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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN AT RESONANCE

JOSE VANTERLER DA C. SOUSA, MARIANE PIGOSSI, NEMAT NYAMORADI

ABSTRACT. In this article, we investigate the existence and multiplicity of solutions for a fractional differential equations with p-Laplacian equation at resonance in the ψ-fractional space $\mathbb{H}_p^{\alpha,\beta;\psi}$. In addition, we show that the energy functional satisfies the Palais-Smale condition.

1. Introduction and motivation

Since the first results on fractional calculus were published, there has been a growing number of researchers who use fractional differential equations to better describe phenomena in mechanics, chemistry, medicine, etc. [20, 22, 30, 36, 43, 44, 45, 46]. Also, the theory of fractional differential equations gained space and strength with the consolidation of fractional calculus. Fractional differential equations have been valuable tools in fields, such as viscoelasticity, engineering, physics and economics, see [1, 3, 34, 38]. In addition, researchers have studied properties such as existence, uniqueness, stability, controllability for different types of fractional differential equations; see [13, 30, 52] and the references therein. On the other hand, we can highlight important works on fractional differential equations with p-Laplacian at resonance; see [23, 27, 31, 50]. Many of these works are done through fractional derivatives of Caputo and Riemann-Liouville type.

In 1999 Drabek and Robinson [15] considered the boundary value problem

$$\begin{gathered} -\Delta_p u - \lambda |u|^{p-2}u + f(x,u) = 0, \quad \text{in } \Omega \\ u|_{\partial\Delta} = 0, \end{gathered}$$

where $\Delta_p u := \nabla\left(|\nabla u|^{p-2}\nabla u\right)$, Ω is a bounded domain in $\mathbb{R}^n$, $p > 1$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded Caratheodory function. In 2010 Chang and Li [11] investigated the existence and multiplicity of nontrivial solutions for the semilinear elliptic Dirichlet boundary value problem

$$\begin{gathered} -\Delta_p u = f(x,u), \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Delta, \end{gathered}$$

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where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is an open bounded domain with smooth boundary $\partial\Omega$ and $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. Other interesting works on the existence and multiplicity of solutions involving p-Laplacian at resonance, without resonance, can be found in [8, 9, 14, 16, 17, 33, 47, 51].

Motivated by fractional operators (integrals and derivatives), problems of differential equations with p-Laplacian at resonance have gained prominence. In principle, the investigated results generalize the integer cases, and have possible particular cases other than fractional derivatives. For more readings, we refer the reader to [2, 5, 24, 25, 26, 28, 37, 49].

Jiang [27] studied the solvability of fractional differential equation with p-Laplacian at resonance,

$$\begin{gathered} D^{\beta}_{o+}\left(\varphi_p(D^{\alpha}_{0+}u)\right)(t) + f(t, u(t), D^{\alpha-1}_{0+}u(t), D^{\alpha}_{0+}u(t)) = 0 \\ u(0) = D^{\alpha}_{0+}(0) = 0, \\ u(t) = \int_0^1 u(t)h(t)dt = 1, \end{gathered}$$

where $0 \leq \beta \leq 1$, $1 \leq \alpha \leq 2$, $\int_0^1 h(t)t^{\alpha-1}dt = 1$, $\varphi_p(s) = |s|^{p-2}$, $p > 1$, and $D^{\beta}_{0+}(\cdot)$, $D^{\alpha}_{0+}(\cdot)$ are the Riemann-Liouville fractional derivatives.

In 2017, Hu and Zhang [23] investigated the existence of positive solutions of the fractional differential equation with periodic boundary value,

$$\begin{gathered} D^{\alpha}_{o+}u(t) = f(t, u(t)) \ 0 < t < 1 \\ u(0) = u(1), \quad u'(0) = 1, \quad u''(0) = u''(1), \end{gathered}$$

where $2 < \alpha < 3$, $D^{\alpha}_{o+}(\cdot)$ is the Caputo fractional derivative, and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$.

p-Laplace equations involving double phase have been studied during the previous years and their theory is already well developed. They gained prominence from results involving fractional operators. Since the pioneering work by Landesman and Lazer [32], several works have been devoted to resonant problems for ordinary and partial differential equations. However, some challenging and interesting problems still remain open. Resonance problems for divergence operators have been of interest since the 1970. For the common Laplacian and the p-Laplacian, there are several classical papers and some recent papers exploring resonant problems in $\mathbb{R}^n$; see [3]. On the other hand, the existence and multiplicity of solutions for boundary value problems of two non-singular points in resonance have been extensively addressed in the literature, see [4, 10, 19, 1]. Although there are some works in the literature involving fractional operators, they are still numerous open questions because of the difficulty of working with fractional operators. Research on p-Laplacian singulars at resonance has proceeded very slowly. One of the motivations of this paper is to provide new results and possible tools for future work.

Motivated by works and open questions above, we consider the fractional boundary value problem

$$\begin{gathered} {}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{T}\Big(|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0+}\xi(x)|^{p-2}\,{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0+}\xi(x)\Big) = f(x,\xi) \\ I^{\beta(\beta-1);\psi}_{0+}\xi(0) = I^{\beta(\beta-1);\psi}_{T}\xi(T) = 0, \end{gathered} \tag{1.1}$$

where $\Omega = [0, T]$ is a bounded domain in $\mathbb{R}$, ${}^{H}\mathbf{D}^{\alpha,\beta,\psi}_{0+}(\cdot)$ and ${}^{H}\mathbf{D}^{\alpha,\beta,\psi}_{T}(\cdot)$ are the ψ-Hilfer fractional derivatives of order α $(\frac{1}{p} < \alpha \leq 1)$ and type $0 \leq \beta \leq 1$,

$f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, and $1 < p < \infty$. Besides that, $I_{0+}^{\beta(\beta-1)}(\cdot)$ and $I_{T}^{\beta(\beta-1)}(\cdot)$ are the ψ-Riemann-Liouville fractional integrals of order $\beta(\beta-1)$.

