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PERIODIC SOLUTIONS IN DISTRIBUTION FOR STOCHASTIC LATTICE DIFFERENTIAL EQUATIONS

YUE GAO, XUE YANG

Abstract. In this article, we consider stochastic lattice differential equations (SLDEs) in the weighted space l^2_ρ of infinite sequences. We establish the well-posedness of solutions and prove the existence of periodic solutions in distribution. An example is given to illustrate the validity of our results.

1. Introduction

Lattice differential equations have been extensively studied because the variety of applications in image processing, traffic flow analysis, virus propagation, pattern formation, and so on. For the dynamics of deterministic lattice differential equations, we refer the reader to [2, 6, 13, 24] and references therein. Compared to deterministic systems, stochastic lattice systems not only exhibit discrete spatial characteristics but also account for the influence of random environments. This enables SLDEs to better reveal objective phenomena. For this reasing, SLDEs have attracted extensive attention; see [3, 4, 8, 14, 15, 26].

The concept of periodic solutions plays a crucial role in studying the long-term behavior of random dynamical systems simulated by stochastic differential equations. Since the ground breaking work of Poincaré in [21, 22, 23], periodic solutions have been the subject of research for over a century. In the past decade, many works have been devoted to study periodicity of SDEs. For the existence of periodic solutions for finite-dimensional stochastic systems, we refer the reader to [5, 9, 10, 11, 12, 16, 17, 18, 28].

Similar to the case of finite-dimensional systems, a crucial question is: Under what conditions do SLDEs in weighted space l^2_ρ have the desired periodicity? In this article, we focus on asymptotic behavior and attempt to address this issue. Despite the increasing interest in treating SLDEs, the available results in this regard still scarce. There are two main difficulties. First, because the disturbance from noise, the sample paths of the solutions cannot maintain periodicity. In addition, rigorous convergence analysis is required to ensure the well-posedness of solutions of infinite-dimensional systems. To this end, we consider a weaker periodicity, so-called periodic solutions in distribution. In this paper, we first discuss the well-posedness of SLDEs. Inspired by [5, 12], we provide sufficient conditions for

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the existence of periodic solutions in distribution for general SLDSs in a weighted space of infinite sequences. Furthermore, we provide an illustrative example to demonstrate the simplicity of our conditions via Lyapunov method.

The rest of this article is organized as follows. In Section 2, we give some preliminaries. We introduce the notation and definitions of related concepts. In addition, we discuss the well-posedness of SLDEs. In Section 3, we first give a priori estimate to ensure the rationality of the assumptions. Then, we prove the existence of periodic solutions in distribution. In Section 4, we illustrate our main result by an example.

2. Preliminaries

Basic notation. First, we introduce a weighted space of infinite sequences. Let $\rho : \mathbb{Z} \to (0, M_0] \subset \mathbb{R}^+$ and $p \geq 1$ be a real number. For each $i \in \mathbb{Z}$, we define $\rho(i) = \rho_i$ and

$$l^p_\rho = \big\{u = (u_i)_{i\in\mathbb{Z}}; \sum_{i=1}^{\infty} \rho_i |u_i|^p < \infty\big\}$$

with the norm $\|u\|_{\rho,p} = \big(\sum_{i=1}^{\infty} \rho_i |u_i|^p\big)^{1/p}$ for $u \in l^p_\rho$. If $p = 2$, we denote $\|u\|_{\rho,2} = \|u\|_\rho$. For $u, v \in l^2_\rho$, we denote the inner product in l^2_ρ as $\langle u, v\rangle$, where $\langle u, v\rangle = \sum_{i\in\mathbb{Z}} \rho_i u_i v_i$. The space $L^p(\Omega, l^2_\rho)$ consists of all l^2_ρ-valued random variables ξ such that $\mathbb{E}\|\xi\|^p_\rho = \int_\Omega \|\xi\|^p_\rho \mathrm{d}P < \infty$. For a given l^2_ρ-valued random variable ξ, we denote by $P \circ [\xi]^{-1}$ the distribution of ξ on l^2_ρ. Let $\mathcal{B}(l^2_\rho)$ be the Borel set of space l^2_ρ. For $z \in l^2_\rho$, we use z^T to denote the transpose of z. Let $\mathcal{P}(l^2_\rho)$ be the set of Borel probability measures on l^2_ρ. Denote by J_f the Jacobian matrix of function f with respect to $x \in \mathbb{R}^d$. We define

$$\begin{gathered}
\|h\|_\infty = \sup_{x\in l^2_\rho} |h(x)|,\\
\|h\|_L = \sup\big\{\frac{|h(x) - h(y)|}{\|x - y\|_\rho}; x, y \in l^2_\rho, x \neq y\big\},\\
\|h\|_{BL} = \max\{\|h\|_\infty, \|h\|_L\},\\
d_{BL}(\mu_1, \mu_2) = \sup_{\|h\|_{BL}\leq 1} \big| \int h d(\mu_1 - \mu_2)\big|
\end{gathered}$$

for all Lipschitz continuous real-valued functions $h(x)$ on l^2_ρ and all $\mu_1, \mu_2 \in \mathcal{P}(l^2_\rho)$.

Well-posedness of SLDEs. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing, right continuous and $\mathcal{F}_0$ contains all P-null sets). The first component $u(t)$ satisfies the SLDE

$$\begin{aligned}
\mathrm{d}u_i(t) = &[\nu(u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)) - \lambda u_i(t) + f_i(u(t)) + g_i(t)]\mathrm{d}t\\
&+ \sigma_i(t, u(t))\mathrm{d}W_i(t),
\end{aligned} \tag{2.1}$$

where $i \in \mathbb{Z}$, $u_i \in \mathbb{R}$, ν, and λ are positive constants. We assume that $(f_i)_{i\in\mathbb{Z}}$ are smooth functions, $(g_i(t))_{i\in\mathbb{Z}}, (\sigma_i(t))_{i\in\mathbb{Z}} \in l^2_\rho$ are continuous with respect to $t \in \mathbb{R}_+$, and $\{W_i(t) : i \in \mathbb{Z}\}$ are independent one-dimensional Brownian motions.

