MASSERA TYPE THEOREMS FOR ABSTRACT NON-AUTONOMOUS EVOLUTION EQUATIONS

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ABSTRACT. We establish two fixed point theorems for affine maps in Banach spaces, with weaker assumptions than those in the literature. Then we establish some Massera type results for abstract linear evolution equations without assuming the existence of bounded solutions, which is an indispensable condition in the classical Massera theorem and in the earlier literature. As application, we present an existence result on periodic mild solutions to abstract nonautonomous semilinear evolution equations.

1. INTRODUCTION

Let X be a Banach space. Our aim is to investigate the existence of periodic mild solutions to the nonautonomous evolution equations

$$u'(t) = A(t)u(t) + g(t), \quad t \geq 0, \tag{1.1}$$

and

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \geq 0 \tag{1.2}$$

on X. g and f satisfy conditions specified later, and $\{A(t)\}_{t\geq 0}$ satisfies the following assumptions:

(A1) $D(A(t)) = D \subset X$ for all $t \geq 0$ and $A(t)$ is not necessarily densely defined, i.e., $\overline{D} = X$ is not necessarily true;

(A2) there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A(t))$ for every $t \geq 0$ and

$$\|\prod_{j=1}^{k} (\lambda I - A(t_j))^{-1}\| \leq \frac{M}{(\lambda - \omega)^k}$$

for every $\lambda > \omega$ and every finite sequence $\{t_j\}_{j=1}^{k}$ with $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k$, where $k = 1, 2, \ldots$;

(A3) the mapping $t \mapsto A(t)x$ is continuously differentiable in X for every $x \in D$;

(A4) $A(t+1) = A(t)$ for every $t \geq 0$.

The existence of periodic solutions to (1.1), (1.2) and their variants has been of great interest for many authors (cf. [1, 3, 4, 5, 7, 9, 8, 11, 12, 13, 14, 18, 19]). To establish the existence of periodic solutions to (1.2), one of the key steps is to consider first the existence of periodic solutions to the linear equation (1.1). It is

well-known that to find an initial value $x_0 \in X$ for a 1-periodic mild solution of
(1.1), one only needs to solve the equation

$$x_0 = P(x_0),$$

where

$$P : X \to X; x \mapsto U(1,0)x + \int_0^1 U(1,\tau)g(\tau)d\tau$$

stands for the Poincaré mapping and $\{U(t,s)\}_{t \geq s \geq 0}$ is the evolution system gener-
ated by $\{A(t)\}_{t \geq 0}$. Now the problem is transformed into a fixed point problem of
the Poincaré mapping, an affine map.

For fixed point theorems of Poincaré mapping, one of the celebrated result is by
Chow and Hale [2] who proved that P has a fixed point if the range $\mathcal{R}(I - U(1,0))$
is closed and there exists $z_0 \in X$ such that $\sup_{n \in \mathbb{N}} \|P^n z_0\| < +\infty$. Recently,
Zubelevich [20] made an important progress and got the result that P has a fixed
point if X is reflexive and $\sup_{n \in \mathbb{N}} \|P^n z_0\| < +\infty$ for some $z_0 \in X$. So, Zubelevich
removed the closedness of $\mathcal{R}(I - U(1,0))$ in the case of X being reflexive. Very
recently, Ezzinbi and Taoudi [6] also established several interesting results in this
direction on locally convex spaces and ordered Banach spaces. It is needed to note
that the boundedness of $\{P^n z_0\}$ is a key assumption in all the above literature.
As one will see, in this paper, this key assumption is weakened, i.e., we establish
two fixed point theorems for Poincaré mapping, where $\{P^n z_0\}$ is not necessarily
bounded for some $z_0 \in X$. Moreover, we give two examples, which satisfy our
weakened assumptions but not the boundedness condition in the earlier literature
[2, 20] (see Section 2).

Based on our new fixed point theorems for Poincaré mapping, we discuss the
existence of periodic solutions to (1.1) and obtain two Massera type theorems for
(1.1). It is interesting to note that in our Massera type results, the existence of
bounded solutions is not presupposed as in the classical Massera theorem. In fact,
to the best of our knowledge, of all the latest Massera type results up until now, the
existence of bounded solutions is an indispensable assumption (see, e.g., [6, 7, 9, 10]
for some recent Massera type results). Moreover, compared with other type results
on the existence of periodic solutions for (1.1), our Massera type theorems also
have some improvements to some extent. For example, our Massera type theorems
do not need the assumptions $\omega < 0$ and $Me^\omega < 1$ in [15]. As application of our
Massera type theorems, in the last part of this paper, we establish an existence
result on periodic mild solutions to (1.2).

