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P-MEAN (μ_1, μ_2)-PSEUDO ALMOST PERIODIC PROCESSES AND APPLICATION TO INTEGRO-DIFFERENTIAL STOCHASTIC EVOLUTION EQUATIONS

MOEZ AYACHI, SYED ABBAS

Abstract. In this article, we investigate the existence and stability of p-mean (μ_1, μ_2)-pseudo almost periodic solutions for a class of non-autonomous integro-differential stochastic evolution equations in a real separable Hilbert space. Using stochastic analysis techniques and the contraction mapping principle, we prove the existence and uniqueness of p-mean (μ_1, μ_2)-pseudo almost periodic solutions. We also provide sufficient conditions for the stability of these solutions. Finally, we present three examples with numerical simulations to illustrate the significance of the main findings.

1. Introduction

The concept of almost periodic functions was introduced in 1923 by Harald Bohr [13]. It plays an important role in describing the phenomena that are more or less periodic, which can be observed frequently in many fields, such as biology, celestial mechanics, dynamical population, engineering, and so on. For more details, we refer to [2, 19] and references therein. Almost periodic solutions refer to solutions of differential equations that oscillate over time, but not in a strictly periodic manner. In other words, their oscillations are not exactly periodic, but they exhibit some sort of repetitive behavior. Since the introduction of almost periodicity, several extensions of this concept have been introduced, including pseudo-almost periodicity by Zhang [39], weighted pseudo-almost periodicity (WPAP) by Diagana [20, 21], and μ-pseudo almost periodicity (PAP) by Blot et al. [12]. A more general-class pseudo-almost periodicity called (μ_1, μ_2)-pseudo-almost periodicity was considered by Diagana et al. (see [22]). For more details about the (μ_1, μ_2)-p.a.p. functions and their applications in the qualitative theory of differential equations, we refer the reader to [3, 5, 22, 30, 31, 35, 37, 38].

Further, stochastic perturbations are unavoidable and omnipresent in both real and artificial systems. Accordingly, investigating the dynamical behaviors of the systems described by various types of stochastic perturbations is highly important. For more information about the elementary theories for stochastic differential equations, we refer to [25, 32]. The concept of almost periodicity is important in

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probability for investigating stochastic processes. Recently, Bezandry and Diagana initiated the concept of p-mean almost periodic processes and applied it to the study of the existence and uniqueness of square-mean almost periodic mild solutions to some classes of stochastic differential equations [8, 9, 11]. Since then, the concept has been generalized into square-mean pseudo-almost periodicity by Ch'erif [16], p-mean μ-pseudo almost periodicity by Diop et al. [23], and (μ_1, μ_2)-pseudo almost periodicity by Belmabrouk et al. [4]. Several works have focused on investigating square-mean almost periodic processes, their various extensions, and their applications in stochastic differential equations, we refer to [7, 10, 14, 15, 27, 36, 40].

Stochastic integro-differential equations play a crucial role in the qualitative theory of differential equations due to their application in engineering, dynamical population, neural networks, biology, and so on. As a direct consequence, they have attracted an increasing amount of attention over the past few years. This article deals with the p-mean (μ_1, μ_2)-pseudo almost periodic ((μ_1, μ_2)-s.p.a.p. for short) mild solutions of the following non-autonomous integro-differential stochastic evolution equation in a real separable Hilbert space $\mathcal{E}$:

$$\begin{aligned} Z'(t) = A(t)Z(t) + F_1(t, Z(t)) + \int_{-\infty}^{t} Q(t-\zeta)F_2(\zeta, Z(\zeta))d\zeta \\ + \int_{-\infty}^{t} R(t-\zeta)G(\zeta, Z(\zeta))d\mathcal{W}(\zeta), \quad \forall t \in \mathbb{R}, \end{aligned} \tag{1.1}$$

where $A(t) : D(A(t)) \subset L^p(\mathcal{P}, \mathcal{E}) \to L^p(\mathcal{P}, \mathcal{E})$ is a family of densely defined closed linear operators satisfying the so-called "Acquistapace-Terrani" conditions, Q and R are convolution type kernels in $L^1(0, +\infty)$ and $L^2(0, +\infty)$, respectively, satisfying [24, Assumption 3.2], $(\mathcal{W}(t) : t \in \mathbb{R})$ is a $\mathcal{K}$-valued $\mathcal{Q}$-Brownian motion. Here $F_1, F_2 : \mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}) \to L^p(\mathcal{P}, \mathcal{E})$ and $G : \mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}) \to L^p(\mathcal{P}, \mathbb{L}_2^0)$ are jointly continuous functions satisfying some additional conditions. The spaces $L^p(\mathcal{P}, \mathcal{E})$, $\mathbb{L}_2^0$, and the $\mathcal{Q}$-Brownian motion are defined in the next section.

Equation (1.1) was studied in several special cases. For instance, Bezandry [6] investigated the existence and uniqueness of square-mean almost periodic mild solutions for equation (1.1) for $p = 2$ and $A(t) = A$. Li [27] addressed the problem of the existence, uniqueness, and asymptotic stability of square-mean almost periodic mild solutions of equation (1.1) in the case $p = 2$. More recently, Mbaye [29] considered the problem of the existence of square-mean μ-pseudo almost periodic mild solutions of equation (1.1) when $A(t) = A$. However, to the best of our knowledge, the existence, uniqueness, and stability of p-mean (μ_1, μ_2)-s.p.a.p. mild solutions of equation (1.1) is an untreated topic, which is the main motivation of this work.

Acquistapace-Terrani, which is discussed in this work, is an important condition that ensures the existence of a unique evolution family that is necessary for the corresponding integral form of a given differential equation. The equation considered in this work is very general in nature, and several other equations can be derived as special cases of it. Moreover, the concept of (μ_1, μ_2)-pseudo almost periodicity is a very general concept and can be applicable to situations where other kinds of functions, such as almost periodic and pseudo-almost periodic, cannot be used to describe the underlying dynamics. The conditions obtained for existence are very general in nature, and other conditions can be obtained as a special case. For example, one can choose the second component as zero in order to get the corresponding result for almost periodic. Similar results can be obtained by adjusting the equation and space. Moreover, stability is also established, and the condition obtained holds for any p; the particular case when $p = 2$ is given special emphasis. Application to various fields along with numerical graphs makes this work more useful for researchers, especially those who are interested in application.

This work is structured as follows. In Section 2, we present some basic notations and definitions. In Section 3, we establish some sufficient conditions to support the existence, uniqueness, and stability of the p-mean (μ_1, μ_2)-s.p.a.p. mild solution on $\mathbb{R}$ of equation (1.1). In the last section, three examples with numerical simulations are presented for better illustrations and to validate the analytical findings.

2. Preliminaries

This section introduces relevant notation, definitions and preliminary facts that are needed for the study.

2.1. **Q-Brownian motion.** Let $\beta_n(t)$, $n = 1, 2, 3, \dots$ be a sequence of real valued standard Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathcal{P})$. Set $\mathcal{W}(t) := \sum_{n=1}^{\infty} \sqrt{\xi_n}\beta_n(t)e_n$, $t \geq 0$, where $\xi_n \geq 0$ $(n \geq 1)$, are non-negative real numbers and $(e_n)_{n\geq 1}$ is an orthonormal basis in the Hilbert space $\mathcal{K}$. Let $\mathcal{Q}$ be a non-negative symmetric operator with finite trace defined by $\mathcal{Q}(e_n) = \xi_n e_n$, such that $\mathrm{Tr}[\mathcal{Q}] := \sum_{n=1}^{\infty} \xi_n < \infty$. It is well known that $\mathbb{E}[\mathcal{W}(t)] = 0$ and, for all $t \geq s \geq 0$, the distribution of $\mathcal{W}(t) - \mathcal{W}(s)$ is a Gaussian distribution $\mathcal{N}(0, (t-s)\mathcal{Q})$. The above mentioned $\mathcal{K}$-valued stochastic processes $(\mathcal{W}(t))_{t\geq 0}$ is called $\mathcal{Q}$-Brownian motion. Note that a $\mathcal{K}$-valued $\mathcal{Q}$-Brownian motion $(\mathcal{W}(t))_{t\in\mathbb{R}}$ can be obtained as follows: let $\{\mathcal{W}_i(t) : t \in \mathbb{R}_+\}$, $i = 1, 2$, be independent $\mathcal{K}$-valued $\mathcal{Q}$-Brownian motion. Then

$$\mathcal{W}(t) = \begin{cases} \mathcal{W}_1(t), & \text{if } 0 \leq t, \\ \mathcal{W}_2(-t), & \text{if } 0 \geq t, \end{cases}$$

is a $\mathcal{K}$-valued $\mathcal{Q}$-Brownian motion with $\mathbb{R}$ as time parameters. Let $\mathcal{F}_t = \sigma\{\mathcal{W}(u) : u \leq t\}$.

To define stochastic integrals with respect to the $\mathcal{Q}$-Brownian motion $\mathcal{W}$, let us denote $\mathcal{K}_0 = \mathcal{Q}^{\frac{1}{2}}\mathcal{K}$. Now define $\mathbb{L}_2^0 := \{\varphi \in \mathcal{L}(\mathcal{K}_0, \mathcal{E}) : \mathrm{Tr}[\varphi Q \varphi^*] < \infty\}$ the space of Hilbert-Schmidt operators from $\mathcal{K}_0$ to $\mathcal{E}$ equipped with the norm $\|\varphi\|_{\mathbb{L}_2^0}^2 := \mathrm{Tr}[\varphi Q \varphi^*]$ for any $\varphi \in \mathbb{L}_2^0$. The next result is a particular case of [33, Lemma 2.2].

Lemma 2.1 ([11, 33])**.** *For any $p \geq 2$ and for any arbitrary $\mathbb{L}_2^0$-valued predictable processes $\Psi(t)$, $t \in [0, T]$, there exists a constant $C_p > 0$ such that*

$$\mathbb{E}\Big[\sup_{s\in[0,t]} \Big\|\int_0^s \Psi(s)d\mathcal{W}(s)\Big\|^p\Big] \leq C_p.\mathbb{E}\Big[\int_0^t \|\Psi(s)\|_{\mathbb{L}_2^0}^2 ds\Big]^{p/2}.$$

2.2. **P-mean almost periodic stochastic processes.** Assume that $(\mathcal{E}, \|\cdot\|)$ and $(\mathcal{K}, \|\cdot\|_{\mathcal{K}})$ are real separable Hilbert spaces. $(\Omega, \mathcal{F}, \mathcal{P})$ is supposed to be a complete probability space. Let $p \geq 2$, and denote $L^p(\mathcal{P}, \mathcal{E})$ as the collection of all strongly measurable $\mathcal{E}$-valued random variables Y satisfying $\mathbb{E}\|Y\|^p < +\infty$, where the expectation $\mathbb{E}$ is defined by $\mathbb{E}[Y] := \int_\Omega Y(\omega)d\mathcal{P}(\omega)$. Note that $L^p(\mathcal{P}, \mathcal{E})$ is a Banach space when it is equipped with a norm

$$\|Y\|_{L^p} := [\mathbb{E}\|Y\|^p]^{1/p}.$$

Definition 2.2 ([11])**.** A stochastic processes $Z : \mathbb{R} \to L^p(\mathcal{P}, \mathcal{E})$ is said to be stochastically bounded if there exists a constant $C > 0$ such that $\mathbb{E}\|Z(t)\|^p \leq C$, for all $t \in \mathbb{R}$. The process Z is said to be stochastically continuous if $\lim_{t\to s} \mathbb{E}\|Z(t) - Z(s)\|^p = 0$.

We denote $\mathcal{SBC}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))$ the collection of all stochastically bounded and continuous processes from $\mathbb{R}$ into $L^p(\mathcal{P}, \mathcal{E})$. Then $(\mathcal{SBC}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})), \|\cdot\|_\infty)$ is a Banach space, where

$$\|Z\|_\infty := \sup_{t\in\mathbb{R}} [\mathbb{E}\|Z(t)\|^p]^{1/p}.$$

Definition 2.3 ([11])**.** A continuous stochastic processes $Z : \mathbb{R} \to L^p(\mathcal{P}, \mathcal{E})$ is said to be p-mean almost periodic processes, if for any $\epsilon > 0$ we can find $l = l(\epsilon) > 0$ such that for all $\varrho \in \mathbb{R}$, there exists $r \in [\varrho, \varrho + l]$ satisfying

$$\mathbb{E}\|Z(t + r) - Z(t)\|^p < \epsilon, \quad \forall t \in \mathbb{R}.$$

We denote the collection of all such stochastic processes by $\mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))$.

Proposition 2.4 ([11])**.** *The following properties hold for the stochastic processes* $\mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))$*:*

(1) $(\mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})), \|\cdot\|_\infty)$ *is a Banach space.*
(2) $\mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))$ *is invariant by translation.*
(3) $\mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})) \subset \mathcal{SBC}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))$ *is a closed subspace.*

2.3. **P-mean (μ_1, μ_2)-pseudo almost periodic processes.** Let $\mathfrak{B}$ be the Lebesgue σ-field of $\mathbb{R}$ and $\mathfrak{N}$ be the set of all positive measures μ on $\mathfrak{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([r, s]) < +\infty$ for any $r, s \in \mathbb{R}$ $(r < s)$.

