widetilde{\alpha}_1}|\Omega|(2\widetilde{\beta}_1+\widetilde{\beta}_2)+\frac{L}{\widetilde{\alpha}_1}\mathscr{K}_1(\widetilde{\alpha}_2+1)$.

From (1.11) and (3.5), integrating (3.36) over $[t,t+1]$, we have

$$\begin{aligned} &\int_t^{t+1}(\varepsilon(r)|u_t(r)|_2^2+\varepsilon^2(s)|\nabla u_t(r)|_2^2)dr\\ &\le m_0|a|_\infty\int_t^{t+1}\int_0^\infty \mu(s)|\sqrt{a}\nabla\eta^r|_2^2\,ds\,dr+2\widetilde{\alpha}_2\varepsilon(t)|u(t)|_p^p\\ &\quad +L|g|_2^2+2L|\Omega|(2\widetilde{\beta}_1+\widetilde{\beta}_2). \end{aligned}\tag{3.39}$$

Thus, by Corollary 3.5 and (3.38), we have

$$\int_t^{t+1}(\varepsilon(r)|u_t(r)|_2^2+\varepsilon^2(r)|\nabla u_t(r)|_2^2)dr\le\rho_2,\tag{3.40}$$

where $\rho_2=m_0|a|_\infty\mathscr{K}_0+2\widetilde{\alpha}_2c_5+L|g|_2^2+2L|\Omega|(2\widetilde{\beta}_1+\widetilde{\beta}_2)$. This proof iscomplete. $\square$

To demonstrate asymptotic properties of solutions corresponding to the process $\{U(t,\tau)\}_{t\geq\tau}$, we decompose the solution $z=(u,\eta^t)$ of problem (1.15)-(1.16) with initial data $z_\tau=(u_\tau,\eta^\tau)\in\mathcal{M}_\tau$ into the sum

$$U(t,\tau)z_\tau=U_1(t,\tau)z_\tau+U_2(t,\tau)z_\tau, \tag{3.41}$$

where $z_1=U_1(t,\tau)z_\tau=(v(t),\zeta^t)$ and $z_2=U_2(t,\tau)z_\tau=(w(t),\theta^t)$ are two solutions of the following systems respectively:

$$\begin{gathered} v_t-\varepsilon(t)\Delta v_t-\Delta v-\int_0^\infty \mu(s)div\{a(x)\nabla\zeta^t(s)\}ds+f(u)-f(w)+\tilde{\mu}v=0,\\ \zeta_t^t=v-\zeta_s^t, \end{gathered} \tag{3.42}$$

with initial-boundary conditions

$$\begin{gathered} v(x,t)|_{\partial\Omega}=0,\zeta^t(x,s)|_{\partial\Omega\times\mathbb{R}^+}=0,\quad t\geq\tau,\\ v(x,\tau)=u_\tau(x),\,\zeta^\tau(x,s)=\int_0^s u_\tau(x,\tau-r)dr,\quad (x,s)\in\Omega\times\mathbb{R}^+, \end{gathered} \tag{3.43}$$

where $\tilde{\mu}>l$ (from (1.10)) is a constant, and

$$\begin{gathered} \omega_t-\varepsilon(t)\Delta\omega_t-\Delta\omega-\int_0^\infty div\{a(x)\nabla\theta^t(s)\}ds+f(\omega)-\tilde{\mu}v=g,\\ \theta_t^t=\omega-\theta_s^t. \end{gathered} \tag{3.44}$$

with initial-boundary conditions

$$\begin{gathered} \omega(x,t)|_{\partial\Omega}=0,\theta^t(x,s)|_{\partial\Omega\times\mathbb{R}^+}=0,\quad t\geq\tau,\\ \omega(x,\tau)=0,\,\theta^\tau(x,s)=0,\quad (x,s)\in\Omega\times\mathbb{R}^+. \end{gathered} \tag{3.45}$$

Next, we show that z_1 has $\mathcal{M}_t$-decay, but not necessarily exponential decay.

Lemma 3.12. *Assume that $\|z_\tau\|_{\mathcal{M}_\tau}\leq R$ for any given $R>0$, then the solution $z_1=U_1(t,\tau)z_\tau=(v(t),\zeta^t)$ of problem (3.42)-(3.43) satisfies the decaying property*

$$\lim_{\tau\to-\infty}(\|v\|_{\mathcal{H}_t}+\|\zeta^t\|_{\mu,1}^2)=0 \tag{3.46}$$

for $\tau\leq t$.

Proof. We divide the proof into two steps.

Step 1. Multiplying the first equation of (3.42) by v, then integrating over Ω, we obtain

$$\frac{d}{dt}\mathscr{E}_1+|\nabla v|_2^2+(\tilde{\mu}-l)|v|_2^2\leq 0, \tag{3.47}$$

where $\mathscr{E}_1(t)=\frac{1}{2}(|v|_2^2+\varepsilon(t)|\nabla v|_2^2+\int_0^\infty\mu(s)|\sqrt{a}\nabla\zeta^t|_2^2ds)$. By assuming

$$\mathscr{N}_1(t)=\int_0^{+\infty}k(s)|\sqrt{a}\nabla\zeta^t(s)|_2^2ds,$$

then as in the proof of Lemma 3.3, we obtain that $\mathscr{N}_1(t)$ satisfies the differential inequality

$$\frac{d}{dt}\mathscr{N}_1(t)\leq 2\delta^2 m_0|a|_\infty|\nabla v|_2^2-\frac{1}{2}\int_0^\infty\mu(s)|\sqrt{a}\nabla\zeta^t(s)|_2^2ds, \tag{3.48}$$

and the estimate

$$|\mathscr{N}_1(t)|\leq 2\delta\mathscr{E}_1(t), \tag{3.49}$$

Next, let

$$\mathscr{S}_1(t)=\mathscr{E}_1(t)+\pi\mathscr{N}_1(t).$$

Then for suitably small $\pi=\min\{\frac{1}{2\delta},\frac{1}{2\delta^2m_0|a|_\infty}\}>0$, we have

$$(1-2\pi\delta)\mathscr{E}_1(t)\le\mathscr{S}_1(t)\le(1+2\pi\delta)\mathscr{E}_1(t). \tag{3.50}$$

Combining (3.47)-(3.48) and (3.50), we obtain

$$\frac{d}{dt}\mathscr{S}_1(t)+\tilde{\gamma}_1\mathscr{S}_1(t)\le 0, \tag{3.51}$$

where

$$\tilde{\gamma}_1=\frac{1}{1+2\pi\delta}\min\big\{\frac{2-4\pi\delta^2m_0|a|_\infty}{L},\pi,2(\tilde{\mu}-l)\big\}.$$

Applying Gronwall's Lemma for (3.51), and taking limit about τ, we have

$$\lim_{\tau\to-\infty}\mathscr{S}_1(t,\tau)=0.$$

Thus, from (3.50) it follows that

$$0\le\lim_{\tau\to-\infty}\mathscr{E}_1(t,\tau)\le\frac{1}{1-2\pi\delta}\lim_{\tau\to-\infty}\mathscr{S}_1(t,\tau)=0. \tag{3.52}$$

Then

$$\lim_{\tau\to-\infty}\mathscr{E}_1(t,\tau)=0. \tag{3.53}$$

Step 2. As in Lemma 3.6, suppose that

$$\mathscr{N}_2(t)=\int_0^\infty k(s)|\zeta^t(s)|_2^2ds.$$

Then we have that $\mathscr{N}_2(t)$ satisfies the inequality

$$\frac{d}{dt}\mathscr{N}_2(t)\le 2\delta^2m_0|v|_2^2-\frac{1}{2}\int_0^\infty\mu(s)|\zeta^t(s)|_2^2ds, \tag{3.54}$$

and the estimate

$$|\mathscr{N}_2(t)|\le 2\delta E_2(t), \tag{3.55}$$

where $\mathscr{E}_2(t)=\frac{1}{2}\int_0^\infty\mu(s)|\zeta^t(s)|_2^2ds$.