Throughout this paper, we denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^+$ the
set of non-negative real numbers, by $\mathbb{C}$ the set of complex numbers, by $\mathbb{N}$ the set
of positive integers, by $L^p([a,b], X)$ ($L^p_{\text{loc}}(\mathbb{R}^+, X)$) the space of all (locally) pth
integrable functions, by $C(\mathbb{R}^+, X)$ the space of all continuous functions from $\mathbb{R}^+$ to
X, and $P_1(\mathbb{R}^+, X)$ the space of all 1-periodic functions from $\mathbb{R}^+$ to X.

2. Two fixed point theorems for affine maps

We first present a fixed point theorem, where the boundedness assumption in
the classical results by Chow and Hale is weakened.

Theorem 2.1. *Let X be a Banach space, $B : X \to X$ be a bounded linear operator,
$z \in X$ and $P : X \to X; x \mapsto Bx + z$. Assume that the range $\mathcal{R}(I - B)$ is closed*

and there exists $x_0 \in X$ such that

$$\lim_{n \to +\infty} \|\frac{P^n x_0}{n}\| = 0. \tag{2.1}$$

Then P has a fixed point $\overline{x} \in X$.

Proof. Our proof starts with the observation that P has a fixed point if $z \in \mathcal{R}(I - B)$. Let $x_n = \frac{1}{n} \sum_{k=1}^{n} P^k x_0$ for every $n \in \mathbb{N}$, then by a direct calculation, we have

$$\lim_{n \to +\infty} (I - B)x_n = \lim_{n \to +\infty} [(I - P)x_n + z]$$

$$= \lim_{n \to +\infty} \frac{1}{n}[Px_0 - P^{n+1}x_0] + z = z.$$

Combining this with $\mathcal{R}(I - B)$ is closed, we conclude that $z \in \mathcal{R}(I - B)$. □

Remark 2.2. In the case of $\mathcal{R}(I - B) = X$, the conclusion holds without assumption (2.1). Moreover, Chow and Hale [2] obtained the same conclusion of Theorem 2.1 under the condition $\sup_{n \in \mathbb{N}} \|P^n x_0\| < +\infty$, which implies that (2.1) holds. However, the converse is not necessarily true. In fact, let

$$B : l^2 \to l^2; (x_1, x_2, \dots, x_k, \dots) \mapsto \left(0, \sqrt[4]{2}x_1, \sqrt[4]{\frac{3}{2}}x_2, \dots, \sqrt[4]{\frac{k+1}{k}}x_k, \dots\right),$$

and

$$P : l^2 \to l^2; (x_1, x_2, \dots, x_k, \dots) \mapsto B(x_1, x_2, \dots, x_k, \dots) + (1, 0, 0, \dots).$$

By a direct calculation, we obtain that there exists $x_0 = (1, 0, 0, \dots) \in l^2$ such that

$$\sup_{n \in \mathbb{N}} \|P^n x_0\| \geq \sup_{n \in \mathbb{N}} \sqrt[4]{n+1} = +\infty$$

and

$$0 \leq \lim_{n \to +\infty} \|\frac{P^n x_0}{n}\| \leq \lim_{n \to +\infty} \frac{(n+1)^{\frac{3}{4}}}{n} = 0.$$

If X have some special properties, then the condition that $\mathcal{R}(I - B)$ is closed can be removed.

Theorem 2.3. *Let X be a Banach space, $B : X \to X$ be a bounded linear operator, $z \in X$ and $P : X \to X; x \mapsto Bx + z$. Assume that there exists $x_0 \in X$ such that*

$$\sup_{n \in \mathbb{N}} \|\frac{1}{n} \sum_{k=1}^{n} P^k x_0\| < +\infty, \quad \lim_{n \to +\infty} \|\frac{P^n x_0}{n}\| = 0, \tag{2.2}$$

and one of the following conditions holds:

(i) *X is reflexive;*
(ii) *there exists a separable Banach space Y such that X is the dual space of Y and $B^*Y \subset Y$, where B^* is the dual operator of B and Y is considered as a subspace of Y^{**}.*

