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FORMS OF ENTIRE SOLUTIONS OF PARTIAL DIFFERENTIAL DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract. The purpose of this article is to describe the transcendental entire solutions of quadratic trinomial partial differential equations (PDDEs) with constant coefficients. We establish theorems on the forms of finite order transcendental entire solutions for such PDDEs, which generalize and improve previous theorems. Some examples confirm the existence and the forms of transcendental entire solutions with finite order of such equations.

1. Introduction

In this article, we consider transcendental entire solutions of certain quadratic trinomial partial differential difference equations (PDDEs) in $\mathbb{C}^2$, related to the Fermat type functional equations with constant coefficients. We begin with the Pythagorean functional equation

$$f^2 + g^2 = 1, \tag{1.1}$$

which is frequently studied as analogue of Diophantine equation over number fields. In 1966, Gross [4] proved the classical result that entire solutions of (1.1) are $f = \cos a(z)$, $g = \sin a(z)$, where $a(z)$ is an entire function. In fact, the study of these Fermat type functional equations (1.1) goes back to Montel [12] and Pólya [13]. The Pythagorean functional equations also include the eikonal equation $u_{z_1}^2 + u_{z_2}^2 = 1$, which was considered by Li [7] and by Khavinson [6]. They proved that entire solutions of $u_{z_1}^2 + u_{z_2}^2 = 1$ must be linear in $\mathbb{C}^2$. Clearly, the eikonal equation is a typical partial differential equation of Fermat type.

As is known, partial differential equations (PDEs) occure in various areas of applied mathematics, such as nonlinear acoustic wave propagation, geometric optics, and traffic flow (see [2, 3]). In general, it is difficult for us to find entire and meromorphic solutions of nonlinear PDEs. By employing Nevanlinna theory and other methods of complex analysis, there are a number of publications focusing on the solutions of some PDEs and their variants, see [1]. For instance, Yuan [24] obtained all traveling meromorphic exact solutions of the modified Zakharov-Kuznetsov equation by using a method of complex analysis; Khavinson [6] in 1995 showed that each entire solution of the partial differential equation $u_{z_1}^2 + u_{z_2}^2 = 1$ in $\mathbb{C}^2$ is necessarily linear by using Nevanlinna theory. Later, Saleeby [15, 17] studied

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the forms of entire and meromorphic solutions of some PDEs with several complex variables, and obtained the following result.

Theorem 1.1 ([15, Theorem 1]). *If f is an entire solution of*

$$\Big(\frac{\partial f(z_1,z_2)}{\partial z_1}\Big)^2+\Big(\frac{\partial f(z_1,z_2)}{\partial z_2}\Big)^2=1 \tag{1.2}$$

in $\mathbb{C}^2$, then $f(z_1,z_2)=c_1z_1+c_2z_2+c$, where $c,c_1,c_2\in\mathbb{C}$ and $c_1^2+c_2^2=1$.

Saleeby [16] further investigated entire and meromorphic solutions of the quadratic trinomial equation

$$f^2+2\alpha fg+g^2=1,\quad \alpha^2\neq 1,\ \alpha\in\mathbb{C}, \tag{1.3}$$

and obtained the following result.

Theorem 1.2 ([16, Theorem 2.1]). *Entire and meromorphic solutions of equation* (1.3) *(respectively) have the forms*

$$f=\frac{1}{\sqrt{2}}\Big(\frac{\cos h}{\sqrt{1+\alpha}}+\frac{\sin h}{\sqrt{1-\alpha}}\Big),\quad g=\frac{1}{\sqrt{2}}\Big(\frac{\cos h}{\sqrt{1+\alpha}}-\frac{\sin h}{\sqrt{1-\alpha}}\Big)$$

and

$$f=\frac{\alpha_1-\alpha_2\beta^2}{(\alpha_1-\alpha_2)\beta},\quad g=\frac{1-\beta^2}{(\alpha_1-\alpha_2)\beta},$$

where h is entire, β is mermorphic in $\mathbb{C}^2$ and $\alpha_1=-\alpha+\sqrt{\alpha^2-1}$, $\alpha_2=-\alpha-\sqrt{\alpha^2-1}$.

Liu, Cao and et al.[8-11] further studied entire solutions of some variants of Fermat type equations with more general forms than the difference equation $f(z)^2+f(z+c)^2=1$ and obtained the following result.