Taking the inner product of v and the second equation of (3.42) on $\mathcal{V}_0$, we obtain

$$\frac{d}{dt}\mathscr{E}_2(t)\le\frac{\tilde{\gamma}_2}{4}\int_0^\infty\mu(s)|\zeta^t(s)|_2^2ds+\frac{m_0}{\tilde{\gamma}_2}|v|_2^2, \tag{3.56}$$

where $\tilde{\gamma}_2$ is a constant to be defined later. Let

$$\mathscr{S}_2(t)=\mathscr{E}_2(t)+\tilde{\gamma}_2\mathscr{N}_2(t).$$

Then we have

$$(1-2\tilde{\gamma}_2\delta)\mathscr{E}_2(t)\le\mathscr{S}_2(t)\le(1+2\tilde{\gamma}_2\delta)\mathscr{E}_2(t), \tag{3.57}$$

with $\tilde{\gamma}_2\in(0,\frac{1}{2\delta})$. Combining (3.54) and (3.56), we have

$$\frac{d}{dt}\mathscr{S}_2(t)+\tilde{\gamma}_3\mathscr{S}_2(t)\le(2\tilde{\gamma}_2\delta^2m_0+\frac{m_0}{\tilde{\gamma}_2})|v|_2^2. \tag{3.58}$$

where $\tilde{\gamma}_3=\frac{\tilde{\gamma}_2}{2(1+2\tilde{\gamma}_2\delta)}$.

Applying Gronwall's lemma to (3.58) we obtain

$$\mathscr{S}_2(t)\le(1+2\tilde{\gamma}_2\delta)e^{-\tilde{\gamma}_3(t-\tau)}R+(2\tilde{\gamma}_2\delta^2m_0+\frac{m_0}{\tilde{\gamma}_2})e^{-\tilde{\gamma}_3t}\int_\tau^t e^{\tilde{\gamma}_3r}|v(r)|_2^2dr. \tag{3.59}$$

However, from (3.53), we obtain that $|v|_2^2 \to 0$ as $\tau \to -\infty$, which implies that $\int_t^{t+1} |v(r)|_2^2 dr \to 0$ as $\tau \to -\infty$. Hence

$$\begin{aligned} e^{-\tilde{\gamma}_3 t}\int_\tau^t e^{\tilde{\gamma}_3 r}|v(r)|_2^2 dr &= e^{-\tilde{\gamma}_3 t}(\int_{t-1}^t + \int_{t-2}^{t-1} + \cdots)e^{\tilde{\gamma}_3 r}|v(r)|_2^2 dr \\ &\le \frac{1}{1-e^{-\tilde{\gamma}_3}}\int_t^{t+1}|v(r)|_2^2 dr. \end{aligned} \tag{3.60}$$

Combining (3.59) and (3.60), we have

$$\lim_{\tau\to-\infty} \mathscr{S}_2(t) = 0;$$

so

$$\lim_{\tau\to-\infty} \mathscr{E}_2(t) = 0. \tag{3.61}$$

In conclusion, by (3.53) and (3.61), it follows that (3.46) holds. The proof is complete. □

Lemma 3.13. *For each $R > 0$, let $\|z_\tau\|_{\mathcal{M}_\tau} \le R$, and let $z_2 = U_2(t,\tau)z_\tau = (v(t), \zeta^t)$ be the solution of* (3.44)-(3.45)*. Then there exist constants $\mathscr{K}_2, \rho_2 > 0$, such that*

$$\begin{gathered} \|\omega(t)\|_{\mathcal{H}_t}^2 + \int_0^\infty \mu(s)|\sqrt{a}\theta^t(s)|_2^2 ds + \varepsilon(t)|\omega(t)|_p^p \le \mathscr{K}_2, \\ \int_t^{t+1}(\varepsilon(s)|\omega_t(s)|_2^2 + \varepsilon^2(s)|\nabla\omega_t(s)|_2^2 + |\omega(s)|_p^p)ds \le \rho_3, \end{gathered}$$

for all $t \ge \tau$.

The proof of the above lemma is similar to the proof of Corollaries 3.5 and 3.10, Lemma 3.11, and (3.38) word by word. So we omit it here. Next we show that, for all time, the component z_2 belongs to a subset of $\mathcal{M}_t^1$, uniformly as the initial data z_τ belongs to the absorbing set $\bar{B}_0$, given by (3.28).

Lemma 3.14. *For each $R > 0$, let $\|z_\tau\|_{\mathcal{M}_\tau} \le R$. Then there exists a constant $\mathscr{K}_3 = \mathscr{K}_3(R,t,\tau) > 0$, such that the solution $z_2 = (v(t), \zeta^t)$ of the problem* (3.44)-(3.45) *satisfies*

$$\|\omega(t)\|_{\mathcal{H}_t^1}^2 + \|\theta^t\|_{\mu,2}^2 \le \mathscr{K}_3, \quad \forall t \ge \tau.$$

Proof. Firstly, for degenerate memory term, we have

$$\begin{aligned} &-\int_0^\infty \mu(s)\operatorname{div}\{a(x)\nabla\theta^t(s)\}ds \\ &= -\int_0^\infty \mu(s)\nabla a(x)\nabla\theta^t(s)ds - \int_0^\infty \mu(s)a(x)\Delta\theta^t(s)ds. \end{aligned} \tag{3.62}$$

Then using $-\Delta\omega$ to make inner product over $L^2(\Omega)$ for (3.62). At this time, we only need to deal with the right-hand side of (3.62); that is

$$\begin{aligned} \Big|\int_0^\infty \mu(s)(\nabla a(x)\nabla\theta^t(s), -\Delta\omega)ds\Big| &\le \int_0^\infty \mu(s)|\nabla a(x)\nabla\theta^t(s)|_2|\Delta\omega|_2 ds \\ &\le \frac{m_0}{2}|\nabla a|_\infty^2\int_0^\infty \mu(s)|\nabla\theta^t(s)|_2^2 ds + \frac{1}{2}|\Delta\omega|_2^2. \end{aligned}$$

In addition, by combining with the second equation of (3.44), we obtain

$$\begin{aligned} &-\int_0^\infty \mu(s)(a(x)\Delta\theta^t(s), -\Delta\omega)ds \\ &= \frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(s)|\sqrt{a}\Delta\theta^t|_2^2 ds - \int_0^\infty \mu'(s)|\sqrt{a}\Delta\theta^t|_2^2 ds. \end{aligned} \tag{3.63}$$

