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FORM OF SOLUTIONS TO QUADRATIC TRINOMIAL PARTIAL DIFFERENTIAL EQUATIONS WITH TWO COMPLEX VARIABLES

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ABSTRACT. This article describes the from of entire solutions to quadratic trinomial partial differential equations (PDEs). By applying the Nevanlinna theory and the characteristic equation of PDEs, we extend some of the results obtained in [24] . Also we also provide examples that illustrate our results.

1. Introduction

In 1995, Wiles and Taylor [18, 19] pointed out that the Fermat-type functional equation (also called Pythagorean functional equation)

$$x^m + y^m = 1 \tag{1.1}$$

does not admit non-trivial solutions when $m \geq 3$, but it admits non-trivial solutions when $m = 2$. Actually, the study of (1.1) can be tracked back to Montel [12] and Gross [1]. They proved that the equation $f^m + g^m = 1$ has entire solutions and pointed out that for $m = 2$, the equation has non-constant entire solutions $f = \cos p, g = \sin p$, where p is any non-constant entire function. Recently, with the evolution of Nevanlinna theory, many scholars gained plentiful results about these equations of Fermat-type. Liu, Cao and Cao [8] in 2012 investigated the existence of entire solutions with finite order of Fermat equations and obtained the following result.

Theorem 1.1 ([8]). *Suppose f is a transcendental entire solution of*

$$f'(z)^2 + f(z+c)^2 = 1, \tag{1.2}$$

then f must satisfy $f(z) = \sin(z \pm Bi)$, where $B \in \mathbb{C}$ and $c = 2k\pi$, or $c = (2k+1)\pi$ with k an integer.

In 2013, Saleeby [16] generalized the Pythagorean functional equation $f^2+g^2 = 1$ and studied the quadratic trinomial functional equation

$$f^2 + 2\alpha fg + g^2 = 1, \quad \alpha \in \mathbb{C} - \{1, -1\} \tag{1.3}$$

and obtained the following result.

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Theorem 1.2 ([16]). *If equation* (1.3) *has a transcendental entire solution, then f, g must satisfy*

$$f = \frac{1}{\sqrt{2}}\Big(\frac{\cos h}{\sqrt{1+\alpha}} + \frac{\sin h}{\sqrt{1-\alpha}}\Big), \quad g = \frac{1}{\sqrt{2}}\Big(\frac{\cos h}{\sqrt{1+\alpha}} - \frac{\sin h}{\sqrt{1-\alpha}}\Big)$$

or

$$f = \frac{\alpha_1 - \alpha_2\beta^2}{(\alpha_1-\alpha_2)\beta}, \quad g = \frac{1-\beta^2}{(\alpha_1-\alpha_2)\beta},$$

where h is an entire function, β is a meromorphic function and $\alpha_1 = -\alpha+\sqrt{\alpha^2-1}$, $\alpha_2 = -\alpha - \sqrt{\alpha^2-1}$.

In 2016, Liu and Yang [9] researched the related properties on the meromorphic solutions of the following equations, for $\alpha^2 \neq 0, 1$,

$$f(z)^2 + 2\alpha f(z)f'(z) + f'(z)^2 = 1, \tag{1.4}$$

$$f(z)^2 + 2\alpha f(z)f(z+c) + f(z+c)^2 = 1. \tag{1.5}$$

If $\alpha^2 \neq 0, 1$, then (1.4) has no transcendental meromorphic solutions but (1.5) has transcendental meromorphic solutions with finite order and the order must be equal to one.

Now, let us mention some previous results about the Fermat-type PDEs with two complex variables. In 1995, Khavinson [3] pointed out that any entire solution of the partial differential equations

$$\Big(\frac{\partial u}{\partial z_1}\Big)^2 + \Big(\frac{\partial u}{\partial z_2}\Big)^2 = 1 \tag{1.6}$$

in $\mathbb{C}^2$ is necessarily linear. Later, Saleeby in [15] extended the result by exploring the solutions of Fermat-type functional equations (1.6) and obtain the following result.

Theorem 1.3 ([15]). *The entire solution of* (1.6) *must satisfy $u(z_1, z_2) = c_1z_1 + c_2z_2 + c$, where $c, c_1, c_2 \in \mathbb{C}$ and $c_1^2 + c_2^2 = 1$.*

Later, Li et al. discussed equations (1.6) with more general forms

$$\Big(\frac{\partial f}{\partial z_1}\Big)^2 + \Big(\frac{\partial f}{\partial z_2}\Big)^2 = f^n,$$

$$(\frac{\partial f}{\partial z_1})^2 + (\frac{\partial f}{\partial z_2})^2 = p, \Big(\frac{\partial f}{\partial z_1}\Big)^2 + \Big(\frac{\partial f}{\partial z_2}\Big)^2 = e^g,$$

where $n \in N^+$, p, g are polynomials in $\mathbb{C}^2$ (see [4, 5, 6, 7]). Li in 2005 further investigated the functional equation of Fermat-type

$$\Big(\frac{\partial u}{\partial z_1}\Big)^2 + \Big(\frac{\partial u}{\partial z_2}\Big)^2 = e^g \tag{1.7}$$

and obtained the following result.

Theorem 1.4 ([6]). *If equation* (1.7) *admits an entire solution of $f(z)$ with finite order in $\mathbb{C}^2$, where g is a polynomial, then u is an entire solution of* (1.7) *if and only if*

(i) $u = f(c_1z_1 + c_2z_2)$ *or*
(ii) $u = \phi_1(z_1 + iz_2) + \phi_2(z_1 - iz_2)$,

where f is an entire solution and $f'(c_1z_1 + c_2z_2) = \pm e^{\frac{1}{2}g(z)}$, c_1 and c_2 are two constants satisfying ${c_1}^2 + {c_2}^2 = 1$, and $\phi_1'(z_1 + iz_2) + \phi_2'(z_1 - iz_2) = \frac{1}{4}e^{g(z)}$.

Recently, Lü [10] studied the quadratic trinomial partial differential equation

$$u_{z_1}{}^2 + 2Bu_{z_1}u_{z_2} + u_{z_2}{}^2 = e^{g(z)}, \tag{1.8}$$

where B is a constant and g is a polynomial or an entire function in $\mathbb{C}^2$, and obtained the following result.

Theorem 1.5 ([10]). *Let g be a polynomial in $\mathbb{C}^2$, let t_1 and t_2 be two different roots of the equation $1+2At+t^2=0(A\neq\pm1)$. Then u is an entire solution of the partial differential equation (1.8) if and only if*

(i) $u(z_1,z_2) = G(z_1+t_2z_2+B(z_1+t_1z_2))$ *or*
(ii) $u(z_1,z_2) = \phi_1(z_1+t_1z_2)+\phi_2(z_1+t_2z_2)$,

where G is an entire function in $\mathbb{C}$ satisfying

$$BG'^2(z_1+t_2z_2+B(z_1+t_1z_2)) = e^{g-\log\tau},$$

where $B\in\mathbb{C}-\{0\}$, $\tau=4(1-A^2)$, ϕ_1 and ϕ_2 are entire functions in $\mathbb{C}$ and satisfy

$$\phi_1'(z_1+t_1z_2) = e^{\alpha(z_1+t_1z_2)}, \quad \phi_2'(z_1+t_2z_2) = e^{\beta(z_1+t_2z_2)},$$

where α and β are two polynomials such that

$$\alpha(z_1+t_1z_2)+\beta(z_1+t_2z_2) = g(z_1,z_2)-\log\tau.$$

Theorem 1.6 ([10]). *Let g be an entire function in $\mathbb{C}^2$. Then u is an entire solution of the partial differential equation (1.8) in $\mathbb{C}^2$ if and only if*

$$u(z_1,z_2) = F(z_1,z_2\mp z_1)+f(z_2\mp z_1),$$

where f is an entire function in $\mathbb{C}$ and

$$F(t,s) = \int_0^t \pm e^{\frac{g(t,\pm t+s)}{2}}\,dt.$$

In 2020, Xu, Tu and Wang [24] researched several Fermat-type PDEs and PDDEs with two complex variables

$$\Big(f(z)+\frac{\partial f}{\partial z_1}\Big)^2+\Big(f(z)+\frac{\partial f}{\partial z_2}\Big)^2 = 1, \tag{1.9}$$

$$\Big(f(z)+\frac{\partial f}{\partial z_1}\Big)^2+\Big(f(z)+\frac{\partial^2 f}{\partial z_1\partial z_2}\Big)^2 = 1, \tag{1.10}$$

and obtained interesting results:

Theorem 1.7 ([24]). *If equation (1.9) has an entire solution in $\mathbb{C}^2$, then*

$$f(z_1,z_2) = \pm\frac{\sqrt{2}}{2}+\eta e^{-(z_1+z_2)}$$

or

$$f(z_1,z_2) = \frac{1}{2}sin(z_2-z_1+\eta_1)+\frac{1}{2}cos(z_2-z_1+\eta_1)+\eta_2e^{-(z_1+z_2)},$$

where $\eta,\eta_1,\eta_2\in\mathbb{C}$.

Theorem 1.8 ([24]). *If equation (1.10) has an entire solution in $\mathbb{C}^2$, then*

$$f(z_1,z_2) = \pm\frac{\sqrt{2}}{2}+\eta e^{z_2-z_1},$$

where $\eta\in\mathbb{C}$.

Other results on Fermat-type PDEs with two complex variables can be found in [2, 3, 20, 11, 21, 22, 23, 25, 26]). Inspired by the aforesaid theorems, the following question raises spontaneously.

> What will happen when u_{z_1} is superseded by $f(z)+\frac{\partial f}{\partial z_1}$, and u_{z_2} is superseded by $f(z)+\frac{\partial f}{\partial z_2}$ or $f(z)+\frac{\partial^2 f}{\partial z_1^2}$, or $f(z)+\frac{\partial^2 f}{\partial z_1 \partial z_2}$ in question (1.8), where $g(z)$ is a polynomial?

2. Results and examples

Motivated by the above question, we study the solutions of the following quadratic trinomial partial differential equations, utilizing the Nevanlinna theory and the characteristic equation of partial differential equations:

$$\Big(f(z)+\frac{\partial f}{\partial z_1}\Big)^2+2\alpha\Big(f(z)+\frac{\partial f}{\partial z_1}\Big)\Big(f(z)+\frac{\partial f}{\partial z_2}\Big)+\Big(f(z)+\frac{\partial f}{\partial z_2}\Big)^2=e^{g(z)}, \tag{2.1}$$

$$\Big(f(z)+\frac{\partial f}{\partial z_1}\Big)^2+2\alpha\Big(f(z)+\frac{\partial f}{\partial z_1}\Big)\Big(f(z)+\frac{\partial^2 f}{\partial z_1^2}\Big)+\Big(f(z)+\frac{\partial^2 f}{\partial z_1^2}\Big)^2=e^{g(z)}, \tag{2.2}$$

$$\Big(f(z)+\frac{\partial f}{\partial z_1}\Big)^2+2\alpha\Big(f(z)+\frac{\partial f}{\partial z_1}\Big)\Big(f(z)+\frac{\partial^2 f}{\partial z_1\partial z_2}\Big)+\Big(f(z)+\frac{\partial^2 f}{\partial z_1\partial z_2}\Big)^2=e^{g(z)}, \tag{2.3}$$

where $\alpha^2\in\mathbb{C}-\{0,1\}$ and $g(z)$ be a polynomial with the linear form $g(z_1,z_2)=\alpha_1 z_1+\alpha_2 z_2+\alpha_0$, where $\alpha_1\neq 0,\alpha_2\neq 0,\alpha_0\in\mathbb{C}$.

For simplicity, let $\alpha^2\neq 0,1$, and

$$A_1:=\frac{1}{2\sqrt{1+\alpha}}-\frac{i}{2\sqrt{1-\alpha}},\quad A_2:=\frac{1}{2\sqrt{1+\alpha}}+\frac{i}{2\sqrt{1-\alpha}}. \tag{2.4}$$

Our main results read as follows.