Theorem 1.3 ([10, Theorem 1.15]). *Let a_1,a_2,a_3,a_4 be nonzero constants. If*

$$[a_1f(z+c)+a_2f(z)]^2+[a_3f(z+c)+a_4f(z)]^2=1$$

admits transcendental entire solutions with finite order, then $a_1^2+a_3^2=a_2^2+a_4^2$ and

$$f(z)=\frac{a_2\cos(aiz+bi)+a_1\sin(aiz+bi)}{a_2a_3-a_1a_4},$$

where a is a nonzero constant and b is a constant.

Cao and Xu [1,19-23] investigated the existence of the solutions for some Fermat type partial differential difference equations with several variables by using the difference analogue of the logarithmic derivative lemma of several complex variables and obtained the following result.

Theorem 1.4 ([22, Theorem 1.2]). *Let $c=(c_1,c_2)$ be a constant in $\mathbb{C}^2$. Then any transcendental entire solution with finite order of the partial differential difference equation*

$$\Big(\frac{\partial f(z_1,z_2)}{\partial z_1}\Big)^2+f(z_1+c_1,z_2+c_2)^2=1 \tag{1.4}$$

has the form of $f(z_1,z_2)=\sin(Az_1+B)$, where A is a constant in $\mathbb{C}$ satisfying $Ae^{iAc_1}=1$, and B is a constant in $\mathbb{C}$; in the special case whenever $c_1=0$, we have $f(z_1,z_2)=\sin(z_1+B)$.

Recently, with the help of Nevanlinna theory and its difference analogues with several complex variables, Xu and et al. obtained some interesting results about Fermat type partial differential difference equations with several complex variables (see e.g.[21-24]). Especially, Zheng and Xu [25] in 2022 obtained the following result. When $\frac{\partial f(z_1,z_2)}{\partial z_1}$ and $f(z_1+c_1, z_2+c_2)$ in equation (1.4) are replaced by the partial differential difference polynomials in $\mathbb{C}^2$.

Theorem 1.5 ([25, Theorem 3.1]). *Let $c=(c_1,c_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, and a_1,a_2,a_3,a_4 be constants in $\mathbb{C}$ such that $D:=a_1a_4-a_2a_3\neq 0$. Let $f(z_1,z_2)$ be a transcendental entire solution with finite order of the partial differential difference equation*

$$\big[a_1f(z+c)+a_2\frac{\partial f}{\partial z_1}\big]^2+\big[a_3f(z+c)+a_4\frac{\partial f}{\partial z_1}\big]^2=1. \tag{1.5}$$

Then $f(z_1,z_2)$ is of the form

$$f(z_1,z_2)=-\frac{1}{D}\big(\frac{a_3+ia_1}{2\alpha_1}e^{L(z)+B}-\frac{a_3-ia_1}{2\alpha_1}e^{-L(z)-B}\big),$$

where $L(z)=\alpha_1z_1+\alpha_2z_2, \alpha_1(\neq 0), \alpha_2, B\in\mathbb{C}$ and $L(z)$ satisfies

$$\alpha_1^2=-\frac{a_1^2+a_3^2}{a_2^2+a_4^2},\quad e^{2L(c)}=\frac{(ia_2+a_4)(a_3-ia_1)}{(ia_2-a_4)(a_3+ia_1)}.$$

Theorem 1.6 ([25, Theorem 3.3]). *Let $c=(c_1,c_2)\in\mathbb{C}^2\backslash\{(0,0)\}$, a_1,a_2,a_3,a_4 be nonzero constants in $\mathbb{C}$, and $u_0=c_2+\frac{a_1a_4}{a_2a_3}c_1$. Let $f(z_1,z_2)$ be a transcendental entire solution with finite order of the partial differential difference equation*

$$\big[a_1f(z+c)+a_2\frac{\partial f}{\partial z_1}\big]^2+\big[a_3f(z+c)+a_4\frac{\partial f}{\partial z_2}\big]^2=1. \tag{1.6}$$