To establish our results, we need the following assumptions:

(A1) Let $\mu_1, \mu_2 \in \mathfrak{N}$ such that $\limsup_{m\to+\infty} \frac{\mu_1([-m,m])}{\mu_2([-m,m])} < +\infty$.
(A2) For all $s \in \mathbb{R}$, there exists $\alpha > 0$ and a bounded interval I of $\mathbb{R}$ such that

$$\mu_1(\{c + s : c \in C\}) \leq \alpha\mu_1(C), \quad C \in \mathfrak{B} \text{ satisfies } C \cap I = \emptyset.$$

Definition 2.5. Let $\mu_1, \mu_2 \in \mathfrak{N}$. A stochastic processes $Z : \mathbb{R} \to L^p(\mathcal{P}, \mathcal{E})$ is said to be p-mean (μ_1, μ_2)-ergodic, if $Z \in \mathcal{SBC}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))$ and satisfies

$$\lim_{m\to+\infty} \frac{1}{\mu_2([-m, m])} \int_{-m}^{m} \mathbb{E}\|Z(t)\|^p d\mu_1(t) = 0.$$

The collection of all such stochastic processes is denoted by $\mathcal{SO}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2})$.

Proposition 2.6. *If* $\mu_1, \mu_2 \in \mathfrak{N}$ *satisfy* (A1)*, then* $(\mathcal{SO}(\mathbb{R}, L^p(\Omega, \mathcal{H}), \mu_{1,2}), \|\cdot\|_\infty)$ *is a Banach space.*

Proof. It is easy to see that $\mathcal{SO}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2})$ is a vector subspace of the Banach space $\mathcal{SBC}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))$. To complete the proof, we need to prove that the space $\mathcal{SO}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2})$ is closed in $\mathcal{SBC}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}))$. Let $(Z_n)_n$ be a sequence in $\mathcal{SO}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2})$ such that $\lim_{n\to+\infty} \|Z_n - Z\|_\infty = 0$. Since $\mu_2(\mathbb{R}) = +\infty$, it follows that $\mu_2([-m, m]) > 0$ for m sufficiently large. Then, by using the inequality

$$\begin{aligned}\int_{-m}^{m} \mathbb{E}\|Z(t)\|^p d\mu_1(t) &\leq 2^{p-1} \int_{-m}^{m} \mathbb{E}\|Z_n(t) - Z(t)\|^p d\mu_1(t) \\ &\quad + 2^{p-1} \int_{-m}^{m} \mathbb{E}\|Z_n(t)\|^p d\mu_1(t),\end{aligned}$$

we have

$$\frac{1}{\mu_2([-m,m])}\int_{-m}^{m}\mathbb{E}\|Z(t)\|^p d\mu_1(t) \leq 2^{p-1}\frac{\mu_1([-m,m])}{\mu_2([-m,m])}\|Z_n - Z\|_\infty^p + 2^{p-1}\frac{1}{\mu_2([-m,m])}\int_{-m}^{m}\mathbb{E}\|Z_n(t)\|^p d\mu_1(t).$$

Thus, in view of (A1), and the fact that $(Z_n)_n \in \mathcal{SO}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$, we obtain

$$\limsup_{m\to+\infty}\frac{1}{\mu_2([-m,m])}\int_{-m}^{m}\mathbb{E}\|Z(t)\|^p d\mu_1(t) \leq 2^{p-1}cst.\|Z_n - Z\|_\infty^p, \text{ for all } n\in\mathbb{N}.$$

Finally, since $\lim_{n\to+\infty}\|Z_n - Z\|_\infty^p = 0$, we conclude that

$$\lim_{m\to+\infty}\frac{1}{\mu_2([-m,m])}\int_{-m}^{m}\mathbb{E}\|Z(t)\|^p d\mu_1(t) = 0.$$

The proof is complete. □

Proposition 2.7 ([4]). *Let $\mu_1,\mu_2\in\mathfrak{N}$, J be a bounded interval (eventually $J=\emptyset$). Moreover, suppose that* (A1) *hold and $Z\in\mathcal{SBC}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}))$. Then the following assertions are equivalent:*

1. $Z\in\mathcal{SO}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$.
2. $\lim_{m\to+\infty}\frac{1}{\mu_2([-m,m]\setminus J)}\int_{[-m,m]\setminus J} mathbbE\|Z(t)\|^p d\mu_1(t) = 0$.
3. *For any $\upsilon>0$,* $\lim_{m\to+\infty}\frac{\mu_1\{t\in[-m,m]\setminus J:\mathbb{E}\|Z(t)\|^p>\upsilon\}}{\mu_2\{t\in[-m,m]\setminus J\}} = 0$.

Definition 2.8. Let $\mu_1,\mu_2\in\mathfrak{N}$. A continuous stochastic processes $Z:\mathbb{R}\to L^p(\mathcal{P},\mathcal{E})$ is said to be p-mean (μ_1,μ_2)-pseudo almost periodic processes (p-mean (μ_1,μ_2)-s.p.a.p. for short) if it can be expressed as $Z = Z^a + Z^e$, where $Z^a\in\mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}))$ and $Z^e\in\mathcal{SO}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$. We denote the collection of all such stochastic processes by $\mathcal{SPAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$

Theorem 2.9 ([4]). *Let $\mu_1,\mu_2\in\mathfrak{N}$ satisfy* (A2)*, then $\mathcal{SPAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$ is invariant by translation.*

Theorem 2.10 ([4]). *Let $\mu_1,\mu_2\in\mathfrak{N}$ and $Z\in\mathcal{SPAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$ be such that $Z = Z^a + Z^e$, where $Z^a\in\mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}))$ and $Z^e\in\mathcal{SO}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$. If $\mathcal{SPAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$ is invariant by translation, then*

$$\{Z^a(t): t\in\mathbb{R}\}\subset\overline{\{Z(t): t\in\mathbb{R}\}}.$$

Theorem 2.11. *Let $\mu_1,\mu_2\in\mathfrak{N}$ satisfy (A2). Then the decomposition of a p-mean (μ_1,μ_2)-s.p.a.p. stochastic processes in the form $Z = Z^a + Z^e$, where $Z^a\in\mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}))$ and $Z^e\in\mathcal{SO}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$ is unique.*

Proof. Suppose that $Z = Z_1^a + Z_1^e = Z_2^a + Z_2^e$, where $Z_1^a, Z_2^a\in\mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}))$ and $Z_1^e, Z_2^e\in\mathcal{SO}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$, then

$$0 = (Z^1 - Z_2^a) + (Z_1^e - Z_2^e)\in\mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}),\mu_{1,2}),$$

where $Z_1^a - Z_2^a\in\mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}))$ and $Z_1^e - Z_2^e\in\mathcal{SO}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$. Then from Theorem 2.10, we obtain $(Z_1^a - Z_2^a)(\mathbb{R})\subset\{0\}$. Consequently $Z_1^a = Z_2^a$ and $Z_1^e = Z_2^e$. □

Remark 2.12. Z^a and Z^e in definition 2.8 are called the p-mean almost periodic component and the p-mean (μ_1,μ_2)-ergodic perturbation of the stochastic processes Z respectively.

Theorem 2.13 ([4]). *If $\mu_1, \mu_2 \in \mathfrak{N}$ satisfy* (A1) *and* (A2)*, then*

$$(\mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}), \|\cdot\|_\infty)$$

is a Banach space.

Let $(\mathcal{E}_1, \|\cdot\|_1)$, $(\mathcal{E}_2, \|\cdot\|_2)$ be Banach spaces, and $L^p(\mathcal{P}, \mathcal{E}_1)$, $L^p(\mathcal{P}, \mathcal{E}_2)$ be corresponding L^p-spaces. Consider the following spaces of stochastic processes

$$\begin{aligned}
&\mathcal{SAP}\left(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2)\right) \\
&= \left\{F(\cdot, Y) \in \mathcal{SAP}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2)\right) \text{ for any } Y \in L^p(\mathcal{P}, \mathcal{E}_1)\right\}, \\
&\mathcal{SO}\left(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2}\right) \\
&= \left\{F(\cdot, Y) \in \mathcal{SO}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2}\right) \text{ for any } Y \in L^p(\mathcal{P}, \mathcal{E}_1)\right\}.
\end{aligned}$$

Definition 2.14. Let $\mu_1, \mu_2 \in \mathfrak{N}$. A stochastically continuous processes $F : \mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1) \to L^p(\mathcal{P}, \mathcal{E}_2)$ is said to be p-mean (μ_1, μ_2)-pseudo almost periodic in $t \in \mathbb{R}$ for any $Y \in L^p(\mathcal{P}, \mathcal{E}_1)$, if it can be expressed as $F = F^a + F^e$, where $F^a \in \mathcal{SAP}(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2))$ and $F^e \in \mathcal{SO}\left(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2}\right)$. We denote the collection of all such stochastically continuous processes by

$$\mathcal{SPAP}\left(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2}\right).$$

Theorem 2.15. *Let $\mu_1, \mu_2 \in \mathfrak{N}$ satisfy* (A2)*. Suppose that $F \in \mathcal{SPAP}(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2})$ satisfies Lipschitz condition in the second variable, that is, there exists $L > 0$ such that for any $Y_1, Y_2 \in L^p(\mathcal{P}, \mathcal{E}_1)$ and for all $t \in \mathbb{R}$*

$$\mathbb{E}\|F(t, Y_1) - F(t, Y_2)\|_2^p \le L.\mathbb{E}\|Y_1 - Y_2\|_1^p. \tag{2.1}$$

Then $F(\cdot, Z(\cdot)) \in \mathcal{SPAP}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2}\right)$ for any $Z \in \mathcal{PAP}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_1), \mu_{1,2}\right)$.

Proof. From Definitions 2.8 and 2.14, let $F = F^a + F^e$ and $Z = Z^a + Z^e$, where $F^a \in \mathcal{SAP}(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2))$, $F^e \in \mathcal{SO}(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2})$, $Z^a \in \mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_1), \mu_{1,2})$, and $Z^e \in \mathcal{SO}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_1), \mu_{1,2})$. Then, we can write

$$\begin{aligned}
F(t, Z(t)) &= F^a(t, Z^a(t)) + [F(t, Z(t)) - F(t, Z^a(t))] + [F(t, Z^a(t)) - F^a(t, Z^a(t))] \\
&= F^a(t, Z^a(t)) + [F(t, Z(t)) - F(t, Z^a(t))] + F^e(t, Z^a(t)).
\end{aligned}$$

First, we claim that $F^a(\cdot, Z^a(\cdot)) \in \mathcal{SAP}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2)\right)$. In fact, since $F \in \mathcal{SPAP}\left(\mathbb{R} \times L^p(\mathcal{P}, \mathcal{E}_1), L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2}\right)$, then for all $Y \in L^p(\mathcal{P}, \mathcal{E}_1)$, we have

$$F(\cdot, Y) \in \mathcal{SPAP}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2}\right).$$

Hence, we can write $F(\cdot, Y) = F^a(\cdot, Y) + F^e(\cdot, Y)$ with $F^a(\cdot, Y) \in \mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2))$ and $F^e(\cdot, Y) \in \mathcal{SO}(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2), \mu_{1,2})$. Since $\|F^a(\cdot, Y_1) - F^a(\cdot, Y_2)\|_2$ is almost periodic component of the p-mean (μ_1, μ_2)-s.p.a.p. function $\|F(\cdot, Y_1) - F(\cdot, Y_2)\|_2$, by using Theorem 2.10, we deduce that $\|F^a(\cdot, Y_1) - F^a(\cdot, Y_2)\|_\infty \le \|F(\cdot, Y_1) - F(\cdot, Y_2)\|_\infty$, which implies that, for any $t \in \mathbb{R}$ and $Y_1, Y_2 \in L^p(\mathcal{P}, \mathcal{E}_1)$

$$\mathbb{E}\|F^a(t, Y_1) - F^a(t, Y_2)\|_2^p \le \mathbb{E}\|F(t, Y_1) - F(t, Y_2)\|_2^p \le L.\mathbb{E}\|Y_1 - Y_2\|_1^p.$$

Let us define $\mathbb{K} = \overline{\{Z^a(t) : t \in \mathbb{R}\}}$. Since Z^a belongs to $\mathcal{SAP}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_1)\right)$, it follows that $\mathbb{K}$ is a compact set. Therefore, using [11, Theorem 4.4], we deduce that $G(\cdot, Z^a(\cdot)) \in \mathcal{SAP}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}_2)\right)$.