Equation (1.1) is called a resonant problem at the first eigenvalue if

$$\lim_{|\xi|\to\infty} \frac{f(x,\xi)}{|\xi|^{p-2}\xi} = \lambda_1, \quad \text{uniformly for } x \in \Omega. \tag{1.2}$$

Jiu and Su [29] obtained the existence of multiple solutions of (1.1) with (1.2) and the non-quadratic condition

$$\lim_{|\xi|\to\infty} (\xi f(x,\xi) - p\mathbf{F}(x,\xi)) = -\infty, \quad \text{uniformly for } x \in \Omega. \tag{1.3}$$

With other versions of the non-quadratic conditions, several papers have studied the case

$$\lambda_1 \le a(x) = \liminf_{|\xi|\to\infty} \frac{f(x,\xi)}{|\xi|^{p-2}\xi} \le \lim sup_{|\xi|\to\infty} \frac{f(x,\xi)}{|\xi|^{p-2}\xi} = b(x) < \lambda_2, \tag{1.4}$$

uniformly for $x \in \Omega$. Here we assume that

$$\max_{|s|\leqslant R} f(x,s) \in L^p(\Omega), \quad \forall R > 0. \tag{1.5}$$

We also assume that some uniformity holds in (1.4): for each $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that

$$\lambda_1 \to \varepsilon \leqslant \frac{f(x,s)}{|s|^{p-2}s}, \quad \forall |s| \geqslant \eta(\varepsilon) \text{ a.e. in } [0,T], \tag{1.6}$$

and for each $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that

$$\lambda_1 \to \frac{f(x,s)}{|s|^{p-2}s} \leqslant \lambda_2 + \varepsilon, \quad \forall |s| \geqslant \eta(\varepsilon) \text{ a.e. in } [0,T].$$

Remark 1.1. Note that (1.5) and (1.6) imply the growth condition

$$|f(x,s)| \leqslant a|s|^{p-1} + b(x), \quad \forall s \in \mathbb{R}, \text{ a.e. in } [0,T] \tag{1.7}$$

where $a > 0$ and $b(\cdot) \in L^{p'}$.

Remark 1.2. Inequalities (1.5) and (1.6) also imply that for each $\varepsilon > 0$ there exists $b_\varepsilon \in L^{p'}$ such that

$$|s|^p(\lambda_1 - \varepsilon) - b_\varepsilon(x) \le sf(x,s) \le |s|^p(\lambda_2 + \varepsilon) + b_\varepsilon(x), \quad \forall s \in \mathbb{R} \text{ a.e. in } [0,T]. \tag{1.8}$$

Let $\lambda_1(a)$ be the first eigenvalues of the equation

$${}^H\mathbf{D}_T^{\alpha,\beta,\psi}\Big(|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)|^{p-2}\,{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\Big) - a(x)|\xi|^{p-2}\xi = \lambda|\xi|^{p-2}\xi$$

with Dirichlet boundary condition. It is well know that $\lambda_1(a)$ is simple and isolated. Then the second eigenvalue is well defined as

$$\begin{aligned}\lambda_2(a) = \inf\big\{\lambda > \lambda_1(a) : \lambda \text{ is eigenvalue of } {}^H\mathbf{D}_T^{\alpha,\beta,\psi}\Big(|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)|^{p-2}\,{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\Big) \\ - a(x) \in \mathbb{H}_p^{\alpha,\beta,\psi}\big\}.\end{aligned}$$

By the monotonicity of $\lambda_1(a)$ and $\lambda_2(b)$, condition (1.4) implies

$$\lambda_1(a) \leqslant 0 \leqslant \lambda_2(b)\,.$$

For the first eigenfunctions $\varphi_1(a) > 0$, if we let $V = \operatorname{span}\{\varphi_1(a)\}$, then

$$V^\perp = \Big\{\xi \in \mathbb{H}_p^{\alpha,\beta,\psi} : \int_0^T (\varphi(a))^{p-1}\xi dx = 0\Big\}.$$

Also we have

$$\mathbb{H}_p^{\alpha,\beta,\psi} = V \oplus V^\perp. \tag{1.9}$$

From [7], we know that there exists $\overline{\lambda(a)} \in (\lambda_1(a), \lambda_2(a)]$ such that

$$\int_0^T \Big(\Big|{}^H\mathbf{D}_{0+}^{\alpha,\beta,\psi}\xi(x)\Big|^p - a(x)|\xi|^p\Big)\, dx \geqslant \overline{\lambda(a)} \int_0^T |\xi|^p dx \tag{1.10}$$

for any $\xi \in V^\perp$. Similarly, we can define $\lambda_1(b), \varphi_1(b)$ and $\overline{\lambda(b)}$.

Before presenting our the main results, we list some assumptions.

(A1) For $1 < p < \infty$ we assume that

$$f \in C(\overline{[0,T]} \times \mathbb{R}, \mathbb{R}) \tag{1.11}$$

and satisfies the growth condition

$$|f(x,t)| \leqslant c(1 + |t|^{q-1}) \quad \text{for all } x \in [0,T],\ t \in \mathbb{R}$$

for some $c > 0$ and $q \in [1, p^*)$, where $p_\alpha^* = \frac{p}{1-\alpha p}$ if $p < 1$, and $p_\alpha^* = \infty$ if $1 \leqslant p$.

(A2) There exists a constant $M > 0$ such that

$$a(x) \leqslant \frac{f(x,\xi)}{|\xi|^{p-2}\xi} \leqslant b(x), \quad \text{for } |\xi| \geqslant M,\ x \in \Omega,$$

where a and b are continuous functions.

(A3) $\lim_{|\xi|\to\infty} \int_0^T \big(\mathbf{F}(x,\xi) - \frac{1}{p}b(x)|\xi|^p\big)dx = -\infty$.

(A4) $\lim_{|\xi|\to\infty} \big(\xi f(x,\xi) - p\mathbf{F}(x,\xi)\big) = -\infty$, where $\mathbf{F}(x,\xi) = \int_0^\xi f(x,t)dt$.

In this article, we investigate the existence and multiplicity of solutions for (1.1) under the conditions (A1)–(A4) in two steps. In the first step, we obtain the following result.

Theorem 1.3. *Assume that* (A1) *and* (A2) *hold. If one of the following condition is satisfied*

(a) $\lambda_1(b) > 0$,
(b) $\lambda_1(b) \geqslant 0$ *and (A3) holds,*
(c) $\lambda_1(a) < 0 < \overline{\lambda(b)}$,
(d) $\lambda_1(a) \leqslant 0 \leqslant \overline{\lambda(b)}$ *and (A4) holds,*

then (1.1) *has at least one solution.*

On the second step we use the assumption

(A5) $f(x,0) = 0$ and there is a function $\ell(x)$ such that, $\lim_{|\xi|\to\infty} \frac{p\mathbf{F}(x,\xi)}{\|\xi\|^p} \leq \ell(x)$ with $\lambda_1(\ell) > 0, x \in [0,T]$,

and obtain the following result.

Theorem 1.4. *Assume that* (c) *or* (d) *of Theorem 1.3, and* (A5) *hold. Then* (1.1) *has a nontrivial solution.*

Remark 1.5. Obviously, (A5) is weaker than the condition

$$\lim_{|\xi|\to 0} \frac{p\mathbf{F}(x,\xi)}{\|\xi\|^p} = \ell(x) \leqslant \lambda_1,\ x \in [0,T]$$

which implies that 0 is a local minimum of I.

Note that, taking $\alpha = 1$ and $\psi(t) = t$ in (1.1), we have the integer case problem,

$$\begin{gathered} -\big(|\xi'|^{p-2}\ \xi'\big)' = f(x,\xi) \\ \xi(0) = \xi(T) = 0. \end{gathered}$$

As $0 < \alpha \le 1$ and with the freedom of chosing $\psi(t)$, we have a wide class of possible cases for probem (1.1).