Theorem 2.1. *Suppose equation* (2.1) *admits a transcendental entire solution $f(z)$ of finite order. Then $f(z_1,z_2)$ must satisfy the following:*

(i) $f(z_1,z_2)=\zeta_1(\beta,\alpha,\alpha_1,\alpha_2)e^{g(z)/2}+\eta e^{-(z_1+z_2)}$, where

$$\zeta_1(\beta,\alpha,\alpha_1,\alpha_2)=\begin{cases}\frac{\sqrt{2}(\beta^2-1)}{i\beta(\alpha_1-\alpha_2)\sqrt{1-\alpha}}, & \alpha_1\neq\alpha_2,\\ \pm\frac{\sqrt{2}}{(2+\alpha_1)\sqrt{1+\alpha}}, & \alpha_1=\alpha_2\neq -2,\\ \frac{(\beta^2+1)(z_1+z_2)}{2\beta\sqrt{2(1+\alpha)}}+\frac{(\beta^2-1)(z_1-z_2)}{2i\beta\sqrt{2(1-\alpha)}}, & \alpha_1=\alpha_2=-2,\end{cases}$$

and $\beta,\alpha,\alpha_1,\alpha_2$ satisfy

$$\frac{(4+\alpha_1+\alpha_2)(\beta^2-1)}{i\sqrt{1-\alpha}}=\frac{(\alpha_1-\alpha_2)(\beta^2+1)}{\sqrt{1+\alpha}};$$

(ii) if $B_{11}=B_{21},B_{12}=B_{22}$, then (2.1) *has no transcendental entire solution $f(z)$ of finite order, hence $B_{11}=B_{21},B_{12}=B_{22}$ cannot coexist, then*

$$f(z_1,z_2)=\frac{1}{\sqrt{2}}\vartheta_1(B_{11},B_{12},B_{21},B_{22})+\eta e^{-(z_1+z_2)},$$

where

$$\vartheta_1(B_{11},B_{12},B_{21},B_{22})$$

$$= \begin{cases} \frac{A_1-A_2}{B_{11}-B_{12}}e^{\gamma_1(z)} + \frac{A_2-A_1}{B_{21}-B_{22}}e^{\gamma_2(z)}, & B_{11}\neq B_{12}, B_{21}\neq B_{22},\\ (A_1z_1+A_2z_2)e^{\gamma_1(z)} + \frac{A_2-A_1}{B_{21}-B_{22}}e^{\gamma_2(z)}, & B_{11}= B_{12}, B_{21}\neq B_{22},\\ \frac{A_1-A_2}{B_{11}-B_{12}}e^{\gamma_1(z)} + (A_2z_1+A_1z_2)e^{\gamma_2(z)}, & B_{11}\neq B_{12}, B_{21}= B_{22}, \end{cases}$$

moreover, $\eta \neq 0$, $\gamma_1(z) = B_{11}z_1 + B_{12}z_2 + \beta_1, \gamma_2(z) = B_{21}z_1 + B_{22}z_2 + \beta_2$, $B_{j1}, B_{j2}, \beta_j \in \mathbb{C}(j = 1, 2)$ *satisfy*

$$g(z) = \gamma_1(z) + \gamma_2(z) = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_0,$$

and $B_{11}, B_{12}, B_{21}, B_{22}$ *satisfy*

$$A_2(B_{11} + 1) = A_1(B_{12} + 1), \quad A_1(B_{21} + 1) = A_2(B_{22} + 1).$$

We list several examples to show the forms of solutions in Theorem 2.1 are precise.

Example 2.2. $f(z_1, z_2) = \pm\frac{2}{\sqrt{37}}e^{\frac{z_1+2z_2}{2}} + ie^{-(z_1+z_2)}$ is a solution of (2.1) with $g(z) = z_1 + 2z_2$. Here, $\alpha = 1/2$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_0 = 0$, $\eta = i$.

Example 2.3. $f(z_1, z_2) = \pm\frac{\sqrt{6}}{3}e^{\frac{-z_1-z_2+2}{2}} + \sqrt{2}e^{-(z_1+z_2)}$ is a solution of (2.1) with $g(z) = -z_1 - z_2 + 2$. Here, $\alpha = 2$, $\alpha_0 = 2$, $\alpha_1 = \alpha_2 = -1$, $\eta = \sqrt{2}$.

Example 2.4. $f(z_1, z_2) = \pm\frac{\sqrt{6}}{6}(z_1 + z_2)e^{\frac{-2z_1-2z_2+1}{2}} + \sqrt{2}e^{-(z_1+z_2)}$ is a solution of (2.1) with $g(z) = -2z_1 - 2z_2 + 1$. Here, $\alpha = 2$, $\alpha_0 = 1$, $\alpha_1 = \alpha_2 = -2$, $\eta = \sqrt{2}$.

Example 2.5. Let $\alpha = 1/2$,

$$\gamma_1(z) = \frac{\sqrt{6}-6-3\sqrt{2}i}{6}z_1 + \frac{\sqrt{6}-6+3\sqrt{2}i}{6}z_2,$$
$$\gamma_2(z) = \frac{\sqrt{6}-6+3\sqrt{2}i}{6}z_1 + \frac{\sqrt{6}-6-3\sqrt{2}i}{6}z_2,$$

$\beta_1 = \beta_2 = 0$, and $\eta = \frac{\sqrt{5}}{2}$. Then

$$f(z_1, z_2) = \frac{1}{\sqrt{2}}\Big(e^{\frac{\sqrt{6}-6-3\sqrt{2}i}{6}z_1 + \frac{\sqrt{6}-6+3\sqrt{2}i}{6}z_2} + e^{\frac{\sqrt{6}-6+3\sqrt{2}i}{6}z_1 + \frac{\sqrt{6}-6-3\sqrt{2}i}{6}z_2}\Big) + \frac{\sqrt{5}}{2}e^{-(z_1+z_2)}$$

is a transcendental entire solution of (2.1) with $g(z) = \frac{\sqrt{6}-6}{3}(z_1 + z_2)$.

Example 2.6. Let $\alpha = 1/2$, $\gamma_1(z) = -z_1-z_2$, $\gamma_2(z) = \frac{\sqrt{6}-6+3\sqrt{2}i}{6}z_1 + \frac{\sqrt{6}-6-3\sqrt{2}i}{6}z_2$, $\beta_1 = \beta_2 = 0$, $\eta = i$. Then

$$f(z_1, z_2) = \Big(\frac{\sqrt{3}-3i}{6}z_1 + \frac{\sqrt{3}+3i}{6}z_2 + i\Big)e^{-z_1-z_2} + \frac{1}{\sqrt{2}}e^{\frac{\sqrt{6}-6+3\sqrt{2}i}{6}z_1 + \frac{\sqrt{6}-6-3\sqrt{2}i}{6}z_2}$$

is a transcendental entire solution of (2.1) with $g(z) = \frac{\sqrt{6}+3\sqrt{2}i-12}{6}z_1 + \frac{\sqrt{6}-3\sqrt{2}i-12}{6}z_2$.

Example 2.7. Let $\alpha = 1/2$, $\gamma_1(z) = \frac{\sqrt{6}-6-3\sqrt{2}i}{6}z_1 + \frac{\sqrt{6}-6+3\sqrt{2}i}{6}z_2$, $\gamma_2(z) = -z_1-z_2$, $\beta_1 = \beta_2 = 0$, $\eta = \sqrt{5}$. Then

$$f(z_1, z_2) = \frac{1}{\sqrt{2}}e^{\frac{\sqrt{6}-6-3\sqrt{2}i}{6}z_1 + \frac{\sqrt{6}-6+3\sqrt{2}i}{6}z_2} + \Big(\frac{\sqrt{3}+3i}{6}z_1 + \frac{\sqrt{3}-3i}{6}z_2 + \sqrt{5}\Big)e^{-z_1-z_2}$$

is a transcendental entire solution of (2.1) with $g(z)=\frac{\sqrt{6}-3\sqrt{2}i-12}{6}z_1+\frac{\sqrt{6}+3\sqrt{2}i-12}{6}z_2$.

Theorem 2.8. *Suppose equation* (2.2) *admits a transcendental entire solution $f(z)$ of finite order, then $f(z_1,z_2)$ must satisfy the following:*

(i) $f(z_1,z_2)=\zeta_2(\beta,\alpha,\alpha_1,\alpha_2)e^{g(z)/2}$, where

$$\zeta_2(\beta,\alpha,\alpha_1,\alpha_2)=\begin{cases}\frac{2\sqrt{2}(\beta^2-1)}{i\beta\alpha_1(2-\alpha_1)\sqrt{1-\alpha}}, & \alpha_1\neq 2,\\ \pm\frac{1}{2\sqrt{2(1+\alpha)}}, & \alpha_1=2,\end{cases}$$

and $\beta,\alpha,\alpha_1,\alpha_2$ satisfy

$$\frac{({\alpha_1}^2+2\alpha_1+8)(\beta^2-1)}{i\sqrt{1-\alpha}}=\frac{(2\alpha_1-{\alpha_1}^2)(\beta^2+1)}{\sqrt{1+\alpha}};$$

(ii)

$$f(z_1,z_2)=\frac{1}{2\sqrt{2}}\big[(A_1+A_2-A_1B_{11})e^{\gamma_1(z)}+(A_1+A_2-A_2B_{21})e^{\gamma_2(z)}\big],$$

where $\eta\neq 0$, $\gamma_1(z)=B_{11}z_1+H(z_2)+B_{12}z_2+\beta_1$, $\gamma_2(z)=B_{21}z_1-H(z_2)+B_{22}z_2+\beta_2$, $B_{j1},B_{j2},\beta_j\in\mathbb{C}(j=1,2)$ satisfy

$$g(z)=\gamma_1(z)+\gamma_2(z)=\alpha_1z_1+\alpha_2z_2+\alpha_0,$$
$$A_2(B_{11}+1)=A_1({B_{11}}^2+1),\quad A_1(B_{21}+1)=A_2({B_{21}}^2+1).$$

We give several examples to show the results in Theorem 2.8 are precise to some extent.

Example 2.9. $f(z_1,z_2)=\pm\frac{1}{2}e^{-z_1+2z_2}$ is a solution of (2.2) with $g(z)=-2z_1+4z_2$. Here, $\alpha=1/2$, $\alpha_1=-2$, $\alpha_2=4$, $\alpha_0=0$.

Example 2.10. $f(z_1,z_2)=\pm\frac{1}{2\sqrt{3}}e^{z_1+2z_2}$ is a solution of (2.2) with $g(z)=2z_1+4z_2$. Here, $\alpha=1/2$, $\alpha_1=2$, $\alpha_2=4$, $\alpha_0=0$.

Example 2.11. Let $\alpha=1/2$, $B_{11}=\frac{-\sqrt{3}i-1}{2}$, and $B_{21}=\frac{\sqrt{3}i-1}{2}$. Then

$$f(z_1,z_2)=\frac{\sqrt{3}}{3}e^{\frac{-\sqrt{3}i-1}{2}z_1+{z_2}^3+z_2+1}+\frac{\sqrt{3}}{3}e^{\frac{\sqrt{3}i-1}{2}z_1-{z_2}^3+z_2+3}$$

is a transcendental entire solution of (2.2) with $g(z)=-z_1+2z_2+4$.

Theorem 2.12. *Suppose equation* (2.3) *admits a transcendental entire solution $f(z)$ of finite order. Then $f(z_1,z_2)$ must satisfy the following:*

(i) $f(z_1,z_2)=\zeta_3(\beta,\alpha,\alpha_1,\alpha_2)e^{g(z)/2}+\eta e^{-z_1+z_2}$, where

$$\zeta_3(\beta,\alpha,\alpha_1,\alpha_2)=\begin{cases}\frac{4(\beta^2+1)}{\beta(\alpha_1\alpha_2+2\alpha_1+8)\sqrt{2(1+\alpha)}}, & \alpha_1+\alpha_2\neq 0,\\ \frac{4(\beta^2-1)}{i\beta(2+\alpha_1)\alpha_1\sqrt{2(1-\alpha)}}, & \alpha_1+\alpha_2=0,\end{cases}$$

and $\beta,\alpha,\alpha_1,\alpha_2$ satisfy

$$\frac{(\alpha_1\alpha_2+2\alpha_1+8)(\beta^2-1)}{i\sqrt{1-\alpha}}=\frac{(2\alpha_1-\alpha_1\alpha_2)(\beta^2+1)}{\sqrt{1+\alpha}};$$

(ii) if $B_{11}=B_{21}$ and $B_{12}=B_{22}$, then (2.3) *does not admit any transcendental entire solution with finite order, if $B_{11}=B_{21}$ and $B_{12}=B_{22}$ do not coexist, and if $B_{11}+B_{12}\neq 0, B_{21}+B_{22}\neq 0$, then*

$$f(z_1,z_2)=\frac{1}{\sqrt{2}}\vartheta_2(B_{11},B_{12},B_{21},B_{22})+\eta e^{-z_1+z_2},$$

where

$$\vartheta_2(B_{11},B_{12},B_{21},B_{22}) = \frac{A_1 - A_2 + A_1B_{12}}{B_{11}+B_{12}}e^{\gamma_1(z)} + \frac{A_2 - A_1 + A_2B_{22}}{B_{21}+B_{22}}e^{\gamma_2(z)};$$

if $B_{11}+B_{12}=0, B_{21}+B_{22}\neq 0$, *then*

$$f(z_1,z_2) = \frac{1}{\sqrt{2}}\vartheta_3(B_{11},B_{12},B_{21},B_{22}) + \eta e^{-z_1+z_2},$$

where

$$\begin{aligned}&\vartheta_3(B_{11},B_{12},B_{21},B_{22})\\ &= \begin{cases} \frac{A_2-A_1+A_2B_{22}}{B_{21}+B_{22}}e^{\gamma_2(z)} + \frac{{A_1}^2}{2A_1-A_2}e^{\gamma_1(z)}, & B_{11} = \frac{A_1-A_2}{A_1};\\ (A_1z_2 - A_2z_2 + A_1z_1)e^{\gamma_1(z)} + \frac{A_2-A_1+A_2B_{22}}{B_{21}+B_{22}}e^{\gamma_2(z)}, & B_{11}=-1;\end{cases}\end{aligned}$$

if $B_{11}+B_{12}\neq 0$ *and* $B_{21}+B_{22}=0$, *then*

$$f(z_1,z_2) = \frac{1}{\sqrt{2}}\vartheta_4(B_{11},B_{12},B_{21},B_{22}) + \eta e^{-z_1+z_2},$$

where

$$\begin{aligned}&\vartheta_4(B_{11},B_{12},B_{21},B_{22})\\ &= \begin{cases} \frac{A_1-A_2+A_1B_{12}}{B_{11}+B_{12}}e^{\gamma_1(z)} + \frac{{A_2}^2}{2A_2-A_1}e^{\gamma_2(z)}, & B_{21} = \frac{A_2-A_1}{A_2};\\ \frac{A_1-A_2+A_1B_{12}}{B_{11}+B_{12}}e^{\gamma_1(z)} + (A_2z_2 - A_1z_2 + A_2z_1)e^{\gamma_2(z)}, & B_{21}=-1;\end{cases}\end{aligned}$$

