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EXISTENCE OF SEMI-NODAL SOLUTIONS FOR ELLIPTIC SYSTEMS RELATED TO GROSS-PITAEVSKII EQUATIONS

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Abstract. In this work we consider existence of semi-nodal solutions, i.e., solutions of the form (u, v) with $u > 0$ and $v^{\pm} := \max\{0, \pm v\} \not\equiv 0$ for a class of elliptic systems related to the Gross-Pitaevskii equation.

1. Introduction

This work concerns the elliptic system

$$\begin{gathered}-\Delta u = \lambda_1 u + \mu_1 |u|^{2p-2} u + \beta |u|^{p-2} u |v|^q, \quad \text{in } \Omega \\ -\Delta v = \lambda_2 v + \mu_2 |v|^{2q-2} v + \beta |u|^p |v|^{q-2} v, \quad \text{in } \Omega \\ u = v = 0, \quad \text{on } \partial\Omega\,.\end{gathered} \tag{1.1}$$

For $p = q = 2$ the cubic system (1.1) arises in mathematical models for various physics problems, especially in nonlinear optics and Bose-Einstein condensation, see [14, 17]. In those works present information on the physical significance of non-cubic nonlinearities and on the existence and multiplicity of solutions. Furthermore, when $\lambda_i < 0$, system (1.1) comes from the study of solitary wave solutions of the coupled Gross-Pitaevskii equations,

$$\begin{gathered}-i\frac{\partial}{\partial t}\Phi_1 = \Delta\Phi_1 + \mu_1|\Phi_1|^2\Phi_2 + \beta\Phi_2^2\Phi_1, \quad x \in \Omega,\ t > 0 \\ -i\frac{\partial}{\partial t}\Phi_2 = \Delta\Phi_2 + \mu_2|\Phi_2|^2\Phi_1 + \beta\Phi_1^2\Phi_2, \quad x \in \Omega,\ t > 0 \\ \Phi_j = \Phi_j(x,t) \in \mathbb{C}, \quad j = 1, 2 \\ \Phi_j(x,t) = 0, \quad x \in \partial\Omega,\ t > 0,\ j = 1, 2.\end{gathered} \tag{1.2}$$

When $\Phi_1(x,t) = e^{-i\lambda_1 t}u$ and $\Phi_2(x,t) = e^{-i\lambda_2 t}v$, system (1.2) reduces to (1.1). In the Kerr-like photorefractive media, the solution Φ_j represents the j^{th} element of the beam (see [2]). The self-focusing in the j^{th} component of the beam is related to the positive constant μ_j, whereas the coupling constant $\beta > 0$ signifies the interaction between the two beam components. When $\mu_j = 0$, the self-focusing

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has been suppressed, and this type of situation is also relevant in optics (see for example [15, 16, 20]). The problem denoted by system (1.2) is also encountered in the Hartree-Fock theory for a binary mixture of Bose-Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ (see for example [13]). In this context, each Φ_j represents the corresponding condensate amplitude, while μ_j and β denote the intra and interspecies scattering lengths. The self-interactions of the single state $|j\rangle$ are represented by the sign of μ_j, with $\mu_j > 0$ indicating the focusing case and $\mu_j < 0$ corresponding to the defocusing case. When the intraspecies scattering length μ_j is zero, it means that the interaction between particles of the same species is extremely weak or nonexistent (see for example [22]). In addition, the sign of β plays a crucial role in determining whether the interactions between states $|1\rangle$ and $|2\rangle$ are attractive or repulsive. Specifically, if $\beta > 0$, the interactions are attractive, while $\beta < 0$ implies that the interactions are repulsive. This feature is important in understanding the competition between different states and can have a significant impact on the behavior of the system as a whole.

Recently, there has been growing interest in studying systems of the form (1.1) that are related to the system (1.2) in the cubic case $p = q = 2$. This is evidenced by the increasing number of research papers published on the topic, among which we highlight [1, 3, 8, 9, 10, 18, 19, 21, 25] and references therein. On this subject, we also refer the interested reader to [23].