Then P has a fixed point $\overline{x} \in X$. Moreover,

$$\|\overline{x}\| \leq \sup_{n \in \mathbb{N}} \|\frac{1}{n} \sum_{k=1}^{n} P^k x_0\|. \tag{2.3}$$

Proof. Firstly, we prove case (i). Let $x_n = \frac{1}{n} \sum_{k=1}^{n} P^k x_0$ for $n \in \mathbb{N}$. There exist $\{x_{n_j}\} \subset \{x_n\}$ and $\overline{x} \in X$ such that x_{n_j} converges weakly to $\overline{x}$ by using (2.2) and condition (i). So, for every $x^* \in X^*$

$$
\begin{aligned}
\lim_{j \to +\infty} \langle x^*, Px_{n_j} - P\overline{x} \rangle &= \lim_{j \to +\infty} \langle x^*, Bx_{n_j} - B\overline{x} \rangle \\
&= \lim_{j \to +\infty} \langle B^* x^*, x_{n_j} - \overline{x} \rangle = 0,
\end{aligned}
$$

which means that Px_{n_j} converges weakly to $P\overline{x}$. On the other hand, by (2.2) and a direct calculation, we have

$$
\lim_{j \to +\infty} Px_{n_j} - x_{n_j} = \lim_{j \to +\infty} \frac{1}{n_j} \left[Px_0 - P^{n_j+1} x_0 \right] = \mathbf{0}, \tag{2.4}
$$

where $\mathbf{0}$ is the zero member of X. Therefore,

$$
\begin{aligned}
0 &\leq \lim_{j \to +\infty} |\langle x^*, Px_{n_j} - \overline{x} \rangle| \\
&\leq \lim_{j \to +\infty} \left(|\langle x^*, Px_{n_j} - x_{n_j} \rangle| + |\langle x^*, x_{n_j} - \overline{x} \rangle| \right) = 0
\end{aligned}
$$

for every $x^* \in X^*$, which means that Px_{n_j} converges weakly to $\overline{x}$. From this and Px_{n_j} converges weakly to $P\overline{x}$, we obtain $P\overline{x} = \overline{x}$. In addition, (2.3) is obvious if $\overline{x} = \mathbf{0}$. When $\overline{x} \neq \mathbf{0}$, we know that there exists $\overline{x}^* \in X^*$ such that $\|\overline{x}^*\| = 1$ and $<\overline{x}^*, \overline{x}> = \|\overline{x}\|$. So

$$
\begin{aligned}
\|\overline{x}\| &= \langle \overline{x}^*, \overline{x} \rangle \\
&= \langle \overline{x}^*, \overline{x} - x_{n_j} \rangle + \langle \overline{x}^*, x_{n_j} \rangle \\
&\leq \langle \overline{x}^*, \overline{x} - x_{n_j} \rangle + \|\overline{x}^*\| \|x_{n_j}\| \\
&= \langle \overline{x}^*, \overline{x} - x_{n_j} \rangle + \| \frac{1}{n_j} \sum_{k=1}^{n_j} P^k x_0 \| \\
&\leq \langle \overline{x}^*, \overline{x} - x_{n_j} \rangle + \sup_{n \in \mathbb{N}} \| \frac{1}{n} \sum_{k=1}^{n} P^k x_0 \|
\end{aligned}
$$

for every $j \in \mathbb{N}$. Combining this with x_{n_j} converges weakly to $\overline{x}$, we obtain

$$
\|\overline{x}\| \leq \lim_{j \to +\infty} \left[\langle \overline{x}^*, \overline{x} - x_{n_j} \rangle + \sup_{n \in \mathbb{N}} \| \frac{1}{n} \sum_{k=1}^{n} P^k x_0 \| \right] = \sup_{n \in \mathbb{N}} \| \frac{1}{n} \sum_{k=1}^{n} P^k x_0 \|.
$$

