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EXISTENCE OF PSEUDOSOLUTIONS FOR DYNAMIC FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we consider the existence of pseudosolutions for boundary value problem for fractional differential equations of the form

$$ {}^{C}_{T}\Delta^{\alpha} x(t) = f(t, x(t)), \quad \text{for } t \in I_a = [0, a] \cap T, $$
$$ x(0) = x_0, \quad x_0 \in E, $$

where ${}^{C}_{T}\Delta^{\alpha} x(t)$, $\alpha \in (0, 1]$ denotes the Caputo fractional derivative, T denotes a time scale, and the function f is weakly-weakly sequentially continuous with values in a Banach space E and satisfies some boundary conditions and conditions expressed in terms of measures of weak non-compactness.

1. Introduction

The research on fractional calculus and fractional differential equations in Banach spaces, with a special focus on weak topology initiated in 2005 by Salem and his team (see [23] for Riemann-Liouville type fractional calculus and [22] for Hadamard type), marked the beginning of a new era in this field of mathematics. The publication of articles [22, 23] triggered significant interest, as evidenced by numerous citations such as [1, 4, 6, 10, 11, 16, 17, 19, 21, 24, 25, 26], leading to the development of a series of studies on initial and boundary value problems for various types of fractional differential equations. The introduction of fractional derivatives on time scales, allowing for the simultaneous modeling of discrete and continuous phenomena, provided additional flexibility in modeling phenomena that do not change in a linear manner or at a constant rate. These mathematical tools have found applications in many fields, from physics and engineering to control theory and quantum mechanics, opening up new possibilities in financial market modeling and population dynamics. In 1967, the Italian mathematician Caputo introduced the differential operator known as the Caputo operator, allowing for a deeper understanding of and solutions to problems related to viscoelasticity using fractional derivatives. The relationship between the Caputo Fractional Derivative and other fractional derivatives, such as Riemann-Liouville or Atangana-Baleanu, highlighted the significance of generalized Mittag-Leffler functions in mathematical modeling. By utilizing specific mathematical models, it became possible to

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enhance outcomes and solve problems previously considered difficult or impossible to address, opening new horizons in the theoretical and applied aspects of fractional calculus. In this paper, we consider the existence of a pseudosolution for the boundary value problem for fractional differential equations of the form

$$\begin{gathered} {}^{C}_{T}\Delta^{\alpha} x(t) = f(t, x(t)), \quad \text{for } t \in I_a = [0, a] \cap T, \\ x(0) = x_0, \quad x_0 \in E, \end{gathered} \tag{1.1}$$

where ${}^{C}_{T}\Delta^{\alpha} x(t)$, $\alpha \in (0, 1]$ is the Caputo fractional derivative, T denotes a time scale. We assume that the function f is weakly-weakly sequentially continuous with values in a Banach space and satisfies some regularity conditions expressed in terms of the De Blasi measure of weak noncompactness. We introduce a weakly sequentially continuous operator associated with an integral equation that is equivalent to the initial problem. There exist many important examples of mappings that are weakly sequentially continuous but not weakly continuous. The relations between weakly sequentially continuous and weakly continuous mappings are studied by Ball [5]. Adopting the fixed point theorem for weakly sequentially continuous mappings given by Kubiaczyk [14], and the properties of measures of weak noncompactness, we are able to study the existence results for the problem.

2. Preliminaries

Let $(E, \|\cdot\|)$ be a Banach space and let E^* be the dual space. Denote by $(C(I_a, E), \omega)$ the space of all continuous functions from I_a to E endowed with the topology $\sigma(C(I_a, E), C(I_a, E)^*)$, and by $C_{rd}(I_a, E)$ denote the space of all rd-continuous functions from the time scale interval I_a to E. By μ_Δ we denote the Lebesgue measure on time scale T. For a precise definition and basic properties of this measure we refer the reader to [8].

We now gather some well-known definitions and results from the literature, which we will use throughout this article.

I. To enable the reader to understand the so-called dynamic equations and to follow this paper easily, we present some preliminary definitions and notations of time scales which are very common in the literature (see [2, 3, 7, 12, 13] and references therein). A time scale T is a nonempty closed subset of real numbers $\mathbb{R}$, with the subspace topology inherited from the standard topology of $\mathbb{R}$. By an interval we mean the time scale interval

$$I_a = [0, a] \cap T = \{t \in T : 0 \le t \le a\} = [0, a]_T.$$

Definition 2.1. The forward jump operator $\sigma : T \to T$ and the backward jump operator $\rho : T \to T$ as $\sigma(t) = \inf\{s \in T : s > t\}$ and $\rho(t) = \sup\{s \in T : s < t\}$, respectively. We put $\inf \emptyset = \inf T$ (i.e. $\rho(m) = m$ if T has a minimum m).