The rest of this article is organized as follows: In Section 2, we present some preliminary results of fractional calculus, i.e., the Riemann-Liouville fractional integral with respect to another function and the ψ-Hilfer fractional derivative, and the ψ-fractional space $\mathbb{H}_p^{\alpha,\beta,\psi}$ with its respective norm. In fact, we verify that the space $\mathbb{H}_p^{\alpha,\beta,\psi}$ is uniformly convex. In section 3, we prove the main results of this article, i.e., the existence and multiplicity of solutions for (1.1).

2. Mathematical background and auxiliary results

Let X be a real Banach space and $\Psi \in C^1(X,\mathbb{R})$ satisfying the Palais-Smale condition. Let $A = \{u \in X : \Psi'(u) = 0\}$ be the critical set of Ψ. Let $u \in A$ be an isolated critical point with $\Psi(u) = b \in \mathbb{R}$, and U be an isolated neighborhood of u. The group $C_*(\Psi,u) = H_*(\Psi^c \cap U, \Psi^c \cap U|u)$, $* = 0,1,\dots$ is called the *-th critical group of Ψ at u, where $\Psi^c = \{u \in X : \Psi(u) \le c\}$, $H_*(\cdot,\cdot)$ are the singular relative homology groups with a coefficients group G. By the excision property of the homology groups, the critical groups are independent of the choices of U, then they are well defined. In particular, if u, v are the critical points of Ψ and $C_q(\Psi,u) \neq C_q(\Psi,v)$ for some q then $u \neq v$, see [12].

Let (a,b) $(-\infty \le a < b \le \infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let $\psi(\cdot)$ be an increasing and positive monotone function on $[a,b]$, having a continuous derivative $\psi'(x) \neq 0$ on (a,b). The fractional integrals of a function f with respect to another function ψ on $[a,b]$ is defined in [43],

$$\mathbf{I}_{a+}^{\alpha;\psi}\xi(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)\big(\psi(x)-\psi(t)\big)^{\alpha-1}\xi(t)\,dt. \tag{2.1}$$

Analogously, we define $\mathbf{I}_{b-}^{\alpha;\psi}(\cdot)$.

Let $n-1 < \alpha < n$, with $n \in \mathbb{N}$, $I = [a,b]$ be the interval such that $-\infty \le a < b \le \infty$, and let two functions $f, \psi \in C^n([a,b],\mathbb{R})$ be such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. Them the ψ-Hilfer fractional derivatives ${}^{\mathbf{H}}\mathbf{D}_{a+}^{\alpha,\beta;\psi}(\cdot)$ of order α and type $0 \le \beta \le 1$ are defined by, [43],

$${}^{\mathbf{H}}\mathbf{D}_{a+}^{\alpha,\beta;\psi}\xi(x) = \mathbf{I}_{a+}^{\beta(n-\alpha);\psi}\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n \mathbf{I}_{a+}^{(1-\beta)(n-\alpha);\psi}\xi(x). \tag{2.2}$$

Analogously, we define ${}^{\mathbf{H}}\mathbf{D}_{b-}^{\alpha,\beta;\psi}(\cdot)$.

Next, we present the integration by parts of the ψ-Riemann-Liouville fractional integral and ψ-Hilfer fractional derivative.

As shown in [40], the equality

$$\int_a^b \big(\mathbf{I}_{a+}^{\alpha;\psi}\xi(t)\big)\theta(t)\,dt = \int_a^b \xi(t)\psi'(t)\mathbf{I}_{b-}^{\alpha;\psi}\big(\frac{\theta(t)}{\psi'(t)}\big)\,dt \tag{2.3}$$

holds if $\theta \in L^p$, $\xi \in L^q$, $\frac{1}{p}+\frac{1}{q} \le 1+\alpha$, $p \ge 1$, $q \ge 1$, with $p \ne 1$, $q \ne 1$ in the case $\frac{1}{p}+\frac{1}{q} = q+\alpha$.

Theorem 2.1 ([40]). *Let $\psi(\cdot)$ be an increasing and positive monotone function on $[a,b]$, having a continuous derivative $\psi'(\cdot) \ne 0$ on (a,b). If $0 < \alpha \le 1$ and $0 \le \beta \le 1$, then*

$$\int_a^b \Big({}^{\mathbf{H}}\mathbf{D}_{a+}^{\alpha,\beta;\psi}\xi(t)\Big)\theta(t)\,dt = \int_a^b \xi(t)\psi'(t)\,{}^{\mathbf{H}}\mathbf{D}_{b-}^{\alpha,\beta;\psi}\left(\frac{\theta(t)}{\psi'(t)}\right)dt \tag{2.4}$$

for $\xi \in AC^1$ and $\theta \in C^1$ satisfying the boundary conditions $\xi(a) = 0 = \xi(b)$.

Definition 2.2 ([40]). Let $0 < \alpha \le 1$, $0 \le \beta \le 1$ and $1 < p < \infty$. Let ξ be a weight, $[0,T] \subset \mathbb{R}$ be open, with $\xi \ne 0$ a.e. in $[0,T]$. The ψ-fractional derivative weight space $\mathbb{H}_p^{\alpha,\beta;\psi} := \mathbb{H}_p^{\alpha,\beta;\psi}([0,T],\mathbb{R})$ is defined as the closure of $C_0^\infty([0,T],\mathbb{R})$, and is given by

$$\begin{aligned}
\mathbb{H}_p^{\alpha,\beta;\psi} &= \Big\{\xi \in L^p([0,T],\mathbb{R}) :\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi \in L^p([0,T],\mathbb{R}), \\
&\qquad \mathbf{I}_{0+}^{\beta(\beta-1)}\xi(0) = \mathbf{I}_{T}^{\beta(\beta-1)}\xi(T) = 0\Big\} \\
&= \overline{C_0^\infty([0,T],\mathbb{R})}
\end{aligned} \tag{2.5}$$

with the norm

$$\|\xi\|_{\mathbb{H}_p^{\alpha,\beta;\psi}} = \big(\|\xi\|_{L^p}^p + \|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi\|_{L^p}^p\big)^{1/p}, \tag{2.6}$$

where ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}(\cdot)$ is the ψ-Hilfer fractional derivative with $0 < \alpha \le 1$ and $0 \le \beta \le 1$.

Choosing $p = 2$, in (2.5), we have the ψ-fractional derivative weight space $\mathbb{H}_p^{\alpha,\beta;\psi}$ defined on $\overline{C_0^\infty([0,T],\mathbb{R})}$ with respect to the norm

$$\|\xi\|_{\mathbb{H}_2^{\alpha,\beta;\psi}} = \Big(\int_0^T |\xi(t)|^2\,dt + \int_0^T |{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi(t)|^2\,dt\Big)^{1/2}.$$

The space $\mathbb{H}_2^{\alpha,\beta;\psi}$ is a Hilbert space with the norm

$$\|\xi\|_{\mathbb{H}_2^{\alpha,\beta;\psi}} = \Big(\int_0^T |{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi(t)|^2\,dt\Big)^{1/2}$$

with $0 < \alpha \le 1$ and $0 \le \beta \le 1$.