In this work, we investigate the existence of semi-nodal solutions for system (1.1), that is, solutions where $u > 0$ in Ω and $v^\pm := \max\{0, \pm v\} \not\equiv 0$ in Ω, which has also received attention in recent studies, in particular, we are interested in the case where $\mu_1 = \mu_2 = 0$ and $p + q < 2^*$, $\beta > 0$ and $N \le 5$. Clapp and Soares [11] dealt with the case where $p = q < 2^*/2 = N/(N-2)$, $\lambda_j = -1$, $\mu_j = 0$ and $\Omega = \mathbb{R}^N$ with $N \ge 4$, among other results, they showed the existence of semi-nodal solutions subject to the mentioned conditions. Chen, Lin & Zou [5, 6] dealt with the case $p = q = 2$, $\lambda_j < 0$, $\mu_j > 0$, $\beta > 0$, and $\Omega \subset \mathbb{R}^N$ bounded with $N \in \{1, 2, 3\}$, they showed existence and multiplicity results of nodal solutions for (1.1). In [7], the same authors provided the existence of semi-nodal solutions for (1.1) for the critical case $p = q = 2^*/2$ with $\Omega \subset \mathbb{R}^N$ bounded, $N \ge 6$, $\mu_j > 0$, $\lambda_j \in (0, \lambda_1(\Omega))$ and $\beta < 0$, here $\lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. In this work, we are interested in the case where $\mu_j = 0$, $\lambda_j < \lambda_1(\Omega)$, $\beta > 0$, $p > 1$, $q > 2$ with $p + q < 2^*$. In particular, $3 < p + q < 2^*$ which implies that $3 < 2^*$. Hence our main result applies only for the cases $N \in \{3, 4, 5\}$ where $p + q < 2^*$. Furthermore, assuming that $N \in \{1, 2\}$, it suffices that $p > 1$ and $q > 2$ because $2^* = +\infty$. For the sake of convenience, we will change the notation of system (1.1) to this case, more specifically, we will consider the system

$$\begin{gathered} -\Delta u = \lambda u + \xi u^{p-1}|v|^q, \quad \text{in } \Omega \\ -\Delta v = \mu v + \tau u^p |v|^{q-2} v, \quad \text{in } \Omega \\ u = v = 0, \quad \text{on } \partial\Omega \\ u > 0, \quad v^\pm \not\equiv 0 \quad \text{in } \Omega. \end{gathered} \tag{1.3}$$

Our main result reads as follows.

Theorem 1.1. *Assume that $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $p > 1$, $q > 2$ with $p + q < 2^* = 2N/(N-2)$ for $N \in \{3, 4, 5\}$, and $p + q < +\infty$ for $N \in \{1, 2\}$,*

$\lambda, \mu < \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. Then there exists a pair of solution $u, v \in C^2(\overline{\Omega})$ to (1.3).

Our approach is based on minimization arguments presented in [4, 24] with the necessary technical modifications. The main difficulties in our approach are to avoid semi-trivial solutions (i.e. solution of the form $(u, 0)$ or $(0, v)$) and to construct Palais-Smale sequence that converges to the infimum of the functional associated with system (2.21) restricted to a certain subset of the Nehari manifold.

2. Main result

To present our main result, we use the following notation: $\mathcal{H} := H_0^1(\Omega) \times H_0^1(\Omega)$, $\|u\|_\lambda^2 := \|u\|^2 - \lambda|u|_2^2$, and $\|v\|_\mu^2 := \|v\|^2 - \mu|v|_2^2$, where $\|f\| := \big(\int_\Omega |\nabla f|^2 dx\big)^{1/2}$ is the norm of $H_0^1(\Omega)$, we will write $|f|_s$ as the norm of $L^s(\Omega)$ and $f^\pm(x) := \max\{0, \pm f(x)\}$. Given the condition $\lambda, \mu < \lambda_1(\Omega)$, we see that there exists $C_{\lambda\mu} := C_{\lambda\mu}(\lambda_1(\Omega))$ such that

$$C_{\lambda\mu}\|u\| \le \|u\|_\lambda \le C_{\lambda\mu}^{-1}\|u\| \quad \text{and} \quad C_{\lambda\mu}\|v\| \le \|v\|_\mu \le C_{\lambda\mu}^{-1}\|v\| \quad \text{for all } u, v \in H_0^1(\Omega) \tag{2.1}$$