if $B_{11}+B_{12}=0$ *and* $B_{21}+B_{22}=0$, *then*

$$f(z_1,z_2) = \frac{1}{\sqrt{2}}\vartheta_4(B_{11},B_{12},B_{21},B_{22}) + \eta e^{-z_1+z_2},$$

where

$$\begin{aligned}&\vartheta_4(B_{11},B_{12},B_{21},B_{22})\\ &= \begin{cases} \frac{{A_1}^2}{2A_1-A_2}e^{\gamma_1(z)} + \frac{{A_2}^2}{2A_2-A_1}e^{\gamma_2(z)}, & B_{11} = \frac{A_1-A_2}{A_1}, B_{21} = \frac{A_2-A_1}{A_2};\\ (A_1z_2 - A_2z_2 + A_1z_1)e^{\gamma_1(z)} + \frac{{A_2}^2}{2A_2-A_1}e^{\gamma_2(z)}, & B_{11}=-1, B_{21} = \frac{A_2-A_1}{A_2};\\ \frac{{A_1}^2}{2A_1-A_2}e^{\gamma_1(z)} + (A_2z_2 - A_1z_2 + A_2z_1)e^{\gamma_2(z)}, & B_{11} = \frac{A_1-A_2}{A_1}, B_{21}=-1;\end{cases}\end{aligned}$$

moreover, $\eta\neq 0$, $\gamma_1(z) = B_{11}z_1 + B_{12}z_2 + \beta_1, \gamma_2(z) = B_{21}z_1 + B_{22}z_2 + \beta_2$, *and* $B_{j1}, B_{j2}, \beta_j \in \mathbb{C}(j=1,2)$ *satisfy*

$$\begin{gathered} g(z) = \gamma_1(z) + \gamma_2(z) = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_0,\\ A_2(B_{11}+1) = A_1(B_{11}B_{12}+1), \quad A_1(B_{21}+1) = A_2(B_{21}B_{22}+1).\end{gathered}$$

Now we give some examples to show that the results in Theorem 2.12 are precise.

Example 2.13. $f(z_1,z_2) = \frac{2\sqrt{3}}{9}e^{\frac{z_1+2z_2}{2}} + 3e^{-z_1+z_2}$ is a solution of (2.3) with $g(z) = z_1 + 2z_2$. Here, $\alpha = 1/2$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_0 = 0$, $\eta = 3$.

Example 2.14. $f(z_1,z_2) = \pm\frac{1}{2}e^{z_1-z_2} + 5e^{-z_1+z_2}$ is a solution of (2.3) with $g(z) = 2z_1 - 2z_2$. Here, $\alpha = 1/2$, $\alpha_1 = 2$, $\alpha_2 = -2$, $\alpha_0 = 0$, $\eta = 5$.

Example 2.15. Let $\alpha = 1/2$, $\gamma_1(z) = z_1 + (-2+\sqrt{3}i)z_2$, $\gamma_2(z) = 3z_1 + (-1 - \frac{2\sqrt{3}i}{3})z_2$, $\beta_1=\beta_2=0$, and $\eta = \frac{5}{2}$. Then

$$f(z_1,z_2) = \frac{-3i+\sqrt{3}}{12}e^{z_1+(-2+\sqrt{3}i)z_2} + \frac{3i+\sqrt{3}}{24}e^{3z_1-(1+\frac{2\sqrt{3}i}{3})z_2} + \frac{5}{2}e^{-z_1+z_2}$$

is a transcendental entire solution of (2.3) with $g(z) = 4z_1 + (-3 + \frac{\sqrt{3}i}{3})z_2$.

Example 2.16. Let $\alpha = 1/2$, $\gamma_1(z) = \frac{3-\sqrt{3}i}{2}z_1 + \frac{-3+\sqrt{3}i}{2}z_2$, $\gamma_2(z) = 3z_1 + (-1 - \frac{2\sqrt{3}i}{3})z_2$, $\beta_1 = \beta_2 = 0$, $\eta = 0$, then

$$f(z_1, z_2) = \frac{2\sqrt{3} - 3i}{21}e^{\frac{3-\sqrt{3}i}{2}z_1 + \frac{-3+\sqrt{3}i}{2}z_2} + \frac{3i + \sqrt{3}}{24}e^{3z_1 - (1+\frac{2\sqrt{3}i}{3})z_2}$$

is a transcendental entire solution of (2.3) with $g(z) = \frac{9-\sqrt{3}i}{2}z_1 + \frac{-15-\sqrt{3}i}{6}z_2$.

Example 2.17. Let $\alpha = 1/2$, $\gamma_1(z) = -z_1 + z_2$, $\gamma_2(z) = 3z_1 + (-1 - \frac{2\sqrt{3}i}{3})z_2$, $\beta_1 = \beta_2 = 0$, $\eta = 6$, then

$$f(z_1, z_2) = \Big(\frac{\sqrt{3} - 3i}{6}z_1 - iz_2\Big)e^{-z_1+z_2} + \frac{3i + \sqrt{3}}{24}e^{3z_1 - (1+\frac{2\sqrt{3}i}{3})z_2} + 6e^{-z_1+z_2}$$

is a transcendental entire solution of (2.3) with $g(z) = 2z_1 - \frac{2\sqrt{3}i}{3}z_2$.

Example 2.18. Let $\alpha = 1/2$, $\gamma_1(z) = z_1 + (-2 + \sqrt{3}i)z_2$, $\gamma_2(z) = \frac{3+\sqrt{3}i}{2}z_1 + \frac{-3-\sqrt{3}i}{2}z_2$, $\beta_1 = \beta_2 = 0$, $\eta = 1$, then

$$f(z_1, z_2) = \frac{-3i + \sqrt{3}}{12}e^{z_1 + (-2+\sqrt{3}i)z_2} + \frac{2\sqrt{3} + 3i}{21}e^{\frac{3+\sqrt{3}i}{2}z_1 + \frac{-3-\sqrt{3}i}{2}z_2} + e^{-z_1+z_2}$$

is a transcendental entire solution of (2.3) with $g(z) = \frac{5+\sqrt{3}i}{2}z_1 + \frac{-7+\sqrt{3}i}{2}z_2$.

Example 2.19. Let $\alpha = 1/2$, $\gamma_1(z) = z_1 + (-2 + \sqrt{3}i)z_2$, $\gamma_2(z) = -z_1 + z_2$, $\beta_1 = \beta_2 = 0$, $\eta = 4$, then

$$f(z_1, z_2) = \frac{-3i + \sqrt{3}}{12}e^{z_1 + (-2+\sqrt{3}i)z_2} + \Big(\frac{\sqrt{3} + 3i}{6}z_1 + iz_2\Big)e^{-z_1+z_2} + 4e^{-z_1+z_2}$$

is a transcendental entire solution of (2.3) with $g(z) = -1 + \sqrt{3}iz_2$.

Example 2.20. Let $\alpha = 1/2$, $\gamma_1(z) = \frac{3-\sqrt{3}i}{2}z_1 + \frac{-3+\sqrt{3}i}{2}z_2$, $\gamma_2(z) = \frac{3+\sqrt{3}i}{2}z_1 + \frac{-3-\sqrt{3}i}{2}z_2$, $\beta_1 = \beta_2 = 0$, $\eta = 2i$, then

$$f(z_1, z_2) = \frac{2\sqrt{3} - 3i}{21}e^{\frac{3-\sqrt{3}i}{2}z_1 + \frac{-3+\sqrt{3}i}{2}z_2} + \frac{2\sqrt{3} + 3i}{21}e^{\frac{3+\sqrt{3}i}{2}z_1 + \frac{-3-\sqrt{3}i}{2}z_2} + 2ie^{-z_1+z_2}$$

is a transcendental entire solution of (2.3) with $g(z) = 3z_1 - 3z_2$.

Example 2.21. Let $\alpha = 1/2$, $\gamma_1(z) = -z_1 + z_2$, $\gamma_2(z) = \frac{3+\sqrt{3}i}{2}z_1 + \frac{-3-\sqrt{3}i}{2}z_2$, $\beta_1 = \beta_2 = 0$, $\eta = \sqrt{2}$, then

$$f(z_1, z_2) = \Big(\frac{\sqrt{3} - 3i}{6}z_1 - iz_2\Big)e^{-z_1+z_2} + \frac{2\sqrt{3} + 3i}{21}e^{\frac{3+\sqrt{3}i}{2}z_1 + \frac{-3-\sqrt{3}i}{2}z_2} + \sqrt{2}e^{-z_1+z_2}$$

is a transcendental entire solution of (2.3) with $g(z) = \frac{1+\sqrt{3}i}{2}z_1 + \frac{-1-\sqrt{3}i}{2}z_2$.

Example 2.22. Let $\alpha = 1/2$, $\gamma_1(z) = \frac{3-\sqrt{3}i}{2}z_1 + \frac{-3+\sqrt{3}i}{2}z_2$, $\gamma_2(z) = -z_1 + z_2$, $\beta_1 = \beta_2 = 0$, $\eta = \sqrt{2}i$, then

$$f(z_1, z_2) = \frac{2\sqrt{3} - 3i}{21}e^{\frac{3-\sqrt{3}i}{2}z_1 + \frac{-3+\sqrt{3}i}{2}z_2} + \Big(\frac{\sqrt{3} + 3i}{6}z_1 + iz_2\Big)e^{-z_1+z_2} + \sqrt{2}ie^{-z_1+z_2}$$

is a transcendental entire solution of (2.3) with $g(z) = \frac{1-\sqrt{3}i}{2}z_1 + \frac{-1+\sqrt{3}i}{2}z_2$.

3. Proof of Theorem 2.1

Proof. Suppose that $f(z)$ is a transcendental entire solution of (2.1) with finite order. Let

$$f(z)+\frac{\partial f}{\partial z_1}=\frac{1}{\sqrt{2}}(u+v),\quad f(z)+\frac{\partial f}{\partial z_2}=\frac{1}{\sqrt{2}}(u-v),$$

where u and v are entire functions. Thus, we rewrite (2.1) in the form

$$(1+\alpha)u^2+(1-\alpha)v^2=e^{g(z)}. \tag{3.1}$$

Then it follows from (3.1) that

$$\Big(\frac{\sqrt{1+\alpha}u}{e^{g(z)/2}}\Big)^2+\Big(\frac{\sqrt{1-\alpha}v}{e^{g(z)/2}}\Big)^2=1.$$

This formula leads to

$$\Big(\frac{\sqrt{1+\alpha}u}{e^{g(z)/2}}+i\frac{\sqrt{1-\alpha}v}{e^{g(z)/2}}\Big)\Big(\frac{\sqrt{1+\alpha}u}{e^{g(z)/2}}-i\frac{\sqrt{1-\alpha}v}{e^{g(z)/2}}\Big)=1, \tag{3.2}$$

which implies that both $\frac{\sqrt{1+\alpha}u}{e^{g(z)/2}}+i\frac{\sqrt{1-\alpha}v}{e^{g(z)/2}}$ and $\frac{\sqrt{1+\alpha}u}{e^{g(z)/2}}-i\frac{\sqrt{1-\alpha}v}{e^{g(z)/2}}$ have no zeros. Therefore, in view of [13]-[17], there exist a polynomial $p(z)$ such that

$$\begin{gathered}\frac{\sqrt{1+\alpha}u}{e^{g(z)/2}}+i\frac{\sqrt{1-\alpha}v}{e^{g(z)/2}}=e^{p(z)},\\ \frac{\sqrt{1+\alpha}u}{e^{g(z)/2}}-i\frac{\sqrt{1-\alpha}v}{e^{g(z)/2}}=e^{-p(z)}.\end{gathered} \tag{3.3}$$

We denote

$$\gamma_1(z)=\frac{g(z)}{2}+p(z),\quad \gamma_2(z)=\frac{g(z)}{2}-p(z). \tag{3.4}$$

Then from (3.3), we have

$$f(z)+\frac{\partial f}{\partial z_1}=\frac{1}{\sqrt{2}}\Big(A_1e^{\gamma_1(z)}+A_2e^{\gamma_2(z)}\Big), \tag{3.5}$$

$$f(z)+\frac{\partial f}{\partial z_2}=\frac{1}{\sqrt{2}}\Big(A_2e^{\gamma_1(z)}+A_1e^{\gamma_2(z)}\Big), \tag{3.6}$$

where A_1, A_2 are defined by (2.4). Thus, from (3.5) and (3.6) it follows that

$$\frac{\partial f}{\partial z_1}-\frac{\partial f}{\partial z_2}=\frac{1}{\sqrt{2}}[(A_1-A_2)e^{\gamma_1(z)}+(A_2-A_1)e^{\gamma_2(z)}]. \tag{3.7}$$

On the other hand, from (3.5) and (3.6), by combining with $\frac{\partial^2 f}{\partial z_1\partial z_2}=\frac{\partial^2 f}{\partial z_2\partial z_1}$, we have

$$\frac{\partial f}{\partial z_1}-\frac{\partial f}{\partial z_2}=\frac{1}{\sqrt{2}}\Big[\Big(A_2\frac{\partial\gamma_1}{\partial z_1}-A_1\frac{\partial\gamma_1}{\partial z_2}\Big)e^{\gamma_1(z)}+\Big(A_1\frac{\partial\gamma_2}{\partial z_1}-A_2\frac{\partial\gamma_2}{\partial z_2}\Big)e^{\gamma_2(z)}\Big]. \tag{3.8}$$

Thus, (3.7) and (3.8) yield

$$\Big(A_1-A_2-A_2\frac{\partial\gamma_1}{\partial z_1}+A_1\frac{\partial\gamma_1}{\partial z_2}\Big)e^{\gamma_1(z)-\gamma_2(z)}=A_1-A_2+A_1\frac{\partial\gamma_2}{\partial z_1}-A_2\frac{\partial\gamma_2}{\partial z_2}. \tag{3.9}$$

Now we consider two cases.