Lemma 2.3 ([40]). *Let $0 < \alpha \le 1$, $0 \le \beta \le 1$, and $1 \le p < \infty$. For each $\xi \in L^p([0,T],\mathbb{R})$, we have*

$$\|\mathbf{I}_{0+}^{\alpha,\psi}\|_{L^p[0,T]} \le \frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\|\xi\|_{L^p[0,T]}$$

for all $t \in [0,T]$.

Proposition 2.4 ([40]). *Let $0 < \alpha \le 1$, $0 \le \beta \le 1$, and $1 < \alpha < \infty$. For all $\xi \in \mathbb{H}_p^{\alpha,\beta;\psi}$, if $1-\alpha \ge \frac{1}{p}$ or $\alpha > \frac{1}{p}$, we have*

$$\|\xi\|_{L^p} \le \frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi\|_{L^p}. \tag{2.7}$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\|\xi\|_\infty \le \frac{(\psi(T)-\psi(0))^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi\|_{L^p} \tag{2.8}$$

where $\|\xi\|_\infty = \sup_{t\in[0,T]}|\xi(t)|$.

From (2.8)*, we also have,* [40],

$$\|\xi\|_\infty \le \frac{(\psi(T)-\psi(0))^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi\|_{\mathbb{H}_p^{\alpha,\beta;\psi}},$$

that is $\mathbb{H}_p^{\alpha,\beta;\psi}$ is continuously injected into $C([0,T])$ for $\alpha > 1/p$.

According to (2.7) we can consider $\mathbb{H}_p^{\alpha,\beta;\psi}$ with respect to the equivalent norm

$$\|\xi\| = \|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}\xi\|_{L^p}.$$

Under condition (1.11), it is well know that the weak solutions of (1.1) correspond to the critical points of the functional $I:\mathbb{H}_p^{\alpha,\beta;\psi}\to\mathbb{R}$ defined by

$$I(\xi) = \frac{1}{p}\int_0^T \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\xi(x)\right|^p dx - \int_0^p \mathbf{F}(x,\xi)dx, \tag{2.9}$$

where $\mathbf{F}(x,\xi) = \int_0^\xi f(x,t)dt$. The next result on convexity is important for the main results of this paper.

Theorem 2.5. *The space $(\mathbb{H}_p^{\alpha,\beta,\psi}, \|\cdot\|_{\mathbb{H}_p^{\alpha,\beta,\psi}})$ is uniformly convex.*

Proof. First, let $p\in[2,\infty)$. For each $z,w\in\mathbb{R}$, it holds

$$|\frac{z+w}{2}|^p + |\frac{z-w}{2}|^p \leqslant \frac{1}{2}(|z|^p+|w|^p).$$

Let $\xi,\nu\in\mathbb{H}_p^{\alpha,\beta,\psi}$ satisfy $\|\xi\|_{\mathbb{H}_p^{\alpha,\beta,\psi}} = \|\nu\|_{\mathbb{H}_p^{\alpha,\beta,\psi}} = 1$, and $\|\xi-\nu\|_{\mathbb{H}_p^{\alpha,\beta,\psi}} \geqslant \varepsilon$, $\varepsilon\in(0,2]$. We have

$$\begin{aligned}
\|\frac{\xi+\nu}{2}\|^p_{\mathbb{H}_p^{\alpha,\beta,\psi}} + \|\frac{\xi-\nu}{2}\|^p_{\mathbb{H}_p^{\alpha,\beta,\psi}} &= \int_0^T \Big(\Big|\frac{{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x) +{}^H \mathbf{D}_{0^+}^{\alpha,\beta,\psi}\nu(x)^p}{2}\Big|\Big)dx \\
&\quad + \int_0^T \Big(\Big|\frac{{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x) +{}^H \mathbf{D}_{0^+}^{\alpha,\beta,\psi}\nu(x)^p}{2}\Big|\Big)dx \\
&\leqslant \int_0^T \frac{1}{2}\Big(|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)|^p + |{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\nu(x)|^p\Big)dx \\
&= \frac{1}{2}\big(\|\xi\|_{\mathbb{H}_p^{\alpha,\beta,\psi}} + \|\nu\|_{\mathbb{H}_p^{\alpha,\beta,\psi}}\big) = 1
\end{aligned}$$

which yields

$$\|\frac{\xi+\nu}{2}\|^p_{\mathbb{H}_p^{\alpha,\beta,\psi}} \leqslant 1-\Big(\frac{\varepsilon}{2}\Big)^p. \tag{2.10}$$

On the other hand, if $p\in(1,2)$ then for each $z,w\in\mathbb{R}$ it holds

$$|\frac{z+w}{2}|^{p'} + |\frac{z-w}{2}|^{p'} \leqslant \Big(\frac{1}{2}(|z|^p+|w|^p)\Big)^{\frac{1}{p-1}}. \tag{2.11}$$

Straight forward computations show that if $v\in\mathbb{H}_p^{\alpha,\beta,\psi}$, then $\|\,|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\nu|^p\|_{\mathbb{H}_{p-1}^{\alpha,\beta,\psi}} = \|\nu\|^{p'}_{\mathbb{H}_p^{\alpha,\beta,\psi}}$.

Let $\nu_1,\nu_2\in\mathbb{H}_p^{\alpha,\beta,\psi}$ then $|^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\nu_1|^{p'}$, $|^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\nu_2|^{p'}\in L^{p-1}([0,T])$ with $0<p-1<1$ and from

$$\begin{aligned} &\| \, |^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\nu_1|^{p'}+|^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\nu_2|^{p'}\|_{\mathbb{H}_{p-1}^{\alpha,\beta,\psi}}\\ &\geqslant \| \, |^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\nu_1|^{p'}\|_{\mathbb{H}_{p-1}^{\alpha,\beta,\psi}}+\| \, |^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\nu_2|^{p'}\|_{\mathbb{H}_{p-1}^{\alpha,\beta,\psi}}, \end{aligned} \tag{2.12}$$

we have

$$\begin{aligned} &\|\frac{\nu_1+\nu_2}{2}\|^p_{\mathbb{H}_p^{\alpha,\beta,\psi}}+\|\frac{\nu_1-\nu_2}{2}\|^p_{\mathbb{H}_p^{\alpha,\beta,\psi}}\\ &=\| \, \big|^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\big(\frac{\nu_1+\nu_2}{2}\big)\big|^{p'}\|_{\mathbb{H}_{p-1}^{\alpha,\beta,\psi}}+\| \, big|^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\big(\frac{\nu_1-\nu_2}{2}\big)\big|^{p'}\|_{\mathbb{H}_{p-1}^{\alpha,\beta,\psi}}\\ &\leqslant \| \, \big|^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\big(\frac{\nu_1+\nu_2}{2}\big)\big|^{p'}+\big|^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\big(\frac{\nu_1-\nu_2}{2}\big)\big|^{p'}\|_{\mathbb{H}_{p-1}^{\alpha,\beta,\psi}}\\ &=\Big[\int_0^T\Big(\Big|\frac{^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\nu_1+^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\nu_2}{2}\Big|^{p'}+\Big|\frac{^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\nu_1-^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\nu_2}{2}\Big|^{p'}\Big)^{p-1}dx\Big]^{\frac{1}{p-1}}\\ &\leqslant\Big[\frac12\int_0^T\Big(\big|^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\nu_1\big|^p+\big|^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta,\psi}\nu_2\big|^p\Big)dx\Big]^{\frac{1}{p-1}}\\ &=\Big(\frac12\|\nu_1\|^p_{\mathbb{H}_p^{\alpha,\beta,\psi}}+\frac12\|\nu_2\|^p_{\mathbb{H}_p^{\alpha,\beta,\psi}}\Big)^{\frac{1}{p-1}}. \end{aligned}$$