Then $f(z_1,z_2)$ is of one of the following two forms:
(i)

$$f(z_1,z_2)=\phi(z_2+\frac{a_1a_4}{a_2a_3}z_1),$$

where $\phi(u)$ is a transcendental entire function with finite order in $u:=z_2+\frac{a_1a_4}{a_2a_3}z_1$ satisfying

$$\phi(u+u_0)+\frac{a_4}{a_3}\phi(u)=\pm\frac{1}{\sqrt{a_1^2+a_3^2}};$$

(ii)

$$f(z_1,z_2)=\frac{a_3+ia_1}{2(\alpha_1a_2a_3-\alpha_2a_1a_4)}e^{L(z)+B}-\frac{a_3-ia_1}{2(\alpha_1a_2a_3-\alpha_2a_1a_4)}e^{-L(z)-B}+\varphi(u),$$

where $L(z)=\alpha_1z_1+\alpha_2z_2, \alpha_1,\alpha_2,B\in\mathbb{C}$ and $\varphi(u)$ satisfy

$$\frac{(a_2\alpha_1)^2+(a_4\alpha_2)^2}{a_1^2+a_3^2}=-1,$$
$$e^{L(c)}=-\frac{a_4\alpha_2+ia_2\alpha_1}{a_3+ia_1}=-\frac{a_3-ia_1}{ia_2\alpha_1-a_4\alpha_2},$$
$$\frac{a_4}{a_3}\varphi'(u)+\varphi(u+u_0)=0.$$

Inspired by the above results, the following question can be raised naturally:

> How about replacing binomials in the left-hand sides of both equations (1.5) and (1.6) with trinomials?

2. Main results and examples

Motivated by the above question, our purpose of this paper is to explore the finite order transcendental entire solutions of the following quadratic trinomial partial differential difference equations

$$\mathcal{D}_1(f)^2 + 2\alpha\mathcal{D}_1(f)\mathcal{D}_2(f) + \mathcal{D}_2(f)^2 = 1 \tag{2.1}$$

and

$$\mathcal{D}_1(f)^2 + 2\alpha\mathcal{D}_1(f)\mathcal{D}_3(f) + \mathcal{D}_3(f)^2 = 1, \tag{2.2}$$

where $\alpha(\neq 0, \pm 1)$, λ_j ($j = 1, 2, 3, 4$ and $\lambda_1\lambda_4 - \lambda_2\lambda_3 \neq 0$) are constants in $\mathbb{C}$, and

$$\mathcal{D}_1(f) = \lambda_1 f(z+c)+\lambda_2\frac{\partial f}{\partial z_1}, \mathcal{D}_2(f) = \lambda_3 f(z+c)+\lambda_4\frac{\partial f}{\partial z_1}, \mathcal{D}_3(f) = \lambda_3 f(z+c)+\lambda_4\frac{\partial f}{\partial z_2}.$$

If $\alpha = 0$, then equations (2.1) and (2.2) reduce to equations (1.5) and (1.6) respectively. If $\alpha = \pm 1$, that is, equations (2.1) and (2.2) can be represented as $[\mathcal{D}_1(f) \pm \mathcal{D}_2(f)]^2 = 1$ and $[\mathcal{D}_1(f) \pm \mathcal{D}_3(f)]^2 = 1$ respectively, then we have

$$(\lambda_1 \pm \lambda_3)f(z + c) + (\lambda_2 \pm \lambda_4)\frac{\partial f}{\partial z_1} = \pm 1 \tag{2.3}$$

and

$$(\lambda_1 \pm \lambda_3)f(z + c) + \lambda_2\frac{\partial f}{\partial z_1} \pm \lambda_4\frac{\partial f}{\partial z_2} = \pm 1, \tag{2.4}$$

respectively. Further, equation (2.3) has finite order transcendental entire solutions with the form $f(z) = \pm\frac{1}{\lambda_1 \pm \lambda_3} + e^{\beta_1 z_1+\beta_2 z_2+\beta_0}$, where $(\lambda_1 \pm \lambda_3)e^{\beta_1 c_1+\beta_2 c_2} = -(\lambda_2 \pm \lambda_4)\beta_1$; equation (2.4) has finite order transcendental entire solutions with the form $f(z) = \pm\frac{1}{\lambda_1 \pm \lambda_3} + e^{\gamma_1 z_1+\gamma_2 z_2+\gamma_0}$, where $(\lambda_1 \pm \lambda_3)e^{\gamma_1 c_1+\gamma_2 c_2} = -(\lambda_2\gamma_1 \pm \lambda_4\gamma_2)$.

In the following, we assume that $\alpha \neq 0, \pm 1$ and denote

$$A_1 = \frac{1}{\sqrt{1+\alpha}} + \frac{1}{i\sqrt{1-\alpha}}, \quad A_2 = \frac{1}{\sqrt{1+\alpha}} - \frac{1}{i\sqrt{1-\alpha}}.$$

The first main theorem is about the existence and the forms of transcendental entire solutions of the quadratic trinomial partial differential difference equation (2.1).