Case 1. If $e^{\gamma_1(z)-\gamma_2(z)}$ is a constant, then $\gamma_1(z)-\gamma_2(z)$ is a constant. By combining with $\gamma_1(z)-\gamma_2(z)=2p(z)$, it follows that $p(z)$ is a constant. Let $\beta=e^{p(z)}$, then equations (3.5)-(3.6) can be represented as

$$f(z)+\frac{\partial f}{\partial z_1}=\frac{1}{\sqrt{2}}\Big(\frac{k_1}{\sqrt{1+\alpha}}+\frac{k_2}{\sqrt{1-\alpha}}\Big)e^{g(z)/2}, \tag{3.10}$$

$$f(z)+\frac{\partial f}{\partial z_2}=\frac{1}{\sqrt{2}}\Big(\frac{k_1}{\sqrt{1+\alpha}}-\frac{k_2}{\sqrt{1-\alpha}}\Big)e^{g(z)/2}, \tag{3.11}$$

where $k_1=\frac{\beta+\beta^{-1}}{2}$, $k_2=\frac{\beta-\beta^{-1}}{2i}$, and $k_1^2+k_2^2=1$. This leads to

$$\frac{\partial f}{\partial z_1}-\frac{\partial f}{\partial z_2}=\frac{\sqrt{2}k_2}{\sqrt{1-\alpha}}e^{g(z)/2}. \tag{3.12}$$

On the other hand, differentiating both sides of equations (3.10) and (3.11) with respect to z_2 and z_1, respectively, and combining this with $\frac{\partial^2 f}{\partial z_1\partial z_2}=\frac{\partial^2 f}{\partial z_2\partial z_1}$, we deduce that

$$\begin{aligned} &\frac{\partial f}{\partial z_1}-\frac{\partial f}{\partial z_2}\\ &=\frac{1}{2\sqrt{2}}\Big(\frac{k_1}{\sqrt{1+\alpha}}\frac{\partial g}{\partial z_1}-\frac{k_2}{\sqrt{1-\alpha}}\frac{\partial g}{\partial z_1}-\frac{k_1}{\sqrt{1+\alpha}}\frac{\partial g}{\partial z_2}-\frac{k_2}{\sqrt{1-\alpha}}\frac{\partial g}{\partial z_2}\Big)e^{g(z)/2}. \end{aligned} \tag{3.13}$$

From (3.12) and (3.13) it follows that

$$\frac{4k_2}{\sqrt{1-\alpha}}=\frac{k_1}{\sqrt{1+\alpha}}\frac{\partial g}{\partial z_1}-\frac{k_2}{\sqrt{1-\alpha}}\frac{\partial g}{\partial z_1}-\frac{k_1}{\sqrt{1+\alpha}}\frac{\partial g}{\partial z_2}-\frac{k_2}{\sqrt{1-\alpha}}\frac{\partial g}{\partial z_2}. \tag{3.14}$$

Since $g(z)$ is a polynomial with the linear form $g(z_1,z_2)=\alpha_1z_1+\alpha_2z_2+\alpha_0$, where $\alpha_1\neq 0,\alpha_2\neq 0,\alpha_0\in\mathbb{C}$. Hence, from (3.14) we deduce that

$$\frac{(4+\alpha_1+\alpha_2)k_2}{\sqrt{1-\alpha}}=\frac{(\alpha_1-\alpha_2)k_1}{\sqrt{1+\alpha}}\,. \tag{3.15}$$

The characteristic equations of (3.12) are

$$\frac{dz_1}{dt}=1,\quad \frac{dz_2}{dt}=-1,\quad \frac{df}{dt}=\frac{\sqrt{2}k_2}{\sqrt{1-\alpha}}e^{g(z)/2}.$$

Using the initial conditions: $z_1=0$, $z_2=s$, and $f(z_1,z_2)=f(0,s):=\phi_0(s)$ with a parameter s. Thus, we obtain the following parametric representation for the solutions of the characteristic equations: $z_1=t$, $z_2=-t+s$,

$$f(t,s)=\int_0^t\frac{\sqrt{2}k_2}{\sqrt{1-\alpha}}e^{\frac{\alpha_1t-\alpha_2t+\alpha_2s+\alpha_0}{2}}dt+\phi_0(s), \tag{3.16}$$

where $\phi_0(s)$ is a finite order transcendental entire function in $s=z_1+z_2$.

Subcase 1.1. If $\alpha_1-\alpha_2\neq 0$, it follows from (3.16) that

$$\begin{aligned} f(t,s)&=\int_0^t\frac{\sqrt{2}k_2}{\sqrt{1-\alpha}}e^{\frac{\alpha_1t-\alpha_2t+\alpha_2s+\alpha_0}{2}}dt+\phi_0(s)\\ &=\frac{2\sqrt{2}k_2}{\sqrt{1-\alpha}(\alpha_1-\alpha_2)}e^{\frac{\alpha_1t-\alpha_2t+\alpha_2s+\alpha_0}{2}}+\phi_1(s), \end{aligned} \tag{3.17}$$

where

$$\phi_1(s)=\phi_0(s)-\frac{2\sqrt{2}k_2}{\sqrt{1-\alpha}(\alpha_1-\alpha_2)}e^{\frac{\alpha_2s+\alpha_0}{2}}$$

is a finite order transcendental entire function in s. Thus, from (3.17), it follows that

$$f(z_1,z_2)=\frac{2\sqrt{2}k_2}{\sqrt{1-\alpha}(\alpha_1-\alpha_2)}e^{\frac{\alpha_1 z_1+\alpha_2 z_2+\alpha_0}{2}}+\phi_1(z_1+z_2). \tag{3.18}$$

Substituting (3.18) into (3.10) or (3.11), then by combining with (3.15), it yields that

$$\phi_1(z_1+z_2)+\phi_1'(z_1+z_2)=0, \tag{3.19}$$

which implies $\phi_1(z_1+z_2)=\eta_1 e^{-(z_1+z_2)}$, $\eta_1\in\mathbb{C}\setminus\{0\}$.

Subcase 1.2. If $\alpha_1-\alpha_2=0$, it follows from (3.16) that

$$\begin{aligned} f(t,s)&=\int_0^t \frac{\sqrt{2}k_2}{\sqrt{1-\alpha}}e^{\frac{\alpha_1 t-\alpha_2 t+\alpha_2 s+\alpha_0}{2}}dt+\phi_0(s)\\ &=\frac{\sqrt{2}k_2}{\sqrt{1-\alpha}}e^{\frac{\alpha_2 s+\alpha_0}{2}}t+\phi_2(s), \end{aligned} \tag{3.20}$$

where $\phi_2(s)=\phi_0(s)$ is a transcendental entire function with finite order in s. In view of (3.20), we have

$$f(z_1,z_2)=\frac{\sqrt{2}k_2}{\sqrt{1-\alpha}}e^{\frac{\alpha_2 z_1+\alpha_2 z_2+\alpha_0}{2}}z_1+\phi_2(z_1+z_2). \tag{3.21}$$

From (3.15) and $\alpha_1-\alpha_2=0$, we have $\alpha_1=\alpha_2=-2$ or $k_2=0$.

Subcase 1.2.1. If $\alpha_1=\alpha_2\neq-2$, then we have $k_2=0$, it follows from (3.21) that

$$f(z_1,z_2)=\phi_3(z_1+z_2). \tag{3.22}$$

Substituting (3.22) into the (3.10) or (3.11) yields

$$\phi_3(z_1+z_2)+\phi_3'(z_1+z_2)=\pm\frac{1}{\sqrt{2(1+\alpha)}}e^{\frac{\alpha_1 z_1+\alpha_2 z_2+\alpha_0}{2}},$$

which implies

$$\phi_3(z_1+z_2)=\pm\frac{\sqrt{2}}{(2+\alpha_1)\sqrt{1+\alpha}}e^{g(z)/2}+\eta_3 e^{-(z_1+z_2)},\quad \eta_3\in\mathbb{C}.$$

Subcase 1.2.2. If $\alpha_1=\alpha_2=-2$, it follows from (3.21) that

$$f(z_1,z_2)=\frac{\sqrt{2}k_2}{\sqrt{1-\alpha}}e^{\frac{-2z_1-2z_2+\alpha_0}{2}}z_1+\phi_4(z_1+z_2). \tag{3.23}$$

Substituting (3.23) into the (3.10) or (3.11), yields

$$\phi_4(z_1+z_2)+\phi_4'(z_1+z_2)=\frac{1}{\sqrt{2}}(\frac{k_1}{\sqrt{1+\alpha}}-\frac{k_2}{\sqrt{1-\alpha}})e^{\frac{-2z_1-2z_2+\alpha_0}{2}},$$

which implies

$$\phi_4(z_1+z_2)=\frac{1}{\sqrt{2}}(\frac{k_1}{\sqrt{1+\alpha}}-\frac{k_2}{\sqrt{1-\alpha}})(z_1+z_2)e^{g(z)/2}+\eta_4 e^{-(z_1+z_2)},\quad \eta_4\in\mathbb{C}.$$

The proof of Theorem 2.1(i) is complete.

Case 2. If $e^{\gamma_1(z)-\gamma_2(z)}$ is not a constant, then $p(z)$ is not a constant. It follows from (3.9) that

$$\begin{aligned} A_1 - A_2 - A_2\frac{\partial\gamma_1}{\partial z_1} + A_1\frac{\partial\gamma_1}{\partial z_2} = 0,\\ A_1 - A_2 + A_1\frac{\partial\gamma_2}{\partial z_1} - A_2\frac{\partial\gamma_2}{\partial z_2} = 0. \end{aligned} \tag{3.24}$$

Otherwise, without loss of generality, if $A_1 - A_2 - A_2\frac{\partial\gamma_1}{\partial z_1} + A_1\frac{\partial\gamma_1}{\partial z_2} \neq 0$, we have

$$e^{2p(z)} = \frac{A_1 - A_2 + A_1\frac{\partial\gamma_2}{\partial z_1} - A_2\frac{\partial\gamma_2}{\partial z_2}}{A_1 - A_2 - A_2\frac{\partial\gamma_1}{\partial z_1} + A_1\frac{\partial\gamma_1}{\partial z_2}}. \tag{3.25}$$

Since $p(z)$, $g(z)$ are polynomials, the left-hand side of (3.25) is transcendental, which contradicts with the right-hand side of (3.25) is a rational function. Thus, in view of (3.24), we have

$$\begin{aligned} A_2\frac{\partial\gamma_1}{\partial z_1} - A_1\frac{\partial\gamma_1}{\partial z_2} = A_1 - A_2,\\ A_1\frac{\partial\gamma_2}{\partial z_1} - A_2\frac{\partial\gamma_2}{\partial z_2} = A_2 - A_1. \end{aligned} \tag{3.26}$$

Next, we prove that $\gamma_1(z)$ and $\gamma_2(z)$ are linear forms of z_1, z_2. Similar calculations to the ones in equation (3.12) can be used to (3.26); we can obtain

$$\begin{aligned} \gamma_1 &= \frac{A_1 - A_2}{A_2}z_1 + \varphi_1(z_2 + \frac{A_1}{A_2}z_1),\\ \gamma_2 &= \frac{A_2 - A_1}{A_1}z_1 + \varphi_2(z_2 + \frac{A_2}{A_1}z_1). \end{aligned}$$

Since $g(z)$ is a polynomial with the linear form $g(z_1, z_2) = \gamma_1(z) + \gamma_2(z) = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_0$, where $\alpha_1 \neq 0, \alpha_2 \neq 0, \alpha_0 \in \mathbb{C}$, it follows that

$$\varphi_1 + \varphi_2 = \big[\alpha_1 - \frac{(A_1 - A_2)^2}{A_1A_2}\big]z_1 + \alpha_2 z_2 + \alpha_0.$$

Let

$$\begin{aligned} \varphi_1 &= b_m{s_1}^m + b_{m-1}{s_1}^{m-1} + \cdots + b_0, \quad s_1 = z_2 + \frac{A_1}{A_2}z_1,\\ \varphi_2 &= d_n{s_2}^n + d_{n-1}{s_2}^{n-1} + \cdots + d_0, \quad s_2 = z_2 + \frac{A_2}{A_1}z_1. \end{aligned}$$

If $m \geq 2$, we have $n = m$ and $b_j = -d_j, j = 2, \dots, n$. Furthermore, if $b_j \neq 0$ for $j = 2, \dots, n$, we need consider the coefficient of $z_2^{m-1}z_1$ in φ_1, φ_2, then it yields that

$$C_m^1 b_m\frac{A_1}{A_2}z_2^{m-1}z_1 + C_m^1 d_m\frac{A_2}{A_1}z_2^{m-1}z_1 = 0,$$

further, we can obtain that

$$\frac{A_1}{A_2} = \frac{A_2}{A_1},$$

this is a contradiction with the required condition of theorems. Thus, we deduce that $m = 1$, and the γ_1, γ_2 are linear forms of z_1, z_2. Without loss of generality, we set

$$\gamma_1(z) = B_{11}z_1 + B_{12}z_2 + \beta_1, \quad \gamma_2(z) = B_{21}z_1 + B_{22}z_2 + \beta_2.$$

According to equation (3.7), we obtain that the characteristic equations of (3.7) are

$$\frac{dz_1}{dt} = 1, \quad \frac{dz_2}{dt} = -1, \quad \frac{df}{dt} = \frac{1}{\sqrt{2}}[(A_1 - A_2)e^{\gamma_1} + (A_2 - A_1)e^{\gamma_2}].$$

Similarly, we obtain

$$\begin{aligned} f(t,s) = \int_0^t \frac{1}{\sqrt{2}}\big[&(A_1 - A_2)e^{(B_{11}-B_{12})t+B_{12}s+\beta_1} \\ &+ (A_2 - A_1)e^{(B_{21}-B_{22})t+B_{22}s+\beta_2}\big]dt + \phi_0(s), \end{aligned} \tag{3.27}$$

where $\phi_0(s)$ is a transcendental entire function with finite order in $s = z_1 + z_2$.