For $\xi,\nu\in\mathbb{H}_p^{\alpha,\beta,\psi}$ with $\|\xi\|_{\mathbb{H}_p^{\alpha,\beta,\psi}}=\|\nu\|_{\mathbb{H}_p^{\alpha,\beta,\psi}}=1$ and $\|\xi-\nu\|_{\mathbb{H}_p^{\alpha,\beta,\psi}}\geqslant\varepsilon\in(0,2]$, we obtain

$$\|\frac{\xi+\nu}{2}\|^{p'}\leqslant 1-\Big(\frac{\varepsilon}{2}\Big)^{p'}. \tag{2.13}$$

From (2.10) and (2.13) in either case there exists $\delta(\varepsilon)>0$ such that $\|\xi+\nu\|_{\mathbb{H}_p^{\alpha,\beta,\psi}}\leqslant 2(1-\delta(\varepsilon))$. $\square$

Definition 2.6. A functional I is said to satisfy the Palais-Smale condition at the level $c\in\mathbb{R}$ ($(PS)_c$ for short) if every sequence $\{\xi_n\}\subset\mathbb{H}_p^{\alpha,\beta,\psi}$ with

$$I(\xi_n)\to c,\quad (\|\xi_n\|+1)I(\xi_n)\to 0,\quad \text{as } n\to\infty \tag{2.14}$$

possesses a convergent subsequence. Furthermore, I satisfies the (PS) if I satisfies $(PS)_c$ at each $c\in\mathbb{R}$.

Lemma 2.7. *The sequence*

$$g_n=\frac{f(x,nw_n)}{n^{p-1}}$$

is bounded in $L^{p'}([0,T])$ and consequently a subsequence g_n converges weakly to a function g in $L^{p'}([0,T])$.

The proof of the above lemma is an immediate consequence of (1.7).

Lemma 2.8 ([6])**.** *The g obtained in Lemma 2.7 satisfies $g=0$ almost everywhere in $[0,T]\backslash A$, where $A=\{x\in[0,T]:z(x)\neq0\}$.*

Lemma 2.9 ([6])**.** *Set*

$$m(x)=\begin{cases}\frac{g(x)}{|z(x)|^{p-2}z(x)}, & \text{on } A,\\ \beta, & \text{on } [0,T]\backslash A,\end{cases}$$

where β is a fixed number with $\lambda_1<\beta<\lambda_2$. Then

$$\lambda_1\leqslant m(x)<\lambda_2\quad a.e.\ in\,[0,T]. \tag{2.15}$$

Lemma 2.10 ([47]). *Let E be a vector space such that for subspace X and Y, $E = X \oplus Y$. If Y is a finite dimensional and Z is a subspace of E such that $X \cap Z = \{0\}$ and $\dim Z = \dim Y$, then $E = X \oplus Z$.*

3. Palais-Smale condition

Lemma 3.1. *Under the assumptions of Theorem 1.3, the functional I satisfies the* (PS) *condition.*

Proof. We consider the following 4 cases.

Case 1: Assume (A1), (A2) and (a). We show that I is coercive on $\mathbb{H}_p^{\alpha,\beta,\psi}$. Since $\lambda_1(b) > 0$ and $b \in C(\overline{[0,T]}, \mathbb{R})$ (see inequality (1.10)), we have

$$
\begin{aligned}
&\int_0^T \big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p dx \\
&= \int_0^T \big(\big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p - b(x)|\xi|^p\big)dx + \int_0^T b(x)|\xi(x)|^p\,dx \\
&\leqslant \int_0^T \big(\big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p - b(x)|\xi|^p\big)dx + c\int_0^T |\xi(x)|^p dx \\
&\leqslant \int_0^T \big(\big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p - b(x)|\xi(x)|^p\big)dx \\
&\quad + \frac{c}{\lambda_1(b)}\int_0^T \big(\big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p - b(x)|\xi(x)|^p\big)dx \\
&= \big(1 + \frac{c}{\lambda_1(b)}\big)\int_0^T \big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p dx - 2 + \int_0^T b(x)|\xi(x)|^p dx \\
&\leqslant \tilde{c}\int_0^T \big(\big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p - b(x)|\xi|^p\big)dx.
\end{aligned}
$$

Then there exists a constant $\delta > 0$ such that

$$
\int_0^T \big(\big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p - b(x)|\xi|^p\big)dx \geqslant \delta\int_0^T \big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p dx \tag{3.1}
$$

with $\xi \in \mathbb{H}_p^{\alpha,\beta,\psi}$. Using conditions (A1) and (A2), it follows that

$$
\mathbf{F}(x,\xi) \leqslant \frac{1}{p}b(x)|\xi(x)|^p + c. \tag{3.2}
$$

Then using the inequalities (3.1) and (3.2) yields

$$
I(\xi) = \frac{1}{p}\int_0^T \big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p dx - \int_0^T \mathbf{F}(x,\xi)dx \geqslant \frac{\delta}{p}\int_0^T \big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p dx - c.
$$

Then, we have $I(\xi) \to \infty$ as $\|\xi\| \to \infty$.

Case 2: Assume (A1), (A2), and (b). By contradiction we show that I is coercive on $\mathbb{H}_p^{\alpha,\beta,\psi}$. Consider the sequence $\{\xi_n\} \subset \mathbb{H}_p^{\alpha,\beta,\psi}$ (without loss of generality assume it is the whole sequence) and a constant c_0 such that

$$
I(\xi_n) \leqslant c_0, \quad \text{as } \|\xi_n\| \to \infty. \tag{3.3}
$$

Also, let $\nu_n = \xi_n/\|\xi_n\|$. Then there exists a subsequence $\{\nu_n\}$ and $v \in \mathbb{H}_p^{\alpha,\beta,\psi}$ (without loss of generality assume it is the whole sequence), such that

$$\begin{gathered}
\nu_n \rightharpoonup v \quad \text{weakly in } \mathbb{H}_p^{\alpha,\beta,\psi} \\
\nu_n \to v \quad \text{strongly in } L^p \\
\nu_n \to v \quad \text{for a.e. } x \in [0,T].
\end{gathered} \tag{3.4}$$