To obtain solutions for system (1.3), we define the functional $I_{\lambda\mu} \in C^1(\mathcal{H}, \mathbb{R})$ given by

$$I_{\lambda\mu}(u, v) = \frac{1}{2}\|u\|_\lambda^2 + \frac{1}{2}\|v\|_\mu^2 - \frac{1}{p+q}\int_\Omega |u|^p|v|^q\,dx$$

Here we shall follow same ideas from [4] which allows us to minimize the functional $I_{\lambda\mu}$ over the following subsets of the Nehari manifold

$$\mathcal{N}_\lambda := \{(u, v) \in \mathcal{H} : I'_{\lambda\mu}(u, v)(u, 0) = 0,\ u \not\equiv 0,\ v \not\equiv 0\},$$
$$\mathcal{N}_\mu^\pm := \{(u, v) \in \mathcal{H} : I'_{\lambda\mu}(u, v)(0, v^\pm) = 0,\ u \not\equiv 0,\ v^\pm \not\equiv 0\},$$
$$\mathcal{M}_{\lambda\mu} := \mathcal{N}_\lambda \cap \mathcal{N}_\mu^+ \cap \mathcal{N}_\mu^-$$

The following result is of fundamental importance for constructing a Palais-Smale sequence at the level where we obtain solutions to our problem. This approach is based on an idea presented in the work [24].

Lemma 2.1. *Let $(u, v) \in \mathcal{M}_{\lambda\mu}$ and $z, w \in H_0^1(\Omega) \setminus \{0\}$ then for all $\delta > 0$ there are unique positive numbers $t = t(\delta)$, $r = r(\delta)$, and $s = s(\delta)$ such that*

$$\big(t(u - \delta z), r(v - \delta w)^+ - s(v - \delta w)^-\big) \in \mathcal{M}_{\lambda\mu}.$$

Moreover if $\|z\|, \|w\| \le 1$ and $\|u\|, \|v\| \le M_1$, then there are constants $M_0 = M_0(p, q, \lambda, \mu, \lambda_1(\Omega), \Omega, M_1, N) > 0$, and $C_i = C_i(p, q, \lambda, \mu, \lambda_1(\Omega), \Omega, M_1, N) > 0$ such that

$$|t'(0)|, |r'(0)|, |s'(0)| \in [0, C_2] \quad \textit{and} \quad |t(0)|, |r(0)|, |s(0)| \in [C_1, C_2], \tag{2.2}$$
$$\|u\|, \|v^\pm\| \ge M_0 \tag{2.3}$$

Proof. Firstly, we mention that $(\phi, \varphi) \in \mathcal{M}_{\lambda\mu}$ if and only if

$$\|\phi\|_\lambda^2 = \frac{p}{p+q}\int |\phi|^p|\varphi|^q dx \quad \text{and} \quad \|\varphi^\pm\|_\mu^2 = \frac{q}{p+q}\int |\phi|^p|\varphi^\pm|dx.$$

Therefore, for each $(t(u-\delta z), r(v-\delta w)^+ - s(v-\delta w)^-) \in \mathcal{M}_{\lambda\mu}$, we obtain that

$$\begin{aligned} \|t(u-\delta z)\|_\lambda^2 &= \frac{p}{p+q}\int |t(u-\delta z)|^p |r(v-\delta w)^+ - s(v-\delta w)^-|^q \, dx \\ \|r(v-\delta w)^+\|_\mu^2 &= \frac{q}{p+q}\int |t(u-\delta z)|^p |r(v-\delta w)^+| \, dx \\ \|s(v-\delta w)^-\|_\mu^2 &= \frac{q}{p+q}\int |t(u-\delta z)|^p |s(v-\delta w)^-| \, dx \end{aligned} \tag{2.4}$$