Subcase 2.1. If $B_{11} - B_{12} \neq 0$, $B_{21} - B_{22} \neq 0$, it follows from (3.27) that

$$\begin{aligned} f(t,s) &= \int_0^t \frac{1}{\sqrt{2}}\big[(A_1 - A_2)e^{(B_{11}-B_{12})t+B_{12}s+\beta_1} \\ &\quad + (A_2 - A_1)e^{(B_{21}-B_{22})t+B_{22}s+\beta_2}\big]dt + \phi_0(s) \\ &= \frac{1}{\sqrt{2}}\big[\frac{A_1 - A_2}{B_{11} - B_{12}}e^{(B_{11}-B_{12})t+B_{12}s+\beta_1} \\ &\quad + \frac{A_2 - A_1}{B_{21} - B_{22}}e^{(B_{21}-B_{22})t+B_{22}s+\beta_2}\big] + \phi_5(s). \end{aligned} \tag{3.28}$$

where

$$\phi_5(s) = \phi_0(s) - \frac{1}{\sqrt{2}}\big[\frac{A_1 - A_2}{B_{11} - B_{12}}e^{B_{12}s+\beta_1} + \frac{A_2 - A_1}{B_{21} - B_{22}}e^{B_{22}s+\beta_2}\big]$$

is a finite order transcendental entire function in s. Thus, it follows (3.28) that

$$f(z_1, z_2) = \frac{1}{\sqrt{2}}\big[\frac{A_1 - A_2}{B_{11} - B_{12}}e^{\gamma_1(z)} + \frac{A_2 - A_1}{B_{21} - B_{22}}e^{\gamma_2(z)}\big] + \phi_5(z_1 + z_2). \tag{3.29}$$

Since $g(z)$ is a polynomial with the linear form $g(z_1, z_2) = \gamma_1(z) + \gamma_2(z) = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_0$, where $\alpha_1 \neq 0, \alpha_2 \neq 0, \alpha_0 \in \mathbb{C}$. Hence, we deduce from (3.26) that

$$A_2(B_{11} + 1) = A_1(B_{12} + 1), \quad A_1(B_{21} + 1) = A_2(B_{22} + 1). \tag{3.30}$$

Substituting (3.29) into (3.5) or (3.6), then combining this with (3.30) yields that

$$\phi_5(z_1 + z_2) + \phi_5'(z_1 + z_2) = 0, \tag{3.31}$$

which implies $\phi_5(z_1 + z_2) = \eta_5 e^{-(z_1+z_2)}$, $\eta_5 \in \mathbb{C}$.

Subcase 2.2. If $B_{11} - B_{12} = 0$ and $B_{21} - B_{22} \neq 0$, from (3.27) it follows that

$$\begin{aligned} &f(t,s) \\ &= \int_0^t \frac{1}{\sqrt{2}}\big[(A_1 - A_2)e^{B_{11}s+\beta_1} + (A_2 - A_1)e^{(B_{21}-B_{22})t+B_{22}s+\beta_2}\big]dt + \phi_0(s) \\ &= \frac{1}{\sqrt{2}}\big[(A_1 - A_2)e^{B_{11}s+\beta_1}t + \frac{A_2 - A_1}{B_{21} - B_{22}}e^{(B_{21}-B_{22})t+B_{22}s+\beta_2}\big] + \phi_6(s), \end{aligned} \tag{3.32}$$

where

$$\phi_6(s) = \phi_0(s) - \frac{1}{\sqrt{2}}\frac{A_2 - A_1}{B_{21} - B_{22}}e^{B_{22}s+\beta_2}$$

is a transcendental entire function in s. Thus, in view of (3.32), we obtain

$$f(z_1, z_2) = \frac{1}{\sqrt{2}}\big[(A_1 - A_2)e^{\gamma_1(z)}z_1 + \frac{A_2 - A_1}{B_{21} - B_{22}}e^{\gamma_2(z)}\big] + \phi_6(z_1 + z_2). \tag{3.33}$$

Also, we deduce from (3.26) that

$$B_{11} = B_{12} = -1, \quad A_1(B_{21}+1) = A_2(B_{22}+1). \tag{3.34}$$

Substituting (3.33) into (3.5) or (3.6), then by combining this with (3.34), it yields that

$$\phi_6(z_1+z_2) + \phi_6'(z_1+z_2) = \frac{1}{\sqrt{2}} A_2 e^{\gamma_1(z)}, \tag{3.35}$$

which implies $\phi_6(z_1+z_2) = \frac{1}{\sqrt{2}} A_2 e^{\gamma_1(z)}(z_1+z_2) + \eta_6 e^{-(z_1+z_2)}$, $\eta_6 \in \mathbb{C}$.

Subcase 2.3. If $B_{11} - B_{12} \neq 0$ and $B_{21} - B_{22} = 0$, it follows from (3.27) that

$$\begin{aligned} &f(t,s) \\ &= \int_0^t \frac{1}{\sqrt{2}} \big[(A_1 - A_2) e^{(B_{11}-B_{12})t + B_{12}s + \beta_1} + (A_2 - A_1) e^{B_{21}s + \beta_2}\big] dt + \phi_0(s) \\ &= \frac{1}{\sqrt{2}} \big[\frac{A_1 - A_2}{B_{11} - B_{12}} e^{(B_{11}-B_{12})t + B_{12}s + \beta_1} + (A_2 - A_1) e^{B_{21}s + \beta_2} t\big] + \phi_7(s), \end{aligned} \tag{3.36}$$

where

$$\phi_7(s) = \phi_0(s) - \frac{1}{\sqrt{2}} \frac{A_1 - A_2}{B_{11} - B_{12}} e^{B_{12}s + \beta_1}$$

is a finite order transcendental entire function in s. Thus, from (3.36), it follows that

$$f(z_1, z_2) = \frac{1}{\sqrt{2}} \big[\frac{A_1 - A_2}{B_{11} - B_{12}} e^{\gamma_1(z)} + (A_2 - A_1) e^{\gamma_2(z)} z_1\big] + \phi_7(z_1 + z_2). \tag{3.37}$$

Also, we deduce from (3.26) that

$$A_2(B_{11}+1) = A_1(B_{12}+1), \quad B_{21} = B_{22} = -1. \tag{3.38}$$

Substituting (3.37) into (3.10) or (3.11), then by combining this with (3.38), it yields that

$$\phi_7(z_1+z_2) + \phi_7'(z_1+z_2) = \frac{1}{\sqrt{2}} A_1 e^{\gamma_2(z)}, \tag{3.39}$$

which implies $\phi_7(z_1+z_2) = \frac{1}{\sqrt{2}} A_1 e^{\gamma_2(z)}(z_1+z_2) + \eta_7 e^{-(z_1+z_2)}$, $\eta_7 \in \mathbb{C}$.

Subcase 2.4. If $B_{11} - B_{12} = 0$ and $B_{21} - B_{22} = 0$, it follows from (3.27) that

$$\begin{aligned} f(t,s) &= \int_0^t \frac{1}{\sqrt{2}} \big[(A_1 - A_2) e^{B_{12}s + \beta_1} + (A_2 - A_1) e^{B_{21}s + \beta_2}\big] dt + \phi_0(s) \\ &= \frac{1}{\sqrt{2}} \big[(A_1 - A_2) e^{B_{12}s + \beta_1} + (A_2 - A_1) e^{B_{12}s + \beta_2}\big] t + \phi_8(s), \end{aligned} \tag{3.40}$$

where $\phi_8(s) = \phi_0(s)$ is a transcendental entire function with finite order in s. Then from (3.40) we obtain

$$f(z_1, z_2) = \frac{1}{\sqrt{2}} \big[(A_1 - A_2) e^{\gamma_1(z)} + (A_2 - A_1) e^{\gamma_2(z)}\big] z_1 + \phi_8(z_1 + z_2). \tag{3.41}$$

Also, we deduce from (3.26) that

$$B_{11} = B_{12} = -1, \quad B_{21} = B_{22} = -1,$$

which leads to $p(z)$ being a constant. By the assumption at the begin of Case 2, we obtain a contradiction. Thus, the proof of Theorem 2.1(ii) is complete. □

4. Proof of Theorem 2.8

Proof. Suppose that $f(z)$ is a transcendental entire solution of (2.2) with finite order. By using the same discussion in the proof of Theorem 2.1, we obtain that

$$f(z)+\frac{\partial f}{\partial z_1}=\frac{1}{\sqrt{2}}\Big(A_1e^{\gamma_1(z)}+A_2e^{\gamma_2(z)}\Big), \tag{4.1}$$

$$f(z)+\frac{\partial^2 f}{\partial z_1^2}=\frac{1}{\sqrt{2}}\Big(A_2e^{\gamma_1(z)}+A_1e^{\gamma_2(z)}\Big), \tag{4.2}$$

where A_1, A_2 are defined by (2.4). Thus, it follows from (4.1) and (4.2) that

$$\frac{\partial f}{\partial z_1}-\frac{\partial^2 f}{\partial z_1^2}=\frac{1}{\sqrt{2}}\big[(A_1-A_2)e^{\gamma_1(z)}+(A_2-A_1)e^{\gamma_2(z)}\big]. \tag{4.3}$$

On the other hand, differentiating with respect to z_1 on equation (4.1), in accordance with (4.2), we have

$$f(z)-\frac{\partial f}{\partial z_1}=\frac{1}{\sqrt{2}}\big[(A_2-A_1\frac{\partial \gamma_1}{\partial z_1})e^{\gamma_1(z)}+(A_1-A_2\frac{\partial \gamma_2}{\partial z_1})e^{\gamma_2(z)}\big]. \tag{4.4}$$

Differentiating with respect to z_1 on equation (4.4) yields

$$\begin{aligned}\frac{\partial f}{\partial z_1}-\frac{\partial^2 f}{\partial z_1^2}=\frac{1}{\sqrt{2}}\big[&(A_2\frac{\partial \gamma_1}{\partial z_1}-A_1(\frac{\partial \gamma_1}{\partial z_1})^2-A_1\frac{\partial^2 \gamma_1}{\partial z_1^2})e^{\gamma_1(z)}\\&+(A_1\frac{\partial \gamma_2}{\partial z_1}-A_2(\frac{\partial \gamma_2}{\partial z_1})^2-A_2\frac{\partial^2 \gamma_2}{\partial z_1^2})e^{\gamma_2(z)}\big].\end{aligned} \tag{4.5}$$

Thus, in line with (4.3) and (4.5), it follows that

$$\begin{aligned}&\Big(A_1-A_2-A_2\frac{\partial \gamma_1}{\partial z_1}+A_1(\frac{\partial \gamma_1}{\partial z_1})^2+A_1\frac{\partial^2 \gamma_1}{\partial z_1^2}\Big)e^{2p(z)}\\&=A_1-A_2+A_1\frac{\partial \gamma_2}{\partial z_1}-A_2(\frac{\partial \gamma_2}{\partial z_1})^2-A_2\frac{\partial^2 \gamma_2}{\partial z_1^2}.\end{aligned} \tag{4.6}$$

Now, we consider two cases.