Next, we claim that $F(\cdot,Z(\cdot))-F(\cdot,Z^a(\cdot))\in\mathcal{SO}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}_2),\mu_{1,2})$. By using the Lipschitz condition, we obtain

$$\begin{aligned}
&\lim_{m\to\infty}\frac{1}{\mu_2([-m,m])}\int_{-m}^{m}\mathbb{E}\|F(t,Z(t))-F(t,Z^a(t))\|_2^p d\mu_1(t)\\
&\le \lim_{m\to\infty}\frac{1}{\mu_2([-m,m])}\int_{-m}^{m}L.\mathbb{E}\|Z(t)-Z^a(t)\|_1^p d\mu_1(t)\\
&\le \lim_{m\to\infty}\frac{1}{\mu_2([-m,m])}\int_{-m}^{m}L.\mathbb{E}\|Z^e(t)\|_1^p d\mu_1(t)=0.
\end{aligned}$$

Finally, it remains to show that $F^e(\cdot,Z^a(\cdot))\in\mathcal{SO}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}_2),\mu_{1,2})$. Indeed, in the view of (2.1) and Theorem 2.10, it follows that

$$\begin{aligned}
&\mathbb{E}\|F^e(t,Y_1)-F^e(t,Y_2)\|_2^p\\
&=\mathbb{E}\|F(t,Y_1)-F^a(t,Y_1)-F(t,Y_2)+F^a(t,Y_2)\|_2^p\\
&\le 2^{p-1}\mathbb{E}\|F(t,Y_1)-F(t,Y_2)\|_2^p+2^{p-1}\mathbb{E}\|F^a(t,Y_1)-F^a(t,Y_2)\|_2^p\\
&\le 2^pL.\mathbb{E}\|Y_1-Y_2\|_1^p.
\end{aligned}\tag{2.2}$$

Since $\mathbb{K}=\overline{\{Z^a(t):t\in\mathbb{R}\}}$ is compact, for $\epsilon>0$, there exist $Y_1,\dots,Y_k\in\mathbb{K}$, such that

$$\mathbb{K}\subset\cup_{i=1}^k B\big(Y_i,\frac{\epsilon}{2^{2p-1}\omega L}\big),$$

where

$$\omega:=\limsup_{m\to+\infty}\frac{\mu_1([-m,m])}{\mu_2([-m,m])}<+\infty,$$

$$B\Big(Y_i,\frac{\epsilon}{2^{p-1}\omega L}\Big):=\Big\{Y\in\mathbb{K}:\mathbb{E}\|Y_i-Y\|_1^p\le\frac{\epsilon}{2^{2p-1}\omega L}\Big\}.$$

By using (2.2) along with the above result, we obtain

$$\mathbb{K}\subset\cup_{i=1}^k\Big\{Y\in K:\forall t\in\mathbb{R},\mathbb{E}\|F^e(t,Y)-F^e(t,Y_i)\|_2^p\le\frac{\epsilon}{2^{p-1}\omega}\Big\}.$$

Let $t\in\mathbb{R}$ and $Y\in\mathbb{K}$, then there exists $i_*\in\{1,\dots,k\}$ such that

$$\mathbb{E}\|F^e(t,Y)-F^e(t,Y_{i_*})\|_2^p\le\frac{\epsilon}{2^{p-1}\omega}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}\|F^e(t,Z^a(t))\|_2^p&\le 2^{p-1}\mathbb{E}\|F^e(t,Z^a(t))-F^e(t,Y_{i_*})\|_2^p+2^{p-1}\mathbb{E}\|F^e(t,Y_{i_*})\|_2^p\\
&\le\frac{\epsilon}{\omega}+2^{p-1}\mathbb{E}\|F^e(t,Y_{i_*})\|_2^p\\
&\le\frac{\epsilon}{\omega}+2^{p-1}\sum_{i=1}^k\mathbb{E}\|F^e(t,Y_i)\|_2^p.
\end{aligned}\tag{2.3}$$

Since for all $i\in\{1,\dots,k\}$,

$$\lim_{m\to\infty}\frac{1}{\mu_2([-m,m])}\int_{-m}^{m}\mathbb{E}\|F^e(t,Y_i)\|_2^p d\mu_1(t)=0,$$

by using (A1) and (2.3), we obtain

$$\limsup_{m\to\infty}\frac{1}{\mu_2([-m,m])}\int_{-m}^{m}\mathbb{E}\|F^e(t,Z^a(t))\|_2^p d\mu_1(t)\le\epsilon,\ \text{for all }\epsilon>0,$$

which further implies

$$\lim_{m\to\infty}\frac{1}{\mu_2([-m,m])}\int_{-m}^{m}\mathbb{E}\|F^e(t,Z^a(t))\|_2^p d\mu_1(t)=0.$$

Therefore, $H(\cdot,Z^a(\cdot))\in\mathcal{SO}\left(\mathbb{R},L^p(\mathcal{P},\mathcal{E}_2),\mu_{1,2}\right)$. The proof is complete. □

3. Application to integro-differential stochastic evolution equations

In this section, we establish some sufficient conditions to ensure the existence, uniqueness and stability of p-mean (μ_1,μ_2)-s.p.a.p. mild solution of (1.1).

Definition 3.1. An $\mathcal{F}_t$ progressively measurable processes $(Z(t))_{t\in\mathbb{R}}$ is called mild solution of (1.1) if it satisfies the following stochastic integral equation

$$\begin{aligned}Z(t)&=U(t,a)Z(a)+\int_a^t U(t,s)F_1(s,Z(s))ds\\&\quad+\int_a^t U(t,s)\int_a^s Q(s-\zeta)F_2(\zeta,Z(\zeta))d\zeta ds\\&\quad+\int_a^t U(t,s)\int_a^s R(s-\zeta)G(\zeta,Z(\zeta))d\mathcal{W}(\zeta)ds\end{aligned}\tag{3.1}$$

for all $t\ge a$ and each $a\in\mathbb{R}$.

The Acquistapace-Terreni conditions (ATC, for short), which was firstly introduced in [1], play an important role in the study of non-autonomous evolution equations. We state it below for the readers' convenience.

Definition 3.2. A family of closed linear operators $A(t)$ for $t\in\mathbb{R}$ on a Banach space $(\mathcal{E},\|\cdot\|)$ with domain $D(A(t))$ (possibly not densely defined) satisfies ATC, if there exist constants $w>0$, $\gamma\in(\frac{\pi}{2},\pi)$, $K_1,K_2\ge 0$ and $\nu_1,\nu_2\in(0,1]$ with $\nu_1+\nu_2>1$ such that

$$\begin{gathered}S_\gamma\cup\{0\}\subset\rho(A(t)-w),\\ \|\mathcal{R}(\lambda,A(t)-w)\|\le\frac{K_2}{1+|\lambda|},\\ \|(A(t)-w)\mathcal{R}(\lambda,A(t)-w)[\mathcal{R}(w,A(t))-\mathcal{R}(w,A(s))]\|\le K_2.|t-s|^{\nu_1}|\lambda|^{-\nu_2},\\ \text{for all } t,s\in\mathbb{R},\lambda\in S_\gamma:=\{\lambda\in\mathbb{C}-\{0\}:|arg\lambda|\le\gamma\}.\end{gathered}$$

Lemma 3.3. *Let $A(t)$ be a family of closed linear operators which satisfies ATC. Then there exists a unique evolution family $\{U(t,s)\}_{-\infty<s\le t<+\infty}$ on $L^p(\mathcal{P},\mathcal{E})$, which governs the linear part of Eq (1).*

We shall use the following assumptions:

- (A3) The family of operators $A(t)$ on $L^p(\mathcal{P},\mathcal{E})$ satisfies ATC, and the evolution family associated with $A(t)$ is exponentially stable, that is, there exists two numbers $M,\kappa>0$ such that
$$\|U(t,s)\|\le Me^{-\kappa(t-s)},\quad\text{for all } t,s\in\mathbb{R},\ \text{such that } t\ge s.$$
- (A4) $\mathcal{R}(w,A(\cdot))\in AP(\mathbb{R},\mathcal{L}(L^p(\mathcal{P},\mathcal{E})))$, for w in Definition 3.2.

(A5) The processes $F_i : \mathbb{R}\times L^p(\mathcal{P},\mathcal{E}) \to L^p(\mathcal{P},\mathcal{E})$ $(i=1,2)$ and $G:\mathbb{R}\times L^p(\mathcal{P},\mathcal{E}) \to L^p(\mathcal{P},\mathbb{L}_2^0)$ are p-mean (μ_1,μ_2)-pseudo almost periodic in $t\in\mathbb{R}$ for any $Y\in L^p(\mathcal{P},\mathcal{E})$. Moreover, F_1, F_2, and G are Lipschitz in the following sense : there exists $L_i>0$ $(i=1,2,3)$ such that
$$\mathbb{E}\|F_i(t,Y_1)-F_i(t,Y_2)\|^p \le L_i.\mathbb{E}\|Y_1-Y_2\|^p, \quad i=1,2,$$
$$\mathbb{E}\|G(t,Y_1)-G(t,Y_2)\|^p_{\mathbb{L}_2^0} \le L_3.\mathbb{E}\|Y_1-Y_2\|^p,$$
for all stochastic processes $Y_1,Y_2\in L^p(\mathcal{P},\mathcal{E})$ and $t\in\mathbb{R}$.

The next Lemma, which can be seen as an immediate consequence of [28, Proposition 4.4] is essential to study the existence of p-mean (μ_1,μ_2)-s.p.a.p. mild solutions.

Lemma 3.4. *Suppose that* (A3), (A4) *hold. Then, for any $\epsilon>0$ and $h>0$, there exists $l=l(\epsilon)>0$ such that every interval of length l contains at least a number r satisfying*
$$\|U(t+r,s+r)-U(t,s)\| \le \epsilon e^{-\frac{\kappa}{2}(t-s)}, \quad \text{for all } t-s\ge h.$$

Lemma 3.5. *Let $\mu_1,\mu_2\in\mathfrak{N}$ satisfy* (A1)–(A4) *hold. Furthermore, assume that $Z\in\mathcal{SPAP}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$. Then the function*
$$\Gamma: t\mapsto \int_{-\infty}^t U(t,s)Z(s)ds,$$
belongs to $\mathcal{SPAP}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$.

Proof. For $Z\in\mathcal{SPAP}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$, there exist $Z^a\in\mathcal{SAP}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}))$ and $Z^e\in\mathcal{SO}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$, such that $Z=Z^a+Z^e$. Consequently, we can write
$$\begin{aligned}\Gamma(t)&=\Gamma_1(t)+\Gamma_2(t)\\ &=\int_{-\infty}^t U(t,s)Z^a(s)ds+\int_{-\infty}^t U(t,s)Z^e(s)ds.\end{aligned}$$
First, we show that $\Gamma_1\in\mathcal{SAP}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}))$. In fact, since $Z^a\in\mathcal{SAP}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}))$, so according to Lemma 3.4, for a given $\epsilon>0$, one can find $l(\epsilon)>0$ such that for any interval of length $l(\epsilon)$ contains at least a number r with the property that
$$\|U(t+r,s+r)-U(t,s)\|\le\epsilon e^{-\frac{\kappa}{2}(t-s)}, \quad \text{for all } t-s\ge\epsilon, \tag{3.2}$$
and
$$\mathbb{E}\|Z^a(t+r)-Z^a(t)\|^p<\eta, \quad \text{for all } t\in\mathbb{R}, \tag{3.3}$$
where $\eta=\eta(\epsilon)\to 0$ as $\epsilon\to 0$. By using (3.2), (3.3), Hölder's inequality, and that $|x+y+z|^p\le 3^{p-1}(|z|^p+|y|^p+|z|^p)$, it follows that
$$\begin{aligned}&\mathbb{E}\|\Gamma_1(t+r)-\Gamma_1(t)\|^p\\ &=\mathbb{E}\Big\|\int_{-\infty}^{t+r}U(t+r,s)Z^a(s)ds-\int_{-\infty}^t U(t,s)Z^a(s)ds\Big\|^p\\ &=\mathbb{E}\Big\|\int_0^{+\infty}U(t+r,t+r-s)Z^a(t+r-s)ds-\int_0^{+\infty}U(t,t-s)Z^a(t-s)ds\Big\|^p\\ &\le 3^{p-1}\mathbb{E}\Big[\int_0^{+\infty}\|U(t+r,t+r-s)\|\cdot\|Z^a(t+r-s)-Z^a(t-s)\|ds\Big]^p\end{aligned}$$

$$
\begin{aligned}
&+ 3^{p-1}\mathbb{E}\Big[\int_0^{\epsilon} \|U(t+r,t+r-s) - U(t,t-s))\| \cdot \|Z^a(t-s)\|ds\Big]^p \\
&+ 3^{p-1}\mathbb{E}\Big[\int_{\epsilon}^{+\infty} \|U(t+r,t+r-s) - U(t,t-s))\| \cdot \|Z^a(t-s)\|ds\Big]^p \\
&\le 3^{p-1}M^p\mathbb{E}\Big[\int_0^{+\infty} e^{-\kappa s}\|Z^a(t+r-s) - Z^a(t-s)\|ds\Big]^p \\
&+ 3^{p-1}M^p\mathbb{E}\Big[\int_0^{\epsilon} 2e^{-\kappa s}\|Z^a(t-s)\|ds\Big]^p \\
&+ 3^{p-1}\epsilon^p\mathbb{E}\Big[\int_{\epsilon}^{+\infty} e^{-\frac{\kappa}{2}s}\|Z^a(t-s)\|ds\Big]^p \\
&\le 3^{p-1}M^p\Big[\int_0^{+\infty} e^{-\kappa s}ds\Big]^{p-1}\int_0^{+\infty} e^{-\kappa s}\mathbb{E}\|Z^a(t+r-s) - Z^a(t-s)\|^p ds \\
&+ 3^{p-1}2^pM^p\Big[\int_0^{\epsilon} e^{-\kappa s}ds\Big]^{p-1}\int_0^{\epsilon} e^{-\kappa s}\mathbb{E}\|X^a(t-s)\|^p ds \\
&+ 3^{p-1}\epsilon^p\Big[\int_{\epsilon}^{+\infty} e^{-\frac{\kappa}{2}s}ds\Big]^{p-1}\int_{\epsilon}^{+\infty} e^{-\frac{\kappa}{2}s}\mathbb{E}\|Z^a(t-s)\|^p ds \\
&\le 3^{p-1}M^p\Big[\int_0^{+\infty} e^{-\kappa s}ds\Big]^{p}\sup_{s\in\mathbb{R}}\mathbb{E}\|Z^a(t+r-s) - Z^a(t-s)\|^p \\
&+ 3^{p-1}2^pM^p\Big[\int_0^{\epsilon} e^{-\kappa s}ds\Big]^{p}\sup_{s\in\mathbb{R}}\mathbb{E}\|Z^a(t-s)\|^p \\
&+ 3^{p-1}\epsilon^p\Big[\int_{\epsilon}^{+\infty} e^{-\frac{\kappa}{2}s}ds\Big]^{p}\sup_{s\in\mathbb{R}}\mathbb{E}\|Z^a(t-s)\|^p \\
&\le 3^{p-1}M^p(1/\kappa)^p\eta(\epsilon) + 3^{p-1}2^pM^p\|Z\|_\infty^p\epsilon^p + 3^{p-1}(2/\kappa)^p\|Z\|_\infty^p\epsilon^p,
\end{aligned}
$$

which implies that $\Gamma_1 \in \mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}))$.