To make the presentation clear, we define the following functions:

$$f_1(\delta) = \|u-\delta z\|_\lambda^2, \quad f_2(\delta) = \int_\Omega |u-\delta z|^p [(v-\delta w)^+]^q,$$
$$f_3(\delta) = \int_\Omega |u-\delta z|^p [(v-\delta w)^-]^q, \quad f_4(\delta) = \|(v-\delta w)^+\|_\mu^2,$$
$$f_5(\delta) = \|(v-\delta w)^-\|_\mu^2.$$

It follows from (2.4) that $t(\delta), r(\delta)$, and $s(\delta)$ are precisely the solutions for the system

$$t^2 f_1(\delta) = \frac{p}{p+q} t^p r^q f_2(\delta) + \frac{p}{p+q} t^p s^q f_3(\delta), \tag{2.5}$$
$$r^2 f_4(\delta) = \frac{q}{p+q} t^p r^q f_2(\delta), \tag{2.6}$$
$$s^2 f_5(\delta) = \frac{q}{p+q} t^p s^q f_3(\delta), \tag{2.7}$$

Here we observe that the solution $t(\delta)$ is given explicitly by

$$t(\delta) = \big(1+\frac{p}{q}\big)^{\frac{1}{p+q-2}} [f_1(\delta)]^{-\frac{q-2}{2(p+q-2)}} \Big\{ \frac{[f_4(\delta)]^{\frac{q}{q-2}}}{[f_2(\delta)]^{\frac{2}{q-2}}} + \frac{[f_5(\delta)]^{\frac{q}{q-2}}}{[f_3(\delta)]^{\frac{2}{q-2}}} \Big\}^{\frac{q-2}{2(p+q-2)}}. \tag{2.8}$$

Recall that

$$f_1(0) = \|u\|_\lambda^2, \quad f_2(0) = \int_\Omega |u|^p |v^+|^q = \frac{p+q}{q}\|v^+\|_\mu^2,$$
$$f_3(0) = \int_\Omega |u|^p |v^-|^q = \frac{p+q}{q}\|v^-\|_\mu^2, \quad f_4(0) = \|v^+\|_\mu^2,$$
$$f_5(0) = \|v^-\|_\mu^2$$

Here we used that $(u,v) \in \mathcal{M}_{\lambda\mu}$ to determine the values of $f_2(0)$ and $f_3(0)$. Since $0 < \mu < \lambda_1(\Omega)$, $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, and $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ by Hölder inequality's there exists $C_{pq} = C_{pq}(\Omega) > 0$ such that

$$\Big(1 - \frac{\mu}{\lambda_1(\Omega)}\Big)\|v^\pm\|^2 \le \|v^\pm\|_\mu^2 = \frac{q}{p+q}\int_\Omega |u|^p |v^\pm|^q \le C_{pq}\|u\|^p \|v^\pm\|^q. \tag{2.9}$$

Since $q > 2$ and $\|u\|, \|v\| \le M_1$, expression (2.9) yields a constant $M_0 > 0$ such that $\|u\|, \|v^\pm\|^2 \ge M_0$, hence (2.3) is proved. As $\|u\|, \|v^\pm\| \in [M_0, M_1]$, we easily see from the expressions of $f_i(0)$ that there exist $K_1 = K_1(a,b,p,q,\lambda,\mu,\lambda_1(\Omega), M_1, N) > 0$ and $K_2 = K_2(a,b,p,q,\lambda,\mu,\lambda_1(\Omega), M_1, N) > 0$ such that

$$K_1 \le f_i(0) \le K_2, \quad 1 \le i \le 5. \tag{2.10}$$

A standard calculation shows that

$$f_1'(0) = -\Big(\int_\Omega \nabla u \nabla z - \lambda u z\Big), \quad f_2'(0) = -p\int_\Omega |u|^{p-2} u z |v^+|^q - q\int |u|^p |v^+|^{q-1} w,$$

$$f_3'(0) = -p\int_\Omega |u|^{p-2}uz|v^-|^q - q\int |u|^p|v^-|^{q-1}w,$$
$$f_4'(0) = -\int_\Omega \nabla v^+\nabla w - \mu\int v^+w, \quad f_5'(0) = -\int_\Omega \nabla v^-\nabla w - \mu\int v^-w.$$