For $u \in l^2_\rho$, let A, B, and B^* be linear operators from l^2_ρ to l^2_ρ as follows:

$$(Bu)_i = u_{i+1} - u_i, \quad (B^*u)_i = u_{i-1} - u_i,$$

and $(Au)_i = -u_{i+1} + 2u_i - u_{i-1}$. Then we have $A = BB^* = B^*B$ and $\langle B^*u, v\rangle = \langle u, Bv\rangle$ for all $u, v \in l^2_\rho$. Therefore, $\langle Au, u\rangle \geq 0$ for all $u \in l^2_\rho$. Let e^i denote the element having 1 at position i and all the other components 0. We define

$$W(t) = \sum_{i\in\mathbb{Z}} W_i(t)e^i, \quad f(u(t)) = (f_i(u(t)))_{i\in\mathbb{Z}},$$
$$g(t) = (g_i(t))_{i\in\mathbb{Z}}, \quad \sigma(t, u(t)) = (\widetilde{\sigma}_{ij}(t, u(t)))_{i,j\in\mathbb{Z}},$$

where

$$\widetilde{\sigma}_{ij} = \begin{cases} \sigma_i, & i = j, \\ 0, & i \neq j. \end{cases}$$

Then we can rewrite (2.1) as

$$\mathrm{d}u(t) = [-\nu Au(t) - \lambda u(t) + f(u(t)) + g(t)]\mathrm{d}t + \sigma(t, u(t))\mathrm{d}W(t). \tag{2.2}$$

Note that (2.2) can be interpreted as an integral equation

$$u(t) = u_0 + \int_0^t [-\nu Au(s) - \lambda u(s) + f(u(s)) + g(s)]\mathrm{d}s + \int_0^t \sigma(s, u(s))\mathrm{d}W(s) \tag{2.3}$$

with initial value $u_0 := u(0)$.

We make the following assumptions on the coefficients of the above SLDE.

Assumption 2.1. For every $i \in \mathbb{Z}$, $t \in [0, \infty)$, and $u, v \in l^2_\rho$, there exists positive constants L and K such that

$$\|f(u) - f(v)\|^2_\rho \leq L\|u - v\|^2_\rho, \ \|f(u)\|^2_\rho \leq K(1 + \|u\|^2_\rho),$$
$$\|\sigma(t, u) - \sigma(t, v)\|^2_\rho \leq L\|u - v\|^2_\rho, \ \|\sigma(t, u)\|^2_\rho \leq K(1 + \|u\|^2_\rho).$$

Next, we prove the existence and uniqueness of solutions to SLDEs.

Theorem 2.2. *Let $T > 0$, and suppose that Assumptions 2.1 holds. Then* (2.3) *admits a unique solution $u(t) \in L^2(\Omega, C([0, T], l^2_\rho))$ with initial value $u(0) = u_0 \in L^2(\Omega, l^2_\rho)$.*

Proof. **Step 1.** We show uniqueness. Assume that $u(t)$ and $\widetilde{u}(t)$ are two solution of system (2.3) with initial value $u_0 \in l^2_\rho$. Then we have

$$\begin{aligned} u(t) - \widetilde{u}(t) = \int_0^t &[-\nu A(u(s) - \widetilde{u}(s)) - \lambda u(s) + \lambda\widetilde{u}(s)) \\ &+ f(u(s)) - f(\widetilde{u}(s))]\mathrm{d}s + \int_0^t [\sigma(s, u(s)) - \sigma(s, \widetilde{u}(s))]\mathrm{d}W(s). \end{aligned}$$

Hence, by Itô isometry and Assumption 2.1, we have

$$\begin{aligned} &\mathbb{E}\|u(t) - \widetilde{u}(t)\|^2_\rho \\ &= \mathbb{E}\| \int_0^t [-\nu A(u(s) - \widetilde{u}(s)) - \lambda(u(s) - \widetilde{u}(s)) + f(u(s)) - f(\widetilde{u}(s))]\mathrm{d}s \\ &\quad + \int_0^t [\sigma(s, u(s)) - \sigma(s, \widetilde{u}(s))]\mathrm{d}W(s)\|^2_\rho \\ &\leq 2t\mathbb{E}\int_0^t \| - \nu A(u(s) - \widetilde{u}(s)) - \lambda(u(s) - \widetilde{u}(s)) + f(u(s)) - f(\widetilde{u}(s))\|^2_\rho \mathrm{d}s \\ &\quad + 2\mathbb{E}\int_0^t \|\sigma(s, u(s)) - \sigma(s, \widetilde{u}(s))\|^2_\rho \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\le 6t\mathbb{E}\int_0^t \|-\nu A(u(s)-\widetilde{u}(s))\|_\rho^2 \mathrm{d}s + 6t\mathbb{E}\int_0^t \lambda^2\|u(s)-\widetilde{u}(s)\|_\rho^2\mathrm{d}s \\
&\quad + 6t\mathbb{E}\int_0^t \|f(u(s))-f(\widetilde{u}(s))\|_\rho^2\mathrm{d}s + 2\mathbb{E}\int_0^t \|\sigma(s,u(s))-\sigma(s,\widetilde{u}(s))\|_\rho^2\mathrm{d}s.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\|\nu A(u(s)-\widetilde{u}(s))\|_\rho^2 \\
&= \sum_{i\in\mathbb{Z}} \rho_i\Big(\nu^2 \sum_{i\in\mathbb{Z}} [(u_{i+1}(s)-\widetilde{u}_{i+1}(s)) - 2(u_i(s)-\widetilde{u}_i(s)) + (u_{i-1}(s)-\widetilde{u}_{i-1}(s))]^2\Big) \\
&\le 18\nu^2\|u(s)-\widetilde{u}(s)\|_\rho^2.
\end{aligned}
$$

So,

$$
\mathbb{E}\|u(t)-\widetilde{u}(t)\|_\rho^2 \le [6t(18\nu^2+\lambda^2+L)+2L]\mathbb{E}\int_0^{t\wedge\xi} \|u(s)-\widetilde{u}(s)\|_\rho^2\mathrm{d}s.
$$

By Grownwall's inequality, we obtain

$$
\mathbb{E}\|u(t)-\widetilde{u}(t)\|_\rho^2 = 0.
$$

This means that $P\{u(t)=\widetilde{u}(t)\}=1$ for all $t\ge 0$.

Step 2. We claim that (2.3) admits a solution. Let $u_0(t)=u_0$. For each $n=1,2,\dots$, we define the Picard iterations

$$
\begin{aligned}
u_n(t) = u_0 &+ \int_0^t [-\nu A u_{n-1}(s) - \lambda u_{n-1}(s) + f(u_{n-1}(s)) + g(s)]\mathrm{d}s \\
&+ \int_0^t \sigma(s,u_{n-1}(s))\mathrm{d}W(s).
\end{aligned} \tag{2.4}
$$