Next, applying $-\Delta\omega$ on the first equation of (3.44) on $L^2(\Omega)$, then combining with Hölder inequality, Young inequality, and (3.62)-(3.63), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(|\nabla\omega|_2^2 + \varepsilon(t)|\Delta\omega|_2^2 + \int_0^\infty \mu(s)|\sqrt{a}\Delta\theta^t|_2^2 ds\Big) + \frac{1}{4}|\Delta\omega|_2^2 + (\tilde{\mu} - l)|\nabla\omega|_2^2 \\ &\le \frac{m_0}{2}|\nabla a|_\infty^2 \int_0^\infty \mu(s)|\nabla\theta^t(s)|_2^2 ds + 2\tilde{\mu}^2|u|_2^2 + 2|g|_2^2. \end{aligned} \tag{3.64}$$

Furthermore, using $-\Delta\theta^t$ to make inner product on $\mathcal{V}_0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(s)|\nabla\theta^t(s)|_2^2 ds \le \frac{\tilde{\mu} - l}{2}|\nabla\omega|_2^2 + \frac{m_0}{2(\tilde{\mu} - l)}\int_0^\infty \mu(s)|\nabla\theta^t(s)|_2^2 ds. \tag{3.65}$$

By (3.64) and (3.65), it is easily to obtain

$$\frac{1}{2}\frac{d}{dt}(\|\omega(t)\|_{\mathcal{H}_t^1}^2 + \|\theta^t\|_{\mu,2}^2) \le \frac{m_0}{2}(|\nabla a|_\infty^2 + \frac{1}{\tilde{\mu} - l})\int_0^\infty \mu(s)|\nabla\theta^t(s)|_2^2 ds + 2\tilde{\mu}^2|u|_2^2 + 2|g|_2^2.$$

Then according to Corollary 3.5, we have

$$\frac{d}{dt}(\|\omega(t)\|_{\mathcal{H}_t^1}^2 + \|\theta^t\|_{\mu,2}^2) \le \tilde{\gamma}_4(\|\omega(t)\|_{\mathcal{H}_t^1}^2 + \|\theta^t\|_{\mu,2}^2) + 4\tilde{\mu}^2\mathscr{K}_0 + 4|g|_2^2, \tag{3.66}$$

where $\tilde{\gamma}_4 = 1 + m_0(|\nabla a|_\infty^2 + \frac{1}{\tilde{\mu} - l})$.

Applying Gronwall's lemma to (3.66), we have

$$\|\omega(t)\|_{\mathcal{H}_t^1}^2 + \|\theta^t\|_{\mu,2}^2 \le e^{\tilde{\gamma}_4(t-\tau)}(4\tilde{\mu}^2\mathscr{K}_0 + 4|g|_2^2), \quad \forall t \ge \tau.$$

Letting $\mathscr{K}_3 = e^{\tilde{\gamma}_4(t-\tau)}(4\tilde{\mu}^2\mathscr{K}_0 + 4|g|_2^2)$, the conclusion follows. □

The following result shall be used in the proof of asymptotic structure of time-dependent global attractors.

Lemma 3.15. *For each $t > \tau$, let $\mathscr{C}_t := \mathbb{P}U_2(t,\tau)B_\tau^0$, where $\mathbb{P} : \mathcal{H}_t \times \mathcal{V}_a \to \mathcal{V}_a$. Then there exists a constant $C^* = C^*(\|B_t^0\|_{\mathcal{M}_t}) > 0$, such that*

1. *$\mathscr{C}_t$ is bounded in $L_\mu^2(\mathbb{R}^+; H_a^2(\Omega)) \cap H_\mu^1(\mathbb{R}^+; L^2(\Omega))$;*
2. *$\sup_{\theta^t \in \mathscr{C}_t} \|\theta^t(s)\|_{\mu,1}^2 \le C^*$.*

Therefore, $\mathscr{C}_t$ is relatively compact in $L_\mu^2(\mathbb{R}^+; H_a^1(\Omega))$.

Proof. From Lemma 3.14, it is easy to obtain that $\mathscr{C}_t$ is bounded in $L_\mu^2(\mathbb{R}^+; H_a^2(\Omega))$. Also we know that $\theta^t(s) = \int_{t-s}^t \omega(y)dy$, so $\partial_s\theta^t(s) = \omega(t-s)$. Thus, from and Lemma 3.7 and Lemma 3.13, it follows that θ^t and $\omega(t)$ are bounded in $L_\mu^2(\mathbb{R}^+; L^2(\Omega))$, which shows that $\mathscr{C}_t$ is bounded in $H_\mu^1(\mathbb{R}^+; L^2(\Omega))$. Additionally, by Lemma 3.7 and Lemma 3.14, we obtain

$$\|\theta^t(s)\|_{\mu,1}^2 = \int_0^\infty \mu(s)(|\theta^t(s)|_2^2 + |\sqrt{a}\nabla\partial_s\theta^t(s)|_2^2)ds \le C_0^*,$$

where $C_0^* \in L_\mu^1(\mathbb{R}^+)$. The above formula indicates that $\sup_{\theta^t \in \mathscr{C}_t} \|\theta^t(s)\|_{\mu,1}^2 \le C^* \in L_\mu^1(\mathbb{R}^+)$.

From the above arguments, Lemma 2.16, and ($H_a^2(\Omega) \hookrightarrow\hookrightarrow H_a^1(\Omega) \hookrightarrow L^2(\Omega)$), one can infer that $\mathscr{C}_t$ is relatively compact in $L^2_\mu(\mathbb{R}^+; H_a^1(\Omega))$ for any $t > \tau$. □

After the above preparations, we can prove the existence and uniqueness of time-dependent global attractors. The key to achieve this goal is to demonstrate the pullback asymptotic compactness of process $U(t,\tau)$. We know from [20] that the method of the standard Kuratowski measure of non-compactness may be useful for verifying the asymptotic compactness of solution process generated by equation (1.15). But the contractive function method seems to be more concise for our problem, which is mainly based on our previous research [32, 33, 36]. Thus, we only need to prove that $U(t,\tau)$ is a $\mathcal{M}_t$-contractive process by Theorem 2.14.

Theorem 3.16. *The family of process* $\{U(t,\tau)\}_{t\geq\tau}$ *generated by* (1.15) *with initial-boundary value coditions* (1.16) *is a family of* $\mathcal{M}_t$*-contractive process on* $B_T^0 \in \bar{B}_0$.

Proof. Let $z_i(t) = (u_i(t), \xi_i^t) = U(t,\tau)z_\tau^i (i = 1,2)$ be the solutions of problem (1.15)-(1.16) with initial data $z_\tau^i \in B_\tau^0 \in \bar{B}_0$ $(i = 1,2)$ ($\bar{B}_0$ from (3.28)) respectively. By (3.41), we can establish the decomposition

$$z_i(t) = U(t,\tau)z_\tau^i = U_1(t,\tau)z_\tau^i + U_2(t,\tau)z_\tau^i = (v_i(t), \zeta_i^t) + (\omega_i(t), \theta_i^t).$$

It yields

$$\begin{aligned}
&\|U(t,\tau)z_\tau^1 - U(t,\tau)z_\tau^2\|^2_{\mathcal{M}_t} \\
&\leq 2\|U_1(t,\tau)z_\tau^1 - U_1(t,\tau)z_\tau^2\|^2_{\mathcal{M}_t} + 2(\|\omega_1(t) - \omega_2(t)\|^2_{\mathcal{H}_t} + \|\theta_1^t - \theta_2^t\|^2_{\mu,1}),
\end{aligned} \tag{3.67}$$

and by Lemma 3.12, we have

$$\lim_{\tau\to-\infty} \|U_1(t,\tau)z_\tau^1 - U_1(t,\tau)z_\tau^2\|^2_{\mathcal{M}_t} \leq 2\lim_{\tau\to-\infty}(\|U_1(t,\tau)z_\tau^1\|^2_{\mathcal{M}_t} + \|U_1(t,\tau)z_\tau^2\|^2_{\mathcal{M}_t}) = 0.$$

Hence, for each $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{4}$ such that

$$2\|U(t,T)z_T^1 - U(t,T)z_T^2\|^2_{\mathcal{M}_t} < \varepsilon, \tag{3.68}$$

holds for any $t \geq T = T(\varepsilon)$ fixed.