Theorem 2.1. *Let $c = (c_1, c_2) \in \mathbb{C}^2\setminus\{(0, 0)\}$, and $\lambda_j (j = 1, 2, 3, 4)$ be nonzero constants in $\mathbb{C}$ such that $D := \lambda_1\lambda_4 - \lambda_2\lambda_3 \neq 0$. If equation (2.1) has a transcendental entire solution $f(z_1, z_2)$ with finite order, then $f(z_1, z_2)$ has the form*

$$f(z_1, z_2) = \frac{\sqrt{2}}{4D}[\frac{\lambda_1 A_2 - \lambda_3 A_1}{\alpha_1}e^{L(z)+B} - \frac{\lambda_1 A_1 - \lambda_3 A_2}{\alpha_1}e^{-L(z)-B}],$$

where $L(z) = \alpha_1 z_1 + \alpha_2 z_2, \alpha_1(\neq 0), \alpha_2, B \in \mathbb{C}$ and $L(z)$ satisfies

$$\alpha_1^2 = -\frac{(\lambda_1 A_1 - \lambda_3 A_2)(\lambda_1 A_2 - \lambda_3 A_1)}{(\lambda_4 A_2 - \lambda_2 A_1)(\lambda_4 A_1 - \lambda_2 A_2)},$$
$$e^{2L(c)} = \frac{(\lambda_1 A_1 - \lambda_3 A_2)(\lambda_4 A_1 - \lambda_2 A_2)}{(\lambda_3 A_1 - \lambda_1 A_2)(\lambda_4 A_2 - \lambda_2 A_1)}.$$

The following examples confirm the conclusion about the form of transcendental entire solutions of equation (2.1).

Example 2.2. Let

$$f(z_1, z_2) = \frac{3\sqrt{21} - \sqrt{7}i}{14}e^{\frac{\sqrt{21}}{3}iz_1+\log\sqrt{\frac{13+3\sqrt{3}i}{14e^2}}iz_2+b_0}$$

$$+\frac{3\sqrt{21}+\sqrt{7}i}{14}e^{-\frac{\sqrt{21}}{3}iz_1-\log\sqrt{\frac{13+3\sqrt{3}i}{14e^2}}iz_2-b_0},$$

where $b_0 \in \mathbb{C}$. Then $f(z_1,z_2)$ is a transcendental entire solution of equation (2.1) with $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = \lambda_4 = 1$, $c_1 = -\frac{\sqrt{21}i}{7}$, $c_2 = 1$, $\alpha_1 = \frac{\sqrt{21}i}{3}$, $\alpha_2 = \log\sqrt{\frac{13+3\sqrt{3}i}{14e^2}}i$, $\alpha = \frac{1}{2}$ and $\rho(f) = 1$.

Example 2.3. Let

$$f(z_1,z_2)=\frac{21\sqrt{2}-7\sqrt{6}i}{6\sqrt{51}}e^{\frac{\sqrt{51}i}{14}z_1+(9+\sqrt{3}i)z_2+b_0}+\frac{21\sqrt{2}+7\sqrt{6}i}{6\sqrt{51}}e^{-\frac{\sqrt{51}i}{14}z_1-(9+\sqrt{3}i)z_2-b_0},$$

where $b_0 \in \mathbb{C}$. Then $f(z_1,z_2)$ is a transcendental entire solution of equation (2.1) with $\lambda_1 = 2$, $\lambda_2 = \sqrt{2}$, $\lambda_3 = 1$, $\lambda_4 = \sqrt{2}$, $c_1 = \frac{14}{\sqrt{51}i}$, $c_2 = \frac{3}{14}$, $\alpha_1 = \frac{\sqrt{51}i}{14}$, $\alpha_2 = 9+\sqrt{3}i$, $\alpha = \frac{1}{2}$, and $\rho(f) = 1$.

When $\frac{\partial f}{\partial z_1}$ in equation (2.1) is replaced by $\frac{\partial f}{\partial z_1}+\frac{\partial f}{\partial z_2}$, we obtain the second theorem as follows.