Hence

$$
\begin{aligned}
&\mathbb{E}[\|u_n(t)-u_{n-1}(t)\|_\rho^2] \\
&= \mathbb{E}[\|\int_0^t [-\nu A u_{n-1}(s) - \lambda u_{n-1}(s) + f(u_{n-1}(s)) + g(s) \\
&\quad - (-\nu A u_{n-2}(s) - \lambda u_{n-2}(s) + f(u_{n-2}(s)) + g(s))]\mathrm{d}s \\
&\quad + \int_0^t [\sigma(s,u_{n-1}(s)) - \sigma(s,u_{n-2}(s))]\mathrm{d}W(s)\|_\rho^2] \\
&\le 2t\mathbb{E}[\int_0^t \|-\nu A(u_{n-1}(s)-u_{n-2}(s)) - \lambda(u_{n-1}(s)-u_{n-2}(s)) \\
&\quad + f(u_{n-1}(s)) - f(u_{n-2}(s)))\|_\rho^2\mathrm{d}s] + 2\mathbb{E}\int_0^t \|\sigma(s,u_{n-1}(s)) - \sigma(s,u_{n-2}(s))\|_\rho^2\mathrm{d}s \\
&\le 6t\mathbb{E}[\int_0^t \|-\nu A(u_{n-1}(s)-u_{n-2}(s))\|_\rho^2 + \|-\lambda(u_{n-1}(s)-u_{n-2}(s))\|_\rho^2 \\
&\quad + \|f(u_{n-1}(s)) - f(u_{n-2}(s))\|_\rho^2\mathrm{d}s] + 2\mathbb{E}\int_0^t \|\sigma(s,u_{n-1}(s)) - \sigma(s,u_{n-2}(s))\|_\rho^2\mathrm{d}s \\
&\le [6t(18\nu^2+\lambda^2+L)+2L]\mathbb{E}\int_0^t \|u_{n-1}(s)-u_{n-2}(s)\|_\rho^2\mathrm{d}s.
\end{aligned}
$$

In addition,

$$
\mathbb{E}[\|u_1(t)-u_0(t)\|_\rho^2]
$$

$$
\begin{aligned}
&= \mathbb{E}[\| \int_0^t [-\nu A u_0(s) - \lambda u_0(s) + f(u_0(s)) + g(s)]\mathrm{d}s + \int_0^t \sigma(s, u_0(s))\mathrm{d}W(s)\|_\rho^2] \\
&\le 8t\mathbb{E}\int_0^t [\| -\nu A u_0(s)\|_\rho^2 + \|\lambda u_0(s)\|_\rho^2 + \|f(u_0(s))\|_\rho^2 + \|g(s)\|_\rho^2]\mathrm{d}s \\
&\quad + 2\mathbb{E}\int_0^t \|\sigma(s, u_0(s))\|_\rho^2 \mathrm{d}s \\
&\le 8t\mathbb{E}\int_0^t [18\nu^2 \|u_0(s)\|_\rho^2]\mathrm{d}s + 8t\mathbb{E}\int_0^t \lambda^2 \|u_0(s)\|_\rho^2 \mathrm{d}s + 8t\mathbb{E}\int_0^t K(1 + \|u_0(s)\|_\rho^2)\mathrm{d}s \\
&\quad + 8K_1 t^2 + 2\mathbb{E}\int_0^t K(1 + \|u_0(s)\|_\rho^2)\mathrm{d}s \\
&\le [8t^2(18\nu^2 + \lambda^2 + K) + 2Kt]\mathbb{E}\|u_0(s)\|_\rho^2 + 8(K + K_1)t^2 + 2Kt,
\end{aligned}
$$

where $K_1 = \max_{s\in[0,T]} \|g(s)\|_\rho^2$. Then there exists a positive constant $C_1 < \infty$ such that

$$
\mathbb{E}\|u_1(t) - u_0(t)\|_\rho^2 \le C_1 t,
$$

where C_1 only depends on ν, λ, K, K_1, T. By induction, there exists a positive constant C_2 such that for any $n \ge 0$, $t \in [0, T]$, we have

$$
\mathbb{E}\|u_n(t) - u_{n-1}(t)\|_\rho^2 \le \frac{C_2^n t^n}{n!},
$$

where C_2 only depends on $\nu,\ \lambda,\ K,\ K_1,\ T,\ L$ and $C_2 \ge \max\{C_1, 6T(18\nu^2 + \lambda^2 + L) + 2L\}$. In addition,

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T} \|u_n(t) - u_{n-1}(t)\|_\rho^2\Big) \\
&\le [6T(18\nu^2 + \lambda^2 + L) + 2L]\mathbb{E}\int_0^T \|u_{n-1}(s) - u_{n-2}(s)\|_\rho^2 \mathrm{d}s \\
&\le C_2 \int_0^T \frac{C_2^{n-1} s^{n-1}}{(n-1)!}\mathrm{d}s \\
&= \frac{C_2 T^n}{n!}.
\end{aligned}
$$

By Chebyshev's inequality, we obtain

$$
P\Big\{\sup_{0\le t\le T} \|u_n(t) - u_{n-1}(t)\| \ge \frac{1}{2^n}\Big\} \le \frac{(4C_2 T)^n}{n!}.
$$

Note that $\sum_{n=1}^\infty \frac{(4C_2 T)^n}{n!} < \infty$. Hence by Borel-Cantelli's lemma, for almost all $\omega \in \Omega$, there exists an integer constant n_0 such that

$$
\sup_{0\le t\le T} \|u_n(t) - u_{n-1}(t)\|_\rho^2 \le \frac{1}{2^n}
$$

for $n \ge n_0$. Consequently, $u_n(t)$ converges to $u(t)$ as $n \to \infty$ uniformly in $t \in [0, T]$ for almost all ω. Note also that

$$
\begin{aligned}
\mathbb{E}\|u_n(t)\|_\rho^2 = \mathbb{E}\|u_0 &+ \int_0^t [-\nu A u_{n-1}(s) - \lambda u_{n-1}(s) + f(u_{n-1}(s)) + g(s)]\mathrm{d}s \\
&+ \int_0^t \sigma(s, u_{n-1}(s))\mathrm{d}W(s)\|_\rho^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq 3\mathbb{E}\|u_0\|_\rho^2 + [12T(18\nu^2 + \lambda^2 + K) + 3K]\mathbb{E}\int_0^t \|u_{n-1}(s)\|_\rho^2 \mathrm{d}s \\
&\quad + 12KT^2 + 12K_1T^2 + 3KT.
\end{aligned}
$$

From this inequality, for all $k \geq 1$, we have

$$
\begin{aligned}
&\max_{1\leq n\leq k} \mathbb{E}\|u_n(t)\|_\rho^2 \\
&\leq 3\mathbb{E}\|u_0\|_\rho^2 + [12T(18\nu^2 + \lambda^2 + K) + 3K]\mathbb{E}\int_0^t \max_{1\leq n\leq k} \|u_{n-1}\|_\rho^2 \mathrm{d}s \\
&\quad + 12KT^2 + 12K_1T^2 + 3KT \\
&\leq 3\mathbb{E}\|u_0\|_\rho^2 + [12T(18\nu^2 + \lambda^2 + K) + 3K]\mathbb{E}\int_0^t [\|u_0\|_\rho^2 + \max_{1\leq n\leq k} \|u_n\|_\rho^2] \mathrm{d}s \\
&\quad + 12KT^2 + 12K_1T^2 + 3KT \\
&\leq 3\mathbb{E}\|u_0\|_\rho^2 + 12KT^2 + 12K_1T^2 + 3KT + [12T(18\nu^2 + \lambda^2 + K) + 3KT]\mathbb{E}\|u_0\|_\rho^2 \\
&\quad + [12T(18\nu^2 + \lambda^2 + K) + 3K]\int_0^t [\max_{1\leq n\leq k} \mathbb{E}\|u_n\|_\rho^2] \mathrm{d}s.
\end{aligned}
$$