Moreover, it is easy to check that $(\psi(t), \xi^t) = (\omega_1(t) - \omega_2(t), \theta_1^t - \theta_2^t)$ is the solution of the system

$$\begin{gathered}
\psi_t - \varepsilon(t)\Delta\psi_t - \Delta\psi - \int_0^\infty \mu(s)\operatorname{div}\{a(x)\nabla\xi^t(s)\}ds + f(\omega_1) - f(\omega_2) + \tilde{\mu}\psi \\
= \tilde{\mu}(u_1 - u_2), \\
\xi_t^t = -\xi_s^t + \psi.
\end{gathered} \tag{3.69}$$

with initial-boundary value conditions

$$\begin{gathered}
\psi(x,t)|_{\partial\Omega} = 0, \xi^t(x,s)|_{\partial\Omega\times\mathbb{R}^+} = 0, \quad t \geq \tau, \\
\psi(x,\tau) = 0,\ \xi^\tau(x,s) = 0, \quad (x,s) \in \Omega\times\mathbb{R}^+.
\end{gathered} \tag{3.70}$$

Taking the inner product of ψ and ξ^t on $L^2(\Omega)$ and $\mathcal{V}_0$ for the first and second equations of (3.69) respectively, by Hölder inequality, we have

$$\frac{d}{dt}\Big(\|\psi\|^2_{\mathcal{H}_t} + \int_0^\infty \mu(s)|\sqrt{a}\nabla\xi|_2^2 ds\Big) + 2|\nabla\psi|_2^2 + \frac{3(\tilde{\mu} - l)}{2}|\psi|_2^2 \leq \frac{2\tilde{\mu}^2}{\tilde{\mu} - l}|u_1 - u_2|_2^2, \tag{3.71}$$

and

$$\frac{d}{dt}\int_0^\infty \mu(s)|\xi|_2^2 ds \leq \frac{2m_0}{\tilde{\epsilon}}|\psi|_2^2 + \frac{\tilde{\epsilon}}{2}\int_0^\infty \mu(s)|\xi|_2^2 ds. \tag{3.72}$$

As in Lemma 3.6, we obtain

$$\frac{d}{dt}\int_0^\infty k(s)|\xi|_2^2 ds \le 2\delta^2 m_0 |\psi|_2^2 - \frac{1}{2}\int_0^\infty \mu(s)|\xi|_2^2 ds. \tag{3.73}$$

Subsequently, let

$$\mathscr{G}(t) = \int_0^\infty \mu(s)|\xi|_2^2 ds + \tilde{\epsilon}\int_0^\infty k(s)|\xi|_2^2 ds.$$

Taking $\tilde{\epsilon} \in (0, 1/\delta)$, such that

$$(1-\delta\tilde{\epsilon})\int_0^\infty \mu(s)|\xi|_2^2 ds \le \mathscr{G}(t) \le (1+\delta\tilde{\epsilon})\int_0^\infty \mu(s)|\xi|_2^2 ds. \tag{3.74}$$

by (3.72)-(3.74), we obtain

$$\frac{d}{dt}\mathscr{G}(t) \le \tilde{\gamma}_5 |\psi|_2^2, \tag{3.75}$$

where $\tilde{\gamma}_5 = \frac{2m_0}{\tilde{\epsilon}} + 2\tilde{\epsilon}\delta^2 m_0$.

Now integrating (3.71) and (3.75) from τ to t ($t \ge \tau \ge T$) respectively, and combining with (3.74), it follows that

$$\|\omega_1(t) - \omega_2(t)\|_{\mathcal{H}_t}^2 + \int_0^\infty \mu(s)|\sqrt{a}\nabla\xi|_2^2 ds \le C\int_T^t |u_1(s) - u_2(s)|_2^2 ds, \tag{3.76}$$

$$\int_0^\infty \mu(s)|\xi|_2^2 ds \le C\int_T^t |\omega_1(s) - \omega_2(s)|_2^2 ds, \tag{3.77}$$

where $C = \max\{\frac{2\tilde{\mu}^2}{\tilde{\mu}-l}, \frac{\tilde{\gamma}_5}{1-\delta\tilde{\epsilon}}\}$.

From (3.76) and (3.77), we have

$$\begin{aligned} &\|\omega_1(t) - \omega_2(t)\|_{\mathcal{H}_t}^2 + \|\theta_1^t - \theta_2^t\|_{\mu,1}^2 \\ &\le C\int_T^t |u_1(s) - u_2(s)|_2^2 ds + C\int_T^t |\omega_1(s) - \omega_2(s)|_2^2 ds. \end{aligned} \tag{3.78}$$

Then we let

$$\begin{aligned} \varphi_T^t(z_1, z_2) &= C\int_T^t |u_1(s) - u_2(s)|_2^2 ds + C\int_T^t |\omega_1(s) - \omega_2(s)|_2^2 ds \\ &:= \tilde{\varphi}_T^t(z_1, z_2) + \bar{\varphi}_T^t(z_1, z_2). \end{aligned} \tag{3.79}$$

Combining this, Corollary 3.5, and Lemma 3.11, and applying Lemma 2.15, there exists a subsequence of $\{u_n(s)\}_{n=1}^\infty$ that converges strongly in $L^2(T, t; L^2(\Omega))$. In other words, for any sequences $\{z_{nT} = (u_{nT}, \eta_n^T)\} \subset B_T^0 \in \bar{B}_0$, $\{z_n(t) = (u_n(t), \eta_n^t)\}$, as the solution of problem (1.15) with the initial data $\{z_{nT} = (u_{nT}, \eta_n^T)\}$, includes a subsequence $\{z_{n_k}\}$ satisfying:

$$\lim_{k\to\infty}\lim_{l\to\infty}\tilde{\varphi}_T^t(z_{n_k}, z_{n_l}) = C\lim_{k\to\infty}\lim_{l\to\infty}\int_T^t |u_{n_k}(s) - u_{n_l}(s)|_2^2 ds = 0. \tag{3.80}$$

Similarly, through Lemmas 3.13 and 3.14, it is easy to obtain that the set

$$\{\omega(t) = \Pi_1 U_2(t,\tau) z_\tau : z_\tau \in B_\tau^0 \in \bar{B}_0\}$$

is bounded in $H_0^1(\Omega)$; therefore, $\{\omega(t) = \Pi_1 U_2(t, T) z_T : T \le \tau, z_T \in B_T^0 \in \bar{B}_0\}$ is compact in $L^2(T, t; L^2(\Omega))$, where Π is the projection from $X \times Y$ to X. That is, we have

$$\lim_{k\to\infty}\lim_{l\to\infty}\bar{\varphi}_T^t(z_{n_k}, z_{n_l}) = C\lim_{k\to\infty}\lim_{l\to\infty}\int_T^t |\omega_{n_k}(s) - \omega_{n_l}(s)|_2^2 ds = 0. \tag{3.81}$$