Theorem 2.4. *Let $c = (c_1,c_2) \in \mathbb{C}^2\backslash\{(0,0)\}$, and $\lambda_j (j = 1,2,3,4)$ be nonzero constants in $\mathbb{C}$ such that $D := \lambda_1\lambda_4 - \lambda_2\lambda_3 \neq 0$. Let $f(z_1,z_2)$ be a transcendental entire solution with finite order of the partial differential difference equation*

$$\mathcal{D}_1^*(f)^2 + 2\alpha\mathcal{D}_1^*(f)\mathcal{D}_2^*(f) + \mathcal{D}_2^*(f)^2 = 1, \tag{2.5}$$

where $\alpha(\neq 0,\pm1) \in \mathbb{C}$, and

$$\mathcal{D}_1^*(f) = \lambda_1 f(z+c) + \lambda_2(\frac{\partial f}{\partial z_1}+\frac{\partial f}{\partial z_2}), \quad \mathcal{D}_2^*(f) = \lambda_3 f(z+c) + \lambda_4(\frac{\partial f}{\partial z_1}+\frac{\partial f}{\partial z_2}).$$

Then $f(z_1,z_2)$ is of the form

$$f(z_1,z_2)=\frac{\sqrt{2}}{4D}[\frac{\lambda_1A_2-\lambda_3A_1}{\alpha_1+\alpha_2}e^{L(z)+B}-\frac{\lambda_1A_1-\lambda_3A_2}{\alpha_1+\alpha_2}e^{-L(z)-B}],$$

where $L(z) = \alpha_1z_1 + \alpha_2z_2, \alpha_1(\neq 0), \alpha_2, B \in \mathbb{C}$ and $L(z)$ satisfies

$$(\alpha_1+\alpha_2)^2=-\frac{(\lambda_1A_1-\lambda_3A_2)(\lambda_1A_2-\lambda_3A_1)}{(\lambda_4A_2-\lambda_2A_1)(\lambda_4A_1-\lambda_2A_2)},$$

$$e^{2L(c)}=\frac{(\lambda_1A_1-\lambda_3A_2)(\lambda_4A_1-\lambda_2A_2)}{(\lambda_3A_1-\lambda_1A_2)(\lambda_4A_2-\lambda_2A_1)}.$$

Since the proof of Theorem 2.4 is similar as the one of Theorem 2.1, we omit its proof. The following example confirms the conclusion about the forms of transcendental entire solutions of equation (2.5).

Example 2.5. Let

$$f(z_1,z_2)=\frac{3+9\sqrt{3}}{2\sqrt{66}}e^{\sqrt{22}iz_1+2\sqrt{22}iz_2+b_0}+\frac{3-9\sqrt{3}}{2\sqrt{66}}e^{-\sqrt{22}iz_1-2\sqrt{22}iz_2-b_0},$$

where $b_0 \in \mathbb{C}$. Then $f(z_1,z_2)$ is a transcendental entire solution of equation (2.5) with $\lambda_1 = 2$, $\lambda_3 = 1$, $\lambda_2 = \lambda_4 = \frac{1}{4}$, $c_1 = 1$, $c_2 = \frac{5}{6}$, $\alpha_1 = \sqrt{22}i$, $\alpha_2 = 2\sqrt{22}i$, $\alpha = \frac{1}{2}$, and $\rho(f) = 1$.

When $\lambda_4\frac{\partial f}{\partial z_1}$ in equation (2.1) is replaced by $\lambda_4\frac{\partial f}{\partial z_2}$, that is equation (2.2), we obtain the following theorem.

Theorem 2.6. *Let $c = (c_1, c_2) \in \mathbb{C}^2\backslash(0,0)$, and $\lambda_j (j = 1,2,3,4)$ be nonzero constants in $\mathbb{C}$. If equation (2.2) has a transcendental entire solution $f(z_1, z_2)$ with finite order, then $f(z_1, z_2)$ is of one of the following two forms.*
(i)

$$f(z_1, z_2) = \phi\big(z_2 + \frac{\lambda_1\lambda_4}{\lambda_2\lambda_3} z_1\big),$$

where $\phi(u)$ is a transcendental entire solution with finite order in $u := z_2 + \frac{\lambda_1\lambda_4}{\lambda_2\lambda_3} z_1$ satisfying

$$\phi(u + u_0) + \frac{\lambda_4}{\lambda_3}\phi'(u) = \pm\frac{1}{\sqrt{\lambda_1^2 + \lambda_3^2 + 2\alpha\lambda_1\lambda_3}}, \quad u_0 = c_2 + \frac{\lambda_1\lambda_4}{\lambda_2\lambda_3} c_1.$$