Let $K_3 = 12KT^2 + 12K_1T^2 + 3KT + [12T(18\nu^2 + \lambda^2 + K) + 3 + 3KT]\mathbb{E}\|u_0\|_\rho^2$. Then by Gronwall's inequality, it holds that

$$
\max_{1\leq n\leq k} \mathbb{E}\|u_n(t)\|_\rho^2 \leq K_3 e^{12T^2(18\nu^2+\lambda^2+K)+3KT}.
$$

So, we have

$$
\mathbb{E}\|u_n(t)\|_\rho^2 \leq K_3 e^{12T^2(18\nu^2+\lambda^2+K)+3KT},
$$

for $t \in [0,T]$, $n \geq 1$, which implies that $\mathbb{E}\|u(t)\|_\rho^2 < \infty$ for $t \in [0,T]$.

We proceed to prove that $u(t)$ satisfies system (2.4) with $u_0 \in l_\rho^2 \times \mathbb{S}$. It is not difficult to verify that

$$
\begin{aligned}
&\mathbb{E}[\| \int_0^t [-\nu A(u_n(s) - u(s)) - \lambda(u_n(s) - u(s)) + (f(u_n(s)) - f(u(s)))] \mathrm{d}s \\
&+ \int_0^t \sigma(s, u_n(s)) - \sigma(s, u(s)) \mathrm{d}W(s)\|_\rho^2] \\
&\leq [6T(18\nu^2 + \lambda^2 + L) + 2L]\mathbb{E}\int_0^t \|u_n(s) - u(s)\|_\rho^2 \mathrm{d}s \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

Hence $u(t)$ satisfies (2.3). □

The proof of Theorem 2.2 is inspired by proofs of [19, Theorem 3.1] and [27, Theorem 3.1].

3. Existence of periodic solutions

In this section, we establish the criterion for the existence of the periodic solution in distribution of (2.3) in l_ρ^2. First, we give the definition for the periodic solution in distribution.

Definition 3.1. A solution $u(t)$ of (2.3) is said to be a θ-periodic solution in distribution if for any $t \in \mathbb{R}^+$, $u(t)$ satisfies the following conditions:

(i) $P \circ [u(t)]^{-1} = P \circ [(u(t+\theta))]^{-1}$;

(ii) there exists $\overline{W}(t)$ such that $u(t+\theta)$ is a solution of the equation

$$\mathrm{d}u(t) = [-\nu Au(t) - \lambda u(t) + f(u(t)) + g(t)]\mathrm{d}t + \sigma(t,u)\mathrm{d}\overline{W}(t),$$

where $\overline{W}(t)$ has the same distribution with $W(t)$.

Definition 3.2. A sequence of probability measures $\mu_n \in \mathcal{P}(l^2_\rho)$ is said to be weakly convergent to a probability measure $\mu \in \mathcal{P}(l^2_\rho)$, if

$$\int_{l^2_\rho} f(x)\mu_n(\mathrm{d}x) \to \int_{l^2_\rho} f(x)\mu(\mathrm{d}x) \quad \text{as } n \to \infty,$$

where $f(x)$ is any continuous bounded function on l^2_ρ.

Definition 3.3. A sequence of l^2_ρ-valued stochastic processes $\{X_n(t)\}$ is said to be convergent in distribution to an l^2_ρ-valued stochastic process $X(t)$ if the distribution of $\{X_n(t)\}$ converges weakly to the distribution of $X(t)$ for all $t \in \mathbb{R}^+$.

Next, we estimate the p-th moment of the solution $u(t)$.

Lemma 3.4. *Let $p > 2$ and $\xi \in L^p(\Omega, l^2_\rho)$. Suppose that Assumption 2.1 holds. Then for all $t \in [0,T]$,*

$$\mathbb{E}(\sup_{0\le s\le t} \|u(t)\|^p_\rho) \le (1 + 3^{p-1}\mathbb{E}\|u_0\|^p_\rho)e^{at},$$

where

$$\begin{aligned} a = \max\Big\{&(12T)^{p-1}[3^{p-1}(2+2^p)\nu^p + \lambda^p + 2^{\frac{p}{2}-1}K^{p/2}] \\ &+ 3^{p-1}2^{\frac{p}{2}-1}\big(\frac{p(p-1)}{2}\big)^{p/2}T^{\frac{p-2}{2}},\ (2^{\frac{p}{2}-1}K^{p/2} + K_1)(12T)^{p-1} \\ &+ 3^{p-1}2^{\frac{p}{2}-1}\big(\frac{p(p-1)}{2}\big)^{p/2}T^{\frac{p-2}{2}}\Big\}. \end{aligned}$$

Proof. Note that

$$\begin{aligned} \|-\nu Au(s)\|^p_\rho &= \sum_{i\in\mathbb{Z}} \rho_i[\nu^p(u_{i+1}(s) - 2u_i(s) + u_{i-1}(s))^p] \\ &\le 3^{p-1}\nu^p \sum_{i\in\mathbb{Z}}[|u_{i+1}(s)|^p + 2^p|u_i(s)|^p + |u_{i-1}(s)|^p] \\ &\le 3^{p-1}(2+2^p)\nu^p\|u(s)\|^p_\rho. \end{aligned} \tag{3.1}$$