Case 1: $p(z)$ is a constant. Let $\beta=e^{p(z)}$, then equations (4.1)-(4.2) can be written as

$$f(z)+\frac{\partial f}{\partial z_1}=\frac{1}{\sqrt{2}}\Big(\frac{k_1}{\sqrt{1+\alpha}}+\frac{k_2}{\sqrt{1-\alpha}}\Big)e^{g(z)/2}, \tag{4.7}$$

$$f(z)+\frac{\partial^2 f}{\partial z_1^2}=\frac{1}{\sqrt{2}}\Big(\frac{k_1}{\sqrt{1+\alpha}}-\frac{k_2}{\sqrt{1-\alpha}}\Big)e^{g(z)/2}, \tag{4.8}$$

where $k_1=\frac{\beta+\beta^{-1}}{2}$, $k_2=\frac{\beta-\beta^{-1}}{2i}$, and $k_1^2+k_2^2=1$. This leads to

$$\frac{\partial f}{\partial z_1}-\frac{\partial^2 f}{\partial z_1^2}=\frac{\sqrt{2}k_2}{\sqrt{1-\alpha}}e^{g(z)/2}. \tag{4.9}$$

On the other hand, differentiating with respect to z_1 on equation (4.7), and combining this with (4.8), we have

$$f(z)-\frac{\partial f}{\partial z_1}=\frac{1}{\sqrt{2}}\big[(1-\frac{\alpha_1}{2})\frac{k_1}{\sqrt{1+\alpha}}-(1+\frac{\alpha_1}{2})\frac{k_2}{\sqrt{1-\alpha}}\big]e^{g(z)/2}. \tag{4.10}$$

Differentiating with respect to z_1 on equation (4.10), combining this with (4.9) yields

$$\frac{\partial f}{\partial z_1}-\frac{\partial^2 f}{\partial z_1^2}=\frac{1}{\sqrt{2}}\big[(\frac{\alpha_1}{2}-\frac{\alpha_1^2}{4})\frac{k_1}{\sqrt{1+\alpha}}-(\frac{\alpha_1}{2}+\frac{\alpha_1^2}{4})\frac{k_2}{\sqrt{1-\alpha}}\big]e^{g(z)/2}. \tag{4.11}$$

Since $g(z)$ is a polynomial with the linear form $g(z_1,z_2)=\alpha_1 z_1+\alpha_2 z_2+\alpha_0$, where $\alpha_1\neq 0,\alpha_2\neq 0,\alpha_0\in\mathbb{C}$. Hence, we deduce from (4.9) and (4.11) that

$$\frac{8+2\alpha_1+\alpha_1^2}{\sqrt{1-\alpha}}k_2=\frac{2\alpha_1-\alpha_1^2}{\sqrt{1+\alpha}}k_1. \tag{4.12}$$

According to (4.7) and (4.10), we deduce that

$$f(z_1,z_2)=\frac{1}{2\sqrt{2}}\big[(2-\frac{\alpha_1}{2})\frac{k_1}{\sqrt{1+\alpha}}-\frac{\alpha_1}{2}\frac{k_2}{\sqrt{1-\alpha}}\big]e^{g(z)/2}. \tag{4.13}$$

Subcase 1.1. If $\alpha_1\neq 2$, combining (4.12) and (4.13) yields

$$f(z_1,z_2)=\frac{4\sqrt{2}k_2}{\alpha_1(2-\alpha_1)\sqrt{1-\alpha}}e^{g(z)/2}. \tag{4.14}$$

Subcase 1.2. If $\alpha_1=2$, combining (4.12) and (4.13) yields

$$f(z_1,z_2)=\pm\frac{k_1}{2\sqrt{2(1+\alpha)}}e^{g(z)/2}, \tag{4.15}$$

where $k_1=\pm1,k_2=0$. Thus, the proof of Theorem 2.8(i) is complete.

Case 2. If $p(z)$ is a non-constant, it follows from (4.6) that

$$\begin{gathered}A_1-A_2-A_2\frac{\partial\gamma_1}{\partial z_1}+A_1(\frac{\partial\gamma_1}{\partial z_1})^2+A_1\frac{\partial^2\gamma_1}{\partial z_1^2}=0,\\ A_1-A_2+A_1\frac{\partial\gamma_2}{\partial z_1}-A_2(\frac{\partial\gamma_2}{\partial z_1})^2-A_2\frac{\partial^2\gamma_2}{\partial z_1^2}=0.\end{gathered} \tag{4.16}$$

Otherwise, without loss of generality, if $A_1-A_2-A_2\frac{\partial\gamma_1}{\partial z_1}+A_1(\frac{\partial\gamma_1}{\partial z_1})^2+A_1\frac{\partial^2\gamma_1}{\partial z_1^2}\neq 0$, we have

$$e^{2p(z)}=\frac{A_1-A_2+A_1\frac{\partial\gamma_2}{\partial z_1}-A_2(\frac{\partial\gamma_2}{\partial z_1})^2-A_2\frac{\partial^2\gamma_2}{\partial z_1^2}}{A_1-A_2-A_2\frac{\partial\gamma_1}{\partial z_1}+A_1(\frac{\partial\gamma_1}{\partial z_1})^2+A_1\frac{\partial^2\gamma_1}{\partial z_1^2}}. \tag{4.17}$$

Since $p(z)$ and $g(z)$ are polynomials, the left-hand side of (4.17) is transcendental, which contradicts with the right-hand side of (4.17) being a rational function. Thus, we have

$$\begin{gathered}A_2\frac{\partial\gamma_1}{\partial z_1}-A_1(\frac{\partial\gamma_1}{\partial z_1})^2-A_1\frac{\partial^2\gamma_1}{\partial z_1^2}=A_1-A_2,\\ A_1\frac{\partial\gamma_2}{\partial z_1}-A_2(\frac{\partial\gamma_2}{\partial z_1})^2-A_2\frac{\partial^2\gamma_2}{\partial z_1^2}=A_2-A_1.\end{gathered} \tag{4.18}$$

In view of (4.18), we obtain that γ_1 and γ_2 are of the form

$$\gamma_1(z)=B_{11}z_1+H(z_2)+B_{12}z_2+\beta_1,\quad \gamma_2(z)=B_{21}z_1-H(z_2)+B_{22}z_2+\beta_2,$$

where $H(z_2)=d_n z_2^n+d_{n-1}z_2^{n-1}+\cdots+d_2 z_2^2$. Then (4.18) can be rewritten as

$$\begin{gathered}A_2B_{11}-A_1B_{11}^2=A_1-A_2,\\ A_1B_{21}-A_2B_{21}^2=A_2-A_1.\end{gathered} \tag{4.19}$$

By (4.1) and (4.4) we deduce that

$$f(z_1,z_2)=\frac{1}{2\sqrt{2}}\big[(A_1+A_2-A_1B_{11})e^{\gamma_1(z)}+(A_1+A_2-A_2B_{21})e^{\gamma_2(z)}\big]. \tag{4.20}$$

Thus, the proof of Theorem 2.8(ii) is complete. □

5. Proof of Theorem 2.12

Proof. Suppose that $f(z)$ is a transcendental entire solution of (2.3) with finite order. By using the same argument in the proof of Theorem 2.1, we obtain

$$f(z)+\frac{\partial f}{\partial z_1}=\frac{1}{\sqrt{2}}\Big(A_1e^{\gamma_1(z)}+A_2e^{\gamma_2(z)}\Big), \tag{5.1}$$

$$f(z)+\frac{\partial^2 f}{\partial z_1\partial z_2}=\frac{1}{\sqrt{2}}\Big(A_2e^{\gamma_1(z)}+A_1e^{\gamma_2(z)}\Big), \tag{5.2}$$

where A_1 and A_2 are defined by (2.4). Thus, it follows from (5.1) and (5.2) that

$$\frac{\partial f}{\partial z_1}-\frac{\partial^2 f}{\partial z_1\partial z_2}=\frac{1}{\sqrt{2}}\big[(A_1-A_2)e^{\gamma_1(z)}+(A_2-A_1)e^{\gamma_2(z)}\big]. \tag{5.3}$$

On the other hand, differentiating with respect to z_2 on equation (5.1), and combining this with (5.2), we obtain

$$f(z)-\frac{\partial f}{\partial z_2}=\frac{1}{\sqrt{2}}\Big[\Big(A_2-A_1\frac{\partial\gamma_1}{\partial z_2}\Big)e^{\gamma_1(z)}+\Big(A_1-A_2\frac{\partial\gamma_2}{\partial z_2}\Big)e^{\gamma_2(z)}\Big]. \tag{5.4}$$

Differentiating with respect to z_1 on equation (5.4), then combining this with $\frac{\partial^2 f}{\partial z_1\partial z_2}=\frac{\partial^2 f}{\partial z_2\partial z_1}$ yields

$$\frac{\partial f}{\partial z_1}-\frac{\partial^2 f}{\partial z_1\partial z_2}=\frac{1}{\sqrt{2}}\Big(\Gamma_1e^{\gamma_1(z)}+\Gamma_2e^{\gamma_2(z)}\Big), \tag{5.5}$$

where

$$\Gamma_1=A_2\frac{\partial\gamma_1}{\partial z_1}-A_1\frac{\partial\gamma_1}{\partial z_1}\frac{\partial\gamma_1}{\partial z_2}-A_1\frac{\partial^2\gamma_1}{\partial z_2\partial z_1},$$
$$\Gamma_2=A_1\frac{\partial\gamma_2}{\partial z_1}-A_2\frac{\partial\gamma_2}{\partial z_1}\frac{\partial\gamma_2}{\partial z_2}-A_2\frac{\partial^2\gamma_2}{\partial z_2\partial z_1}.$$

Thus, on the basis of (5.3) and (5.5), we have

$$(A_1-A_2-\Gamma_1)e^{2p(z)}=A_1-A_2+\Gamma_2. \tag{5.6}$$

Now, we consider two cases.

Case 1.=: $p(z)$ is a constant. Let $\beta=e^{p(z)}$, then equations (5.1)-(5.2) can be written as

$$f(z)+\frac{\partial f}{\partial z_1}=\frac{1}{\sqrt{2}}\Big(\frac{k_1}{\sqrt{1+\alpha}}+\frac{k_2}{\sqrt{1-\alpha}}\Big)e^{g(z)/2}, \tag{5.7}$$

$$f(z)+\frac{\partial^2 f}{\partial z_1\partial z_2}=\frac{1}{\sqrt{2}}\Big(\frac{k_1}{\sqrt{1+\alpha}}-\frac{k_2}{\sqrt{1-\alpha}}\Big)e^{g(z)/2}, \tag{5.8}$$

where $k_1=\frac{\beta+\beta^{-1}}{2}$, $k_2=\frac{\beta-\beta^{-1}}{2i}$, and $k_1^2+k_2^2=1$. This leads to

$$\frac{\partial f}{\partial z_1}-\frac{\partial^2 f}{\partial z_1\partial z_2}=\frac{\sqrt{2}k_2}{\sqrt{1-\alpha}}e^{g(z)/2}. \tag{5.9}$$

On the other hand, differentiating with respect to z_2 on equation (5.7), then in line with (5.8), we have

$$f(z)-\frac{\partial f}{\partial z_2}=\frac{1}{\sqrt{2}}\big[(1-\frac{\alpha_2}{2})\frac{k_1}{\sqrt{1+\alpha}}-(1+\frac{\alpha_2}{2})\frac{k_2}{\sqrt{1-\alpha}}\big]e^{g(z)/2}. \tag{5.10}$$

Differentiating with respect to z_1 on equation (5.10), then combining this with (5.9) yields

$$\begin{aligned}&\frac{\partial f}{\partial z_1}-\frac{\partial^2 f}{\partial z_1\partial z_2}\\&=\frac{1}{\sqrt{2}}\big[(\frac{\alpha_1}{2}-\frac{\alpha_1\alpha_2}{4})\frac{k_1}{\sqrt{1+\alpha}}-(\frac{\alpha_1}{2}+\frac{\alpha_1\alpha_2}{4})\frac{k_2}{\sqrt{1-\alpha}}\big]e^{g(z)/2}.\end{aligned} \tag{5.11}$$

Since $g(z)$ is a polynomial with the linear form $g(z_1,z_2)=\alpha_1z_1+\alpha_2z_2+\alpha_0$, where $\alpha_1\neq 0,\alpha_2\neq 0,\alpha_0\in\mathbb{C}$. Hence, we deduce from (5.9) and (5.11) that

$$\frac{8+2\alpha_1+\alpha_1\alpha_2}{\sqrt{1-\alpha}}k_2=\frac{2\alpha_1-\alpha_1\alpha_2}{\sqrt{1+\alpha}}k_1. \tag{5.12}$$

From (5.7) and (5.10), we deduce that

$$\frac{\partial f}{\partial z_1}+\frac{\partial f}{\partial z_2}=\big[\frac{\alpha_2k_1}{2\sqrt{2(1+\alpha)}}+\frac{(\alpha_2+4)k_2}{2\sqrt{2(1-\alpha)}}\big]e^{g(z)/2}. \tag{5.13}$$

The characteristic equations of (5.13) are

$$\frac{dz_1}{dt}=1,\quad \frac{dz_2}{dt}=1,\quad \frac{df}{dt}=\big[\frac{\alpha_2k_1}{2\sqrt{2(1+\alpha)}}+\frac{(\alpha_2+4)k_2}{2\sqrt{2(1-\alpha)}}\big]e^{g(z)/2}.$$

Similarly, we obtain

$$f(t,s)=\int_0^t\big[\frac{\alpha_2k_1}{2\sqrt{2(1+\alpha)}}+\frac{(\alpha_2+4)k_2}{2\sqrt{2(1-\alpha)}}\big]e^{g(z)/2}dt+\phi_0(s), \tag{5.14}$$

where $\phi_0(s)$ is a finite order transcendental entire function in $s=z_2-z_1$.

Subcase 1.1. If $\alpha_1+\alpha_2\neq 0$, it follows from (5.14) that

$$\begin{aligned}&f(t,s)\\&=\int_0^t\big[\frac{\alpha_2k_1}{2\sqrt{2(1+\alpha)}}+\frac{(\alpha_2+4)k_2}{2\sqrt{2(1-\alpha)}}\big]e^{\frac{(\alpha_1+\alpha_2)t+\alpha_2s+\alpha_0}{2}}dt+\phi_0(s)\\&=\frac{2}{\alpha_1+\alpha_2}\big[\frac{\alpha_2k_1}{2\sqrt{2(1+\alpha)}}+\frac{(\alpha_2+4)k_2}{2\sqrt{2(1-\alpha)}}\big]e^{\frac{(\alpha_1+\alpha_2)t+\alpha_2s+\alpha_0}{2}}+\phi_9(s),\end{aligned} \tag{5.15}$$

where

$$\phi_9(s)=\phi_0(s)-\frac{2}{\alpha_1+\alpha_2}\big[\frac{\alpha_2k_1}{2\sqrt{2(1+\alpha)}}+\frac{(\alpha_2+4)k_2}{2\sqrt{2(1-\alpha)}}\big]e^{\frac{\alpha_2s+\alpha_0}{2}}.$$

It follows (5.15) that

$$f(z_1,z_2)=\frac{1}{\alpha_1+\alpha_2}\big[\frac{\alpha_2k_1}{\sqrt{2(1+\alpha)}}+\frac{(\alpha_2+4)k_2}{\sqrt{2(1-\alpha)}}\big]e^{\frac{\alpha_1z_1+\alpha_2z_2+\alpha_0}{2}}+\phi_9(z_2-z_1). \tag{5.16}$$

Substituting (5.16) into the (5.7) or (5.8), then combining this with (5.12) yields

$$\phi_9(z_2-z_1)-\phi_9'(z_2-z_1)=0, \tag{5.17}$$

which implies $\phi_9(z_2-z_1)=\eta_9e^{-z_1+z_2}$, $\eta_9\in\mathbb{C}$.