By (3.80) and (3.81), we have

$$\lim_{k\to\infty}\lim_{l\to\infty}\varphi_T^t(z_{n_k},z_{n_l})=0. \tag{3.82}$$

This implies that $\varphi_T^t\in\mathfrak{E}(B_T)$. Combining (3.67)-(3.67) and (3.71)-(3.80), one obtain

$$\|U(t,T)x-U(t,T)y\|_{\mathcal{M}_t}^2\le\varepsilon+\varphi_T^t(x,y).$$

From Definitions 2.12 and 2.13, we know that φ_T^t is contractive function in B_T^0. Therefore, it's easy to obtain that the process $U(t,\tau)$ is a $\mathcal{M}_t-$contractive process on $B_T^0\in\bar{B}_0$. □

Theorem 3.17. *The process $U(t,\tau)$ defined by* (3.3) *possesses a time-dependent global attractor $\mathscr{A}=\{A(t)\}_{t\in\mathbb{R}}$ in $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$, and $\mathscr{A}$ is non-empty, compact, invariant in $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ and pullback attracting every bounded set in $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$.*

Proof. By Theorems 3.8 and 3.16, the existence and uniqueness of time-dependent global attractor $\mathscr{A}$ for the process $U(t,\tau)$ generated by equation (1.1) in time-dependent product spaces $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$. In addition, from Lemma 2.11 and (3.2) of Lemma 3.1, we can obtain the invariance of time-dependent global attractor $\mathscr{A}$. □

Remark 3.18. (i) By fully utilizing the method in this paper, if the condition of the nonlinearity $f(u)$ in [15] can be weakened to polynomial growth of arbitrary order (such as (1.9)), then the existence of global attractors for the reaction-diffusion equation with degenerate memory (i.e., the equation (4.1)) can still be obtained.

(ii) Similar to the study of [12], for sufficiently small δ, we can also obtain the regularity of time-dependent attractor $\mathscr{A}$ by Lemmas 3.12 and 3.14, i.e., $\mathscr{A}\subset\{\mathcal{M}_t^1\}_{t\in\mathbb{R}}$. This is so because we can obtain the uniform boundedness of $\|u(t,\tau)z_\tau\|_{\mathcal{M}_t^1}^2$ with respect to τ in Lemma 3.14 by introducing the functional of Lemma 3.3 when δ is sufficiently small.

4. Asymptotic structure of the attractor

Following the idea in [11], we study the relationship between the time-dependent attractor for problem (1.1)-(1.3) and the global attractor for the following limit system (4.1) with initial-boundary (1.2)-(1.3) as $t\to\infty$ (i.e., $\varepsilon(t)=0$):

$$\bar{u}_t-\Delta\bar{u}-\int_0^\infty k(s)\operatorname{div}\{a(x)\nabla\bar{u}(t-s)\}ds+f(\bar{u})=g(x), \tag{4.1}$$

where $\int_0^\infty k(s)\operatorname{div}\{a(x)\nabla\bar{u}(t-s)\}ds=\int_0^\infty\mu(s)\operatorname{div}\{a(x)\nabla\bar{\eta}^t(s)\}ds$. For this purpose, we introduce the following conclusions about the completely bounded trajectories (CBT); for more details see [11].

Definition 4.1. *A function $z:t\mapsto z(t)\in X_t$ is a CBT of $U(t,\tau)$ if and only if*

(i) $\sup_{t\in\mathbb{R}}\|z(t)\|_{X_t}<\infty$, *and*
(ii) $z(t)=U(t,\tau)z(\tau)$, *for all $t\ge\tau\in\mathbb{R}$.*

Theorem 4.2. *Let $\tilde{\mathscr{A}}=\{\mathcal{A}_t\}_{t\in\mathbb{R}}$ be the time-dependent global attractor of $U(t,\tau)$, if $\tilde{\mathscr{A}}$ is invariant, then*

$$\mathcal{A}_t=\{z(t)\in X_t:z\text{ is CBT of }U(t,\tau)\}.$$

Consequently, we can write

$$\mathcal{A}_t=\{z:t\mapsto z(t)\in X_t\text{ with }z\,CBT\text{ of }U(t,\tau)\}.$$

Theorem 4.3. *Suppose that, for any sequence $z_n = (x_n, y_n)$ of CBT of the process $U(t,\tau)$ and any $t_n \to \infty$, there is a CBT w of the semigroup $S(t)$ and any $s \in \mathbb{R}$ for which*
$$\lim_{n\to\infty} \|x_n(s+t_n) - w(s)\|_X = 0,$$
up to a subsequence. Then
$$\lim_{t\to\infty} dist_X(\Pi_t \mathcal{A}_t, \mathcal{A}_\infty) = 0,$$
where $\Pi_t : X_t \to X$ is the projection on the first component of X_t, i.e., for $(u,\eta) \in Z_t$, $\Pi_t(u,\eta) = u$. Accordingly, if $Y_t \in X_t$, then $\Pi_t Y_t = \{u \in X : (u,\eta) \in Y_t\}$. Specially, if $\mathscr{Y} = \{Y_t\}_{t\in\mathbb{R}}$, denote $\Pi\mathscr{Y} = \{\Pi_t Y_t\}_{t\in\mathbb{R}}$.

Let (H1)–(H3) hold, then the unique solution $\bar{z}(t) = (\bar{u}(t), \bar{\eta}^t)$ can be obtained from [15]. In particular, one knows from Remark 3.18 that the semigroup $\{S(t)\}_{t\geq 0}$ generated by (4.1) possesses global attractors on $\mathcal{M}_0 = L^2(\Omega) \times \mathcal{V}_a$ from Remark 3.18, and $\bar{z}(t) = S(t)\bar{z}_\tau$. It should be pointed out that $\{S(t)\}_{t\geq 0}$ has a global attractor $\mathscr{A}_\infty$ in $\mathcal{M}_0$. Additionally, for fixed $s \in \mathbb{R}$, there is
$$\mathscr{A}_\infty = \{\bar{z}(s) : \mathbb{R} \to \mathcal{M}_0 \text{ with } \bar{z} \text{ CBT of } S(t)\}.$$

Next, we establish the asymptotic closeness of the time-dependent global attractor $\mathscr{A} = \{A(t)\}_{t\in\mathbb{R}}$ of the process $\{U(t,\tau)\}_{t\geq\tau}$ generated by (1.1) and the global attractor $\mathscr{A}_\infty$ of the semigroup $\{S(t)\}_{t\geq 0}$. For this purpose, we first give the following Lemma.