By Hölder inequality, [20, Theorem 1.7.2], Assumption 2.1, and (3.1), we obtain

$$\begin{aligned} &\mathbb{E}\big(\sup_{0\le s\le t} \|u(t)\|^p_\rho\big) \\ &= \mathbb{E}\big(\sup_{0\le s\le t} \|u_0 + \int_0^s [-\nu Au(s) - \lambda u(s) + f(u(s)) + g(r)]\mathrm{d}r \\ &\quad + \int_0^s \sigma(r,u(r))\mathrm{d}W(r)\|^p_\rho\big) \\ &\le 3^{p-1}\mathbb{E}\|u_0\|^p_\rho + (12t)^{p-1}\mathbb{E}[\int_0^t \|-\nu Au(s) - \lambda u(s) + f(u(s)) + g(s)\|^p_\rho \mathrm{d}s] \\ &\quad + 3^{p-1}\mathbb{E}\big(\sup_{0\le s\le t} \|\int_0^s \sigma(r,u(r))\mathrm{d}W(r)\|^p_\rho\big) \end{aligned}$$

$$
\begin{aligned}
&\leq 3^{p-1}\mathbb{E}\|u_0\|_\rho^p + (12t)^{p-1}[\mathbb{E}\int_0^t \|-\nu Au(s)\|_\rho^p \mathrm{d}s + \mathbb{E}\int_0^t \|-\lambda u(s)\|_\rho^p \mathrm{d}s \\
&\quad + \mathbb{E}\int_0^t \|f(u(s))\|_\rho^p \mathrm{d}s + \mathbb{E}\int_0^t \|g(s)\|_\rho^p \mathrm{d}s] \\
&\quad + 3^{p-1}\big(\frac{p(p-1)}{2}\big)^{p/2} T^{\frac{p-2}{2}} 3^{p-1}\mathbb{E}\int_0^t \|\sigma(s,u(s))\|_\rho^p \mathrm{d}s \\
&\leq 3^{p-1}\mathbb{E}\|u_0\|_\rho^p + (12T)^{p-1}[3^{p-1}(2+2^p)\nu^p + \lambda^p + 2^{\frac{p}{2}-1}K^{p/2}]\mathbb{E}\int_0^t \|u(t)\|_\rho^p \mathrm{d}s \\
&\quad + (12T)^{p-1}2^{\frac{p}{2}-1}K^{p/2}t + (12T)^{p-1}K_1 t \\
&\quad + 3^{p-1}2^{\frac{p}{2}-1}\big(\frac{p(p-1)}{2}\big)^{p/2} T^{\frac{p-2}{2}} K^{p/2}\mathbb{E}\int_0^t (1+\mathbb{E}\|u(s)\|_\rho^2)\mathrm{d}s \\
&\leq 3^{p-1}\mathbb{E}\|u_0\|_\rho^p + a\int_0^t (1+\mathbb{E}\|u(s)\|_\rho^p)\mathrm{d}s,
\end{aligned}
$$

where

$$
\begin{aligned}
a = \max\Big\{&(12T)^{p-1}[3^{p-1}(2+2^p)\nu^p + \lambda^p + 2^{\frac{p}{2}-1}K^{p/2}] \\
&+ 3^{p-1}2^{\frac{p}{2}-1}\big(\frac{p(p-1)}{2}\big)^{p/2} T^{\frac{p-2}{2}},\ (2^{\frac{p}{2}-1}K^{p/2} + K_1)(12T)^{p-1} \\
&+ 3^{p-1}2^{\frac{p}{2}-1}\big(\frac{p(p-1)}{2}\big)^{p/2} T^{\frac{p-2}{2}}\Big\}.
\end{aligned}
$$

Hence

$$
1+\mathbb{E}\Big(\sup_{0\leq s\leq t}\|u(t)\|_\rho^p\Big) \leq 1 + 3^{p-1}\mathbb{E}\|u_0\|_\rho^p + a\int_0^t \Big[1+\mathbb{E}\Big(\sup_{0\leq r\leq s}\|u(r)\|_\rho^p\Big)\Big]\mathrm{d}s.
$$

It follows from Gronwall's inequality that

$$
1+\mathbb{E}\Big(\sup_{0\leq s\leq t}\|u(t)\|_\rho^p\Big) \leq (1+3^{p-1}\mathbb{E}\|u_0\|_\rho^p)e^{at}
$$

for $t \in [0,T]$. Therefore, we obtain

$$
\mathbb{E}\Big(\sup_{0\leq s\leq t}\|u(t)\|_\rho^p\Big) \leq (1+3^{p-1}\mathbb{E}\|u_0\|_\rho^p)e^{at}
$$

for $t \in [0,T]$. □

For (2.3), we make the following assumptions.

Assumption 3.5. Suppose that all the time-dependent coefficient functions are θ-periodic in $t \in \mathbb{R}^+$; that is, for all $t \in \mathbb{R}^+$, $i \in \mathbb{Z}$, $u \in l_\rho^2$

$$
g_i(t+\theta) = g_i(t), \quad \sigma_i(t+\theta, u) = \sigma_i(t,u).
$$

Assumption 3.6. For some $p > 2$ and $n = 0, 1, 2, \dots$, there exists a positive constant C independent of n such that

$$
\mathbb{E}\|u(n\theta)\|_\rho^p \leq C.
$$

Assumption 3.7. The distribution $P \circ [u(t)]^{-1}$ with respect to $u(t)$ satisfies

$$
\lim_{k\to\infty}\frac{1}{n_k+1}\sum_{m=0}^{n_k} d_{BL}(P\circ[u((m+1)\theta)]^{-1}, P\circ[u(m\theta)]^{-1}) = 0,
$$

where $\{n_k\}$ is a sequence of integers tending to $+\infty$.

Lemma 3.4 ensures that Assumption 3.6 is reasonable. In Section 4, we give an example to verify Assumptions 3.6 and 3.7.

Definition 3.8. A family of random variables $\mathcal{H}$ in $L^1(\Omega, l^2_\rho)$ is uniformly integrable, if it satisfies

$$\sup_{\xi\in\mathcal{H}} \int_{\|\xi\|_\rho \geq M} \|\xi\|_\rho \mathrm{d}P \to 0 \quad \text{as } M \to \infty.$$

Theorem 3.9. *Suppose that Assumptions 2.1-3.7 hold. Then there exists a θ-periodic solution in distribution of* (2.3).

Proof. Let γ_k be a random variable independent of $W(t)$ and $u(0,\omega)$ such that

$$P\{\gamma_k = N\theta\} = \frac{1}{k+1}, \quad N = 0, 1, \dots, k,$$

for each $k \in \mathbb{Z}^+$. We define a sequence of stochastic processes

$$\begin{gathered} v_k(t) = u(t+\gamma_k), \\ v_k(0) = u(\gamma_k). \end{gathered}$$

Then $v_k(t)$ is a weak solution of (2.3). In fact, for $C \in \mathcal{B}(l^2_\rho)$ and $t \in \mathbb{R}^+$, we define

$$\overline{W}(t) = W(t+\gamma_k) - W(\gamma_k),$$

where $\overline{W}(t)$ has the same distribution with $W(t)$. Hence we obtain

$$\begin{aligned} &u(t+\gamma_k) \\ &= u(0) + \int_0^{\gamma_k} [-\nu Au(s) - \lambda u(s) + f(u(s)) + g(s)]\mathrm{d}s + \int_0^{\gamma_k} \sigma(s,u(s))\mathrm{d}W(s) \\ &\quad + \int_{\gamma_k}^{t+\gamma_k} [-\nu Au(s) - \lambda u(s) + f(u(s)) + g(s)]\mathrm{d}s + \int_{\gamma_k}^{t+\gamma_k} \sigma(s,u(s))\mathrm{d}W(s) \\ &= u(\gamma_k) + \int_{\gamma_k}^{t+\gamma_k} [-\nu Au(s) - \lambda u(s) + f(u(s)) + g(s)]\mathrm{d}s + \int_{\gamma_k}^{t+\gamma_k} \sigma(s,u(s))\mathrm{d}W(s) \\ &= u(\gamma_k) + \int_0^t [-\nu Au(s+\gamma_k) - \lambda u(s+\gamma_k) + f(u(s+\gamma_k)) \\ &\quad + g(s+\gamma_k)]\mathrm{d}s + \int_0^t \sigma(s+\gamma_k, u(s+\gamma_k))\mathrm{d}\overline{W}(s). \end{aligned}$$