Subcase 1.2. If $\alpha_1+\alpha_2=0$, then from (5.14) it follows that

$$\begin{aligned} f(t,s) &= \int_0^t \big[\frac{\alpha_2 k_1}{2\sqrt{2(1+\alpha)}} + \frac{(\alpha_2+4)k_2}{2\sqrt{2(1-\alpha)}}\big] e^{\frac{(\alpha_1+\alpha_2)t+\alpha_2 s+\alpha_0}{2}} dt + \phi_0(s) \\ &= \big[\frac{\alpha_2 k_1}{2\sqrt{2(1+\alpha)}} + \frac{(\alpha_2+4)k_2}{2\sqrt{2(1-\alpha)}}\big] e^{\frac{\alpha_2 s+\alpha_0}{2}} t + \phi_{10}(s), \end{aligned} \tag{5.18}$$

where $\phi_{10}(s)=\phi_0(s)$. In view of (5.18), we have

$$f(z_1,z_2) = \big[\frac{\alpha_2 k_1}{2\sqrt{2(1+\alpha)}} + \frac{(\alpha_2+4)k_2}{2\sqrt{2(1-\alpha)}}\big] e^{\frac{\alpha_2(z_2-z_1)+\alpha_0}{2}} z_1 + \phi_{10}(z_2-z_1). \tag{5.19}$$

Substituting (5.19) into the (5.7) or (5.8), then combining this with (5.12) yields

$$\phi_{10}(z_2-z_1) - \phi'_{10}(z_2-z_1) = \frac{4k_2}{\alpha_1\sqrt{2(1-\alpha)}} e^{\frac{\alpha_2(z_2-z_1)+\alpha_0}{2}}, \tag{5.20}$$

which implies $\phi_{10}(z_2-z_1) = \frac{8k_2}{(2+\alpha_1)\alpha_1\sqrt{2(1-\alpha)}} e^{g(z)/2} + \eta_{10} e^{-z_1+z_2}$, $\eta_{10}\in\mathbb{C}$. The proof of Theorem 2.12(i) is complete.

Case 2. If $p(z)$ is a non-constant, it follows from (5.6) that

$$\begin{aligned} A_1 - A_2 - A_2\frac{\partial\gamma_1}{\partial z_1} + A_1\frac{\partial\gamma_1}{\partial z_1}\frac{\partial\gamma_1}{\partial z_2} + A_1\frac{\partial^2\gamma_1}{\partial z_2\partial z_1} &= 0, \\ A_1 - A_2 + A_1\frac{\partial\gamma_2}{\partial z_1} - A_2\frac{\partial\gamma_2}{\partial z_1}\frac{\partial\gamma_2}{\partial z_2} - A_2\frac{\partial^2\gamma_2}{\partial z_2\partial z_1} &= 0. \end{aligned} \tag{5.21}$$

Otherwise, without loss of generality, if

$$A_1 - A_2 - A_2\frac{\partial\gamma_1}{\partial z_1} + A_1\frac{\partial\gamma_1}{\partial z_1}\frac{\partial\gamma_1}{\partial z_2} + A_1\frac{\partial^2\gamma_1}{\partial z_2\partial z_1} \neq 0,$$

we have

$$e^{2p(z)} = \frac{A_1 - A_2 + A_1\frac{\partial\gamma_2}{\partial z_1} - A_2\frac{\partial\gamma_2}{\partial z_1}\frac{\partial\gamma_2}{\partial z_2} - A_2\frac{\partial^2\gamma_2}{\partial z_2\partial z_1}}{A_1 - A_2 - A_2\frac{\partial\gamma_1}{\partial z_1} + A_1\frac{\partial\gamma_1}{\partial z_1}\frac{\partial\gamma_1}{\partial z_2} + A_1\frac{\partial^2\gamma_1}{\partial z_2\partial z_1}}. \tag{5.22}$$

Since $p(z)$ and $g(z)$ are polynomials, the left-hand side of (5.22) is transcendental, which contradicts with the right-hand side of (5.22) being a rational function. In view of (5.21), we have

$$\begin{aligned} A_2\frac{\partial\gamma_1}{\partial z_1} - A_1\frac{\partial\gamma_1}{\partial z_1}\frac{\partial\gamma_1}{\partial z_2} - A_1\frac{\partial^2\gamma_1}{\partial z_2\partial z_1} &= A_1 - A_2, \\ A_1\frac{\partial\gamma_2}{\partial z_1} - A_2\frac{\partial\gamma_2}{\partial z_1}\frac{\partial\gamma_2}{\partial z_2} - A_2\frac{\partial^2\gamma_2}{\partial z_2\partial z_1} &= A_2 - A_1. \end{aligned} \tag{5.23}$$

Next, we discuss the forms of γ_1 and γ_2. Set

$$\gamma_1 = \sum_{k=0}^{n} \alpha_k(z_2) z_1^k = \alpha_n(z_2) z_1^k + \alpha_{n-1}(z_2) z_1^{k-1} + \cdots + \alpha_0(z_2),$$

where $\alpha_n(z_2), \alpha_{n-1}(z_2), \dots, \alpha_0(z_2)$ are polynomials is z_2 and notice that z_2 does not have a degree n. Differentiating with respect to z_1 and z_2 on γ_1 respectively, and substituting $\frac{\partial\gamma_1}{\partial z_1}$, $\frac{\partial\gamma_1}{\partial z_2}$ and $\frac{\partial^2\gamma_1}{\partial z_2\partial z_1}$ into (5.23), we have

$$\begin{aligned} &A_2\sum_{k=1}^{n}k\alpha_k(z_2)z_1^{k-1}-A_1\sum_{k=1}^{n}k\alpha_k(z_2)z_1^{k-1}\sum_{k=0}^{n}\alpha_k'(z_2)z_1^{k}\\ &-A_1\sum_{k=1}^{n}k\alpha_k'(z_2)z_1^{k-1}=A_1-A_2. \end{aligned} \tag{5.24}$$

Considering the highest degree of z_1, if $k\geq 1$, then $\alpha_k'(z_2)\equiv 0$, $\alpha_k(z_2)$ is a constant. Otherwise, the left of (5.24) is a non-constant polynomial, which contradicts the right-hand side of (5.24) being a constant. Hence, equation (5.24) can be rewritten as

$$A_2\sum_{k=1}^{n}k\alpha_k{z_1}^{k-1}-A_1\sum_{k=1}^{n}k\alpha_k{z_1}^{k-1}\alpha_0'(z_2)=A_1-A_2. \tag{5.25}$$

Obviously, ${\alpha_0}'(z_2)$ is a constant. Otherwise, considering the coefficients on both sides of z_2 leads to a contradiction. Hence, we let ${\alpha_0}'(z_2)=c$, where c is a constant. Further, if $k\geq 2$, then we have

$$A_2k\alpha_k-A_1ck\alpha_k=0,\quad A_2\alpha_1-A_1\alpha_1c=A_1-A_2.$$

The formula above yields $A_1=A_2$ which is a contradiction. If $k=0$, the left-hand side of (5.23) is zero, which contradicts with the right-hand side t of (5.23) being a nonzero constant. Hence, $k=1$. Whereupon, $\gamma_1=\alpha_0(z_2)+\alpha_1z_1$, where ${\alpha_0}'(z_2)$ is a constant. Similar to the arguments in γ_2, we have the same form for γ_1. Without loss of generality, we set

$$\gamma_1(z)=B_{11}z_1+B_{12}z_2+\beta_1,\quad \gamma_2(z)=B_{21}z_1+B_{22}z_2+\beta_2.$$

In view of (5.23), this can be rewritten as

$$\begin{aligned} A_2B_{11}-A_1B_{11}B_{12}&=A_1-A_2,\\ A_1B_{21}-A_2B_{21}B_{22}&=A_2-A_1. \end{aligned} \tag{5.26}$$

According to equation (5.1) and (5.4), we deduce that

$$\frac{\partial f}{\partial z_1}+\frac{\partial f}{\partial z_2}=\frac{1}{\sqrt{2}}\big[(A_1-A_2+A_1B_{12})e^{\gamma_1}+(A_2-A_1+A_2B_{22})e^{\gamma_2}\big]. \tag{5.27}$$

The characteristic equations of (5.27) are

$$\frac{dz_1}{dt}=1,\quad \frac{dz_2}{dt}=1,\quad \frac{df}{dt}=\frac{1}{\sqrt{2}}[(A_1-A_2+A_1B_{12})e^{\gamma_1}+(A_2-A_1+A_2B_{22})e^{\gamma_2}].$$

Similarly, we obtain

$$\begin{aligned} f(t,s)=\int_0^t\frac{1}{\sqrt{2}}\big[&(A_1-A_2+A_1B_{12})e^{(B_{11}+B_{12})t+B_{12}s+\beta_1}\\ &+(A_2-A_1+A_2B_{22})e^{(B_{21}+B_{22})t+B_{22}s+\beta_2}\big]dt+\phi_0(s), \end{aligned} \tag{5.28}$$

where $\phi_0(s)$ is a transcendental entire function with finite order in $s=z_2-z_1$.

Subcase 2.1. If $B_{11}+B_{12}\neq 0$, $B_{21}+B_{22}\neq 0$, it follows from (5.28) that

$$\begin{aligned} f(t,s) &= \int_0^t \frac{1}{\sqrt{2}}\big[(A_1-A_2+A_1B_{12})e^{(B_{11}+B_{12})t+B_{12}s+\beta_1} \\ &\quad + (A_2-A_1+A_2B_{22})e^{(B_{21}+B_{22})t+B_{22}s+\beta_2}\big]dt+\phi_0(s) \\ &= \frac{1}{\sqrt{2}}\big[\frac{A_1-A_2+A_1B_{12}}{B_{11}+B_{12}}e^{(B_{11}+B_{12})t+B_{12}s+\beta_1} \\ &\quad + \frac{A_2-A_1+A_2B_{22}}{B_{21}+B_{22}}e^{(B_{21}+B_{22})t+B_{22}s+\beta_2}\big]+\phi_{11}(s), \end{aligned} \tag{5.29}$$

where

$$phi_{11}(s)=\phi_0(s)-\frac{1}{\sqrt{2}}\big[\frac{A_1-A_2+A_1B_{12}}{B_{11}+B_{12}}e^{B_{12}s+\beta_1}+\frac{A_2-A_1+A_2B_{22}}{B_{21}+B_{22}}e^{B_{22}s+\beta_2}\big].$$

Thus, from (5.29), it follows that

$$\begin{aligned} f(z_1,z_2) &= \frac{1}{\sqrt{2}}\big[\frac{A_1-A_2+A_1B_{12}}{B_{11}+B_{12}}e^{\gamma_1(z)}+\frac{A_2-A_1+A_2B_{22}}{B_{21}+B_{22}}e^{\gamma_2(z)}\big] \\ &\quad +\phi_{11}(z_2-z_1). \end{aligned} \tag{5.30}$$

Substituting (5.30) into (5.1) or (5.2), then combining this with (5.26) yields

$$\phi_{11}(z_2-z_1)-\phi_{11}'(z_2-z_1)=0, \tag{5.31}$$

which implies $\phi_{11}(z_2-z_1)=\eta_{11}e^{-z_1+z_2}$, $\eta_{11}\in\mathbb{C}$.

Subcase 2.2. If $B_{11}+B_{12}=0$, $B_{21}+B_{22}\neq 0$, then it follows from (5.28) that

$$\begin{aligned} f(t,s) &= \int_0^t \frac{1}{\sqrt{2}}\big[(A_1-A_2+A_1B_{12})e^{B_{12}s+\beta_1} \\ &\quad + (A_2-A_1+A_2B_{22})e^{(B_{21}+B_{22})t+B_{22}s+\beta_2}\big]dt+\phi_0(s) \\ &= \frac{1}{\sqrt{2}}\big[(A_1-A_2+A_1B_{12})e^{B_{12}s+\beta_1}t \\ &\quad + \frac{A_2-A_1+A_2B_{22}}{B_{21}+B_{22}}e^{(B_{21}+B_{22})t+B_{22}s+\beta_2}\big]+\phi_{12}(s), \end{aligned} \tag{5.32}$$

where

$$phi_{12}(s)=\phi_0(s)-\frac{1}{\sqrt{2}}\frac{A_2-A_1+A_2B_{22}}{B_{21}+B_{22}}e^{B_{22}s+\beta_2}.$$

Thus, in view of (5.32), it follows that

$$\begin{aligned} f(z_1,z_2) &= \frac{1}{\sqrt{2}}\big[(A_1-A_2+A_1B_{12})e^{\gamma_1(z)}z_1+\frac{A_2-A_1+A_2B_{22}}{B_{21}+B_{22}}e^{\gamma_2(z)}\big] \\ &\quad +\phi_{12}(z_2-z_1). \end{aligned} \tag{5.33}$$

Since $B_{11}+B_{12}=0$, $B_{21}+B_{22}\neq 0$, in view of (5.26), we can deduce that

$$B_{11}=\frac{A_1-A_2}{A_1},\ B_{12}=\frac{A_2-A_1}{A_1},\quad \text{or}\quad B_{11}=-1,\ B_{12}=1. \tag{5.34}$$

Thus, there exist several cases as follows.