It is important to observe that $\|z\|, \|w\| \le 1$ and $\|u\|, \|v\| \le M_1$. By the Sobolev embedding theorems, there exists a constant $K_3 = K_3(p,q,\lambda,\mu,\lambda_1(\Omega),M_1,N) > 0$ such that

$$|f_i'(0)| \le K_3, \quad 1 \le i \le 5. \tag{2.11}$$

From the explicit expression of $t(\delta)$ given by in (2.8) it follows that for a certain $\Psi \in C^1(\mathbb{R}^5_+)$ (and $\Psi \notin C^1(\overline{\mathbb{R}^5_+})$) we can write $t(\delta) = \Psi(f_1(\delta),\dots,f_5(\delta))$. Therefore $t'(0) = \sum f_i'(0)\Psi_{x_i}(f_1(0),\dots,f_5(0))$, and from this equality, together with (2.10) and (2.11) we conclude that there exist $C_i = C_i(p,q,\lambda,\mu,\lambda_1(\Omega),\Omega,M_1,N) > 0$ such that $|t'(0)| \le C_2$ and $C_1 \le t(0) \le C_2$. The other inequalities can be derived from combining the last estimates with (2.6) and (2.7). ◻

The following proposition shows the existence of a Palais-Smale sequence that converges to the infimum of $I_{\lambda\mu}$ over $\mathcal{M}_{\lambda\mu}$. Notice also that $p+q>2$ and

$$I_{\lambda\mu}(u,v) = \big(\frac{1}{2} - \frac{1}{p+q}\big)\left(\|u\|_\lambda^2 + \|v\|_\mu^2\right) \quad \text{for all } (u,v)\in\mathcal{M}_{\lambda\mu}. \tag{2.12}$$

Hence, $\inf_{(u,v)\in\mathcal{M}_{\lambda\mu}} I_{\lambda\mu}(u,v) > -\infty$. In what follows, we will use the notation $\|(\varphi,\phi)\| := \|\varphi\| + \|\phi\|$ for all $\varphi,\phi\in H_0^1(\Omega)$.

Proposition 2.2. *Let $c_{\lambda\mu} := \inf_{(u,v)\in\mathcal{M}_{\lambda\mu}} I_{\lambda\mu}(u,v)$. Then there exists a sequence $(u_n,v_n)\in\mathcal{M}_{\lambda\mu}$ such that*

$$I_{\lambda\mu}(u_n,v_n)\to c_{\lambda\mu} \quad \text{and} \quad I_{\lambda\mu}'(u_n,v_n)\to 0.$$

Moreover, there exist $M_0, M_1 > 0$ such that $\|u_n\|, \|v_n^\pm\| \in [M_0, M_1]$ for all $n\in\mathbb{N}$.

Proof. By applying the Ekeland's variational principle [12], we construct a sequence $(u_n,v_n)\in\mathcal{M}_{\lambda\mu}$ such that

$$\begin{gathered} I_{\lambda\mu}(u_n,v_n)\to c_{\lambda\mu},\\ I_{\lambda\mu}(u_n,v_n) < I_{\lambda\mu}(\varphi,\phi) + \frac{1}{n}\|(u_n-u, v_n-v)\|, \quad \forall(\varphi,\phi)\in\mathcal{M}_{\lambda\mu}. \end{gathered} \tag{2.13}$$

As $(u_n,v_n)\in\mathcal{M}_{\lambda\mu}$, then $\|u_n\|_\lambda^2 + \|v_n\|_\mu^2 = \int_\Omega |u_n|^p|v_n|^q$ which leads us to

$$\begin{aligned} \frac{p+q-1}{2(p+q)}\left(\|u_n\|_\lambda^2 + \|v_n\|_\mu^2\right) &= \frac{1}{2}\left(\|u_n\|_\lambda^2 + \|v_n\|_\mu^2\right) - \frac{1}{p+q}\int_\Omega |u_n|^p|v_n|^q\\ &= I_{\lambda\mu}(u_n,v_n) = c_{\lambda\mu} + o_n(1). \end{aligned}$$