From the construction of $v_k(t)$ and the independence of γ_k, we have

$$\begin{aligned} P\{v_k(t) \in A\} &= P\{u(t+\gamma_k) \in A\} \\ &= P\{u(t+\gamma_k) \in A | \gamma_k = 0\} P\{\gamma_k = 0\} \\ &\quad + P\{u(t+\gamma_k) \in A | \gamma_k = \theta\} P\{\gamma_k = \theta\} + \dots \\ &\quad + P\{u(t+\gamma_k) \in A | \gamma_k = k\theta\} P\{\gamma_k = k\theta\} \\ &= \frac{1}{k+1} \sum_{N=0}^{k} P\{u(t+N\theta) \in A\} \end{aligned} \tag{3.2}$$

for each $A \in \mathcal{B}(l_\rho^2)$. From (3.2), Assumption 3.6, and Chebyshev's inequality, it follows that uniformly in k,

$$\begin{aligned} P\{\|v_k(0,\omega)\|_\rho > R\} &= \frac{1}{k+1}\sum_{N=0}^{k} P\{\|u(N\theta)\|_\rho > R\} \\ &\le \frac{1}{k+1}\sum_{N=0}^{k} \frac{\mathbb{E}\|u(N\theta)\|_\rho^2}{R^2} \to 0 \quad \text{as } R \to \infty. \end{aligned}$$

So, $v_k(0,\omega)$ satisfies conditions of [7, Theorem 7.2]. According to Skorohod theorem [25, p 13] in another probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$, there exists a sequence $\widetilde{v}_k(0,\widetilde{\omega})$ $(k = 0, 1, \dots)$ with the same distribution as $v_k(0,\omega)$. Furthermore, there exists a subsequence $\widetilde{v}_{n_k}(0,\widetilde{\omega})$ that converges to $\widetilde{v}(0,\widetilde{\omega})$ in probability. We can construct l_ρ^2-valued random variables $v(0,\omega)$ and $v_{n_k}(0,\omega)$ on $(\Omega, \mathcal{F}, P)$ with the same distribution as $\widetilde{v}(0,\widetilde{\omega})$ and $\widetilde{v}_{n_k}(0,\widetilde{\omega})$, respectively. From Assumption 3.6, we have

$$\mathbb{E}\|\widetilde{v}_{n_k}(0,\widetilde{\omega})\|_\rho^p = \mathbb{E}\|v_{n_k}(0,\omega)\|_\rho^p \le C < \infty.$$

for some $p > 2$. By [1, Proposition 2.5.7], $\|\widetilde{v}_{n_k}(0,\widetilde{\omega})\|_\rho^2$ is uniformly integrable. It follows from [1, Theorem 2.5.9] that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $A \in \mathcal{F}$ with $P(A) \le \delta$, we have $\sup_{\xi \in \mathcal{H}} \int_A \|\widetilde{v}_{n_k}(0,\widetilde{\omega})\|_\rho^2 \mathrm{d}P \le \varepsilon$. According to Vitali's convergence theorem, we have

$$\mathbb{E}\|\widetilde{v}_{n_k}(0,\widetilde{\omega}) - \widetilde{v}(0,\widetilde{\omega})\|_\rho^2 \to 0 \quad \text{as } n_k \to \infty.$$

Let $\widetilde{v}_{n_k}(t)$ be the solution of the equation

$$\mathrm{d}u(t) = [-\nu A u(t) - \lambda u(t) + f(u(t)) + g(t)]\mathrm{d}t + \sigma(t, u(t))\mathrm{d}\overline{W}(t),$$

with initial condition $\widetilde{v}_{n_k}(0,\widetilde{\omega}) = \widetilde{v}_{n_k}(\widetilde{\omega})$ on the probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$. By Cauchy-Schwarz's inequality, Itô's isometry, and Assumption 2.1, we obtain

$$\begin{aligned} &\mathbb{E}\|\widetilde{v}_{n_k}(t) - \widetilde{v}(t)\|_\rho^2 \\ &\le 3\mathbb{E}\|\widetilde{v}_{n_k}(0,\widetilde{\omega}) - \widetilde{v}(0,\widetilde{\omega})\|_\rho^2 + [9t(18\nu^2 + \lambda^2 + L) + 3L]\mathbb{E}\int_0^t \|\widetilde{v}_{n_k}(s) - \widetilde{v}(s)\|_\rho^2 \mathrm{d}s. \end{aligned}$$

By Gronwall's inequality, we have

$$\mathbb{E}\|\widetilde{v}_{n_k}(t) - \widetilde{v}(t)\|_\rho^2 \le 3\mathbb{E}\|\widetilde{v}_{n_k}(0,\widetilde{\omega}) - \widetilde{v}(0,\widetilde{\omega})\|_\rho^2 e^{9t^2(18\nu^2+\lambda^2+L)+3Lt} \to 0$$

as $n_k \to \infty$. It follows from the uniqueness of weak solution that

$$P \circ [v_{n_k}(t)]^{-1} = P \circ [\widetilde{v}_{n_k}(t)]^{-1} \to P \circ [\widetilde{v}(t)]^{-1} \tag{3.3}$$

uniformly on $[0,\theta]$. In addition, $v(0,\omega)$ on $(\Omega, \mathcal{F}, P)$ has the same distribution as $\widetilde{v}(0,\omega)$ on $(\widetilde{\Omega}, \widetilde{F}, \widetilde{P})$. From the uniqueness of the weak solution of (2.3), $v(t)$ admits the same distribution with $\widetilde{v}(t)$. By (3.3), (3.2) and Assumption 3.7, we derive