Next, we check that $\Gamma_2 \in \mathcal{SO}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$, that is

$$
\lim_{m\to+\infty}\frac{1}{\mu_2([-m,m])}\int_{-m}^{m}\mathbb{E}\|\Gamma_2(t)\|^p d\mu_1(t) = 0.
$$

Let us denote $A := \{s : s \le t\}$ and $B := \{v : v \ge 0\}$. Applying Hölder's inequality and Fubini's theorem, for $m > 0$, we obtain

$$
\begin{aligned}
&\frac{1}{\mu_2([-m,m])}\int_{-m}^{m}\mathbb{E}\|\Gamma_2(t)\|^p d\mu_1(t) \\
&= \frac{1}{\mu_2([-m,m])}\int_{-m}^{m}\mathbb{E}\Big\|\int_{-\infty}^{t} U(t,s)Z^e(s)ds\Big\|^p d\mu_1(t) \\
&\le \frac{1}{\mu_2([-m,m])}\int_{-m}^{m}\mathbb{E}\Big[\int_{-\infty}^{t} \|U(t,s)\|\|Z^e(s)\|ds\Big]^p d\mu_1(t) \\
&\le \frac{M^p}{\mu_2([-m,m])}\int_{-m}^{m}\Big\{\Big[\int_{-\infty}^{t} e^{-\kappa(t-s)}ds\Big]^{p-1}\int_{-\infty}^{t} e^{-\kappa(t-s)}\mathbb{E}\|Z^e(s)\|^p ds\Big\}d\mu_1(t) \\
&\le \frac{M^p}{\kappa^{p-1}}\frac{1}{\mu_2([-m,m])}\int_{-m}^{m}\Big\{\int_{-\infty}^{+\infty} e^{-\kappa(t-s)}\mathbb{E}\|Z^e(s)\|^p\chi_A ds\Big\}d\mu_1(t) \\
&\le \frac{M^p}{\kappa^{p-1}}\frac{1}{\mu_2([-m,m])}\int_{-\infty}^{\infty}\Big\{\int_{-m}^{+m} e^{-\kappa(t-s)}\mathbb{E}\|Z^e(s)\|^p\chi_A d\mu_1(t)\Big\}ds.
\end{aligned}
$$

By making change of variables $v = t - s$, it follows that

$$
\begin{aligned}
&\frac{1}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\|\Gamma_2(t)\|^p d\mu_1(t) \\
&\le \frac{M^p}{\kappa^{p-1}} \frac{1}{\mu_2([-m,m])} \int_{+\infty}^{-\infty} \Big\{ \int_{-m}^{+m} e^{-\kappa v} \mathbb{E}\|Z^e(t-v)\|^p \chi_B d\mu_1(t) \Big\} (-dv) \\
&\le \frac{M^p}{\kappa^{p-1}} \frac{1}{\mu_2([-m,m])} \int_{0}^{+\infty} \Big\{ \int_{-m}^{+m} e^{-\kappa v} \mathbb{E}\|Z^e(t-v)\|^p d\mu_1(t) \Big\} dv \\
&\le \frac{M^p}{\kappa^{p-1}} \int_{0}^{+\infty} \Big\{ \frac{e^{-\kappa v}}{\mu_2([-m,m])} \int_{-m}^{+m} \mathbb{E}\|Z^e(t-v)\|^p d\mu_1(t) \Big\} dv.
\end{aligned}
$$

One can see that

$$
\Big| \frac{e^{-\kappa v}}{\mu_2([-m,m])} \int_{-m}^{+m} \mathbb{E}\|Z^e(t-v)\|^p d\mu_1(t) \Big| \le e^{-\kappa v} \|X^e\|_\infty^p \frac{\mu_1([-m,m])}{\mu_2([-m,m])},
$$

for all $v \ge 0$. Since μ_1 and μ_2 satisfy (A2), from Theorem 2.9, we have

$$
[t \mapsto X^e(t-v)] \in \mathcal{SO}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}\right).
$$

Then, by using (A2) and the Lebesgue dominate convergence theorem, we obtain

$$
\lim_{m \to +\infty} \frac{M^p}{\kappa^{p-1}} \int_{0}^{+\infty} \Big\{ \frac{e^{-\kappa v}}{\mu_2([-m,m])} \int_{-m}^{+m} \mathbb{E}\|Z^e(t-v)\|^p d\mu_1(t) \Big\} dv = 0,
$$

which implies that

$$
\lim_{m \to +\infty} \frac{1}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\|\Gamma_2(t)\|^p d\mu_1(t).
$$

Finally, we obtain $\Gamma = \Gamma_1 + \Gamma_2 \in \mathcal{SPAP}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}\right)$. □

Lemma 3.6. *Let $\mu_1, \mu_2 \in \mathfrak{N}$ satisfy* (A1), (A2) *and $Z \in \mathcal{SPAP}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}\right)$. Then the function*

$$
\Lambda : t \mapsto \int_{-\infty}^{t} Q(t-\zeta) Z(\zeta) d\zeta,
$$

belongs to $\mathcal{SPAP}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}\right)$.

Proof. For $Z \in \mathcal{SPAP}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}\right)$, there exist $Z^a \in \mathcal{SAP}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})\right)$ and $Z^e \in \mathcal{SO}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2}\right)$, such that $Z = Z^a + Z^e$. Hence, we can write

$$
\begin{aligned}
\Lambda(t) &= \Lambda_1(t) + \Lambda_2(t) \\
&= \int_{-\infty}^{t} Q(t-\zeta) Z^a(\zeta) d\zeta + \int_{-\infty}^{t} Q(t-\zeta) Z^e(\zeta) d\zeta \\
&= \int_{0}^{+\infty} Q(\zeta) Z^a(t-\zeta) d\zeta + \int_{0}^{+\infty} Q(\zeta) Z^e(t-\zeta) d\zeta.
\end{aligned}
$$

First, let us show that $\Lambda_1 \in \mathcal{SAP}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})\right)$. Since $Z^a \in \mathcal{SAP}\left(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E})\right)$, so for any given $\epsilon > 0$, one can find $l(\epsilon) > 0$ such that for any interval of length $l(\epsilon)$ contains at least a number r such that

$$
\mathbb{E}\|Z^a(t+r) - Z^a(t)\|^p < \frac{\epsilon}{\|Q\|_{L^1(0,+\infty)}^p}, \quad \text{for all } t \in \mathbb{R}.
$$

Now by using Hölder's inequality, we obtain

$$
\mathbb{E}\|\Lambda_1(t+r) - \Lambda_1(t)\|^p
$$

$$
\begin{aligned}
&= \mathbb{E}\Big\| \int_0^{+\infty} Q(\zeta)Z^a(t+r-\zeta)d\zeta - \int_0^{+\infty} Q(\zeta)Z^a(t-\zeta)d\zeta\Big\|^p \\
&\leq \mathbb{E}\Big[\int_0^{+\infty} \|Q(\zeta)\|\|Z^a(t+r-\zeta) - Z^a(t-\zeta)\|d\zeta\Big]^p \\
&\leq \Big[\int_0^{+\infty} \|Q(\zeta)\|d\zeta\Big]^{p-1} \int_0^{+\infty} \|Q(\zeta)\|\mathbb{E}\|Z^a(t+r-\zeta) - Z^a(t-\zeta)\|^p d\zeta \\
&\leq \|Q\|^p_{L^1(0,+\infty)} \sup_{\zeta\in\mathbb{R}} \mathbb{E}\|Z^a(t+r-\zeta) - Z^a(t-\zeta)\|^p \leq \epsilon,
\end{aligned}
$$

which implies that $\Lambda_1 \in \mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}))$.

Next, we check that $\Lambda_2 \in \mathcal{SO}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}), \mu_{1,2})$; that is

$$
\lim_{m\to+\infty} \frac{1}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\|\Lambda_2(t)\|^p d\mu_1(t) = 0.
$$

By using Hölder's inequality and Fubini's theorem, for $m > 0$, we obtain

$$
\begin{aligned}
&\frac{1}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\|\Lambda_2(t)\|^p d\mu_1(t) \\
&= \frac{1}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\ big\| \int_0^{+\infty} Q(\zeta)Z^e(t-\zeta)d\zeta\Big\|^p d\mu_1(t) \\
&\leq \frac{1}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\Big[\int_0^{+\infty} \|Q(\zeta)\|\cdot\|Z^e(t-\zeta)\|d\zeta\Big]^p d\mu_1(t) \\
&\leq \frac{1}{\mu_2([-m,m])} \int_{-m}^{m} \Big\{\Big[\int_0^{+\infty} \|Q(\zeta)\|d\zeta\Big]^{p-1} \int_0^{+\infty} \|Q(\zeta)\|\mathbb{E}\|Z^e(t-\zeta)\|^p d\zeta\Big\} d\mu_1(t) \\
&\leq \|Q\|^{p-1}_{L^1(0,+\infty)} \int_0^{+\infty} \Big\{ \frac{\|Q(\zeta)\|}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\|Z^e(t-\zeta)\|^p d\mu_1(t)\Big\} d\zeta.
\end{aligned}
$$

Since

$$
\Big| \frac{\|Q(\zeta)\|}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\|X^e(t-\zeta)\|^p d\mu_1(t)\Big| \leq \|Q(\zeta)\|\|Z^e\|^p_\infty \frac{\mu_1([-m,m])}{\mu_2([-m,m])},
$$

for all $\zeta \geq 0$, by using the Lebesgue dominate convergence theorem, (A1), and that the space $\mathcal{SO}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}), \mu_{1,2})$ is invariant by translation, we obtain

$$
\lim_{m\to+\infty} \int_0^{+\infty} \Big\{ \frac{\|Q(\zeta)\|}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\|Z^e(t-\zeta)\|^p d\mu_1(t)\Big\} d\zeta = 0,
$$

which implies that

$$
\lim_{m\to+\infty} \frac{1}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\|\Lambda_2(t)\|^p d\mu_1(t).
$$

Finally, we obtain $\Lambda = \Lambda_1 + \Lambda_2 \in \mathcal{SPAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}), \mu_{1,2})$. □

Lemma 3.7. *Let $\mu_1, \mu_2 \in \mathfrak{N}$ satisfy* (A1), (A2) *and $Z \in \mathcal{SPAP}\big(\mathbb{R}, L^p(\mathcal{P}, \mathbb{L}_2^0), \mu_{1,2}\big)$. Then the function*

$$
\Delta : t \mapsto \int_{-\infty}^{t} R(t-\zeta)Z(\zeta)d\mathcal{W}(\zeta),
$$

belongs to $\mathcal{SPAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}), \mu_{1,2})$.

Proof. Let us assume that $Z \in \mathcal{SPAP}\big(\mathbb{R}, L^p(\mathcal{P},\mathbb{L}_2^0),\mu_{1,2}\big)$. Then, there exist $Z^a \in \mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P},\mathbb{L}_2^0))$ and $Z^e \in \mathcal{SO}(\mathbb{R}, L^p(\mathcal{P},\mathbb{L}_2^0),\mu_{1,2})$, such that $Z = Z^a + Z^e$. Hence, we can write

$$\begin{aligned}\Delta(t) &= \Delta_1(t) + \Delta_2(t)\\ &= \int_{-\infty}^t R(t-\zeta)Z^a(\zeta)d\mathcal{W}(\zeta) + \int_{-\infty}^t R(t-\zeta)Z^e(\zeta)d\mathcal{W}(\zeta).\end{aligned}$$

First, let us show that $\Delta_1 \in \mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}))$. Since $Z^a \in \mathcal{SAP}\big(\mathbb{R}, L^p(\mathcal{P},\mathbb{L}_2^0)\big)$, for given $\epsilon > 0$, one can find $l(\epsilon) > 0$ such that for any interval of length $l(\epsilon)$ contains at least r with the property that

$$\mathbb{E}\|Z^a(t+r) - Z^a(t)\|_{\mathbb{L}_2^0}^p < \frac{\epsilon}{C_p.\|R\|_{L^2(0,+\infty)}^p}, \quad \text{for all } t \in \mathbb{R}.$$

Let $\overline{\mathcal{W}}(\zeta) = \mathcal{W}(\zeta + r) - \mathcal{W}(r)$ for each $\zeta \in \mathbb{R}$. Then, $\overline{\mathcal{W}}$ is also a Brownian motion and has the same distribution as $\mathcal{W}$. By making change of variable $v = \zeta - r$, and using Lemma 2.1, we obtain

$$\begin{aligned}&\mathbb{E}\|\Delta_1(t+r) - \Delta_1(t)\|^p\\ &= \mathbb{E}\Big\|\int_{-\infty}^{t+r} R(t+r-\zeta)X^a(\zeta)d\mathcal{W}(\zeta) - \int_{-\infty}^t R(t-\zeta)Z^a(\zeta)d\mathcal{W}(\zeta)\Big\|^p\\ &= \mathbb{E}\Big\|\int_{-\infty}^{t} R(t-v)X^a(v+r)d\overline{\mathcal{W}}(v) - \int_{-\infty}^t R(t-v)Z^a(v)d\overline{\mathcal{W}}(v)\Big\|^p\\ &= \mathbb{E}\Big\|\int_{-\infty}^{t} R(t-v)\Big[Z^a(v+r) - Z^a(v)\Big]d\overline{\mathcal{W}}(v)\Big\|^p\\ &\le C_p\mathbb{E}\Big[\int_{-\infty}^t \|R(t-v)\|^2\|Z^a(v+r) - Z^a(v)\|_{\mathbb{L}_2^0}^2 dv\Big]^{p/2}\\ &\le C_p\mathbb{E}\Big[\int_{-\infty}^t \|R(t-v)\|^{2\frac{2}{q}}\|R(t-v)\|^{2\frac{2}{p}}\|Z^a(v+r) - Z^a(v)\|_{\mathbb{L}_2^0}^2 dv\Big]^{p/2},\end{aligned}$$

where $q > 0$ solves $\frac{1}{p/2} + \frac{1}{q/2} = \frac{2}{p} + \frac{2}{q} = 1$. Then, from Hölder's inequality, we obtain

$$\begin{aligned}&\mathbb{E}\|\Delta_1(t+r) - \Delta_1(t)\|^p\\ &\le C_p\Big[\int_{-\infty}^t \|R(t-v)\|^2 dv\Big]^{\frac{p-2}{2}}\int_{-\infty}^t \|R(t-v)\|^2\mathbb{E}\|Z^a(v+r) - Z^a(v)\|_{\mathbb{L}_2^0}^p dv\\ &\le C_p\|R\|_{L^2(0,+\infty)}^p \sup_{v\in\mathbb{R}}\mathbb{E}\|Z^a(v+r) - Z^a(v)\|_{\mathbb{L}_2^0}^p \le \epsilon.\end{aligned}$$

Which implies that $\Delta_1 \in \mathcal{SAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}))$.