The above expression and (2.1) gives us

$$2C_{\lambda\mu}\left(\|u_n\|^2 + \|v_n\|^2\right) \le \left(\|u_n\|_\lambda^2 + \|v_n\|_\mu^2\right) = \frac{2(p+q)c_{\lambda\mu}}{p+q-1} + o_n(1);$$

therefore (u_n,v_n) is a bounded sequence, i.e, there exists $M_1>0$ such that

$$\|(u_n,v_n)\| \le M_1 \quad \text{for all } n\in\mathbb{N} \tag{2.14}$$

By the Riesz Representation Theorem, it follows that for every fixed $n\in\mathbb{N}$ there exist z_n, $w_n\in H_0^1(\Omega)$ such that $(z_n,w_n)\cong I_{\lambda\mu}'(u_n,v_n)/\|I_{\lambda\mu}'(u_n,v_n)\|$; moreover

$$\|(z_n,w_n)\| = 1 \quad \text{and} \quad I_{\lambda\mu}'(u_n,v_n)(z_n,w_n) = \|I_{\lambda\mu}'(u_n,v_n)\|. \tag{2.15}$$

From now on, we will assume that $n \in \mathbb{N}$ is fixed. Let $t(\delta) := t_n(\delta), r(\delta) := r_n(\delta), s(\delta) := s_n(\delta)$ be given as stated in Lemma 2.1 and $u := u_n$, $v := v_n$, $z := z_n$, $w := w_n$ we will define $(u(\delta), v(\delta)) := (u_n(\delta), v_n(\delta))$ by

$$(u(\delta), v(\delta)) := \big(t(\delta)[u - \delta z], \quad r(\delta)[v - \delta w]^+ - s(\delta)[v - \delta w]^-\big) \in \mathcal{M}_{\lambda\mu}.$$

Recall also that $I_{\lambda\mu} \in C^1(\mathcal{H}, \mathbb{R})$, where $\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega)$. Setting $R(X - Y) := I_{\lambda\mu}(X) - I_{\lambda\mu}(Y) - I'_{\lambda\mu}(X)(X - Y)$ for any $X, Y \in \mathcal{H}$, we have $\lim_{X\to Y}(R(X - Y)/\|X - Y\|) = 0$. Since $\lim_{\delta\to 0^+}(u(\delta), v(\delta)) = (u, v)$, it is not difficult to see that $(\|u(\delta) - u\| + \|v(\delta) - v\|)/\delta \to \|t'(0)u - z\| + \|r'(0)v^+ - s'(0)v^- - w\|$ as $\delta \to 0^+$. Therefore, $o(\delta) := R(u(\delta) - u, v(\delta) - v)$ satisfies $o(\delta)/\delta \to 0$ as $\delta \to 0^+$ and

$$I_{\lambda\mu}(u(\delta), v(\delta)) = I_{\lambda\mu}(u, v) + I'_{\lambda\mu}(u(\delta), v(\delta))(u(\delta) - u, v(\delta) - v) + o(\delta) \tag{2.16}$$

setting $T_\delta(\varphi, \phi) := I'_{\lambda\mu}(u(\delta), v(\delta))(\varphi, \phi)$, by (2.13) and (2.16) we have