Now, we prove case (ii). Let $x_n = \frac{1}{n} \sum_{k=1}^{n} P^k x_0$ for $n \in \mathbb{N}$. We conclude from (2.2) and condition (ii) that there exist $\{x_{n_j}\} \subset \{x_n\}$ and $\overline{x} \in X$ such that x_{n_j} weak*-convergent to $\overline{x}$, hence that

$$
\begin{aligned}
\lim_{j \to +\infty} \langle Px_{n_j} - P\overline{x}, y \rangle &= \lim_{j \to +\infty} \langle Bx_{n_j} - B\overline{x}, y \rangle \\
&= \lim_{j \to +\infty} \langle x_{n_j} - \overline{x}, B^* y \rangle = 0
\end{aligned}
$$

for every $y \in Y$, which means that Px_{n_j} weak*-convergent to $P\overline{x}$. Furthermore, we obtain $\{Px_{n_j} - x_{n_j}\}$ weakly*-convergent to $\mathbf{0}$ by (2.4). Then

$$
\begin{aligned}
0 &\leq \lim_{j \to +\infty} |\langle Px_{n_j} - \overline{x}, y \rangle| \\
&\leq \lim_{j \to +\infty} \left(|\langle Px_{n_j} - x_{n_j}, y \rangle| + |\langle x_{n_j} - \overline{x}, y \rangle| \right) = 0
\end{aligned}
$$

for every $y \in Y$, which means that Px_{n_j} weak*-convergent to $\overline{x}$. From this and Px_{n_j} weak*-convergent to $P\overline{x}$, we have $P\overline{x} = \overline{x}$. Moreover, since x_{n_j} is weak*-convergent to $\overline{x}$,

$$|\langle \overline{x}, y \rangle| = \lim_{j \to +\infty} |\langle x_{n_j}, y \rangle| \leq \limsup_{j \to +\infty} \|x_{n_j}\| \|y\| = \limsup_{j \to +\infty} \|x_{n_j}\| \leq \sup_{n \in \mathbb{N}} \|x_n\|$$

for every $y \in Y$ with $\|y\| = 1$. Therefore,

$$\|\overline{x}\| \leq \sup_{n \in \mathbb{N}} \|x_n\| = \sup_{n \in \mathbb{N}} \|\frac{1}{n} \sum_{k=1}^{n} P^k x_0\|.$$

Hence, the conclusion is valid.   □

Zubelevich [20] obtained the conclusion of the above theorem assuming that $\sup_{n \in \mathbb{N}} \|P^n x_0\| < +\infty$ and X is reflexive. We note that $\sup_{n \in \mathbb{N}} \|P^n x_0\| < +\infty$ implies

$$\sup_{n \in \mathbb{N}} \|\frac{1}{n} \sum_{k=1}^{n} P^k x_0\| < +\infty \quad \text{and} \quad \lim_{n \to +\infty} \|\frac{P^n x_0}{n}\| = 0.$$

However, the converse is not necessarily true. For example, let

$$P : l^2 \to l^2; (x_1, x_2, \ldots, x_k, \ldots) \mapsto \left(0, \sqrt[4]{2}x_1, \sqrt[4]{\frac{3}{2}}x_2, \ldots, \sqrt[4]{\frac{k+1}{k}}x_k, \ldots\right).$$

It is easy to see that P is a bounded linear operator and there exists $x_0 = (1, 0, \ldots) \in l^2$ such that

$$\sup_{n \in \mathbb{N}} \|P^n x_0\| = \sup_{n \in \mathbb{N}} \sqrt[4]{n+1} = +\infty,$$

$$\lim_{n \to +\infty} \|\frac{P^n x_0}{n}\| = \lim_{n \to +\infty} \frac{\sqrt[4]{n+1}}{n} = 0,$$

$$\sup_{n \in \mathbb{N}} \|\frac{1}{n} \sum_{k=1}^{n} P^k x_0\| \leq \sup_{n \in \mathbb{N}} \frac{(n+1)^{\frac{3}{4}}}{n} \leq 2.$$

3. MASSERA TYPE THEOREMS FOR LINEAR EVOLUTION EQUATIONS

In this section, we consider the existence of 1-periodic mild solutions to (1.1) with the help of our new fixed point theorems for affine maps. Throughout this section, we assume that $g \in P_1(\mathbb{R}^+, X) \cap L^1_{\text{loc}}(\mathbb{R}^+, X)$. Next, we first recall the following definitions and results which are taken from [15, 17, 16].

Proposition 3.1 ([17]). *Assume that $\{A(t)\}_{t \geq 0}$ satisfies conditions* (A1)–(A3) *and $t_0 > 0$. If $z \in \overline{D}$ and $h \in L^1([0, t_0], X)$, then the limit*

$$u(t) := \lim_{\lambda \to 0^+} \left(U_\lambda(t, 0)z + \int_0^{[\frac{t}{\lambda}]\lambda} U_\lambda(t, r)h(r)dr\right) \tag{3.1}$$

exists uniformly for $t \in [0, t_0]$, and u is a continuous function on $[0, t_0]$, where

$$U_\lambda(t, 0) = \prod_{j=1}^{[\frac{t}{\lambda}]} (I - \lambda A(j\lambda))^{-1}$$

for every $t \in [0, t_0]$ and $\lambda > 0$.