Next, we check that $\Delta_2 \in \mathcal{SO}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$. Let us denote $A := \{\zeta : \zeta \le t\}$ and $B := \{v : v \ge 0\}$. By using Hölder's inequality and Fubini's theorem, we obtain, for $m > 0$;

$$\begin{aligned}&\frac{1}{\mu_2([-m,m])}\int_{-m}^m \mathbb{E}\|\Delta_2(t)\|^p d\mu_1(t)\\ &= \frac{1}{\mu_2([-m,m])}\int_{-m}^m \mathbb{E}\Big\|\int_{-\infty}^t R(t-\zeta)Z^e(\zeta)d\mathcal{W}(\zeta)\Big\|^p d\mu_1(t)\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\mu_2([-m,m])} C_p \int_{-m}^{m} \mathbb{E}\Big[\int_{-\infty}^{t} \|R(t-\zeta)\|^2 \|Z^e(\zeta)\|^2_{\mathbb{L}_2^0} d\zeta\Big]^{p/2} d\mu_1(t) \\
&\le \frac{1}{\mu_2([-m,m])} C_p \int_{-m}^{m} \Big\{\Big[\int_{-\infty}^{t} \|R(t-\zeta)\|^2 d\zeta\Big]^{\frac{p-2}{2}} \\
&\quad \times \int_{-\infty}^{t} \|R(t-\zeta)\|^2 \mathbb{E}\|Z^e(\zeta)\|^p_{\mathbb{L}_2^0} d\zeta\Big\} d\mu_1(t) \\
&\le \frac{1}{\mu_2([-m,m])} C_p \|R\|^{p-2}_{L^2(0,+\infty)} \int_{-m}^{m} \Big\{\int_{-\infty}^{+\infty} \|R(t-\zeta)\|^2 \mathbb{E}\|Z^e(\zeta)\|^p_{\mathbb{L}_2^0} \chi_A d\zeta\Big\} d\mu_1(t) \\
&\le \frac{1}{\mu_2([-m,m])} C_p \|R\|^{p-2}_{L^2(0,+\infty)} \int_{-\infty}^{+\infty} \Big\{\int_{-m}^{m} \|R(t-\zeta)\|^2 \mathbb{E}\|Z^e(\zeta)\|^p_{\mathbb{L}_2^0} \chi_A d\mu_1(t)\Big\} d\zeta.
\end{aligned}
$$

By making change of variables $v = t - \zeta$, we obtain

$$
\begin{aligned}
&\frac{1}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\|\Delta_2(t)\|^p d\mu_1(t) \\
&\le \frac{1}{\mu_2([-m,m])} C_p \|R\|^{p-2}_{L^2(0,+\infty)} \int_{-\infty}^{+\infty} \Big\{\int_{-m}^{m} \|R(v)\|^2 \mathbb{E}\|Z^e(t-v)\|^p_{\mathbb{L}_2^0} \chi_B d\mu_1(t)\Big\} dv \\
&= C_p \|R\|^{p-2}_{L^2(0,+\infty)} \int_{0}^{+\infty} \Big\{\frac{\|R(v)\|^2}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\|Z^e(t-v)\|^p_{\mathbb{L}_2^0} d\mu_1(t)\Big\} dv.
\end{aligned}
$$

Since, for all $v \ge 0$

$$
\Big|\frac{\|R(v)\|^2}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\|X^e(t-v)\|^p_{\mathbb{L}_2^0} d\mu_1(t)\Big| \le \|R(v)\|^2 \|Z^e\|^p_\infty \frac{\mu_1([-m,m])}{\mu_2([-m,m])},
$$

by using the Lebesgue dominate convergence theorem, (A1), and that the space $\mathcal{SO}(\mathbb{R}, L^p(\mathcal{P}, \mathbb{L}_2^0), \mu_{1,2})$ is invariant by translation, we obtain

$$
\lim_{m\to+\infty} \int_{0}^{+\infty} \Big\{\frac{\|R(v)\|^2}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\|Z^e(t-v)\|^p_{\mathbb{L}_2^0} d\mu_1(t)\Big\} dv = 0,
$$

which implies that

$$
\lim_{m\to+\infty} \frac{1}{\mu_2([-m,m])} \int_{-m}^{m} \mathbb{E}\|\Delta_2(t)\|^p d\mu_1(t).
$$

Finally $\Delta = \Delta_1 + \Delta_2 \in \mathcal{SPAP}\,(\mathbb{R}, L^p(\mathcal{P}, \mathcal{E}), \mu_{1,2})$. □

Theorem 3.8. *Let $\mu_1, \mu_2 \in \mathfrak{N}$ satisfy* (A1), (A2). *Suppose that* (A3)–(A5) *hold. Then* (1.1) *has a unique p-mean (μ_1, μ_2)-s.p.a.p. mild solution, which can be explicitly expressed as*

$$
\begin{aligned}
Z(t) &= \int_{-\infty}^{t} U(t,s) F_1(s, Z(s)) ds + \int_{-\infty}^{t} U(t,s) \int_{-\infty}^{s} Q(s-\zeta) F_2(\zeta, Z(\zeta)) d\zeta ds \\
&\quad + \int_{-\infty}^{t} U(t,s) \int_{-\infty}^{s} R(s-\zeta) G(\zeta, Z(\zeta)) d\mathcal{W}(\zeta) ds, \quad t \in \mathbb{R},
\end{aligned}
$$

whenever

$$
\Theta_p := M^p (1/\kappa)^p \Big[L_1 + L_2 \|Q\|^p_{L^1(0,+\infty)} + C_p L_3 \|R\|^p_{L^2(0,+\infty)}\Big] < (1/3)^{p-1}, \quad (3.4)
$$

for $p > 2$, and

$$
\Theta_2 := M^2 (1/\kappa)^2 \Big[L_1 + L_2 \|Q\|^2_{L^1(0,+\infty)} + L_3 \|R\|^2_{L^2(0,+\infty)}\Big] < 1/3, \quad (3.5)
$$

for $p=2$.

Proof. First of all, it is not difficult to see that the stochastic processes

$$Z(t)=\int_{-\infty}^{t}U(t,s)F_1(s,Z(s))ds+\int_{-\infty}^{t}U(t,s)\int_{-\infty}^{s}Q(s-\zeta)F_2(\zeta,Z(\zeta))d\zeta ds \\ +\int_{-\infty}^{t}U(t,s)\int_{-\infty}^{s}R(s-\zeta)G(\zeta,Z(\zeta))d\mathcal{W}(\zeta)ds$$

is well defined and satisfies (3.1). Hence it is a mild solution of (1.1). Let us consider a nonlinear operator Γ given by

$$(\Gamma Z)(t)=\int_{-\infty}^{t}U(t,s)F_1(s,Z(s))ds+\int_{-\infty}^{t}U(t,s)\int_{-\infty}^{s}Q(s-\zeta)F_2(\zeta,Z(\zeta))d\zeta ds \\ +\int_{-\infty}^{t}U(t,s)\int_{-\infty}^{s}R(s-\zeta)G(\zeta,Z(\zeta))d\mathcal{W}(\zeta)ds.$$

We prove that Γ is a self mapping from $\mathcal{SPAP}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$ to itself and it is a contraction. Hence, according to the Banach contraction principle, we can conclude that Γ has a unique fixed point $Z^*\in\mathcal{SPAP}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$, that is $\Gamma Z^*=Z^*$ which satisfies (3.1).

Let us first show that Γ is a self mapping. Let $Z\in\mathcal{SPAP}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$. Then, using (A5) and Theorem 2.15, we can easily see that $G(\cdot,Z(\cdot))$ belongs to $\mathcal{SPAP}\big(\mathbb{R},L^p(\mathcal{P},\mathbb{L}_2^0),\mu_{1,2}\big)$, and $F_1(\cdot,Z(\cdot))$, $F_2(\cdot,Z(\cdot))\in\mathcal{SPAP}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$. Consequently, in view of Lemmas 3.5, 3.6 and 3.7, we can conclude that $\Gamma Z\in\mathcal{SPAP}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$.

Now, to complete the proof, we have to check that Γ is a strict contraction mapping on $\mathcal{SPAP}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$. In fact, for each $t\in\mathbb{R}$, and $Z_1,Z_2\in\mathcal{SPAP}(\mathbb{R},L^p(\mathcal{P},\mathcal{E}),\mu_{1,2})$, we have

$$\begin{aligned}
&\mathbb{E}\|(\Gamma Z_1)(t)-(\Gamma Z_2)(t)\|^p\le 3^{p-1}\mathbb{E}\|\int_{-\infty}^{t}U(t,s)\left[F_1(s,Z_1(s))-F_1(s,Z_2(s))\right]ds\|^p\\
&\quad+3^{p-1}\mathbb{E}\|\int_{-\infty}^{t}U(t,s)\int_{-\infty}^{s}Q(s-\zeta)\left[F_2(\zeta,Z_1(\zeta))-F_2(\zeta,Z_2(\zeta))\right]d\zeta ds\|^p\\
&\quad+3^{p-1}\mathbb{E}\|\int_{-\infty}^{t}U(t,s)\int_{-\infty}^{s}R(s-\zeta)\left[G(\zeta,Z_1(\zeta))-G(\zeta,Z_2(\zeta))\right]d\mathcal{W}(\zeta)ds\|^p\\
&\le 3^{p-1}\mathbb{E}\Big[\int_{-\infty}^{t}\|U(t,s)\|\|F_1(s,Z_1(s))-F_1(s,Z_2(s))\|ds\Big]^p\\
&\quad+3^{p-1}\mathbb{E}\Big[\int_{-\infty}^{t}\|U(t,s)\|\|\int_{-\infty}^{s}Q(s-\zeta)[F_2(\zeta,Z_1(\zeta))-F_2(\zeta,Z_2(\zeta))]d\zeta\|ds\Big]^p\\
&\quad+3^{p-1}\mathbb{E}\Big[\int_{-\infty}^{t}\|U(t,s)\|\|\int_{-\infty}^{s}R(s-\zeta)[G(\zeta,Z_1(\zeta))-G(\zeta,Z_2(\zeta))]d\mathcal{W}(\zeta)ds\|\Big]^p.
\end{aligned}$$

Now, we evaluate the first term of the right-hand side with the help of Hölder's inequality as follows

$$\begin{aligned}
&3^{p-1}\mathbb{E}\Big[\int_{-\infty}^{t}\|U(t,s)\|\|F_1(s,Z_1(s))-F_1(s,Z_2(s))\|ds\Big]^p\\
&\le 3^{p-1}M^p\mathbb{E}\Big[\int_{-\infty}^{t}e^{-\kappa(t-s)}\|F_1(s,Z_1(s))-F_1(s,Z_2(s))\|ds\Big]^p
\end{aligned}$$

$$
\begin{aligned}
&\le 3^{p-1}M^p\Big[\int_{-\infty}^{t} e^{-\kappa(t-s)}ds\Big]^{p-1}\int_{-\infty}^{t} e^{-\kappa(t-s)}\mathbb{E}\|F_1(s,Z_1(s))-F_1(s,Z_2(s))\|^p ds\\
&\le 3^{p-1}M^pL_1\Big[\int_{-\infty}^{t} e^{-\kappa(t-s)}ds\Big]^{p-1}\int_{-\infty}^{t} e^{-\kappa(t-s)}\mathbb{E}\|Z_1(s)-Z_2(s)\|^p ds\\
&\le 3^{p-1}M^pL_1\Big[\int_{-\infty}^{t} e^{-\kappa(t-s)}ds\Big]^{p-1}\Big[\int_{-\infty}^{t} e^{-\kappa(t-s)}ds\Big]\sup_{s\in\mathbb{R}}\mathbb{E}\|Z_1(s)-Z_2(s)\|^p\\
&\le 3^{p-1}M^pL_1(1/\kappa)^p\|Z_1-Z_2\|_\infty^p.
\end{aligned}
$$