$$\begin{aligned}
&\frac{1}{n}\|u(\delta) - u, v(\delta) - v\| \\
&\geq I_{\lambda\mu}(u, v) - I_{\lambda\mu}(u(\delta), v(\delta)) \\
&= T_\delta(u - u(\delta), v - v(\delta)) + o(\delta) = \\
&= (1 - t(\delta))T_\delta(u - \delta z, 0) + T_\delta(u, 0) - T_\delta(u - \delta z, 0) \\
&\quad + (1 - r(\delta))T_\delta(0, [v_n - \delta w]^+) + T_\delta(0, v^+) - T_\delta(0, [v - \delta w]^+) \\
&\quad - (1 - s(\delta))T_\delta(0, [v - \delta w]^-) - T_\delta(0, v^-) + T_\delta(0, [v - \delta w]^-) + o(\delta) \\
&= (1 - t(\delta))T_\delta(u - \delta z, 0) + (1 - r(\delta))T_\delta(0, [v - \delta w]^+) \\
&\quad - (1 - s(\delta))T_\delta(0, [v - \delta w]^-) + \delta T_\delta(z, w) + o(\delta).
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\frac{1}{n}\|\frac{u(\delta) - u}{\delta}, \frac{v(\delta) - v}{\delta}\| \geq \big(\frac{1 - t(\delta)}{\delta}\big)T_\delta(u - \delta z, 0) + \big(\frac{1 - r(\delta)}{\delta}\big)T_\delta(0, [v - \delta w]^+) \\
- \big(\frac{1 - s(\delta)}{\delta}\big)T_\delta(0, [v - \delta w]^-) + T_\delta(z, w) + \frac{o(\delta)}{\delta}.
\end{aligned}$$

Given that $\lim_{\delta\to 0^+} t(\delta) = \lim_{\delta\to 0^+} r(\delta) = \lim_{\delta\to 0^+} s(\delta) = 1$, taking the limit as $\delta \to 0^+$, the above inequality gives us

$$\begin{aligned}
&\frac{1}{n}\|t'(0)u - z, r'(0)v^+ - s'(0)v^- - w\| \\
&\geq -t'(0)T_0(u, 0) - r'(0)T_0(0, v^+) + s'(0)T_0(0, v^-) + T_0(z, w) \\
&= -t'(0)T_0(u, 0) - r'(0)T_0(0, v^+) + s'(0)T_0(0, v^-) + T_0(z, w).
\end{aligned} \tag{2.17}$$

Since $(u, v) \in \mathcal{M}_{\lambda\mu}$, it follows that $T_0(u, 0) = T_0(0, v^+) = T_0(0, v^-) = 0$; therefore, from (2.17) and (2.15) we conclude that

$$\begin{aligned}
\frac{|t'(0)| + |r'(0)| + |s'(0)|}{n}(\|u, v\| + \|z, w\|) &\geq T_0(z, w) \\
&= I'_{\lambda\mu}(u, v)(z, w) \\
&= \|I'_{\lambda\mu}(u, v)\|.
\end{aligned} \tag{2.18}$$

By Lemma 2.1 there exists $C_2 > 0$ (that does not depend on the index n) such that $|t'(0)| + |r'(0)| + |s'(0)| \leq 3C_2$. From (2.18), (2.14) and (2.15) we obtain $\|I'_{\lambda\mu}(u_n, v_n)\| \leq 3(2M_1 + 1)C_2/n$. The existence of the constant $M_0 > 0$ is guaranteed by Lemma 2.1. This completes the proof. $\square$

Proof of Theorem 1.1. Firstly, we deal with the case $N \in \{3,4,5\}$. Let (u_n, v_n) the sequence obtained in Proposition 2.2, by (2.12), $\|u_n\|, \|v_n^{\pm}\| \in [M_0, M_1]$, $M_0 > 0$ and $\lambda, \mu < \lambda_1(\Omega)$, we can deduce that $c_{\lambda\mu} > 0$. It follows from the boundedness of u_n, v_n in $H_0^1(\Omega)$ that there exists $u_0, v_0 \in H_0^1(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup u_0$ and $v_n \rightharpoonup v_0$ weakly in $H_0^1(\Omega)$. Since $I'_{\lambda\mu}(u_n, v_n) \to 0$ and $(v_n^{\pm} - v_0^{\pm})$ is bounded in $H_0^1(\Omega)$, it follows that

$$\|v_n^{\pm}\|_{\mu}^2 - \int_{\Omega} \big(\nabla v_n \nabla v_0^{\pm} - \mu v_n v_0^{\pm}\big) - \Gamma_n = I'_{\lambda\mu}(u_n, v_n)(0, v_n^{\pm} - v_0^{\pm}) \to 0, \tag{2.19}$$

where

$$\Gamma_n := \frac{q}{p+q} \int_{\Omega} |u_n|^p |v_n^{\pm}|^{q-2} v_n^{\pm}(v_n^{\pm} - v_0^{\pm}) dx.$$