Lemma 4.4. *Let $g \in L^2(\Omega)$, $\varepsilon(t)$ satisfy* (1.4)-(1.5), *and the conditions $(H_1)-(H_3)$ hold. Then, for any $z_\tau = (u_\tau, \eta^\tau) \in \mathscr{B}_\tau(R) \subset \mathcal{M}_\tau$, there exist positive constant $\mathscr{K}_i\,(i = 4,5)$ such that*
$$\sup_{z_\tau \in A(\tau)} \sup_{t\geq\tau} \|u\|^2_{\mathcal{H}^1_t} \leq \mathscr{K}_3, \tag{4.2}$$
and
$$\int_\tau^\infty |u_t(r)|_2^2 dr \leq \mathscr{K}_4. \tag{4.3}$$

Proof. By Lemma 3.17, we can obtain (4.2). For the conclusion (4.3), as in the proof of Lemma 3.11, we only need to use the inner product of u_t and (1.15) on $L^2(\Omega)$. Then (4.2) and (4.3) can be obtained by using slightly different estimates. □

Remark 4.5. From Theorem 3.17, we know that $\mathscr{A}$ is invariant. Thus, by Lemma 4.2, we have
$$\mathscr{A} = \{z : t \mapsto z(t) = (u(t), \eta^t) \in \mathcal{M}_t \textit{ with } z \text{ CBT of } U(t,\tau)\},$$

Lemma 4.6. *Under the assumptions of Lemma 4.4, for every sequence $z_n = (u_n, \eta^t_n)$ of CBT for the process $U(t,\tau)$ generated by* (1.1) *and any $t_n \to \infty$ as $n \to \infty$, there exists a CBT $\bar{z} = (\bar{u}, \bar{\eta}^t)$ of the semigroup $S(t)$ generated by* (4.1) *such that, for each $\mathcal{T} > 0$, up to a subsequence,*
$$\begin{gathered} \lim_{n\to\infty} \sup_{t\in[-\mathcal{T},\mathcal{T}]} |u_n(t+t_n) - \bar{u}|_2^2 = 0, \\ \lim_{n\to\infty} \sup_{t\in[-\mathcal{T},\mathcal{T}]} \|\eta_n^{t+t_n}(s) - \bar{\eta}^t(s)\|^2_{\mu,1} = 0. \end{gathered} \tag{4.4}$$

Proof. This lemma will be proved exactly as in [11, Lemma 6.2]. Combining (4.2) and (4.3) in Lemma 4.4, for any $\mathcal{T} > 0$, we know that

$$u_n(\cdot + t_n) \text{ is bounded in } L^\infty([-\mathcal{T},\mathcal{T}], H_0^1(\Omega)),$$
$$\partial_t u_n(\cdot + t_n) \text{ is bounded in } L^2([-\mathcal{T},\mathcal{T}], L^2(\Omega)).$$

Thus, combining this with Lemma 2.15, it yields, up to a subsequence, that

$$u_n(\cdot + t_n) \text{ is relatively compact in } C([-\mathcal{T},\mathcal{T}], L^2(\Omega)).$$

Then there exists a function $\bar{u}$ of $L^2(\Omega)$, such that $u_n(\cdot + t_n) \to \bar{u}(\cdot)$ in the sense of (4.4). Particularly, $\bar{u} \in C([-\mathcal{T},\mathcal{T}], L^2(\Omega))$. Also from (4.2), there exists $\mathscr{K}_5 > 0$ such that

$$\sup_{t\in\mathbb{R}} |\nabla \bar{u}|_2^2 \leq \mathscr{K}_5. \tag{4.5}$$

On the other hand, for any $z \in A(t)$, by Theorem 3.17 we can obtain that

$$\sup_{t\geq\tau}(\|u\|_{\mathcal{H}_t^1} + \|\eta^t\|_{\mu,2}^2 + \int_\tau^t |\Delta u(s)|_2^2 ds) \leq C. \tag{4.6}$$

By Lemma 3.15 and (4.6), we obtain that the sequence $\eta_t^{\cdot+t_n}(s)$ is bounded in

$$L^\infty([-\mathcal{T},\mathcal{T}]; L_\mu^2(\mathbb{R}^+; H_a^2(\Omega)) \cap H_\mu^1(\mathbb{R}^+; L^2(\Omega))),$$

which indicates that

$$\eta_t^{\cdot+t_n}(s) \text{ is relatively compact in } C([-\mathcal{T},\mathcal{T}], L_\mu^2(\mathbb{R}^+; H_a^1(\Omega)).$$

Then there exists a function $\bar{\eta}^\cdot \in L_\mu^2(\mathbb{R}^+; H_a^1(\Omega)$, such that, up to a subsequence, $\eta_t^{\cdot+t_n}(s) \to \eta_t^{\cdot+t_n}(s)$.

Next we need to verify that $\bar{z} = (\bar{u}, \bar{\eta}^t)$ solves (4.1). For this purpose, let

$$\upsilon_n(t) = u_n(t+t_n), \quad \varepsilon_n(t) = \varepsilon(t+t_n), \quad \eta_n^{t+t_n}(s) = \theta_n^t(s).$$

Then (1.1) can be rewritten as

$$\partial_t \upsilon_n = \varepsilon_n(t)\Delta\partial_t \upsilon_n + \Delta\upsilon_n + \int_0^\infty k(s)\operatorname{div}\{a(x)\nabla\upsilon_n(t-s)\}ds - f(\upsilon_n) + g,$$

or

$$\partial_t \upsilon_n = \varepsilon_n(t)\Delta\partial_t \upsilon_n + \Delta\upsilon_n + \int_0^\infty \mu(s)\operatorname{div}\{a(x)\nabla\theta_n^t(s)\}ds - f(\upsilon_n) + g.$$

Next, we handle the first term at the right end of the above formula. For a fixed $\mathcal{T} > 0$ and every smooth function ϕ with L^2-value and supported on $(-\mathcal{T},\mathcal{T})$, through the processing method in [30, 11, section 6], we can obtain that there exists positive constant χ_0 fixed later, such that

$$\Big|\int_{-\mathcal{T}}^{\mathcal{T}}(\varepsilon_n(t)\Delta\partial_t\upsilon_n(t), \phi(t))dt\Big| \leq \chi_0\mathcal{T}\sup_{t\in[-\mathcal{T},\mathcal{T}]}\sqrt{\varepsilon_n(t)} + \chi_0(\sqrt{\varepsilon_n(\mathcal{T})} - \sqrt{\varepsilon_n(-\mathcal{T})}). \tag{4.7}$$

By (4.2), we have

$$\begin{aligned}
&\Big|\int_{-\mathcal{T}}^{\mathcal{T}}(\varepsilon_n(t)\Delta\partial_t\upsilon_n(t), \phi(t))dt\Big| \\
&= \Big| -\int_{-\mathcal{T}}^{\mathcal{T}}\varepsilon_n(t)(\Delta\upsilon(t), \phi'(t))dt - \int_{-\mathcal{T}}^{\mathcal{T}}\varepsilon_n'(t)(\Delta\upsilon_n(t), \phi(t))dt\Big|
\end{aligned}$$

$$
\begin{aligned}
&\le \mathcal{Q}(|\phi'|_2)\int_{-\mathcal{T}}^{\mathcal{T}}\varepsilon_n|\Delta\upsilon|_2dt+\mathcal{Q}_1(|\phi|_2)\int_{-\mathcal{T}}^{\mathcal{T}}\varepsilon_n'|\Delta\upsilon_n(t)|_2dt\\
&\le \mathcal{Q}(|\phi'|_2)\Big(\int_{-\mathcal{T}}^{\mathcal{T}}\varepsilon_n(t)|\Delta\upsilon|_2^2dt\Big)^{1/2}\Big(\int_{-\mathcal{T}}^{\mathcal{T}}\varepsilon_n(t)dt\Big)^{1/2}\\
&\quad+\mathcal{Q}(|\phi|_2)_1\int_{-\mathcal{T}}^{\mathcal{T}}\frac{\varepsilon_n'(t)}{\sqrt{\varepsilon_n(t)}}\sqrt{\varepsilon_n(t)}|\Delta\upsilon_n(t)|_2dt\\
&\le \mathcal{Q}(\sqrt{\mathscr{K}_3},|\phi'|_2,\sqrt{\mathcal{T}})\sup_{t\in[-\mathcal{T},\mathcal{T}]}\sqrt{\varepsilon_n(t)}+\mathcal{Q}_1(\sqrt{\mathscr{K}_3},|\phi'|_2)(\sqrt{\varepsilon_n(\mathcal{T})}-\sqrt{\varepsilon_n(-\mathcal{T})}),
\end{aligned}
$$