Subcase 2.2.1. If $B_{11}=\frac{A_1-A_2}{A_1}$, it follows from (5.33) that

$$f(z_1,z_2)=\frac{1}{\sqrt{2}}\frac{A_2-A_1+A_2B_{22}}{B_{21}+B_{22}}e^{\gamma_2(z)}+\phi_{12}(z_2-z_1). \tag{5.35}$$

Substituting (5.35) into the (5.1) or (5.2), then combining this with (5.34) yields

$$\begin{aligned}\phi_{12}(z_2-z_1)-\phi_{12}'(z_2-z_1)&=\frac{1}{\sqrt{2}}A_1e^{\gamma_1(z)},\\ \phi_{12}(z_2-z_1)-\phi_{12}''(z_2-z_1)&=\frac{1}{\sqrt{2}}A_2e^{\gamma_1(z)},\end{aligned}$$

which implies

$$\phi_{12}(z_2-z_1)=\frac{1}{\sqrt{2}}\frac{{A_1}^2}{2A_1-A_2}e^{\gamma_1(z)}+\eta_{12}e^{-z_1+z_2},\eta_{12}\in\mathbb{C}.$$

Subcase 2.2.2. If $B_{11}=-1$, it follows from (5.33) that

$$\begin{aligned}f(z_1,z_2)=\frac{1}{\sqrt{2}}\big[&(2A_1-A_2)e^{\gamma_1(z)}z_1+\frac{A_2-A_1+A_2B_{22}}{B_{21}+B_{22}}e^{\gamma_2(z)}\big]\\ &+\phi_{13}(z_2-z_1).\end{aligned}\tag{5.36}$$

Substituting (5.36) into the (5.1) or (5.2), thencombining this with (5.34) yields

$$\begin{aligned}\phi_{13}(z_2-z_1)-\phi_{13}'(z_2-z_1)&=\frac{1}{\sqrt{2}}(A_2-A_1)e^{\gamma_1(z)},\\ \phi_{13}(z_2-z_1)-\phi_{13}''(z_2-z_1)&=\frac{1}{\sqrt{2}}(2A_2-2A_1)e^{\gamma_1(z)},\end{aligned}$$

which implies

$$\phi_{13}(z_2-z_1)=\frac{1}{\sqrt{2}}(A_2-A_1)e^{\gamma_1(z)}(z_1-z_2)+\eta_{13}e^{-z_1+z_2},\eta_{13}\in\mathbb{C}.$$

Subcase 2.3. If $B_{11}+B_{12}\neq 0$ and $B_{21}+B_{22}=0$, then it follows from (5.28) that

$$\begin{aligned}f(t,s)&=\int_0^t\frac{1}{\sqrt{2}}\big[(A_1-A_2+A_1B_{12})e^{(B_{11}+B_{12})t+B_{12}s+\beta_1}\\ &\quad+(A_2-A_1+A_2B_{22})e^{B_{22}s+\beta_2}\big]dt+\phi_0(s)\\ &=\frac{1}{\sqrt{2}}\big[\frac{A_1-A_2+A_1B_{12}}{B_{11}+B_{12}}e^{(B_{11}+B_{12})t+B_{12}s+\beta_1}\\ &\quad+(A_2-A_1+A_2B_{22})e^{B_{22}s+\beta_2}t\big]+\phi_{14}(s),\end{aligned}\tag{5.37}$$

where

$$\phi_{14}(s)=\phi_0(s)-\frac{1}{\sqrt{2}}\frac{A_1-A_2+A_1B_{12}}{B_{11}+B_{12}}e^{B_{12}s+\beta_1}.$$

Thus, in view of (5.37), it follows that

$$\begin{aligned}f(z_1,z_2)=\frac{1}{\sqrt{2}}\big[&\frac{A_1-A_2+A_1B_{12}}{B_{11}+B_{12}}e^{\gamma_1(z)}+(A_2-A_1+A_2B_{22})e^{\gamma_2(z)}z_1\big]\\ &+\phi_{14}(z_2-z_1).\end{aligned}\tag{5.38}$$

Since $B_{11}+B_{12}\neq 0$ and $B_{21}+B_{22}=0$, in view of (5.26), we deduce that

$$B_{21}=\frac{A_2-A_1}{A_2},\ B_{22}=\frac{A_1-A_2}{A_2}\quad\text{or}\quad B_{21}=-1,\ B_{22}=1.$$

There exists several cases as follows.

Subcase 2.3.1. If $B_{21} = \frac{A_2 - A_1}{A_2}$, then it follows from (5.38) that

$$f(z_1, z_2) = \frac{1}{\sqrt{2}} \frac{A_1 - A_2 + A_1 B_{12}}{B_{11} + B_{12}} e^{\gamma_1(z)} + \phi_{14}(z_2 - z_1). \tag{5.39}$$

Substituting (5.39) into (5.1) or (5.2), then combining this with (5.26) yields

$$\phi_{14}(z_2 - z_1) - \phi'_{14}(z_2 - z_1) = \frac{1}{\sqrt{2}} A_2 e^{\gamma_2(z)},$$
$$\phi_{14}(z_2 - z_1) - \phi''_{14}(z_2 - z_1) = \frac{1}{\sqrt{2}} A_1 e^{\gamma_2(z)},$$

which implies

$$\phi_{14}(z_2 - z_1) = \frac{1}{\sqrt{2}} \frac{{A_2}^2}{2A_2 - A_1} e^{\gamma_2(z)} + \eta_{14} e^{-z_1 + z_2} \eta_{14} \in \mathbb{C}.$$

Subcase 2.3.2. If $B_{21} = -1$, then it follows from (5.33) that

$$\begin{aligned} f(z_1, z_2) = \frac{1}{\sqrt{2}} [&\frac{A_1 - A_2 + A_1 B_{12}}{B_{11} + B_{12}} e^{\gamma_1(z)} + (2A_2 - A_1) e^{\gamma_2(z)} z_1] \\ &+ \phi_{15}(z_2 - z_1). \end{aligned} \tag{5.40}$$

Substituting (5.40) into the (5.1) or (5.2), then combining this with (5.26) yields

$$\phi_{15}(z_2 - z_1) - \phi'_{15}(z_2 - z_1) = \frac{1}{\sqrt{2}} (A_1 - A_2) e^{\gamma_2(z)},$$
$$\phi_{15}(z_2 - z_1) - \phi''_{15}(z_2 - z_1) = \frac{1}{\sqrt{2}} (2A_1 - 2A_2) e^{\gamma_2(z)},$$

which implies

$$\phi_{15}(z_2 - z_1) = \frac{1}{\sqrt{2}} (A_1 - A_2) e^{\gamma_2(z)} (z_1 - z_2) + \eta_{15} e^{-z_1 + z_2}, \quad \eta_{15} \in \mathbb{C}.$$

Subcase 2.4. If $B_{11} + B_{12} = 0$ and $B_{21} + B_{22} = 0$, then it follows from (5.28) that

$$\begin{aligned} f(t, s) &= \int_0^t \frac{1}{\sqrt{2}} \big[(A_1 - A_2 + A_1 B_{12}) e^{B_{12} s + \beta_1} \\ &\quad + (A_2 - A_1 + A_2 B_{22}) e^{B_{22} s + \beta_2}\big] dt + \phi_0(s) \\ &= \frac{1}{\sqrt{2}} \big[(A_1 - A_2 + A_1 B_{12}) e^{B_{12} s + \beta_1} t \\ &\quad + (A_2 - A_1 + A_2 B_{22}) e^{B_{22} s + \beta_2} t\big] + \phi_{16}(s), \end{aligned} \tag{5.41}$$

where $\phi_{16}(s) = \phi_0(s)$ is a transcendental entire function with finite order in s. Thus, in view of (5.41), it follows that

$$\begin{aligned} f(z_1, z_2) = \frac{1}{\sqrt{2}} &\big[(A_1 - A_2 + A_1 B_{12}) e^{\gamma_1(z)} \\ &+ (A_2 - A_1 + A_2 B_{22}) e^{\gamma_2(z)}\big] z_1 + \phi_{16}(z_2 - z_1). \end{aligned} \tag{5.42}$$

Since $B_{11} + B_{12} = 0$, $B_{21} + B_{22} = 0$, in view of (5.26), we deduce that

$$B_{11} = \frac{A_1 - A_2}{A_1}, \ B_{12} = \frac{A_2 - A_1}{A_1} \quad \text{or} \quad B_{11} = -1, \ B_{12} = 1;$$

$$B_{21}=\frac{A_2-A_1}{A_2},\ B_{22}=\frac{A_1-A_2}{A_2}\quad\text{or}\quad B_{21}=-1,\ B_{22}=1.$$

Thus, there exist several cases, as follows/.

Subcase 2.4.1. If $B_{11}=\frac{A_1-A_2}{A_1}$ and $B_{21}=\frac{A_2-A_1}{A_2}$, it follows from (5.42) that

$$f(z_1,z_2)=\phi_{16}(z_2-z_1). \tag{5.43}$$

Substituting (5.43) into (5.1) or (5.2), and then combining this with (5.26) yields

$$\begin{gathered}\phi_{16}(z_2-z_1)-\phi_{16}'(z_2-z_1)=\frac{1}{\sqrt{2}}(A_1e^{\gamma_1(z)}+A_2e^{\gamma_2(z)}),\\ \phi_{16}(z_2-z_1)-\phi_{16}''(z_2-z_1)=\frac{1}{\sqrt{2}}(A_2e^{\gamma_1(z)}+A_1e^{\gamma_2(z)}),\end{gathered}$$

which implies

$$phi_{16}(z_2-z_1)=\frac{1}{\sqrt{2}}\big[\frac{{A_1}^2}{2A_1-A_2}e^{\gamma_1(z)}+\frac{{A_2}^2}{2A_2-A_1}e^{\gamma_2(z)}\big]+\eta_{16}e^{-z_1+z_2},$$

with $\eta_{16}\in\mathbb{C}$.

Subcase 2.4.2. If $B_{11}=-1$ and $B_{21}=\frac{A_2-A_1}{A_2}$, it follows from (5.42) that

$$f(z_1,z_2)=\frac{1}{\sqrt{2}}(2A_1-A_2)e^{\gamma_1(z)}z_1+\phi_{17}(z_2-z_1). \tag{5.44}$$

Substituting (5.44) into (5.1) or (5.2), then combining this with (5.26) yields

$$\begin{gathered}\phi_{17}(z_2-z_1)-\phi_{17}'(z_2-z_1)=\frac{1}{\sqrt{2}}\big[(A_2-A_1)e^{\gamma_1(z)}+A_2e^{\gamma_2(z)}\big],\\ \phi_{17}(z_2-z_1)-\phi_{17}''(z_2-z_1)=\frac{1}{\sqrt{2}}\big[(2A_2-2A_1)e^{\gamma_1(z)}+A_1e^{\gamma_2(z)}\big],\end{gathered}$$

which implies

$$\phi_{17}(z_2-z_1)=\frac{1}{\sqrt{2}}\big[(A_2-A_1)e^{\gamma_1(z)}(z_1-z_2)+\frac{{A_2}^2}{2A_2-A_1}e^{\gamma_2(z)}\big]+\eta_{17}e^{-z_1+z_2},$$

with $\eta_{17}\in\mathbb{C}$.

Subcase 2.4.3. If $B_{11}=\frac{A_1-A_2}{A_1}$ and $B_{21}=-1$, it follows from (5.28) that

$$f(z_1,z_2)=\frac{1}{\sqrt{2}}(2A_2-A_1)e^{\gamma_2(z)}z_1+\phi_{18}(z_2-z_1). \tag{5.45}$$

Substituting (5.45) into (5.1) or (5.2), then combining this with (5.26) yields

$$\begin{gathered}\phi_{18}(z_2-z_1)-\phi_{18}'(z_2-z_1)=\frac{1}{\sqrt{2}}\big[A_1e^{\gamma_1(z)}+(A_1-A_2)e^{\gamma_2(z)}\big],\\ \phi_{18}(z_2-z_1)-\phi_{18}''(z_2-z_1)=\frac{1}{\sqrt{2}}\big[A_2e^{\gamma_1(z)}+(2A_1-2A_2)e^{\gamma_2(z)}\big],\end{gathered}$$

which implies

$$\phi_{18}(z_2-z_1)=\frac{1}{\sqrt{2}}\big[\frac{{A_1}^2}{2A_1-A_2}e^{\gamma_1(z)}+(A_1-A_2)e^{\gamma_2(z)}(z_1-z_2)\big]+\eta_{18}e^{-z_1+z_2},$$

with $\eta_{18}\in\mathbb{C}$.

Subcase 2.4.4. If $B_{11}=-1$ and $B_{21}=-1$, lead to $p(z)$ being a constant. By the assumption at the begin of Case 2, we obtain a contradiction. The proof of Theorem 2.12(ii) is complete. □

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