The following theorem can be derived from (A1)–(A4) and Proposition 3.1 (see also [15, 16]).

Proposition 3.2. *Assume that $\{A(t)\}_{t\geq 0}$ satisfies conditions* (A1)–(A3). *Then the limit*

$$U(t,s)z = \lim_{\lambda\to 0^+} U_\lambda(t,s)z$$

exists for every $z \in \overline{D}$ and $(t,s) \in \triangle$, where

$$U_\lambda(t,s) = \prod_{j=[\frac{s}{\lambda}]+1}^{[\frac{t}{\lambda}]} (I - \lambda A(j\lambda))^{-1}$$

and $\triangle = \{(t,s) | t \geq s \geq 0\}$. Also, the families $\{U(t,s)\}_{t\geq s\geq 0}$ and $\{U_\lambda(t,s)\}_{t\geq s\geq 0}$ satisfy the following properties:

(i) $U_\lambda(t,t)z = z$ *and* $U_\lambda(t,r)U_\lambda(r,s)z = U_\lambda(t,s)z$ *for every $z \in \overline{D}$, $\lambda > 0$ and $t \geq r \geq s \geq 0$;*

(ii) *for every $\lambda > 0$ and $(t,s) \in \triangle$,*

$$\|U_\lambda(t,s)\| \leq M\Big(\frac{1}{1-\lambda w}\Big)^{\frac{t-s}{\lambda}+1}; \tag{3.2}$$

(iii) $U(t,s) : \overline{D} \to \overline{D}$ *is a bounded linear operator for every $(t,s) \in \triangle$;*

(iv) $U(t,t)z = z$ *and* $U(t,r)U(r,s)z = U(t,s)z$ *for every $z \in \overline{D}$ and $t \geq r \geq s \geq 0$;*

(v) *for every $z \in \overline{D}$ and $(t,s) \in \triangle$,*

$$\|U(t,s)z\| \leq Me^{\omega(t-s)}\|z\|; \tag{3.3}$$

(vi) *for every $h \in L^1_{\text{loc}}(\mathbb{R}^+, X)$ and $(t,s) \in \triangle$,*

$$U(t,s)\lim_{\lambda\to 0^+}\int_0^s U_\lambda(s,r)h(r)dr = \lim_{\lambda\to 0^+}\int_0^s U_\lambda(t,r)h(r)dr. \tag{3.4}$$

(vii) *for every $h \in L^1_{\text{loc}}(\mathbb{R}^+, X)$ and $t \geq 0$,*

$$\lim_{\lambda\to 0^+}\int_0^{[\frac{t}{\lambda}]\lambda} U_\lambda(t,r)h(r)dr = \lim_{\lambda\to 0^+}\int_0^t U_\lambda(t,r)h(r)dr; \tag{3.5}$$

In addition, if $\{A(t)\}_{t\geq 0}$ satisfies the condition (A4), *then*

(viii) *for every $z \in \overline{D}$, $k \in \mathbb{N}$ and $(t,s) \in \triangle$,*

$$U_{\frac{1}{k}}(t+1,s+1)z = U_{\frac{1}{k}}(t,s)z; \tag{3.6}$$

(ix) *for every $z \in \overline{D}$ and $(t,s) \in \triangle$,*

$$U(t+1,s+1)z = U(t,s)z. \tag{3.7}$$

Definition 3.3 ([15]). A continuous function $u : [0,+\infty) \to \overline{D}$ is called a mild solution of (1.2) if u satisfies

$$u(t) = U(t,s)u(s) + \lim_{\lambda\to 0^+}\int_s^t U_\lambda(t,r)f(r,u(r))dr$$

for every $(t,s) \in \triangle$. Then, a continuous function $u : [0,+\infty) \to \overline{D}$ is called a mild solution of (1.1) if u satisfies

$$u(t) = U(t,s)u(s) + \lim_{\lambda\to 0^+}\int_s^t U_\lambda(t,r)g(r)dr \tag{3.8}$$

for every $(t,s) \in \triangle$.

Remark 3.4. Let $g \in L^1_{\text{loc}}(\mathbb{R}^+, X)$. By Proposition 3.1 and (3.5), we note that u is continuous if u satisfies (3.8).