Furthermore, by using the Hölder's inequality for the second term, we have

$$
\begin{aligned}
&3^{p-1}\mathbb{E}\Big[\int_{-\infty}^{t}\|U(t,s)\|\|\int_{-\infty}^{s} Q(s-\zeta)[F_2(\zeta,Z_1(\zeta))-F_2(\zeta,Z_2(\zeta))]d\zeta\|ds\Big]^p\\
&\le 3^{p-1}M^p\mathbb{E}\Big[\int_{-\infty}^{t} e^{-\kappa(t-s)}\|\int_{-\infty}^{s} Q(s-\zeta)[F_2(\zeta,Z_1(\zeta))-F_2(\zeta,Z_2(\zeta))]d\zeta\|ds\Big]^p\\
&\le 3^{p-1}M^p\Big[\int_{-\infty}^{t} e^{-\kappa(t-s)}ds\Big]^{p-1}\int_{-\infty}^{t} e^{-\kappa(t-s)}\mathbb{E}\|\int_{-\infty}^{s} Q(s-\zeta)[F_2(\zeta,Z_1(\zeta))\\
&\quad -F_2(\zeta,Z_2(\zeta))]d\zeta\|^p ds\\
&\le 3^{p-1}M^p(1/\kappa)^{p-1}\int_{-\infty}^{t} e^{-\kappa(t-s)}\mathbb{E}\Big[\int_{-\infty}^{s}\|Q(s-\zeta)\|\,\|F_2(\zeta,Z_1(\zeta))\\
&\quad -F_2(\zeta,Z_2(\zeta))\|d\zeta\Big]^p ds\\
&\le 3^{p-1}M^p(1/\kappa)^{p-1}\int_{-\infty}^{t} e^{-\kappa(t-s)}\Big[\int_{-\infty}^{s}\|Q(s-\zeta)\|d\zeta\Big]^{p-1}\int_{-\infty}^{s}\|Q(s-\zeta)\|\\
&\quad\times\mathbb{E}\|F_2(\zeta,Z_1(\zeta))-F_2(\zeta,Z_2(\zeta))\|^p d\zeta ds\\
&\le 3^{p-1}M^pL_2(1/\kappa)^{p-1}\int_{-\infty}^{t} e^{-\kappa(t-s)}\Big[\int_{-\infty}^{s}\|Q(s-\zeta)\|d\zeta\Big]^{p-1}\int_{-\infty}^{s}\|Q(s-\zeta)\|\\
&\times\mathbb{E}\|Z_1(\zeta)-Z_2(\zeta)\|^p d\zeta ds\\
&\le 3^{p-1}M^pL_2(1/\kappa)^{p-1}\|Q\|_{L^1(0,+\infty)}^p\Big[\int_{-\infty}^{t} e^{-\kappa(t-s)}ds\Big]\sup_{\zeta\in\mathbb{R}}\mathbb{E}\|Z_1(\zeta)-Z_2(\zeta)\|^p\\
&\le 3^{p-1}M^pL_2(1/\kappa)^p\|Q\|_{L^1(0,+\infty)}^p\|Z_1-Z_2\|_\infty^p.
\end{aligned}
$$

For the third term, we use Hölder's inequality and Lemma 2.1 to obtain

$$
\begin{aligned}
&3^{p-1}\mathbb{E}\Big[\int_{-\infty}^{t}\|U(t,s)\|\|\int_{-\infty}^{s} R(s-\zeta)[G(\zeta,Z_1(\zeta))\\
&\quad -G(\zeta,Z_2(\zeta))]d\mathcal{W}(\zeta)\|ds\Big]^p\\
&\le 3^{p-1}M^p\mathbb{E}\Big[\int_{-\infty}^{t} e^{-\kappa(t-s)}\|\int_{-\infty}^{s} R(s-\zeta)[G(\zeta,Z_1(\zeta))-G(\zeta,Z_2(\zeta))]d\mathcal{W}(\zeta)\|ds\Big]^p\\
&\le 3^{p-1}M^p\Big[\int_{-\infty}^{t} e^{-\kappa(t-s)}ds\Big]^{p-1}\int_{-\infty}^{t} e^{-\kappa(t-s)}\mathbb{E}\|\int_{-\infty}^{s} R(s-\zeta)[G(\zeta,Z_1(\zeta))\\
&\quad -G(\zeta,Z_2(\zeta))]d\mathcal{W}(\zeta)\|^p ds\\
&\le 3^{p-1}M^pC_p(1/\kappa)^{p-1}\int_{-\infty}^{t} e^{-\kappa(t-s)}\mathbb{E}\Big[\int_{-\infty}^{s}\|R(s-\zeta)\|^2\|G(\zeta,Z_1(\zeta))
\end{aligned}
$$

$$
\begin{aligned}
&- G(\zeta, Z_2(\zeta))\|^2_{\mathbb{L}_2^0} d\zeta\Big]^{p/2} ds \\
&\le 3^{p-1} M^p C_p (1/\kappa)^{p-1} \int_{-\infty}^{t} e^{-\kappa(t-s)} \Big[\int_{-\infty}^{s} \|R(s-\zeta)\|^2 d\zeta\Big]^{\frac{p-2}{2}} \int_{-\infty}^{s} \|R(s-\zeta)\|^2 \\
&\quad \times \mathbb{E}\|G(\zeta, Z_1(\zeta)) - G(\zeta, Z_2(\zeta))\|^p_{\mathbb{L}_2^0} d\zeta ds \\
&\le 3^{p-1} M^p C_p L_3 (1/\kappa)^{p-1} \int_{-\infty}^{t} e^{-\kappa(t-s)} \Big[\int_{-\infty}^{s} \|R(s-\zeta)\|^2 d\zeta\Big]^{\frac{p-2}{2}} \int_{-\infty}^{s} \|R(s-\zeta)\|^2 \\
&\quad \times \mathbb{E}\|Z_1(\zeta) - Z_2(\zeta)\|^p d\zeta ds \\
&\le 3^{p-1} M^p C_p L_3 (1/\kappa)^{p-1} \int_{-\infty}^{t} e^{-\kappa(t-s)} \Big[\int_{-\infty}^{s} \|R(s-\zeta)\|^2 d\zeta\Big]^{\frac{p}{2}} d\zeta ds \\
&\quad \times \sup_{\zeta\in\mathbb{R}} \mathbb{E}\|Z_1(\zeta) - Z_2(\zeta)\|^p \\
&\le 3^{p-1} M^p C_p L_3 (1/\kappa)^{p} \|R\|^p_{L^2(0,+\infty)} \|Z_1 - Z_2\|^p_\infty .
\end{aligned}
$$

Therefore, for each $t \in \mathbb{R}$, we can deduce that

$$
\begin{aligned}
&\mathbb{E}\|(\Gamma Z_1)(t) - (\Gamma Z_2)(t)\|^p \\
&\le 3^{p-1} M^p (1/\kappa)^p \Big[L_1 + L_2\|Q\|^p_{L^1(0,+\infty)} + C_p L_3 \|R\|^p_{L^2(0,+\infty)}\Big] \|Z_1 - Z_1\|^p_\infty .
\end{aligned}
$$

Hence,

$$
\|\Gamma Z_1 - \Gamma Z_2\|^p_\infty \le 3^{p-1}\Theta_p \|Z_1 - Z_2\|^p_\infty .
$$

For the case $p = 2$, by using the same arguments used to prove [27, Theorem 3.3], we obtain

$$
\|\Gamma Z_1 - \Gamma Z_1\|^2_\infty \le 3\Theta_2 \|Z_1 - Z_2\|^2_\infty .
$$

Which implies that Γ is a contraction. Hence, by the Banach contraction principle, we can deduce that Γ has a unique fixed point $Z^* \in \mathcal{SPAP}(\mathbb{R}, L^p(\mathcal{P},\mathcal{E}), \mu_{1,2})$, which correspond to the unique p-mean (μ_1,μ_2)-s.p.a.p. mild solution on $\mathbb{R}$ of (1.1). This completes the proof. □

Finally, we investigate the stability of solution of (1.1) obtained in the previous Theorem. First, we recall the definition of stability.

Definition 3.9. The unique p-mean (μ_1,μ_2)-s.p.a.p. mild solution $Z^*(t)$ of (1.1) is said to be stable in p-mean sense, if for arbitrary $\epsilon > 0$, there exists $\eta > 0$ such that

$$
\mathbb{E}\|Z(t) - Z^*(t)\|^p < \epsilon, \text{ for all } t \ge 0,
$$

whenever $\mathbb{E}\|Z(0) - Z^*(0)\|^p < \eta$, where $Z(t)$ stands for a solution of(1.1), with initial value $Z(0)$.

Theorem 3.10. *Let $\mu_1, \mu_2 \in \mathfrak{N}$ satisfy* (A1), (A2). *Suppose that* (A3)–(A5) *hold. Then* (1.1) *has a unique p-mean (μ_1,μ_2)-s.p.a.p. mild solution which is stable provided that*

$$
\Theta_p := M^p (1/\kappa)^p \big[L_1 + L_2\|Q\|^p_{L^1(0,+\infty)} + C_p L_3 \|R\|^p_{L^2(0,+\infty)}\big] < (1/4)^{p-1}, \quad (3.6)
$$

for $p > 2$, and

$$
\Theta_2 := M^2 (1/\kappa)^2 \Big[L_1 + L_2\|Q\|^2_{L^1(0,+\infty)} + L_3 \|R\|^2_{L^2(0,+\infty)}\Big] < 1/4, \quad (3.7)
$$

for $p = 2$.

Proof. Note that condition (3.7) (resp. (3.6)) implies condition (3.5) (resp. (3.4)). Then, from Theorem 3.8, we know that (1.1) has a unique p-mean (μ_1, μ_2)-s.p.a.p. mild solution Z^*, whose integral form is given by (3.1). Let $Z(t)$ be an arbitrary solution of (1.1) with initial value $Z(0)$. Then

$$\begin{aligned}
\mathbb{E}\|Z(t) - Z^*(t)\|^p = \mathbb{E}\big\|&U(t,0)\,[Z(0) - Z^*(0)] \\
&+ \int_0^t U(t,s)\,[F_1(s, Z(s)) - F_1(s, Z^*(s))]\,ds \\
&+ \int_0^t U(t,s) \int_0^s Q(s-\zeta)[F_2(\zeta, Z(\zeta)) - F_2(\zeta, Z^*(\zeta))]d\zeta ds \\
&+ \int_0^t U(t,s) \int_0^s R(s-\zeta)[G(\zeta, Z(\zeta)) - G(\zeta, Z^*(\zeta))]d\mathcal{W}(\zeta) ds\big\|^p.
\end{aligned}$$

Now assume that $p > 2$. With the help of Hölder's inequality, for any $t \geq 0$, we have

$$\begin{aligned}
&\mathbb{E}\|Z(t) - Z^*(t)\|^p \\
&\leq 4^{p-1}\mathbb{E}\|U(t,0)\,[Z(0) - Z^*(0)]\,\|^p \\
&\quad + 4^{p-1}\mathbb{E}\|\int_0^t U(t,s)[F_1(s, Z(s)) - F_1(s, Z^*(s))]ds\|^p \\
&\quad + 4^{p-1}\mathbb{E}\|\int_0^t U(t,s) \int_0^s Q(s-\zeta)[F_2(\zeta, Z(\zeta)) - F_2(\zeta, Z^*(\zeta))]d\zeta ds\|^p \\
&\quad + 4^{p-1}\mathbb{E}\|\int_0^t U(t,s) \int_0^s R(s-\zeta)[G(\zeta, Z(\zeta)) - G(\zeta, Z^*(\zeta))]d\mathcal{W}(\zeta) ds\|^p \\
&\leq 4^{p-1}\|U(t,0)\|^p\mathbb{E}\|Z(0) - Z^*(0)\|^p \\
&\quad + 4^{p-1}\mathbb{E}\Big[\int_0^t \|U(t,s)\|\|F_1(s, Z(s)) - F_1(s, Z^*(s))\|ds\Big]^p \\
&\quad + 4^{p-1}\mathbb{E}\Big[\int_0^t \|U(t,s)\| \cdot \|\int_0^s Q(s-\zeta)[F_2(\zeta, Z(\zeta)) - F_2(\zeta, Z^*(\zeta))]d\zeta\|ds\Big]^p \\
&\quad + 4^{p-1}\mathbb{E}\Big[\int_0^t \|U(t,s)\| \cdot \|\int_0^s R(s-\zeta)[G(\zeta, Z(\zeta)) - G(\zeta, Z^*(\zeta))]d\mathcal{W}(\zeta)\|ds\Big]^p \\
&\leq 4^{p-1}M^p e^{-\kappa pt}\mathbb{E}\|Z(0) - Z^*(0)\|^p \\
&\quad + 4^{p-1}M^p\mathbb{E}\Big[\int_0^t e^{-\kappa(t-s)}\|F_1(s, Z(s)) - F_1(s, Z^*(s))\|ds\Big]^p \\
&\quad + 4^{p-1}M^p\mathbb{E}\Big[\int_0^t e^{-\kappa(t-s)}\|\int_0^s Q(s-\zeta)(F_2(\zeta, Z(\zeta)) - F_2(\zeta, Z^*(\zeta)))d\zeta\|ds\Big]^p \\
&\quad + 4^{p-1}M^p\mathbb{E}\Big[\int_0^t e^{-\kappa(t-s)}\|\int_0^s R(s-\zeta)(G(\zeta, Z(\zeta)) - G(\zeta, Z^*(\zeta)))d\mathcal{W}(\zeta)\|ds\Big]^p \\
&\leq 4^{p-1}M^p e^{-\kappa pt}\mathbb{E}\|Z(0) - Z^*(0)\|^p \\
&\quad + 4^{p-1}M^p\Big[\int_0^t e^{-\kappa(t-s)}ds\Big]^{p-1}\int_0^t e^{-\kappa(t-s)}\mathbb{E}\|F_1(s, Z(s)) - F_1(s, Z^*(s))\|^p ds \\
&\quad + 4^{p-1}M^p\Big[\int_0^t e^{-\kappa(t-s)}ds\Big]^{p-1}\int_0^t e^{-\kappa(t-s)}\mathbb{E}\|\int_0^s Q(s-\zeta)(F_2(\zeta, Z(\zeta))
\end{aligned}$$

$$
\begin{aligned}
&- F_2(\zeta, Z^*(\zeta)))d\zeta\|^p ds \\
&+ 4^{p-1}M^p\Big[\int_0^t e^{-\kappa(t-s)}ds\Big]^{p-1}\int_0^t e^{-\kappa(t-s)}\mathbb{E}\|\int_0^s R(s-\zeta)(G(\zeta, Z(\zeta)) \\
&- G(\zeta, Z^*(\zeta)))d\mathcal{W}(\zeta)\|^p ds \\
\le{}& 4^{p-1}M^p\mathbb{E}\|Z(0)-Z^*(0)\|^p + 4^{p-1}\Theta_p \sup_{t\in\mathbb{R}}\mathbb{E}\|Z(t)-Z^*(t)\|^p,
\end{aligned}
$$

which implies that

$$
\sup_{t\in\mathbb{R}}\mathbb{E}\|Z(t)-Z^*(t)\|^p \le 4^{p-1}M^p\mathbb{E}\|Z(0)-Z^*(0)\|^p + 4^{p-1}\Theta_p \sup_{t\in\mathbb{R}}\mathbb{E}\|Z(t)-Z^*(t)\|^p.
$$