The jump operators σ and ρ allow the classification of points in time scale in the following way: t is called right dense, right scattered, left dense, left scattered, dense and isolated if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, $\rho(t) = t = \sigma(t)$ respectively.

Definition 2.2. We say that is right-dense continuous (rd-continuous) if k is continuous at every right-dense point $t \in T$ and $\lim_{s \to t^-} k(s)$ exists and is finite at every left-dense point $t \in T$.

Definition 2.3. Fix $t \in T$. Let $f : I_a \to E$. Then we define Δ-derivative of f by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

The function f is called Δ-differentiable on T, if for each $t \in T$ there exists $f^{\Delta}(t)$.

Note that

1. $f^{\Delta} = f'$ is the usual derivative if $T = R$,
2. $f^{\Delta} = \Delta f$, is the usual forward difference operator if $T = Z$,
3. $f^{\Delta} = D_q f$ is the q-derivative if $T = q^{N_0} = \{q^t : t \in N_0, q > 1\}$.

Hence, the time scale allows us to consider the unification of differential, difference and q-difference equations as particular cases. However, our results also hold for more exotic time scales, which appear in fields such as mathematical biology or economics (see [7], for instance).

II. As in classical case, we need to introduce vector - valued Henstock-Kurzweil Δ-integrals. Definitions and basic properties of non absolute integrals were presented in [9]. We will use the notation $\eta(t) := \sigma(t) - t(t)$ where η is called the graininess function and $v(t) := t - \rho(t)$, where v is called the left - graininess function. We say that $\delta = (\delta_L, \delta_R)$ is a Δ-gauge for time scale interval $[a, b]$ provided $\delta_L(t) > 0$ on $(a, b]$, $\delta_R(t) > 0$ on $[a, b)$, $\delta_L(t) \geq 0$, $\delta_R(t) \geq 0$ and $\delta_R(t) \geq \eta(t)$ for all $t \in [a, b)$. We say that a partition D for a time scale interval $[a, b]$ given by

$$D = \{a = t_0 \leq \xi_1 \leq t_1 \leq \cdots \leq t_{n-1} \leq \xi_n \leq t_n = b\}$$

with $t_i > t_{i-1}$, for $1 \leq i \leq n$ and $t_i, \xi_i \in T$ is δ-fine if $\xi_i - \delta_L(\xi_i) \leq t_{i-1} < t_i \leq \xi_i + \delta_R(\xi_i)$, for $1 \leq i \leq n$.

Definition 2.4. A function $f : [a, b] \to E$ is the Henstock-Kurzweil Δ-integrable on $[a, b]$ (HK Δ-integrable in short) if there exists a function $F : [a, b] \to E$, defined on the subintervals of $[a, b]$, satisfying the following property: given $\epsilon > 0$ there exists a positive function δ on $[a, b]$ such that $D = \{[u, v], \xi\}$ is δ-fine division of a $[a, b]$, we have

$$\| \sum_{D} f(\xi)(v - u) - (F(v) - F(u)) \| < \epsilon$$

Definition 2.5. *A function $f : I_a \to E$ is Henstock-Kurzweil-Pettis Δ-integrable (HKP Δ-integrable for short) if*

1. *for all $x^* \in E^*$, $x^* f$ is Henstock-Kurzweil Δ-integrable on I_a,*
2. *forall $t \in I_a$ and all $x^* \in E^*, x^* g(t) = (Delta\text{-}HK) \int_0^t x^* f(s) \Delta s$.*

The function g will be called a primitive of f and by $g(t) = (Delta\text{-}HK)) \int_0^t f(s) \Delta s$ we will denote the Henstock-Kurzweil-Pettis Δ-integral of f on the interval I_a.

In [9] the author give examples of Henstock-Kurzweil-Pettis Δ-integrable functions which are not integrable in the sense of Pettis and Henstock-Kurzweil on time scales.

Theorem 2.6. *Suppose that $f, f_n : [a, b] \to E$, $n = 1, 2, \dots$ are $HKP\Delta$-integrable functions. Let F_n be a primitive of f_n. If one assumes that:*

1. *for all $x^* \in E^*$, $x^* f_n(x) \to x^* f(x)$ μ_Δ almost everywhere on I_a,*
2. *for all $x^* \in E^*$ the family $G = \{x^* F_n : n = 1, 2, \dots\}$ is uniformly ACG^* on I_a (i.e., weakly uniformly ACG^* on I_a),*
3. *for each $x^* \in E^*$ the set G is equicontinuous on I_a*

then f is Δ-HKP integrable on I_a and $\int_0^t f_n(s)\Delta s$ tends weakly in E to $\int_0^t f(s)\Delta s$ for each $t \in I_a$.