$$\begin{aligned} &d_{BL}(P \circ [v(\theta)]^{-1}, P \circ [v(0)]^{-1}) \\ &= \lim_{k\to\infty} d_{BL}(P \circ [v_{n_k}(\theta)]^{-1}, P \circ [v_{n_k}(0)]^{-1}) \\ &= \lim_{k\to\infty} \sup_{\|\varphi\|_{BL}\le 1} \Big(\int_{l_\rho^2} \varphi \mathrm{d}P \circ [v_{n_k}(\theta)]^{-1} - \int_{l_\rho^2} \varphi \mathrm{d}P \circ [v_{n_k}(0)]^{-1} \Big) \\ &= \lim_{k\to\infty} \sup_{\|\varphi\|_{BL}\le 1} \Big(\int_\Omega \varphi(v_{n_k}(\theta))\mathrm{d}P - \int_\Omega \varphi(v_{n_k}(0))\mathrm{d}P \Big) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \sup_{\|\varphi\|_{BL}\le 1} \Big(\frac{1}{n_k+1} \sum_{N=0}^{n_k} \int_\Omega [\varphi(u((N+1)\theta)) - \varphi(u(N\theta))] \mathrm{d}P \Big) \\
&= \lim_{k\to\infty} \frac{1}{n_k+1} \sum_{N=0}^{n_k} d_{BL}(P\circ [u((N+1)\theta)]^{-1}, P\circ [u(N\theta)]^{-1}) = 0.
\end{aligned}
$$

That is to say, $v(\theta)$ has the same distribution as $v(0)$.

We define $z(t) : \mathbb{R}^+ \to l^2_\rho$ by

$$
z(t) = v(t - n_t\theta),
$$

where $n_t = \max\{n \in \mathbb{N} | n\theta < t\}$. Hence, $z(t)$ is a θ-periodic solution in distribution of (2.3). $\square$

4. Applications

It is worth noting that Lyapunov's method can also be applied to prove the existence of periodic solution in distribution.

Example 4.1. Consider the equation of motion of Hooke's law in l^2_ρ:

$$
\mathrm{d}u_i(t) = [\nu(u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)) - \lambda u_i(t)]\mathrm{d}t + \sigma_i(t, u(t))\mathrm{d}W_i(t), \tag{4.1}
$$

where $\lambda > 0$ is a constant. Suppose that Assumption 2.1 holds. Here $-\lambda u_i$ describes the strength of negative feedback, where $\lambda > 2L$. We make the following assumptions:

- (A1) $\rho(i) \le c_0\rho(i\pm 1)$, for all $i \in \mathbb{Z}$, where c_0 is a positive constant with $c_0 < 1 + \frac{\lambda - 2L}{2\nu}$.
- (A2) there exists a positive constant c_1 such that
$$
(2\nu c_0 - 2\nu - \lambda)\|u\|^4_\rho + \|u\|^2_\rho |\sigma(t,u)|^2 + |\langle u, \sigma(t,u)\rangle|^2 \le -c_1\|u\|^4_\rho
$$
for all $u \in l^2_\rho$, $t \in \mathbb{R}^+$.
- (A3) For all $u, v \in l^2_\rho$, $t \in \mathbb{R}^+$, there exists a constant c_2 such that
$$
\sum_{i=1}^{\infty} J^T_{\sigma_i}(t,u) J_{\sigma_i}(t,v) \le -c_2 I
$$
where $c_2 > 2(2c_0\nu - 2\nu - \lambda)/3$.

Then (4.1) admits a periodic solution in distribution.

We define $f(t,u) = \|u\|^4_\rho$. By (A1) and (A2), we have

$$
\begin{aligned}
&\mathcal{L}f(t,u) \\
&= f_t(t,u) + \langle f_u(t,u), -\nu Au - \lambda u\rangle + \frac{1}{2}\mathrm{trace}(\sigma^T(t,u) f_{uu}(t,u)\sigma(t,u)) \\
&\le 4\nu\|u\|^2_\rho \sum_{i\in\mathbb{Z}} \rho_i u_i(u_{i+1} - 2u_i + u_{i-1}) - 4\lambda\|u\|^4_\rho + 2\|u\|^2_\rho|\sigma(t,u)|^2 + 4|\langle u, \sigma(t,u)\rangle|^2 \\
&\le 4\nu\|u\|^2_\rho \sum_{i\in\mathbb{Z}} \rho_i u_i(c_0 u_i - 2u_i + c_0 u_i) - 4\lambda\|u\|^4_\rho + 2\|u\|^2_\rho|\sigma(t,u)|^2 + 4|\langle u, \sigma(t,u)\rangle|^2 \\
&= 4(2\nu c_0 - 2\nu - \lambda)\|u\|^4_\rho + 2\|u\|^2_\rho|\sigma(t,u)|^2 + 4|\langle u, \sigma(t,u)\rangle|^2 \\
&\le -4c_1\|u\|^4_\rho.
\end{aligned}
$$

So, we have

$$\begin{aligned}\mathbb{E}[f(t,u_\xi(t))] &= \mathbb{E}[f(0,\xi)] + \mathbb{E}\int_0^t \mathcal{L}_i f(s,u(s))\mathrm{d}s\\ &\le \mathbb{E}[f(0,\xi)] - 4c_1\int_0^t \mathbb{E}[f(s,u(s))]\mathrm{d}s.\end{aligned}$$

It follows from Gronwall's inequality that for all $\mathbb{E}\|\xi\|_\rho^4 < \infty$, and $t \in \mathbb{R}^+$, that

$$\mathbb{E}\|u_\xi(t)\|_\rho^4 \le \mathbb{E}[f(t,u_\xi(t))] \le \mathbb{E}[f(0,\xi)]e^{-4c_1 t}.$$

Therefore, Assumption 3.6 is fulfilled. Furthermore, by (A1),

$$\begin{aligned}&\mathcal{L}f(t,u-v)\\ &= f_t(t,u-v) + \langle f_u(t,u-v), -\nu A(u-v) - \lambda(u-v)\rangle\\ &\quad + \frac{1}{2}\mathrm{trace}[(\sigma(t,u)-\sigma(t,v))^T f_{uu}(t,u-v)(\sigma(t,u)-\sigma(t,v))]\\ &= 4\nu\|u-v\|_\rho^2\sum_{i\in\mathbb{Z}}\rho_i(u_i-v_i)[(u_{i+1}-v_{i+1}) - 2(u_i-v_i) + (u_{i-1}-v_{i-1})]\\ &\quad - 4\lambda\|u-v\|_\rho^4 + 6\|u-v\|_\rho^2(u-v)^T\Big[\sum_{i=1}^\infty\Big(\int_0^1 J_{\sigma_i}^T(t,v+s(u-v))\mathrm{d}s\Big)\\ &\quad\times\Big(\int_0^1 J_{\sigma_i}^T(t,v+s(u-v))\mathrm{d}s\Big)\Big](u-v)\\ &\le 4\nu(2c_0-2)\|u-v\|_\rho^4 - 4\lambda\|u-v\|_\rho^4\\ &\quad + 6\|u-v\|_\rho^2(u-v)^T\Big[\sum_{i=1}^\infty\Big(\int_0^1 J_{\sigma_i}^T(t,v+s(u-v))\mathrm{d}s\Big)\\ &\quad\times\Big(\int_0^1 J_{\sigma_i}^T(t,v+s(u-v))\mathrm{d}s\Big)\Big](u-v)\\ &= [4(2c_0\nu - 2\nu - \lambda) - 6c_2]\,\|u-v\|_\rho^4.\end{aligned}$$