That is

$$
\sup_{t\in\mathbb{R}}\mathbb{E}\|Z(t)-Z^*(t)\|^p \le \frac{4^{p-1}M^p}{1-4^{p-1}\Theta_p}\mathbb{E}\|Z(0)-Z^*(0)\|^p.
$$

So, for all $t>0$, we obtain

$$
\mathbb{E}\|Z(t)-Z^*(t)\|^p \le \frac{4^{p-1}M^p}{1-4^{p-1}\Theta_p}\mathbb{E}\|Z(0)-Z^*(0)\|^p.
$$

For $\epsilon>0$, choosing $0<\eta<\epsilon\frac{1-4^{p-1}\Theta_p}{4^{p-1}M^p}$, we obtain

$$
\mathbb{E}\|Z(0)-Z^*(0)\|^p<\eta \Longrightarrow \mathbb{E}\|Z(t)-Z^*(t)\|^p\le\epsilon, \text{ for all } t>0,
$$

According to Definition 3.9, we can conclude that (1.1) has a unique p-mean (μ_1,μ_2)-s.p.a.p. mild solution which is stable in p-mean sense.

Now assume that $p=2$. By help of the Cauchy-Schwartz inequality, the Ito's isometry and the Fubini's theorem, we have

$$
\begin{aligned}
&\mathbb{E}\|Z(t)-Z^*(t)\|^2 \\
&\le 4\mathbb{E}\|U(t,0)\left[Z(0)-Z^*(0)\right]\|^2 \\
&\quad + 4\mathbb{E}\|\int_0^t U(t,s)\left[F_1(s,Z(s))-F_1(s,Z^*(s))\right]ds\|^2 \\
&\quad + 4\mathbb{E}\|\int_0^t U(t,s)\int_0^s Q(s-\zeta)\left[F_2(\zeta,Z(\zeta))-F_2(\zeta,Z^*(\zeta))\right]d\zeta ds\|^2 \\
&\quad + 4\mathbb{E}\|\int_0^t U(t,s)\int_0^s R(s-\zeta)\left[G(\zeta,Z(\zeta))-G(\zeta,Z^*(\zeta))\right]d\mathcal{W}(\zeta)ds\|^2 \\
&\le 4M^p\mathbb{E}\|Z(0)-Z^*(0)\|^2 + 4\Theta_2\times\sup_{t\in\mathbb{R}}\mathbb{E}\|Z(t)-Z^*(t)\|^2,
\end{aligned}
$$

Thus, for $t>0$, we obtain

$$
\mathbb{E}\|Z(t)-Z^*(t)\|^2 \le \frac{4M^2}{1-4\Theta_2}\mathbb{E}\|Z(0)-Z^*(0)\|^2.
$$

For $\epsilon>0$, choosing $0<\eta<\epsilon\frac{1-4\Theta_2}{4M^2}$, we obtain

$$
\mathbb{E}\|Z(0)-Z^*(0)\|^2<\eta \Longrightarrow \mathbb{E}\|Z(t)-Z^*(t)\|^2\le\epsilon, \text{ for all } t>0.
$$

According to Definition 3.9, we can conclude that (1.1) has a unique square-mean (μ_1,μ_2)-s.p.a.p. mild solution which is stable in square-mean sense. This completes the proof. □

4. Examples of applications

In this section, we provide some examples to illustrate the practical usefulness of of our results established in the preceding section. We can use $\mathcal{W}(t)dt = dB(t)$, where the derivative is taken in a stochastic sense. It is evident that periodicity is very idealistic situation, and several systems may not show exactly periodic behavior. To capture such behavior, other generalized functions may be used, such as almost periodic, pseudo-almost periodic, etc. This paper analyzes even a more general class. of functions that can capture the not-so usual dynamics of several systems

Example 4.1. Let us consider the generalized stochastic equation with time varying coefficient with Dirichlet boundary conditions,

$$\begin{gathered}\frac{\partial}{\partial t}u(t,x) = -a(t)\frac{\partial^2}{\partial x^2}u(t,x) + f(t,x) + b(t)u(t,x) + h\frac{dB(t)}{dt},\\ u(t,0) = u(t,\pi) = 0, u(0,x) = u_0(x), \quad x \in (0,\pi),\ t \in \mathbb{R}^+.\end{gathered} \tag{4.1}$$

Let us define $\mathcal{E} = L^2(0,\pi)$ and $A(t)v = -a(t)Av$, $Av = v^{''}$ with $D(A) = H^2(0,\pi) \cap H_0^1(0,\pi)$. It is well known that A generates an analytic semigroup $T(t) : t \geq 0$. Thus $U(t,s) = e^{-\int_s^t a(\xi)d\xi}T(t-s)$. As the semigroup $T(t)$ is bounded, we obtain the condition $\|U(t,s)\| \leq Me^{-a(t-s)}$. Let $u(t,x) = u(t)x$, after these setup, the above problem can be written in the abstract form

$$du = (A(t)u + F_1(t,u))dt + hdB(t) \quad \text{or} \quad \frac{du}{dt} = (A(t)u + F_1(t,u)) + h\mathcal{W}(t).$$

We assume that the functions f, b and h are (μ_1,μ_2) s.p.a.p. In particular, we can consider

$$a(t) = c_1e^{-t} + |\sin t|,\ b(t) = c_2\frac{1}{1+t^2} + c_4(\sin t + \sin\sqrt{2}t).$$

We see that

$$E\|F_1(t,x) - F_1(t,y)\| \leq E\|f(t,x) - f(t,y)\| + |b(t)|E\|x-y\|.$$

So under the assumption that f is Lipschitz, we obtain F_1 Lipschitz, here $L_1 = L_f + (2c_4 + c_2)$. Also $k = (1+c_1)$. Thus we can always choose constants c_1, c_2, c_4, so that the required condition of Theorem holds. Hence the existence and uniqueness of (μ_1,μ_2)-s.p.a.p. solution is ensured.

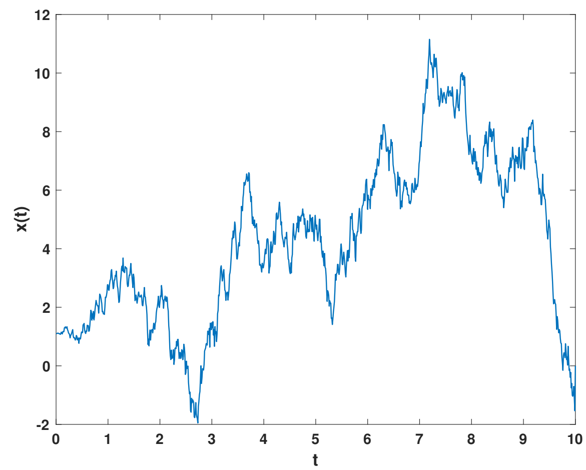
Example 4.2 (Stochastic logistic model with distributed delay)**.** Let us consider the stochastic logistic model with distributed delay and time varying rate,

$$dx(t) = \Big(x(t)(-a(t) - b(t)x(t)) + \int_{-\infty}^{t} k(t-s)x(s)ds\Big)dt + \sigma dB(t). \tag{4.2}$$

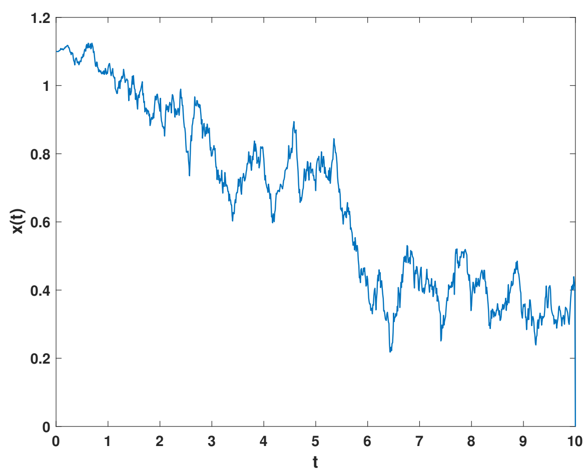
We assume that $a(\cdot), b(\cdot)$ are (μ_1,μ_2)-s.p.a.p. and $k(\cdot) \in L^1([0,\infty))$. In this case, we can see $A(t) = a(t)$, $F_1(t,x(t)) = -b(t)x^2(t)$, $Q(t) = k(t)$, $F_2(t,x(t)) = x(t)$, $R(t) = \sigma$, $G = 1$. In this case, we can see that $U(t,s) = e^{-\int_s^t a(\xi)d\xi}$. To satisfies the required condition, we assume that $a(t)$ is positive and bounded. Hence $\|U(t,s)\| \leq e^{-a(t-s)}$ for some positive constant a. We choose

$$a(t) = c_1e^{-t} + |\sin t|, \quad b(t) = c_2\frac{1}{1+t^2} + c_4(\sin t + \sin\sqrt{2}t), \quad k(t) = e^{-t}.$$

We can see in Figures 1 and 2 that the solution is (μ_1, μ_2)-s.p.a.p. One can see that fluctuations are less when drift coefficient σ is small.



$c_1 = 2$, $c_2 = 1$, $c_4 = 1$, $\sigma = 3$



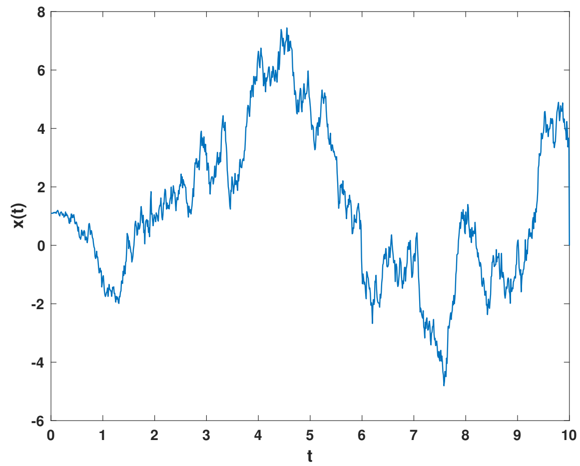
$c_1 = 2$, $c_2 = 1$, $c_4 = 1$, $\sigma = 0.2$

FIGURE 1. Numerical solutions of (4.2) for different parameters.

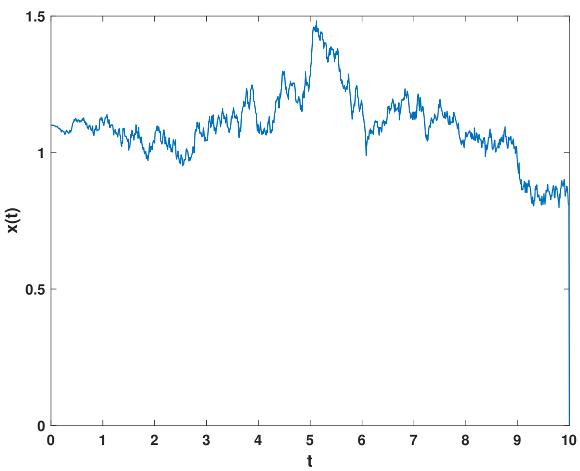
Example 4.3 (Stochastic cellular neural networks with distributed delays)**.** Let us consider the system of stochastic differential equations with distributed delays

$$dx_i(t) = \Big(- c_i(t)x_i(t) + \sum_{j=1}^{n} b_{ij} \int_{-\infty}^{t} k_{ij}(t-s)g_j(x_j(s))ds + I_i\Big)dt + \sigma_i dB(t). \quad (4.3)$$

In model (4.3), n correspond to the number of neurons in the network. For $i, j = 1, \dots, n$; x_i represent the i^{th} neuron state, $c_i(t)$ represent the decay rate,



$c_1 = 0.2,\ c_2 = 0.1,\ c_4 = 1,\ \sigma = 3$



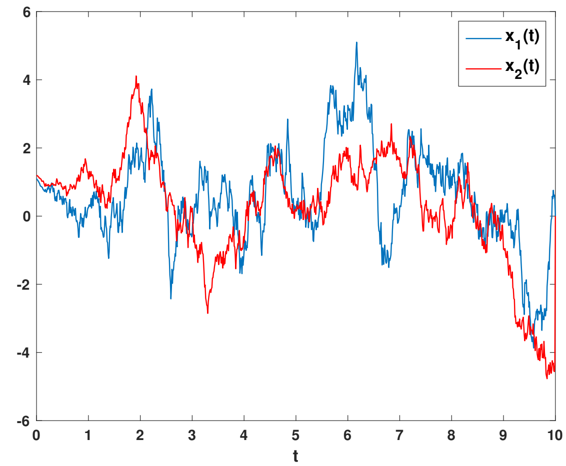
$c_1 = 0.2,\ c_2 = 0.1,\ c_4 = 1,\ \sigma = 0.2$

FIGURE 2. Numerical solutions of (4.2) for different parameters.