Theorem 2.7 ((Mean Value Theorem). *For each Δ-subinterval $[c,d] \subset [a,b]$, if the integral (Δ-HKP)$\int_c^d y(s)\Delta s$ exists, then we have*

$$(\Delta\text{-}HKP)\int_c^d y(s)\Delta s \in \mu_\Delta([c,d]) \cdot \overline{\text{conv}} y([c,d]),$$

where $\overline{\text{conv}} y([c,d])$ denotes the close convex hull of the set $y([c,d])$.

For completeness we introduce the definitions of the Caputo derivative of fractional order.

Definition 2.8. *Suppose that T is a time scale. The Caputo fractional derivative of g is defined by*

$${}_T^C\Delta^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{\Delta^n}(s)\Delta s, t \in I_a,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denote the integer part of α and integral is taken in the sense of Delta-HKP, Γ is the Gamma function.

Definition 2.9. *Suppose that T is a time scale, $g : I \to E$ is Δ-HKP integrable function. The fractional Δ-HKP integral of the order $\alpha \in R^+$ of g is defined by*

$$I^\alpha g(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)\Delta s,$$

where integral is taken in the sense of Δ-HKP and Γ is the Gamma function.

III. Our fundamental tools is the deBlasi measure of weak noncompactness $\beta(A)$. The deBlasi measure of weak noncompactness $\beta(A)$ is defined by

$$\beta(A) = \inf\{t > 0 : \text{ there exists } C \in K^\omega \text{ such that } A \subset C + tB_0\}$$

where K^ω is the set of weakly compact subsets of E and B_0 is the norm unit ball in E. The properties of the measure of noncompactness $\beta(A)$ are as follows:

(i) if $A \subset B$ then $\beta(A) \le \beta(B)$;
(ii) $\beta(A) = 0$ if and only if A is relatively weakly compact;
(iii) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;
(iv) $\beta(\bar{A}^\omega) = \beta(A)$, where $\bar{A}^\omega$ denotes the weak closure of A
(v) $\beta(\lambda A) = |\lambda|\beta(A)$, $(\lambda \in R)$;
(vi) $\beta(A + B) \le \beta(A) + \beta(B)$;
(vii) $\beta(\text{conv}(A)) = \beta(A)$, where $\text{conv}(A)$ denotes the convex extension of A.

Theorem 2.10 ([15]). *Let $H \subset C(I_a, E)$ be a family of strongly equicontinuous functions. Let $H(t) = \{h(t) \in E, h \in H\}$, for $t \in I_a$ and $H(I_a) = \bigcup_{t\in I_a} H(t)$. Then*

$$\beta_C(H) = \sup_{t\in I_a} \beta(H(t)) = \beta(H(I_a))$$

where $\beta_C(H)$ denotes the measure of weak noncompactness in $C(I_a, E)$, and the function $t \mapsto \beta(H(t))$ is continuous.

Definition 2.11. A function $f : I_a \to E$ is said to be weakly continuous if it is continuous from I_a to E endowed with its weak topology. A function $g : E \to E_1$ where E and E_1 are Banach spaces, is said to be weakly sequentially continuous if for each weakly convergent sequence (x_n) in E, the sequence $(g(x_n))$ is weakly convergent in E_1. When the sequence x_n tends weakly to x_0 in E, we will write $x_n \overset{\omega}{\to} x_0$.

Definition 2.12 ([12])**.** A family $\mathbf{F}$ of functions F is said to be uniformly absolutely continuous in the restricted sense on A or in short uniformly $AC_*(A)$, if for every $\epsilon > 0$ there is $\eta > 0$, such that for every F in $\mathbf{F}$ and for every finite or infinite sequence of nonoverlapping intervals $\{[a_i, b_i]\}$ with $a_i, b_i \in A$,and satisfying $\sum_i |b_i - a_i| < \eta$, we have $\sum_i \omega(F, [a_i, b_i]) < \epsilon$ where ω denotes the oscillation of F over $[a_i, b_i]$.

A family $\mathbf{F}$ of functions F is said to be uniformly generalized absolutely continuous in the restricted sense on $[a, b]$ or uniformly $ACG_*([a, b])$ if $[a, b]$ is the union of a sequence of closed sets A_i such that on each A_i the function F is uniformly $AC_*(A_i)$.

In the proof of the main theorem we will apply the following fixed point theorem.