Remark 3.5. Let $t_0 > 0$. According to [17, Theorem 4.2], if $g \in W^{1,1}([0, t_0], X)$, $x \in D$, and $A(0)x + g(0) \in \overline{D}$, then

$$u : [0, t_0] \to \overline{D}; t \mapsto U(t, 0)x + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t, r)g(r)dr$$

is a classical solution to (1.1) on $[0, t_0]$, where

$$W^{1,1}([0, t_0], X) = \left\{ u \in L^1([0, t_0], X) : u(t) = u_0 + \int_0^t v(s)ds \text{ for some } u_0 \in X \right.$$
$$\left. \text{and } v \in L^1([0, t_0], X), \ t \in [0, t_0] \right\}.$$

Using Theorems 2.1 and 2.3, we obtain the following results.

Theorem 3.6. *Assume that there exists a mild solution u_0 of* (1.1) *such that*

$$\lim_{n \to +\infty} \|\frac{u_0(n)}{n}\| = 0$$

and $\mathcal{R}(I - U(1, 0))$ is closed, where I is the identity map and $\mathcal{R}(I - U(1, 0))$ is the range of $I - U(1, 0)$. Then (1.1) *has a 1-periodic mild solution u satisfying*

$$\|u\| \le Me^{|\omega|}\left(\|u(0)\| + \int_0^1 \|g(\sigma)\|d\sigma \right) \tag{3.9}$$

Furthermore, the 1-periodic mild solution of (1.1) *is unique if*

$$\lim_{t \to +\infty} \|U(t, 0)x\| = 0 \quad \text{for every } x \in \overline{D} \text{ with } \sup_{t \ge 0} \|U(t, 0)x\| < +\infty. \tag{3.10}$$

Proof. Let

$$P : \overline{D} \to \overline{D}; x \mapsto U(1, 0)x + \lim_{\lambda \to 0^+} \int_0^1 U_\lambda(1, r)g(r)dr.$$

Take $X = \overline{D}$, $B = U(1, 0)$, $z = \lim_{\lambda \to 0^+} \int_0^1 U_\lambda(1, r)g(r)dr$, and $x_0 = u_0(0)$ in Theorem 2.1. Note that $P^k(x_0) = u_0(k)$ for every $k \in \mathbb{N}$. By Theorem 2.1, P has a fixed point $\overline{x} \in \overline{D}$. Let

$$u : [0, +\infty) \to X; \ t \mapsto U(t, 0)\overline{x} + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t, r)g(r)dr. \tag{3.11}$$

By using (3.4), Remark 3.4 and (iv) in Proposition 3.1, we have

$$U(t, s)u(s) + \lim_{\lambda \to 0^+} \int_s^t U_\lambda(t, r)g(r)dr$$

$$= U(t, s)\left[U(s, 0)\overline{x} + \lim_{\lambda \to 0^+} \int_0^s U_\lambda(s, r)g(r)dr \right] + \lim_{\lambda \to 0^+} \int_s^t U_\lambda(t, r)g(r)dr$$

$$= U(t, 0)\overline{x} + U(t, s) \lim_{\lambda \to 0^+} \int_0^s U_\lambda(s, r)g(r)dr + \lim_{\lambda \to 0^+} \int_s^t U_\lambda(t, r)g(r)dr$$

$$= U(t, 0)\overline{x} + \lim_{\lambda \to 0^+} \int_0^s U_\lambda(t, r)g(r)dr + \lim_{\lambda \to 0^+} \int_s^t U_\lambda(t, r)g(r)dr$$

$$= U(t, 0)\overline{x} + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t, r)g(r)dr = u(t)$$

for every $(t, s) \in \triangle$, which means that u is a mild solution to (1.1). Moreover, by $g \in P_1(\mathbb{R}^+, X)$, (3.4), (3.6), (3.7) and (iv) in Proposition 3.2, we have

$$u(t) = U(t, 0)\overline{x} + \lim_{\lambda \to 0^+} \int_0^t U_\lambda(t, r)g(r)dr$$

$$= U(t, 0)P\overline{x} + \lim_{k \to +\infty} \int_0^t U_{\frac{1}{k}}(t, r)g(r)dr$$

$$= U(t, 0)\Big[U(1, 0)\overline{x} + \lim_{\lambda \to 0^+} \int_0^1 U_\lambda(1, r)g(r)dr\Big]$$