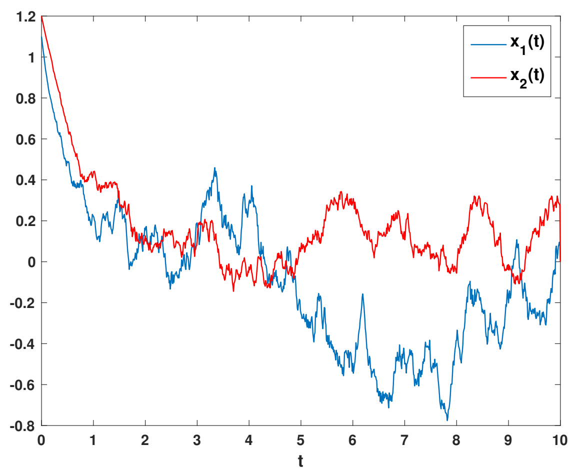
g_j represent the activation function of the j^{th} neuron, I_i represent the external input in the i^{th} neuron, b_{ij} represent the connection weight and the kernel function $k_{ij} \in L^1([0,\infty))$. For more details on neural networks, we refer to [18, 34] and references therein. In this equation, we suppose that $c_i(\cdot)$ are positive bounded function. Also, we choose

$$c_i(t) = a_i e^{-t} + |\sin t|, \quad g_j(x_j(t)) = \Big(a_i \frac{1}{1+t^2} + c_4((\sin t + \sin \sqrt{2}t)\Big) x_j(t),$$

and $k_{ij}(t) = e^{-t}$. The corresponding plots for $i = 1, 2$ are depicted in Figures 3 and 4. One can clearly see that the graphs show (μ_1, μ_2)-s.p.a.p. behavior. In this case also, one can see that fluctuations are less when drift coefficients σ_1, σ_2 are small.



$a_1 = 2,\ a_2 = 1,\ c_4 = 1,\ \sigma_1 = 3,\ \sigma_2 = 2.$



$a_1 = 2,\ a_2 = 1,\ c_4 = 1,\ \sigma_1 = 0.3,\ \sigma_2 = 0.2$

FIGURE 3. Numerical solutions of (4.3) for different parameters.

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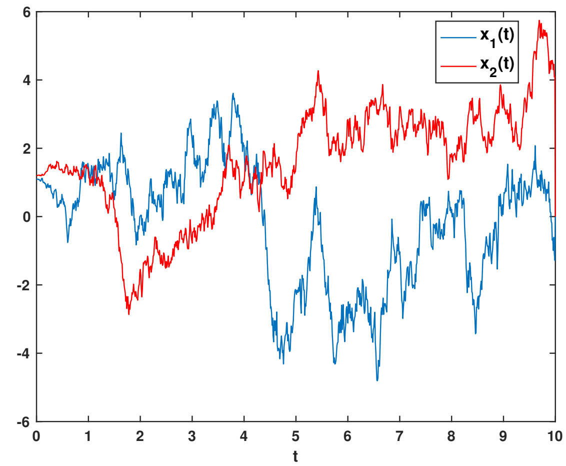
REFERENCES

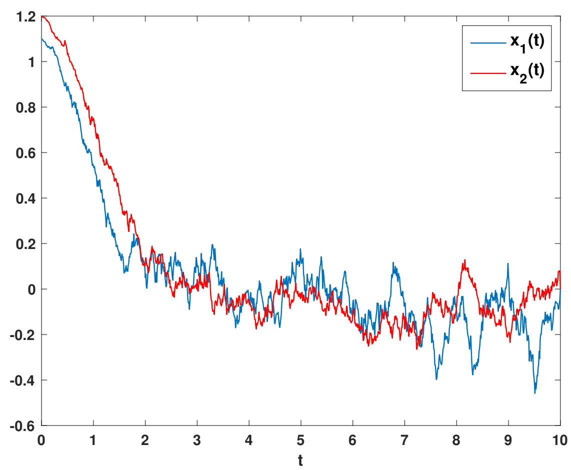
[1] P. Acquistapace, B. Terreni; *A unified approach to abstract linear nonautonomous parabolic equations*, Rendiconti del Seminario Matematico della Universitaa di Padova, 78, 2 (1987), 47-107.

[2] M. Akhmet; *Almost periodicity, chaos, and asymptotic equivalence*, Nonlinear Syst. Complex., 27, Springer, Cham, 2020, xvii+360 pp. ISBN: 978-3-030-19916-6; 978-3-030-20572-0.

[3] M. Ayachi; *Dynamics of fuzzy genetic regulatory networks with leakage and mixed delays in doubly-measure pseudo-almost periodic environment.* Chaos, Solitons & Fractals, 154 (2022), 111659.



$a_1 = 0.2$, $a_2 = 0.1$, $c_4 = 1$, $\sigma_1 = 3$, $\sigma_2 = 2$



$a_1 = 0.2$, $a_2 = 0.1$, $c_4 = 1$, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$

FIGURE 4. Numerical solutions of (4.3) for different parameters.

[4] N. Belmabrouk, M. Damak, M. Miraoui; *Measure pseudo almost periodic solution for a class of nonlinear delayed stochastic evolution equations driven by Brownian motion.* Filomat, 35(2) (2021), 515-534.

[5] M. Ben-Salah, M. Miraoui, A. Rebey; *New results for some neutral partial functional differential equations*, Results in Mathematics, 74(4) (2019), 1-22.

[6] P. H. Bezandry; *Existence of almost periodic solutions to some functional integro-differential stochastic evolution equations*, Statistics and Probability Letters, 78(17) (2008), 2844-2849.

[7] P. H. Bezandry; *Existence of almost periodic solutions for semilinear stochastic evolution equations driven by fractional Brownian motion*, Electronic Journal of Differential Equations, 2012 (156) (2012), 1-21.

[8] P. H. Bezandry, T. Diagana; *Square-mean almost periodic solutions nonautonomous stochastic differential equations.* Electron. J. Differ. Equ., 2007(117) (2007), 1-10.

[9] P. H. Bezandry, T. Diagana; *Existence of almost periodic solutions to some stochastic differential equations*, Appl. Anal., 86 (2007), 819-827.

[10] P. H. Bezandry, T. Diagana; *Existence of quadratic-mean almost periodic solutions to some stochastic hyperbolic differential equations*, Electron. J. Differential Equations, 2009 (2009), No. 111, 14 pp.

[11] P. H. Bezandry, T. Diagana; *Almost Periodic Stochastic Processes*, Springer, New York, 2011, xvi+235 pp. ISBN: 978-1-4419-9475-2.

[12] J. Blot, P. Cieutat, K. Ezzinbi; *New approach for weighted pseudo-almost periodic functions under the light of measure theory, basic results and applications*, Applicable Analysis, 92 (3) (2013), 493-526.

[13] H. Bohr; *Almost periodic functions*, Chelsea Publishing Co., New York, 1947, ii+114 pp.

[14] J. Cao, Q. Yang, Z. Huang; *On almost periodic mild solutions for stochastic functional differential equations*, Nonlinear Analysis: Real World Applications, 13 (1) (2012), 275-286.

[15] Y. K. Chang, Z. H. Zhao; *Almost periodic solutions to a stochastic differential equation in Hilbert spaces*, Results in Mathematics, 63 (1) (2013), 435-449.

[16] F. Cherif; *Quadratic-mean pseudo almost periodic solutions to some stochastic differential equations in a Hilbert space*, Journal of Applied Mathematics and Computing, 40(1) (2012), 427-443.

[17] J. Cui, W. Rong; *Existence and stability of μ-pseudo almost automorphic solutions for stochastic evolution equations*, Frontiers of Mathematics in China, 14(2) (2019), 261-280.

[18] S. Dhama, S. Abbas; *Square-mean almost automorphic solution of a stochastic cellular neural network on time scales.* J. Integral Equations Appl. 32(2) (2020), 151-170.

[19] H. G. Diaz-Marin, O. Osuna; *Tumor spheroid with almost periodic nutrient and inhibitor supplies*, ZAMM Z. Angew. Math. Mech. 104(1) (2024), Paper No. e202200228, 19 pp.

[20] T. Diagana; *Weighted pseudo almost periodic functions and applications*, C R Acad. Sci. Paris, Ser. I, 343(10) (2006), 643-646.

[21] T. Diagana; *Weighted pseudo-almost periodic solutions to some differential equation*, Nonlinear Anal., 68(8) (2008), 2250-2260.

[22] T. Diagana, K. Ezzinbi, M. Miraoui; *Pseudo-almost periodic and pseudo-almost automorphic solutions to some evolution equations involving theoretical measure theory*, Cubo (Temuco), 16(2) (2014), 01-32.

[23] M. A. Diop, K. Ezzinbi, M. M. Mbaye; *Existence and global attractiveness of a pseudo almost periodic solution in p-th mean sense for stochastic evolution equation driven by a fractional Brownian motion*, Stochastics An International Journal of Probability and Stochastic Processes, 87(6) (2015), 1061-1093.

[24] D. Kannan, A. T. Bharucha-Reid; *On a stochastic integro-differential evolution equation of Volterra type*, Journal of Integral Equations, 10 (1985), 351-379.

[25] H. Kuo; *Introduction to stochastic integration*, Springer, New York, 2006. xiv+278 pp. ISBN:978-0387-28720-1

[26] L. Li, Y. Feng, W. Pan; *Existence of Almost Periodic Solutions to a Class of Non-autonomous Functional Integro-differential Stochastic Equations*, International Conference on Information Computing and Applications, (2013), 605-614.

[27] X. L. Li; *Square-mean almost periodic solutions to some stochastic evolution equations*, Acta Mathematica Sinica, English Series, 30(5) (2014), 881-898.

[28] L. Maniar, S. Roland; *Almost Periodicity of Inhomogeneous Parabolic Evolution Equations*, Lecture Notes in Pure and Appl. Math., 234 (2003), 299-318.

[29] M. M. Mbaye; *Square-mean μ-pseudo almost periodic and automophic solutions for a class of semilinear integro-differential stochastic evolution equations*, Afrika Matematika, 28(3-4) (2017), 643-660.

[30] M. Miraoui; *Measure pseudo almost periodic solutions for differential equations with reflection*, Appl. Anal. 101 (2022), no. 3, 938-951.

[31] M. Miraoui, D. D. Repovs; *Dynamics and oscillations of models for differential equations with delays*, Boundary Value Problems, 2020(1), 1-17.

[32] B. B. Oksendal; *Stochastic differential equations*, Springer-Verlag, Berlin, 2003, xxiv+360 pp. ISBN: 3-540-04758-1.

[33] J. Seidler; *Da Prato-Zabczyk's maximal inequality revisited. I.*, Math. Bohem., 118(1) (1993), 67-106.

[34] S. Tyagi, S. Abbas, M. Pinto; *Global exponential stability in Lagrange sense for periodic neural networks with various activation functions and time-varying delays.* Appl. Math. (Warsaw) 46 (2) (2019), 229-252.

[35] D. Votsia, I. Zabsonre; *Pseudo almost periodic solutions of infinite class under the light of measure theory.* Silesian Journal of Pure and Applied Mathematics, 10 (2020), 15-43.

[36] Z. Yan, X. Jia; *Pseudo almost periodicity and its applications to impulsive nonautonomous partial functional stochastic evolution equations*, International Journal of Nonlinear Sciences and Numerical Simulation, 19(5), (2018), 511-529.

[37] I. Zabsonre, H. Toure; *Pseudo-almost periodic and pseudo-almost automorphic solutions of class r under the light of measure theory*, African Diaspora Journal of Mathematics, New Series, 19(1) (2016), 58-86.

[38] I. Zabsonre, D. Votsia; *Pseudo almost periodic solutions of infinite class under the light of measure theory and applications*, Mathematical Methods in the Applied Sciences, 44(6) (2021), 4580-4597.

[39] C. Zhang; *Almost periodic type functions and ergodicity*, Science Press Beijing, BeijingKluwer Academic Publishers, Dordrecht, 2003. xii+355 pp. ISBN:7-03-010489-7.

[40] H. Zhou, W. Jiang; *Existence and stability of positive almost periodic solution for stochastic Lasota-Wazewska model*, Journal of Applied Mathematics and Computing, 47(1) (2015), 61-71.

Moez Ayachi
Laboratory of Mathematics and Applications (LR17ES11), Faculty of Sciences of Gabes, University of Gabes, Gabes 6072, Tunisia
Email address: Moez.Ayachi@fsg.rnu.tn

Syed Abbas (corresponding author)
School of Mathematical and Statistical Sciences, Indian Institute of Technology Mandi, Mandi, H.P., 175005, India
Email address: abbas@iitmandi.ac.in