Theorem 2.13 ([14])**.** *Let X be a metrizable locally convex topological vector space. Let D be a closed convex subset of X, and let F be a weakly-weakly sequentially continuous map from D into itself. If for some $x \in D$ the implication that*

$$\bar{V} = \overline{\operatorname{conv}}(\{x\} \cup F(V)) \implies V \text{ is relatively weakly compact}, \tag{2.1}$$

holds for every subset V of D, then F has a fixed point.

3. Main problem

Now we will consider the integral problem

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, x(s)) \Delta s, \text{ for } t \in I_a, \tag{3.1}$$

where $f : I_a \times E \to E$, T denotes a time scale (nonempty closed subset of real numbers R), $0 \in T$, I_a denotes a time scale interval, $(E, \| \cdot \|)$ is a Banach space and integral is taken in the sense of $\Delta - HKP$. Fix $x^* \in E^*$ and consider the problem

$$ {}^C_T\Delta^\alpha (x^* x)(t) = x^*(f(t, x(t))) \tag{3.2}$$

Definition 3.1. *Let $F : I \to E$ and let $A \subset I$. The function $f : A \to E$ is a fractional pseudo Δ-derivative of F on A if for each $x^* \in E^*$ the real-valued function $x^* F$ is ${}^C_T\Delta^\alpha$-differentiable μ_Δ almost everywhere on A and ${}^C_T\Delta^\alpha(x^* F) = x^* f$ μ_Δ almost everywhere on A.*

Regarding the above definition it is clear that the left-hand side of (3.2) can be rewritten to the form $x^*({}^C_T\Delta^\alpha x(t))$, where ${}^C_T\Delta^\alpha$ denotes the fractional pseudo Δ-derivative.

To obtain the existence result for our problem it is necessary to define a notion of a solution.

Definition 3.2. A function $x : I_a \to E$ is said to be a pseudosolution of problem (1.1) if it satisfies the following conditions:

1. $x(\cdot)$ is ACG^* function,
2. $x(0) = x_0$,

(3) for each $x^* \in E^*$ there exists a set $A(x^*)$ with μ_Δ measure zero, such that for each $t \notin A(x^*)$,
$$ {}_T^C\Delta^\alpha (x^*x)(t) = x^*(f(t, x(t)))\,. $$

Definition 3.3. A continuous function $x : I_a \to E$ is said to be a solution to problem (3.1) if it satisfies (3.1) for every $t \in I_a$.

Let
$$ \begin{gathered} B = \{x \in E : \|x\| \le \|x_0\| + p, p > 0\}, \\ \tilde{B} = \{x \in (C(I_a, E), \omega) : x(0) = x_0, \|x\| \le \|x_0\| + p, p > 0\}, \\ F(x)(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\Delta s, \text{ for } t \in I_a, \\ K = \{F(x) : x \in B\}\,. \end{gathered} $$

Theorem 3.4. *Assume that for each ACG_* function $x : I_a \to E, f(\cdot, x(\cdot))$ is fractionale Δ-HKP integrable, $f(t, \cdot)$ is weakly-weakly sequentially continuous. Suppose, that there exists a constans $c > 0$ such that*
$$ \beta(f(I \times X)) \le c\beta(X), 0 < \frac{c}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Delta s < 1, \quad t \in I, \tag{3.3} $$
for each bounded subset $X \subset B$ and for each subinterval I of I_a. Suppose that the set K is equicontinuous, equibounded and weakly uniformly ACG_* on I_a. Then there exists at least one pseudo solution of problem (1.1) on I_d, for some number $d \in T$, $0 < d \le a$.

Proof. We will prove, in fact, the existence of a solution for problem (3.1) because each solution of problem (3.1) is a solution of problem (1.1). Let x be a continuous solution of (3.1).

Fix an arbitrary $p \ge 0$. Recall, that the set K of continuous function $F(x) \in K$ defined on a time scale interval I_a is equicontinuous on I_a if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\|F(x)(t) - F(x)(\tau)\| < \epsilon$ for all $x \in \tilde{B}$ whenever $|t - \tau| < \delta$, $t, \tau \in I_a$, for each $F(x) \in K$. Thus, for each $\epsilon > 0$ there exists $\delta > 0$ such that $\| \int_\tau^t (t-s)^{\alpha-1} f(s, x(s))\Delta s\| < \epsilon$, for all $x \in \tilde{B}$, whenever $|t-\tau| < \delta$ and $t, \tau \in I_a$. As a result, there exists a number $d, 0 < d \le a$, such that $\| \int_0^t (t-s)^{\alpha-1} f(s, x(s))\Delta s\| \le p$, $t \in I_d$, $x \in \tilde{B}$.