Let $c_3 = 4(2c_0\nu - 2\nu - \lambda) - 6c_2$. For any given $\xi,\eta \in L^2(\Omega, l_\rho^2)$, applying Itô's formula to $f(t,u_\xi(t)-u_\eta(t))e^{-c_3 t}$, we obtain

$$\begin{aligned}\mathbb{E}[f(t,u_\xi(t)-u_\eta(t))e^{-c_3 t}] &= \mathbb{E}[f(0,\xi-\eta)] + \int_0^t -c_3 e^{-c_3 s}\mathbb{E}[f(s,u_\xi(s)-u_\eta(s))]\mathrm{d}s\\ &\quad + \int_0^t e^{-c_3 s}\mathbb{E}[\mathcal{L}_i f(s,u_\xi(s)-u_\eta(s))]\mathrm{d}s\\ &\le \mathbb{E}[f(0,\xi-\eta)].\end{aligned}$$

Hence $\mathbb{E}[f(t,u_\xi(t)-u_\eta(t))] \le \mathbb{E}[f(0,\xi-\eta)]e^{c_3 t}$ for each $t \in \mathbb{R}^+$. In addition, for each $t \in \mathbb{R}^+$, there exists $k \in \mathbb{N}$ such that $t \in [k\theta, k\theta+\theta]$. By Assumption 2.1 and [20, Theorem 1.7.1], we have

$$\begin{aligned}&\mathbb{E}\|u_\xi(t)-u_\eta(t)\|_\rho^4\\ &\le 27\mathbb{E}\|u_\xi(k\theta)-u_\eta(k\theta)\|_\rho^4 + 27t^3\mathbb{E}\int_{k\theta}^t \| -\nu A(u_\xi(s)-u_\eta(s)) - \lambda(u_\xi(s)-u_\eta(s))\\ &\quad + f(u_\xi(s)) - f(u_\eta(s))\|_\rho^4\mathrm{d}s + 972T\mathbb{E}\int_{k\theta}^t \|\sigma(s,u_\xi(s))-\sigma(s,u_\eta(s))\|_\rho^4\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\le 27\mathbb{E}\|u_\xi(k\theta)-u_\eta(k\theta)\|^4+(9T)^3\mathbb{E}\int_{k\theta}^t\|-\nu A(u_\xi(s)-u_\eta(s))\|_\rho^4\mathrm{d}s\\
&\quad+(9T)^3\mathbb{E}\int_{k\theta}^t\|-\lambda(u_\xi(s)-u_\eta(s))\|_\rho^4\mathrm{d}s+(9T)^3\mathbb{E}\int_{k\theta}^t\|f(u_\xi(s))-f(u_\eta(s))\|_\rho^4\mathrm{d}s\\
&\quad+972T\mathbb{E}\int_{k\theta}^t\|\sigma(s,u_\xi(s))-\sigma(s,u_\eta(s))\|_\rho^4\mathrm{d}s\\
&=27\mathbb{E}\|u_\xi(k\theta)-u_\eta(k\theta)\|^4+486(9T)^3\nu^4\int_{k\theta}^t\mathbb{E}\|u_\xi(s)-u_\eta(s))\|_\rho^4\mathrm{d}s\\
&\quad+(9T)^3\lambda^4\int_{k\theta}^t\mathbb{E}\|u_\xi(s)-u_\eta(s))\|_\rho^4\mathrm{d}s+(9T)^3L^2\int_{k\theta}^t\mathbb{E}\|u_\xi(s)-u_\eta(s))\|_\rho^4\mathrm{d}s\\
&\quad+972TL^2\int_{k\theta}^t\mathbb{E}\|u_\xi(s)-u_\eta(s))\|_\rho^4\mathrm{d}s\\
&=27\mathbb{E}\|u_\xi(k\theta)-u_\eta(k\theta)\|_\rho^4+c_4\int_{k\theta}^t\mathbb{E}\|u_\xi(k\theta)-u_\eta(k\theta)\|_\rho^4\mathrm{d}s,
\end{aligned}
$$

where $c_4 := (9T)^3(486\nu^4+\lambda^4+L^2\})+972L^2T$. Applying Gronwall's inequality, we have

$$
\begin{aligned}
\mathbb{E}\|u_\xi(t)-u_\eta(t)\|_\rho^4 &\le 27\mathbb{E}\|u_\xi(k\theta)-u_\eta(k\theta)\|_\rho^4 e^{c_4T}\\
&\le 27e^{c_4T}\mathbb{E}[f(k\theta,u_\xi(k\theta)-u_\eta(k\theta))]\\
&\le 27e^{c_4T+c_3k\theta}\mathbb{E}[f(0,\xi-\eta)]\\
&=\varphi(k\theta)\mathbb{E}\|\xi-\eta\|_\rho^2,
\end{aligned}
\tag{4.2}
$$

where $\varphi(t):=27e^{c_4T+c_3k\theta}$. Note that $c_3<0$, we have $\lim_{k\to\infty}\varphi(k\theta)=0$. Hence, there exists a $k_0>0$ such that $\sup_{k>k_0}\varphi(k\theta)<1$. By the contraction mapping fixed point theorem, there exists a unique fixed point $\xi_*\in L^4(\Omega,l_\rho^4)$ such that $u_{\xi_*}(k\theta)=\xi^*$ for any $k>k_0$. Thus,

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^n d_{BL}(P\circ[u_{\xi^*}((k+1)\theta)]^{-1},P\circ[u_{\xi^*}(k\theta)]^{-1})\\
&=\lim_{n\to\infty}\frac{1}{n+1}\Big(\sum_{k=0}^{k_0} d_{BL}(P\circ[u_{\xi^*}((k+1)\theta)]^{-1},P\circ[u_{\xi^*}(k\theta)]^{-1})\\
&\quad+\sum_{k=k_0}^n d_{BL}(P\circ[u_{\xi^*}((k+1)\theta)]^{-1},P\circ[u_{\xi^*}(k\theta)]^{-1})\Big)=0
\end{aligned}
$$

Hence Assumption 3.7 is satisfied. Therefore, (4.1) admits a periodic solution in distribution.

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Yue Gao
School of Mathematics, Jilin University, Changchun 130012, China
Email address: `gaoyue19972022@163.com`

Xue Yang
School of Mathematics, Jilin University, Changchun 130012, China
Email address: xueyang@jlu.edu.cn