Using inequalities (3.2) and (3.3), and dividing by $\|\xi_n\|^p$, we obtain

$$\begin{aligned}
I(\xi_n) &= \frac{1}{p}\int_0^T \left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi_n(x)\right|^p dx - \int_0^T \mathbf{F}(x,\xi_n)dx \\
&\leqslant \frac{1}{p}\int_0^T \Big(\left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi_n(x)\right|^p - b(x)|\xi_n(x)|^p\Big)dx + c
\end{aligned}$$

which implies that

$$\frac{c_0}{\|\xi_n\|^p} \geqslant \frac{1}{p}\int_0^T \left(\left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\nu_n(x)\right|^p - b(x)|\nu_n(x)|^p\right)dx + \frac{c}{\|\xi_n\|^p}. \tag{3.5}$$

From this inequality, it follows that

$$\limsup_{n\to\infty}\int_0^T \left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\nu_n(x)\right|^p dx \leqslant \int_0^T b(x)|\nu(x)|^p dx \quad \text{as } n\to\infty. \tag{3.6}$$

Moreover, since $\lambda_1(b) \geqslant 0$, from the lower semi-continuity of the norm, we obtain

$$\int_0^T b(x)|\nu(x)|^p dx \leqslant \int_0^T \left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\nu(x)\right|^p dx \leqslant \liminf_{n\to\infty}\int_0^T \left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\nu_n(x)\right|^p dx,$$

this together with (3.6) gives $\|\nu_n\| \to \|\nu\|$, as $n\to\infty$. Since $\mathbb{H}_p^{\alpha,\beta,\psi}$ is uniformly convex (see Theorem 2.5), we have $\nu_n \to v$ in $\mathbb{H}_p^{\alpha,\beta,\psi}$, as $n\to\infty$ with $\|\nu\|=1$ and

$$\int_0^T b(x)|\nu(x)|^p dx = \int_0^T \left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\nu(x)\right|^p dx.$$

Without loss of generally, we assume that $\lambda_1(b) = 0$. Tthen we can take $v = \pm\varphi_1(b)$ which implies that $|\xi_n(x)| \to \infty$, almost everywhere in $[0,T]$. Using condition (A3) it follows that

$$\lim_{n\to\infty}\int_0^T \Big(\mathbf{F}(x,\xi_n) - \frac{1}{p}b(x)\|\xi_n\|^p\Big)dx = -\infty.$$

Then

$$\begin{aligned}
I(\xi_n) &= \frac{1}{p}\int_0^T \left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi_n(x)\right|^p dx - \int_0^T \mathbf{F}\ (x,\xi_n)dx \\
&\geqslant \int_0^T \Big(\mathbf{F}(x,\xi_n) - \frac{1}{p}b(x)|\xi_n(x)|^p\Big)dx \to \infty
\end{aligned}$$

as $n\to\infty$. This contradicts (3.3).

Case 3: Assume (A1), (A2), and (c). We show that I is coercive on $\mathbb{H}_p^{\alpha,\beta,\psi}$. Assume that $\{\xi_n\} \subset \mathbb{H}_p^{\alpha,\beta,\psi}$ (without loss of generality assume it is the whole sequence) and satisfies (2.14), by (A1) it suffices to show that $\{\xi_n\}$ is bounded. We prove this by contradiction. Assume that $\|\xi_n\| \to \infty$ as $n\to\infty$. Let $z_n = \xi_n/\|\xi_n\|$. Then there exists a subsequence $\{z_n\}$ and $z \in \mathbb{H}_p^{\alpha,\beta,\psi}$ (without loss of generality assume it is the whole sequence) such that

$$\begin{gathered} z_n \rightharpoonup z \quad \text{weakly in } \mathbb{H}_{0+}^{\alpha,\beta,\psi} \\ z_n \to z \quad \text{strongly in } L^p \\ z_n \to z \quad \text{for a.e. } x \in [0,T]. \end{gathered}$$

Let $g_n(x) = f(x,\xi_n)/\|\xi_n\|^{p-1}$, then g_n is bounded in L^p with $\frac{1}{p} + \frac{1}{p'} = 1$. Furthermore, for a subsequence of $\{g_n\}$ without loss of generality assume it is whole sequence such that

$$g_n \rightharpoonup g \quad \text{weakly in } L^{p'}. \tag{3.7}$$

Next, we discuss 4 claims.

Claim 1: (See Lemma 2.8) $g = 0$ almost everywhere in $[0,T]\backslash A$, where $A = \{x \in [0,T] : z(x) \neq 0\}$.

Claim 2: (See Lemma 2.9) Set

$$m(x) = \begin{cases} \frac{g(x)}{|z(x)|^{p-2}z(x)}, & \text{on } A \\ \beta, & \text{on } [0,T]\backslash A \end{cases}$$

where β is a fixed number with $\lambda_1 < \beta < \lambda_2$. Then

$$\lambda_1 \leqslant m(x) < \lambda_2 \quad \text{a.e. in } [0,T]. \tag{3.8}$$

Claim 3: $z_n \to z$ in $\mathbb{H}_p^{\alpha,\beta,\psi}$ and z is a nontrivial solution of

$$\begin{gathered} {}^H\mathbf{D}_T^{\alpha,\beta,\psi}(|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)|^{p-2}\,{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)) = m(x)|\xi|^{p-2}u, \quad \text{in } \Omega = [0,T] \\ I_{0^+}^{\beta(\beta-1),\psi}u(0) = I_T^{\beta(\beta-1),\psi}u(T) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.9}$$

From (2.14) for each $\phi \in \mathbb{H}_p^{\alpha,\beta,\psi}$ it follows that

$$\begin{aligned} &\int_0^T |{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}z_n(x)|^{p-2}\,{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}z_n(x)\,{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\phi(x)dx \\ &- \int_0^T \frac{f(x,\xi_n)}{\|\xi_n\|^{p-1}}\phi dx = o(1)\|\phi\|. \end{aligned} \tag{3.10}$$

Let $\phi = z_n - z$ and note that

$$\lim_{n\to\infty}\int_0^T \frac{f(x,\xi_n)}{\|\xi_n\|^{p-1}}(z_n - z)dx = 0. \tag{3.11}$$

Using (3.10) and (3.11), it follows that

$$\lim_{n\to\infty}\int_0^T \left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}z_n(x)\right|^{p-2}\,{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}z_n(x)\,{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}(z_n - z)dx = 0.$$

From the fact that ${}^H\mathbf{D}_T^{\alpha,\beta,\psi}\big(\left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}(\cdot)\right|^{p-2}\,{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}(\cdot)\big)$ is of type S^+, we conclude that $z_n \to z$ in $\mathbb{H}_p^{\alpha,\beta,\psi}$ with $\|z\| = 1$.