We will show that the operator F is well defined and maps $\tilde{B}$ into $\tilde{B}$. To see this, note for any $x^* \in E^*$, such that $\|x^*\| \le 1$, for each $x \in \tilde{B}$ and $t \in I_d$ we have
$$ \begin{aligned} |x^*F(x)(t)| &= |x^*x_0| + \left|x^*\Big(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\Delta s\Big)\right| \\ &\le \|x^*\|\|x_0\| + \|x^*\| \Big\|\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\Delta s\Big\| \\ &\le \|x_0\| + |\frac{1}{\Gamma(\alpha)}|p \le \|x_0\| + p \end{aligned} $$
So
$$ \sup\{|x^*F(x)(t)| : x^* \in E^*, \|x^*\| \le 1\} \le \|x_0\| + p. $$
and as a result $\|F(x)(t)\| \le \|x_0\| + p$. Thus $F(x)(t) \in \tilde{B}$. We will show, that the operator F is weakly-weakly sequentially continuous. By [18, Lemma 9] a sequence

$x_n(\cdot)$ is weakly convergent in $C(I_d,E)$ to $x(\cdot)$ if and only if $x_n(t)$ tends weakly to $x(t)$ for each $t\in I_d$, so if $x_n \overset{\omega}{\to} x$ in $C(I_d,E)$ then $f(s,x_n(s)) \overset{\omega}{\to} f(s,x(s))$ in E for $t\in I_d$ and by Theorem 2.6 we have $F(x_n)(t)\to F(x)(t)$ weakly in E for each $t\in I_d$, so $F(x_n)\to F(x)$ in $C(I_d,E)$ with its weak topology. Suppose that $V\subset\tilde B$ satisfies the condition $\overline{V}=\overline{\text{conv}}(\{x\}\cup F(V))$. We will prove that V is relatively weakly compact and so (2.1) is satisfied. Since $V\subset\tilde B, F(V)\subset K$. Then $V\subset\overline{V}=\overline{\text{conv}}(\{x\}\cup F(V))$ is equicontinuous. By Theorem 2.10 $t\mapsto v(t)=\beta(V(t))$ is continuous on I_d. For fixed $t\in I_d$ we divide the interval $[0,t]$ into m parts in the following way:

$$
\begin{gathered}
t_0=0,\quad t_1=sup_{s\in I_a}\{s: s\ge t_0, s-t_0<\delta\},\\
t_2=\sup_{s\in I_a}\{s: s\ge t_1, s-t_1<\delta\},\dots,\\
t_m=\sup_{s\in I_a}\{s: s\ge t_{m-1}, s-t_{m-1}<\delta\}.
\end{gathered}
$$

Since T is closed, we have $t_i\in I_a$. If some $t_{i+1}=t_i$ then $t_{i+2}=\{\inf\ t\in T: t\ge t_{i+1}\}$.

$$
\begin{aligned}
F(x)(t)&=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s))\Delta s\\
&=x_0+\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{m-1}\int_{J_i}(t-s)^{\alpha-1}f(s,x(s))\Delta s\\
&\in x_0+\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{m-1}\mu_\Delta(J_i)\sup_{s_i\in J_i}(t-s_i)^{\alpha-1}\overline{\text{conv}}(f(J_i,V(J_i)))
\end{aligned}
$$

where $J_i=[t_i,t_{i+1}]$, $i=0,1,\dots,m-1$. Using (3.3) and properties of the measure of weak noncompactness we obtain

$$
\begin{aligned}
\beta(F(V(t)))&\le\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{m-1}\mu_\Delta(J_i)(t-q_i)^{\alpha-1}\beta(f(J_i,V(J_i)))\\
&\le\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{m-1}\mu_\Delta(J_i)(t-q_i)^{\alpha-1}\cdot c\cdot\beta(V(I_d))\\
&\le\frac{c\cdot\beta(V(I_d))}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Delta s\,.
\end{aligned}
$$

Since $V\subset\overline{V}=\overline{\text{conv}}(\{x\}\cup F(V))$, $\beta(V(t))\le\frac{c\cdot\beta(V(I_d))}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Delta s$. Using Theorem 2.6 we obtain

$$
\beta(V(I_d))\le\frac{c\cdot\beta(V(I_d))}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Delta s
$$

Since $\frac{c}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\Delta s<1$, we obtain $v(t)=\beta(V(t))=0$, for $t\in I_d$. Using Ascoli‘s theorem, V is relatively weakly compact. By Theorem 2.13 the operator F has a fixed point. Then there exists a pseudosolution to problem (1.1). □

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