$$\quad + \lim_{k \to +\infty} \int_0^t U_{\frac{1}{k}}(t+1, r+1)g(r+1)dr$$

$$= U(t+1, 1)\Big[U(1, 0)\overline{x} + \lim_{\lambda \to 0^+} \int_0^1 U_\lambda(1, r)g(r)dr\Big]$$

$$\quad + \lim_{k \to +\infty} \int_1^{t+1} U_{\frac{1}{k}}(t+1, r)g(r)dr$$

$$= U(t+1, 0)\overline{x} + \lim_{\lambda \to 0^+} \int_0^1 U_\lambda(t+1, r)g(r)dr + \lim_{\lambda \to 0^+} \int_1^{t+1} U_\lambda(t+1, \tau)g(\tau)d\tau$$

$$= U(t+1, 0)\overline{x} + \lim_{\lambda \to 0^+} \int_0^{t+1} U_\lambda(t+1, r)g(r)dr = u(t+1)$$

for every $t \geq 0$, which means that u is 1-periodic. In addition, let u_1 and u_2 be two 1-periodic mild solutions to (1.1). Note that $u_1(0) - u_2(0) \in \overline{D}$ and

$$\sup_{t \geq 0} \|U(t, 0)(u_1(0) - u_2(0))\| = \sup_{t \geq 0} \|u_1(t) - u_2(t)\| < +\infty.$$

So, $\lim_{t \to +\infty} \|u_1(t) - u_2(t)\| = \lim_{t \to +\infty} \|U(t, 0)(u_1(0) - u_2(0))\| = 0$ by (3.10). Combining this with $u_1 - u_2 \in P_1(\mathbb{R}^+, X)$, we obtain $u_1 = u_2$. Thus, the 1-periodic mild solution to (1.1) is unique. Furthermore, we obtain (3.9) by using (3.2) and (3.3). $\qquad\square$

Theorem 3.7. *Assume that there exists a mild solution u_0 of (1.1) such that*

$$\sup_{n \in \mathbb{N}} \|\frac{1}{n} \sum_{k=1}^n u_0(k)\| < +\infty, \quad \lim_{n \to +\infty} \|\frac{u_0(n)}{n}\| = 0,$$

and one of the following conditions holds:

(i) *X is reflexive;*
(ii) *there exists a separable Banach space Y such that $\overline{D}$ is the dual space of Y and $U^*(1, 0)Y \subset Y$, where $U^*(1, 0)$ is the dual operator of $U(1, 0)$ and Y is considered as a subspace of Y^{**}.*

Then (1.1) has a 1-periodic mild solution u satisfying

$$\|u\| \leq Me^{|\omega|}\Big(\sup_{n \in \mathbb{N}} \|\frac{1}{n} \sum_{k=1}^n u_0(k)\| + \int_0^1 \|g(\sigma)\|d\sigma\Big). \tag{3.12}$$

Furthermore, the 1-periodic mild solution of (1.1) is unique if (3.10) holds.

Proof. By using Theorem 2.3, similar to the proof of Theorem 3.6, one can show that (1.1) has a 1-periodic mild solution u given by (3.11) and u satisfies (3.9). In

addition, according to (2.3), we have

$$\|u(0)\| = \|\overline{x}\| \leq \sup_{n \in \mathbb{N}} \|\frac{1}{n} \sum_{k=1}^{n} P^k(x_0)\| = \sup_{n \in \mathbb{N}} \|\frac{1}{n} \sum_{k=1}^{n} u_0(k)\|.$$

Combining this with (3.9), we obtain (3.12). □

4. Application to semilinear evolution equations

Next, we establish the existence of 1-periodic mild solutions for (1.2) by our Massera type results obtained in the previous section. For convenience, we list some assumptions:

(A5) $f(\cdot, x) \in P_1(\mathbb{R}^+, X) \cap L^1_{\text{loc}}(\mathbb{R}^+, X)$ for every $x \in X$;
(A6) there exist $\varrho > 0$ and $L > 0$ such that

$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|$$

for every $t \geq 0$ and $x_1, x_2 \in X$ with $\|x_1\|, \|x_2\| \leq \varrho$.

From now on, we denote $\gamma = \int_0^1 \|f(\sigma, \mathbf{0})\| d\sigma$.