Using (3.7), we have

$$\lim_{n\to\infty}\int_0^T \frac{f(x,\xi_n)}{\|\xi_n\|^{p-1}}\phi dx = \int_0^T g\phi\, dx.$$

Then, from (3.10) and the 3 claims, we obtain

$$\int_0^T \left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi} z_n(x)\right|^{p-2} {}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi} z_n(x) {}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\phi(x)dx = \int_0^T m(x)|z(x)|^{p-2}z(x)dx,$$

which implies (3.9). On the other hand, using inequality (2.15), the monotonicity of $\lambda_1(a)$ and $\lambda_2(b)$, it follows that

$$\lambda_1(m) \leqslant \lambda_1(a) < 0, \quad \lambda_2(m) \geqslant \lambda_2(b) \geqslant \overline{\lambda(b)} > 0.$$

Then 0 is not an eigenvalue of ${}^H\mathbf{D}_T^{\alpha,\beta,\psi}\big(\left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}(\cdot)\right|^{p-2} {}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}(\cdot)\big) - m(x)$, which the contradicts (3.9).

Case 4: Assume (A1), (A2), and (d). By contradiction we show that I is coercive on $\mathbb{H}_p^{\alpha,\beta,\psi}$. We assume that $\{\xi_n\} \subset \mathbb{H}_p^{\alpha,\beta,\psi}$ (without loss of generality assume it is the whole sequence) and satisfies (2.14), but $\|\xi_n\| \to \infty$ as $n \to \infty$. Let $z_n = \frac{\xi_n}{\|\xi_n\|}$. The there exists a subsequence $\{z_n\}$ and $z \in \mathbb{H}_p^{\alpha,\beta,\psi}$ (without loss of generality assume it is the whole sequence), such that

$$\begin{gathered} z_n \rightharpoonup z \quad \text{weakly in } \mathbb{H}_{0+}^{\alpha,\beta,\psi} \\ z_n \to z \quad \text{strongly in } L^p \\ z_n \to z \quad \text{for a.e. } x \in [0,T]. \end{gathered}$$

Now, using the conditions (A1) and (A4), it follows that

$$\mathbf{F}(x,\xi) \leqslant C|\xi|^p + C, \tag{3.12}$$

with $C > 0$. Combining (2.14) and (3.12), we obtain

$$\frac{1}{p}\|\xi_n\|^p - C\|\xi_n\|^p - C \leqslant C \tag{3.13}$$

which implies that

$$\frac{1}{p} - \|z\|_p^p \leqslant 0, \quad \text{so } z \neq 0. \tag{3.14}$$

If we define $\Omega' = \{x \in [0,T] | z(x) \neq 0\}$, then

$$\operatorname{meas}(\Omega') > 0, \quad |\xi_n(x)| \to \infty, \quad \text{as } n \to \infty$$

with $x \in \Omega'$, which implies that

$$\lim_{n\to\infty} (p\mathbf{F}(x,\xi_n) - \xi_n\mathbf{F}(x,\xi_n)) = \infty, \quad x \in \Omega'.$$

By Fatou's lemma,

$$\lim_{n\to\infty} \int_0^T (p\mathbf{F}(x,\xi_n) - \xi_n\mathbf{F}(x,\xi_n)) = \infty.$$

However, using (2.14), it follows that

$$\lim_{n\to\infty} \int_0^T (p\mathbf{F}(x,\xi_n) - \xi_n\mathbf{F}(x,\xi_n)) = -pC.$$

This contradiction completes the proof. $\square$

4. Existence and multiplicity

Let $\varphi_1(a)$ and $\varphi_1(b)$ be the eigenfunctions corresponding to $\lambda_1(a)$ and $\lambda_1(b)$, respectively. If we set $E_1 = \operatorname{span}\{\varphi_1(a)\}$ and $E_2 = \operatorname{span}\{\varphi_1(b)\}$ then as for (1.9) we have

$$\begin{aligned}\mathbb{H}_p^{\alpha,\beta,\psi} &= E_1 \oplus E_1^{\perp},\\ \mathbb{H}_p^{\alpha,\beta,\psi} &= E_2 \oplus E_2^{\perp}.\end{aligned}$$

Lemma 4.1. *If the continuous functions a and b satisfy $a(x) \leqslant b(x)$ for $x \in [0,T]$, and*

$$\lambda_1(a) \leqslant 0 \leqslant \overline{\lambda(b)},$$

then $\mathbb{H}_p^{\alpha,\beta,\psi} = E_1 \oplus E_2^{\perp}$.

Proof. Using Lemma 2.10 we only need to prove that $E_1 \cap E_2^{\perp} = \{0\}$. Without loss generally, we assume that $\{x \in [0,T] : a(x) \neq b(x)\}$ is not empty so it is easy to see that if

$$\begin{aligned}\xi \in \ker &\big\{{}^H\mathbf{D}_T^{\alpha,\beta,\psi}\Big(\big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}(\cdot)\big|^{p-2}\,{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}(\cdot)\Big) - a(\cdot)\big\}\\ \cap \ker &\big\{{}^H\mathbf{D}_T^{\alpha,\beta,\psi}\Big(\big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}(\cdot)\big|^{p-2}\,{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}(\cdot)\Big) - b(\cdot)\big\},\end{aligned}$$

then we obtain $\xi = 0$. For each $\xi_0 \in E_1 \cap E_2^{\perp}$, it follows that (see inequality (1.10))

$$\begin{aligned}0 \geqslant \lambda_1(a)\int_0^T |\xi_0|^p dx &= \int_0^T \Big(\big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi_0\big|^p - a(x)|\xi_0|^p\Big)dx\\ &\geqslant \int_0^T \Big(\big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi_0\big|^p - b(x)|\xi_0|^p\Big)dx\\ &\geqslant \overline{\lambda(b)}\int_0^T |\xi_0|^p dx \geqslant 0,\end{aligned}$$

which implies that

$$\begin{aligned}\xi_0 \in \ker &\big\{{}^H\mathbf{D}_T^{\alpha,\beta,\psi}\Big(\big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}(\cdot)\big|^{p-2}\,{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}(\cdot)\Big) - a(\cdot)\big\}\\ \cap \ker &\big\{{}^H\mathbf{D}_T^{\alpha,\beta,\psi}\Big(\big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}(\cdot)\big|^{p-2}\,{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}(\cdot)\Big) - b(\cdot)\big\},\end{aligned}$$

thefore $\xi_0 = 0$. □

Proof of Theorem 1.3. For items (a) and (b), since in each case the functional I is coercive on $\mathbb{H}_p^{\alpha,\beta,\psi}$, the existence of a solution is trivial.