where $\mathcal{Q}_1(\cdot,\cdot)$ and $\mathcal{Q}(\cdot,\cdot,\cdot)$ denote positive constants associated with some certain parameters. Let

$$
\chi_0=\chi_0(\mathscr{K}_3,|\phi|_2,|\phi'|_2,\sqrt{\mathcal{T}})=\max\{\mathcal{Q}(\sqrt{\mathscr{K}_3},|\phi'|_2,\sqrt{\mathcal{T}}),\mathcal{Q}_1(\sqrt{\mathscr{K}_3},|\phi'|_2)\}.
$$

Then (4.7) holds.

Because

$$
\lim_{n\to\infty}(\sup_{t\in[-\mathcal{T},\mathcal{T}]}\varepsilon_n(t))=0,
$$

we have

$$
\lim_{n\to\infty}(\sup_{t\in[-\mathcal{T},\mathcal{T}]}\sqrt{\varepsilon_n(t)})=0.
$$

Therefore, from (4.8), we have

$$
\lim_{n\to\infty}\Big|\int_{-\mathcal{T}}^{\mathcal{T}}(\varepsilon_n(t)\Delta\partial_t\upsilon_n(t),\phi(t))dt\Big|=0. \tag{4.8}
$$

Moreover, by the continuity of f (see (H2)), and $\upsilon_n\to\bar{u}$ is almost every convergent in Ω as $n\to\infty$, this yields (up to a subsequence)

$$
f(\upsilon_n)\to f(\bar{u})\quad\text{and}\quad\Delta\upsilon_n\to\Delta\bar{u} \tag{4.9}
$$

in $L^\infty([-\mathcal{T},\mathcal{T}];H^{-1}(\Omega))$ for any $\mathcal{T}>0$. In particular, we can also have

$$
\int_0^\infty\mu(s)\operatorname{div}\{a(x)\nabla\theta_n^t(s)\}ds\to\int_0^\infty\mu(s)\operatorname{div}\{a(x)\nabla\bar{\eta}^t(s)\}ds \tag{4.10}
$$

in the sense of distributions. In fact, for any $\bar{\phi}\in C_0^\infty(\Omega)$, when $n\to\infty$, we have

$$
\begin{aligned}
&\Big|\int_{-\mathcal{T}}^{\mathcal{T}}(\int_0^\infty\mu(s)\operatorname{div}\{a(x)\nabla\theta_n^t(s)\}ds-\int_0^\infty\mu(s)\operatorname{div}\{a(x)\nabla\bar{\eta}^t(s)\}ds,\bar{\phi})dt\Big|\\
&=\Big|\int_{-\mathcal{T}}^{\mathcal{T}}(\int_0^\infty\mu(s)(a(x)\nabla\theta_n^t(s)-a(x)\nabla\bar{\eta}^t(s))ds,\nabla\bar{\phi})dt\Big|\\
&\le\Big|\int_{-\mathcal{T}}^{\mathcal{T}}\int_0^\infty\mu(s)|a(x)\nabla\theta_n^t(s)-a(x)\nabla\bar{\eta}^t(s)ds|_2|\nabla\bar{\phi}|_2dt\Big|\\
&\le|\nabla\bar{\phi}|_2\sqrt{m_0}|a(x)|_\infty\Big|\int_{-\mathcal{T}}^{\mathcal{T}}\Big(\int_0^\infty\mu(s)|\nabla\theta_n^t(s)-\nabla\bar{\eta}^t(s)|_2^2ds\Big)^{1/2}dt\Big|\to0.
\end{aligned} \tag{4.11}
$$

From the definition of θ_n^t we obtain that

$$
\partial_s\theta_n^t=\begin{cases}u_n(t-s), & \tau<s\le t\\ 0, & s>t,\end{cases} \tag{4.12}
$$

which together with Theorem 3.8, Lemma 3.12, Lemma 3.14, Theorem 3.17 and Lemma 2.16 implies that $\theta_n^t(s)$ is relatively compact in $L^2_\mu(\mathbb{R}^+; H_0^1(\Omega))$ (the proof is similar to Lemma 3.15). Thus, we have

$$\int_0^\infty \mu(s)|\nabla\theta_n^t(s) - \nabla\bar{\eta}^t(s)|_2^2 ds \to 0, \quad \text{as } n \to \infty,$$

this implies that (4.11) holds.

Meanwhile, in the distributional sense, there exists a subsequence such that

$$\partial_t \upsilon_n \to \bar{u}_t.$$

In conclusion, we can obtain the equality

$$\bar{u}_t - \Delta\bar{u} - \int_0^\infty \mu(s)\operatorname{div}\{a(x)\nabla\bar{\eta}^t(s)\}ds + f(\bar{u}) = g,$$

i.e., in the sense of Dafermos' transformation,

$$\bar{u}_t - \Delta\bar{u} - \int_0^\infty k(s)\operatorname{div}\{a(x)\nabla\bar{u}(t-s)\}ds + f(\bar{u}) = g,$$

which implies that $\bar{z} = (\bar{u}, \bar{\eta}^t)$ is solution of (4.1). Combining (1.12) and (4.5), it is clear that $\bar{z}$ is a CBT for the semigroup $\{S(t)\}_{t\geq 0}$. $\square$

By Lemma 4.6 and Theorem 4.3, the following conclusion can be obtained at once.

Theorem 4.7. *If* $\mathscr{A} := \{A_t\}_{t\in\mathbb{R}} = \{A(t)\}_{t\in\mathbb{R}}$ *and* $\mathscr{A}_\infty$ *is time-dependent global attractors and global attractors of* $\{U(t,\tau)\}_{t\geq\tau}$ *and* $\{S(t)\}_{t\geq 0}$ *generated by* (1.1) *and* (4.1) *respectively. Then there is*

$$\lim_{t\to\infty} \operatorname{dist}_{\mathcal{M}_0}(\Pi_t A_t, \mathscr{A}_\infty) = 0.$$

Acknowledgments. This research was supported by the Key Technologies R&D Program of CNBM (No. 2021HX1617), and by the General Project of Education Department of Hunan Province (Nos. 21C0660).

References

[1] F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli; Carleman estimates for degenerate parabolic operators with applications to null controllability, *J. Evol. Equ.,* 6 (2006), 161-204.

[2] G. Barenblatt, Y. Zheltov, I. Kochina; Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks, *J. Appl. Math. Mec.*, 24 (1960), 1286-1803.

[3] S. Borini, V. Pata; Uniform attractors for a strongly damped wave equations with linear memory, *Asymptotic Anal.*, 20 (1999), 263-277.

[4] M. Campiti, G. Metafune, D. Pallara; Degenerate self-adjoint evolution equations on the unit interval, *Semigroup Forum*, 57 (1998), 1-36.