Then Hölder's inequality gives us

$$\Big| \int_{\Omega} |u_n|^p |v_n^{\pm}|^{q-2} v_n^{\pm}(v_n^{\pm} - v_0^{\pm}) dx \Big| \le |u_n|_{p+q}^p |v_n^{\pm}|_{p+q}^{q-1} |v_n^{\pm} - v_0^{\pm}|_{p+q}. \tag{2.20}$$

Since $2 < p+q < 2^*$, it follows that $H_0^1(\Omega) \hookrightarrow L^{p+q}(\Omega)$ is a compact embedding; therefore $v_n \rightharpoonup v_0$ weakly in $H_0^1(\Omega)$ imply that $|v_n^{\pm} - v_0^{\pm}|_{p+q} \to 0$, which combined with the boundedness of u_n, v_n in $H_0^1(\Omega)$ and (2.20) gives us $\Gamma_n \to 0$. As

$$\int_{\Omega} (\nabla v_n \nabla v_0^{\pm} - \mu v_n v_0^{\pm}) dx \to \|v_0^{\pm}\|_{\mu}^2$$

from (2.19) we obtain $\|v_n^{\pm}\|_{\mu}^2 \to \|v_0^{\pm}\|_{\mu}^2$. Since $v_n^{\pm} \to v_0^{\pm}$ strongly in $L^2(\Omega)$ it follows that $\|v_n^{\pm}\|^2 \to \|v_0^{\pm}\|^2$ and therefore $v_n^{\pm} \to v_0^{\pm}$ strongly in $H_0^1(\Omega)$. In a completely analogous manner, we can conclude that $u_n \to u_0$ strongly in $H_0^1(\Omega)$. It follows from Proposition 2.2 that $I_{\lambda\mu}(u_0, v_0) = c_{\lambda\mu}$, $I'_{\lambda\mu}(u_0, v_0) = 0$ and $\|u_0\| > 0$ and $\|v_0^{\pm}\| > 0$. Notice also that we can replaced u_n by $|u_n|$ and still have $I_{\lambda\mu}(|u_n|, v_n) \to c_{\lambda\mu}$. Without loss of generality we assume that $u_n \ge 0$ which implies that $u_0 \ge 0$. Since $I'_{\lambda\mu}(u_0, v_0) = 0$ we deduce that u_0, v_0 are the weak solutions of the system

$$\begin{gathered} -\Delta u_0 = \lambda u_0 + \frac{p}{p+q} |u_0|^{p-2} u_0 |v_0|^q, \quad \text{in } \Omega, \\ -\Delta v_0 = \mu v_0 + \frac{q}{p+q} |u_0|^p |v_0|^{q-2} v_0, \quad \text{in } \Omega, \\ u_0 = v_0 = 0, \quad \text{on } \partial\Omega, \\ u_0 \ge 0, \quad v_0^{\pm} \not\equiv 0 \quad \text{in } \Omega. \end{gathered} \tag{2.21}$$

It follows from the standard theory of elliptic regularity and a bootstrap argument that $u_0, v_0 \in C^2(\overline{\Omega})$. Furthermore, by using that $-\Delta u_0 \ge 0$ in Ω and $\|u_0\| > 0$, the Strong Maximum Principle implies that $u_0 > 0$ in Ω. Once again by using that $p+q > 2$, there exist $t, s > 0$ satisfying

$$\big(\frac{p}{p+q}\big) \frac{1}{t^{p-2} s^q} = \xi \quad \text{and} \quad \big(\frac{q}{p+q}\big) \frac{1}{t^p s^{q-2}} = \tau$$

It is easy to verify that $u = tu_0$ and $v = sv_0$ satisfy (1.3). The cases $N = 1, 2$ with $p+q < +\infty$ follows in a similar way by using that $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is a compact embedding for each $q \ge 1$. □

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