For item (c), we consider the next three sub-items:

(1) $I(\xi) \to -\infty$ as $\|\xi\| \to \infty$, $\xi \in E_1$. Using conditions (A1) and (A2), if we set $\mathbf{G}(x,\xi) = \mathbf{F}(x,\xi) - \frac{1}{p}a(x)|\xi|^p$, then

$$\mathbf{G}(x,\xi) \geqslant -C. \tag{4.1}$$

Since $\lambda_1(a) < 0$ and $\dim(E_1) < \infty$, (4.1), assuming that (see inequality (1.10))

$$\begin{aligned}I(\xi) &= \frac{1}{p}\int_0^T \big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p dx - \int_0^T \mathbf{F}(x,\xi)dx\\ &= \frac{1}{p}\int_0^T \big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p dx - \frac{1}{p}\int_0^T a(x)|\xi(x)|^p dx - \int_0^T \mathbf{G}(x,\xi)dx\end{aligned}$$

$$\begin{aligned}&\leq \frac{\lambda_1(a)}{p}\int_0^T |\xi(x)|^p dx + C\\&\leqslant -C\|\xi\|^p + C,\end{aligned}$$

we have that $I(\xi) \to -\infty$ as $\|\xi\| \to \infty$ and $\xi \in E_1$.

(2) $I(\xi)$ is bounded from below on $E_2^{\perp}$. Similarly, if we set $\mathbf{G}_1(x,\xi) = \mathbf{F}(x,\xi) - \frac{1}{p}b(x)|\xi|^p$, then $\mathbf{G}(x,\xi) \leqslant C$, which implies that, for any $\xi \in E_2^{\perp}$, we have (see inequality (1.10))

$$\begin{aligned}I(\xi) &= \frac{1}{p}\int_0^T \left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\right|^p dx - \int_0^T \mathbf{F}(x,\xi)dx\\&= \frac{1}{p}\int_0^T \left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\right|^p dx - \frac{1}{p}\int_0^T b(x)|\xi(x)|^p dx - \int_0^T \mathbf{G}(x,\xi)dx\\&\geqslant \frac{\overline{\lambda(b)}}{p}\int_0^T |\xi(x)|^p dx - C\\&\geqslant -C,\end{aligned}$$

so $I(\xi)$ is bounded from below on $E_2^{\perp}$.

(3) Now, we fix an l such that $\sup_{\xi\in\partial C(l)\cap E_1} I(\xi) \leqslant \beta - 1$ where $\beta = \inf_{\xi\in E_2^{\perp}} I(\xi)$, and $C(l) = \{\xi \in \mathbb{H}_p^{\alpha,\beta,\psi} : \|\xi\| \leqslant l\}$. Set

$$\Gamma = \{\gamma : C(l)\cap E_1 \to \mathbb{H}_p^{\alpha,\beta,\psi} : \gamma(\xi) = i \text{ if } \xi \in E_1, \|\xi\| = l\},$$
$$c = \inf_{\gamma\in\Gamma}\max_{\xi\in C(l)} I(\xi).$$

Since $\partial C(l)\cap E$ and $E_2^{\perp}$ are linking and the (PS) condition holds for I, $c \geqslant \beta$ is a critical value of I, so there is a critical point $\xi_0 \in \mathbb{H}_p^{\alpha,\beta,\psi}$, such that $I(\xi_0) = c$.

For item (d), as for item (c) we only need to prove that $I(\xi) \to -\infty$ as $\|\xi\| \to \infty$, $\xi \in E_1$. Indeed, from $\mathbf{G}(x,\xi) = \mathbf{F}(x,\xi) - \frac{1}{p}a(x)|\xi|^{p-2}\xi$, $g(x,\xi) = f(x,\xi) - a(x)|\xi|^{p-2}\xi$, and (A4), it follows that

$$\lim_{|\xi|\to\infty} G(x,\xi) = \infty, \quad \text{for } x \in [0,T]. \tag{4.2}$$

Then for each $\xi \in E_1$, from (4.2) and the fact that $\dim(E_1) < \infty$, we have

$$\begin{aligned}I(\xi) &= \frac{1}{p}\int_0^T \left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\right|^p dx - \int_0^T \mathbf{F}(x,\xi)dx\\&= \frac{1}{p}\int_0^T \left|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\right|^p dx - \frac{1}{p}\int_0^T a(x)|\xi(x)|^p dx - \int_0^T \mathbf{G}(x,\xi)dx\\&= \frac{\lambda_1(a)}{p}\int_0^T |\xi(x)|^p dx - \int_0^T \mathbf{G}(x,\xi)dx \to -\infty, \quad \text{as } \|\xi\| \to \infty. \qquad \square\end{aligned}$$

We are now interested in obtaining multiple nontrivial solutions of (1.1). For that, we need some results of Morse theory.

Lemma 4.2. *Under conditions* (A1)–(A5), 0 *is a local minimum of the functional* I.

Proof. Since $\lambda_1(\ell) > 0$ there exists a constant $\varepsilon > 0$ such that $\lambda_1(\ell + \varepsilon) > 0$. Using the condition (A5) there exists $\delta = \delta(\varepsilon)$ such that

$$\mathbf{F}(x,t) \leqslant \frac{1}{p}(\ell(x)+\varepsilon)|t|^p, \quad \text{for } |t| \leqslant \delta,\ x \in [0,T].$$

Moreover, for $p < s \leqslant p^*$ we can find a $C > 0$ such that

$$\mathbf{F}(x,t) \leqslant C|t|^s, \quad \text{for } |t| > \delta,\ x \in [0,T].$$

Then, we obtain

$$\mathbf{F}(x,t) \leqslant \frac{1}{p}(\ell(x)+\varepsilon)|t|^p + C|t|^s \tag{4.3}$$

for $t \in \mathbb{R}$, $x \in [0,T]$.

Finally, using (3.1) combining with inequality (4.3) and the embedding theorem, we have

$$\begin{aligned} I(\xi) &= \frac{1}{p}\int_0^T \big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p dx - \int_0^T \mathbf{F}(x,\xi)dx \\ &\geqslant \frac{1}{p}\int_0^T \big|{}^H\mathbf{D}_{0^+}^{\alpha,\beta,\psi}\xi(x)\big|^p dx - \frac{1}{p}\int_0^T (\ell(x)+\varepsilon)|\xi(x)|^p - C\int_0^T |\xi(x)|^s dx \\ &\geqslant C\|\xi\|^p - C\|\xi\|^s > 0, \end{aligned}$$

as $0 < \|\xi\| < 1$, which implies that is 0 is a local minimum of I. □

To conclude this article, we prove the second main result.

Proof of Theorem 1.4. From Remark 1.5, we have

$$C_*(I,0) = \delta_{0,*}\mathbf{G}.$$

Using a result from [8], the solution ξ_0 obtained in Theorem 1.3 satisfies

$$C_1(I,\xi_0) \neq 0.$$

Hence ξ_0 is the nontrivial critical point of I. □

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Jose Vanterler da C. Sousa
Centro de Matemática, Computação e Cognição, Universidade Federal do ABC-UFABC, Santo André, SP, Brazil
Email address: jose.vanterler@ufabc.edu.br

Mariane Pigossi
Departamento de Matemática, Universidade Federal do Espírito Santo, Av. Fernando Ferrari, 514, UFES, 29075-910, Vitória, ES, Brazil
Email address: mariane.pigossi@ufes.br

Nemat Nyamoradi
Department of Mathematics, Faculty of Sciences Razi University, Kermanshah 67149, Iran
Email address: neamat80@yahoo.com