Theorem 4.1. *Assume that*

(a) *there exists $\alpha > 0$ with*

$$Me^{|\omega|}(\alpha + 1) < \min \left\{ \frac{1}{L}, \frac{\varrho}{\varrho L + \gamma} \right\} \tag{4.1}$$

such that (1.1) has a mild solution u_0^g satisfies

$$\sup_{n \in \mathbb{N}} \|\frac{1}{n} \sum_{k=1}^{n} u_0^g(k)\| \leq \alpha \int_0^1 \|g(\sigma)\| d\sigma \quad \text{and} \quad \lim_{n \to +\infty} \|\frac{u_0^g(n)}{n}\| = 0 \tag{4.2}$$

for every $g \in P_1(\mathbb{R}^+, X) \cap L^1_{\text{loc}}(\mathbb{R}^+, X)$;
(b) $\lim_{t \to +\infty} \|U(t, 0)x\| = 0$ *for every $x \in \overline{D}$ with $\sup_{t \geq 0} \|U(t, 0)x\| < +\infty$;*
(c) *one of the following conditions holds:*
   (i) *X is reflexive;*
   (ii) *there exists a separable Banach space Y such that $\overline{D}$ is the dual space of Y and $U^*(1, 0)Y \subset Y$, where $U^*(1, 0)$ is the dual operator of $U(1, 0)$ and Y is considered as a subspace of Y^{**}.*

Then (1.2) has a 1-periodic mild solution u.

Proof. Let $B_\varrho = \{v \in P_1(\mathbb{R}^+, X) \cap C(\mathbb{R}^+, X) | \sup_{t \in \mathbb{R}^+} \|v(t)\| \leq \varrho\}$ and

$$T : B_\varrho \to B_\varrho; \ v \mapsto u^v,$$

where u^v is the 1-periodic mild solution of (1.1) with $g = f(\cdot, v(\cdot))$ by Theorem 3.7.

Step 1. We show that $Tv \in B_\varrho$ for every $v \in B_\varrho$. Let $v \in B_\varrho$. Then we know from Theorem 3.7 that (1.1) has a unique 1-periodic mild solution u^v satisfying

$$\|u^v\| \leq Me^{|\omega|} \left[\sup_{n \in \mathbb{N}} \|\frac{1}{n} \sum_{k=1}^{n} u_0^g(k)\| + \int_0^1 \|g(\sigma)\| d\sigma \right],$$

Combining this with (4.2), (A5) and (A6), we obtain

$$\|u^v\| \leq Me^{|\omega|}(\alpha + 1) \int_0^1 \|f(\sigma, v(\sigma))\| d\sigma,$$

$$\leq Me^{|\omega|}(\alpha + 1)\Big(L \sup_{t\in\mathbb{R}^+} \|v(t)\| + \int_0^1 \|f(\sigma, \mathbf{0})\| d\sigma\Big)$$

$$\leq Me^{|\omega|}(\alpha + 1)(L\varrho + \gamma) \leq \varrho.$$

So, $Tv = u^v \in B_\varrho$ by (4.1).

Step 2. We show that T is a contraction. Let $v_1, v_2 \in B_\varrho$. Then

$$\begin{aligned}
\|Tv_1 - Tv_2\| &= \sup_{t\in\mathbb{R}^+} \|u^{v_1}(t) - u^{v_2}(t)\| \\
&= \sup_{t\in[0,1]} \|U(t,0)(u^{v_1}(0) - u^{v_2}(0)) \\
&\quad + \lim_{\lambda\to 0^+} \int_0^t U_\lambda(t,r)(f(r, v_1(r)) - (f(r, v_2(r)))dr\| \\
&\leq Me^{|\omega|}(\alpha + 1)L \sup_{t\in\mathbb{R}^+} \|v_1(t) - v_2(t)\| \\
&\leq Me^{|\omega|}(\alpha + 1)L\|v_1 - v_2\|,
\end{aligned}$$

where u^{v_1} and u^{v_2} are the corresponding 1-periodic mild solution to (1.1) with $g_1 = f(\cdot, v_1(\cdot))$ and $g_2 = f(\cdot, v_2(\cdot))$, respectively. By (4.1), we conclude that T is a contraction. So T has a unique fixed point $u \in B_\varrho$, i.e.,

$$u(t) = U(t,s)u(s) + \lim_{\lambda\to 0^+} \int_s^t U_\lambda(t,\tau)f(\tau, u(\tau))d\tau, \quad t \geq s \geq 0,$$

which is a 1-periodic mild solution for (1.2). $\square$

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