[5] P. Cannarsa, D. Rocchetti, J. Vancostenoble; Generation of analytic semi-groups in L^2 for a class of second order degenerate elliptic operators, *Control Cybern.*, 37 (2008), 831-878.

[6] M. M. Cavalcanti, V. N. Domingos Cavalcanti, M. A. Jorge Silva, A. Y. Souza Franco; Exponential stability for the wave model with localized memory in a past history framework, *J. Differential Equations*, 264 (2018), 6535-6584.

[7] M. M. Cavalcanti, L. H. Fatori, T. F. Ma; Attractors for wave equations with degenerate memory, *J. Differential Equations*, 260 (2016), 56-83.

[8] P. J. Chen, M. E. Gurtin; On a theory of heat conduction involving two temperatures, *Z. Angew. Math. Phys.*, 19 (1968), 614-627.

[9] V. V. Chepyzhov, A. Miranville; On trajectory and global attractors for semilinear heat equations with fading memory, *Indiana U. Math. J.*, 55 (2006), 119-167.

[10] M. Conti, F. Dell'Oro, V. Pata; Nonclassical diffusion with memory lacking instantaneous damping, *Commun. Pur. Appl. Anal.*, 19 (2020), 2035-2050.

[11] M. Conti, V. Pata; Asymptotic structure of the attractor for processes on time-dependent spaces, *Nonlinear Anal. Real World Appl.*, 19 (2014), 1-10.

[12] M. Conti, V. Pata, R. Temam; Attractors for process on time-dependent spaces: Applications to wave equations, *J. Differential Equations*, 255 (2013), 1254-1277.

[13] M. Conti, E. M. Marchini, V. Pata; Nonclassical diffusion with memory, *Math. Method. Appl. Sci.*, 38 (2015), 948-958.

[14] C. M. Dafermos; Asymptotic stability in viscoelasticity, *Arch. Ration. Mech. Anal.*, 37 (1970), 297-308.

[15] J. C. O. Faria, C. M. Webler; Existence of a global attractor for the heat equation with degenerate memory, *J. Dynam. Differential Equations*, 35 (2023), 845-864.

[16] H. Gao, L. Li, Z. Liu; Stability of degenerate heat equation in non-cylindrical/cylindrical domain, *Z. Angew. Math. Phys.*, 70 (2019), 1-17.

[17] S. Gatti, M. Grasselli, V. Pata; Lyapunov functionals for reaction-diffusion equations with memory, *Math. Method. Appl. Sci.*, 28 (2005), 1725-1735.

[18] C. Giorgi, M. G. Naso, V. Pata; Exponential stability in linear heat conduction with memory: a semigroup approach, *Commun. Appl. Anal.*, 5 (2001), 121-133.

[19] C. Giorgi, V. Pata, A. Marzocchi; Asymptotic behavior of a semilinear problem in heat conduction with memory, *NoDEA-Nonlinear Diff. Equ. Appl.*, 5 (1998), 333-354.

[20] M. Grasselli, V. Pata; Uniform attractors for nonautonomous dynamical systems with memory, *Progress Nonlinear Differ. Equ. Appl.*, 50 (2002), 155-178.

[21] J. Jackle; Heat conduction and relaxation in liquids of high viscosity, *Physica A*, 162 (1990), 377-404.

[22] P. Kloeden, T. Lorenz; Pullback incremental attraction, *Nonauton. Dynam. Syst.*, 1 (2004), 53-60.

[23] Y. Liu; Time-dependent global attractor for the nonclassical diffusion equation, *Appl. Anal.*, 94 (2015), 1439-1449.

[24] Q. Ma, X. Wang, L. Xu; Existence and regularity of time-dependent global attractors for the nonclassical reaction-diffusion equations with lower forcing term, *Bound. Value Probl.*, 1 (2016), 1-11.

[25] F. Meng, M. Yang, C. Zhong; Attractors for wave equation with nonlinear damping on time-dependent space, *Discrete Contin. Dyn. Syst. Ser. B*, 21 (2016), 205-225.

[26] J. C. Robinson; *Infinite-Dimensional Dynamical Dystems*, Cambridge: Cambridge University Press, 2001.

[27] J. L. Shomberg; Regular global attractors for wave equations with degenerate memory, *Ural Math. J.*, 5 (2019), 59-82.

[28] C. Sun, D. Cao, J. Duan; Non-autonomous dynamics of wave equations with nonlinear damping and critical nonlinearity, *Nonlinearity*, 19 (2006), 2645-2665.

[29] N. D. Toan; Uniform attractors of nonclassical diffusion equations lacking instantaneous damping on $\mathbb{R}^N$ with Memory, *Acta Appl. Math.*, 170 (2020), 789-822.

[30] J. Wang, Q. Ma; Asymptotic dynamic of the nonclassical diffusion equation with time-dependent coefficient, *J. Appl. Anal. Comput.*, 11 (2020), 445-463.

[31] X. Wang, L. Yang, C. Zhong; Attractors for the nonclassical diffusion equations with fading memory, *J. Math. Anal. Appl.*, 362 (2010), 327-337.

[32] Y. Xie, D. Liu, J. Zhang, X. Liu; Uniform attractors for nonclassical diffusion equations with perturbed parameter and memory, *J. Math. Phys.*, 64 (2023), 022701.

[33] Z. Xie, J. Zhang, Y. Xie; Asymptotic behavior of quasi-linear evolution equations on time-dependent product spaces, *Discrete Contin. Dyn. Syst. Ser. B*, 28 (2023), 2316-2334.

[34] Y. Xie, Q. Li, K. Zhu; Attractors for nonclassical diffusion equations with arbitrary polynomial growth nonlinearity, *Nonlinear Anal. Real World Appl.*, 31 (2016), 23-37.

[35] Y. Xie, Y. Li, Y. Zeng; Uniform attractors for nonclassical diffusion equations with memory, *J. Funct. Space*, 3-4 (2016), 1-11.

[36] J. Yuan, S. Zhang, Y. Xie, J. Zhang; Attractors for a class of perturbed nonclassical diffusion equations with memory, *Discrete Contin. Dyn. Syst. Ser. B*, 27 (2022), 4995-5007.

[37] J. Zhang, Y. Xie, Q. Luo, Z. Tang; Asymptotic behavior for the semilinear reaction-diffusion equations with memory, *Adv. Differ. Equations*, 1 (2019), 510.

[38] J. Zhang, Z. Liu, J. Huang; Upper semicontinuity of pullback $\mathscr{D}$-attractors for nonlinear parabolic equation with nonstandard growth condition, *Math. Nachr.*, 296 (2023), 5593-5616.

[39] J. Zhang, Z. Xie, Y. Xie; Long-time behavior of nonclassical diffusion equations with memory on time-dependent spaces, *Asymptotic Anal.*, (2023), doi: 10.3233/ASY-231887.

[40] K. Zhu, Y. Xie, F. Zhou; Attractors for the nonclassical reaction-diffusion equations on time-dependent spaces, *Bound. Value Probl.*, 95 (2020), 1-14.

Jiangwei Zhang
College of Science, National University of Defense Technology, Changsha 410073, China
Email address: zjwmath@163.com

Zhe Xie
Research and Development Center, Sinoma Wind Power Blade Co. Ltd, Beijing 100192, China
Email address: xiezhe890917@126.com

Yongqin Xie
School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha 410114, China
Email address: xieyqmath@126.com