(ii) If $c_1\lambda_4 \neq \pm c_2\lambda_2$ and $\lambda_2\lambda_3\alpha_1 - \lambda_1\lambda_4\alpha_2 = 0$, then

$$f(z_1, z_2) = \frac{\sqrt{2}}{4\lambda_2\lambda_3} z_1[(\lambda_3 A_1 - \lambda_1 A_2)e^{\alpha_2 u + B} + (\lambda_3 A_2 - \lambda_1 A_1)e^{-\alpha_2 u - B}] + \varphi(u);$$

if $\lambda_2\lambda_3\alpha_1 - \lambda_1\lambda_4\alpha_2 \neq 0$, then

$$f(z_1, z_2) = \frac{\sqrt{2}}{4}\Big[\frac{\lambda_3 A_1 - \lambda_1 A_2}{\lambda_2\lambda_3\alpha_1 - \lambda_1\lambda_4\alpha_2} e^{L(z)+B} - \frac{\lambda_3 A_2 - \lambda_1 A_1}{\lambda_2\lambda_3\alpha_1 - \lambda_1\lambda_4\alpha_2} e^{-L(z)-B}\Big] + \varphi(u),$$

where $L(z) = \alpha_1 z_1 + \alpha_2 z_2, \alpha_1, \alpha_2, B \in \mathbb{C}$ and $L(z)$ satisfies

$$e^{L(c)} = \frac{\lambda_3 A_2 - \lambda_1 A_1}{\lambda_4 A_2\alpha_2 - \lambda_2 A_1\alpha_1} = \frac{\lambda_2 A_2\alpha_1 - \lambda_4 A_1\alpha_2}{\lambda_3 A_1 - \lambda_1 A_2},$$

and $\varphi(u)$ is an entire function with finite order in $u = z_2 + \frac{\lambda_1\lambda_4}{\lambda_2\lambda_3} z_1$ satisfying $\varphi(u + u_0) + \frac{\lambda_4}{\lambda_3}\varphi'(u) = 0$, $u_0 = c_2 + \frac{\lambda_1\lambda_4}{\lambda_2\lambda_3} c_1$.

We also give two examples to confirm the conclusion about the forms of transcendental entire solutions of equation (2.2).

Example 2.7. Let

$$f(z_1, z_2) = \pm\frac{\sqrt{7}}{7} + e^{2z_1 + z_2}.$$

Then $f(z_1, z_2)$ is a transcendental entire solution of equation (2.2) with $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = \lambda_4 = 1$, $c_1 = c_2 = \pi i$, $\alpha = 1/2$.

Example 2.8. Let

$$f(z_1, z_2) = \frac{\sqrt{6}}{12} e^{L(z)+B} - \frac{\sqrt{6}}{12} e^{-(L(z)+B)} + e^{2z_1 + z_2},$$

where $L(z) = \frac{1}{2} z_1 + (2 - \sqrt{3}) z_2, B \in \mathbb{C}$. Then $f(z_1, z_2)$ is a transcendental entire solution of equation (2.2) with $\lambda_1 = \lambda_3 = \lambda_4 = 1$, $\lambda_2 = 2$,

$$c_1 = (1 + \sqrt{3}) \log\frac{2\sqrt{3}}{1 + 3\sqrt{3} - (3 + \sqrt{3})i} + (1 - \sqrt{3})\pi i,$$

$$c_2 = -\frac{1 + \sqrt{3}}{2}\Big(\log\frac{2\sqrt{3}}{1 + 3\sqrt{3} - (3 + \sqrt{3})i} - \pi i\Big),$$

$\alpha_1 = 1/2$, $\alpha_2 = 2 - \sqrt{3}i, \alpha = 2$, and $\rho(f) = 1$.

Next, we give some lemmas which play the key role in proving our results.

Lemma 2.9 ([14, 18]). *For an entire function F in $\mathbb{C}^n$, $F(0) \neq 0$ put $\rho(n_F) = \rho < \infty$. Then there exist a canonical function f_F and a function $g_F(z) \in \mathbb{C}^n$ such that $F(z) = f_F(z)e^{g_F(z)}$. For the special case $n = 1$, f_F is the canonical product of Weierstrass.*

Here we denote by $\rho(n_F)$ the order of the counting function of zeros of F.

Lemma 2.10 ([13]). *If g and h are entire functions in the complex plane and $g(h)$ is an entire function of finite order, then there are only two possible cases: either*

(i) *the internal function h is a polynomial and the external function g is of finite order; or*
(ii) *the internal function h is not a polynomial but a function of finite order, and the external function g is of zero order.*

Lemma 2.11 ([5]). *Let $f_j(\not\equiv 0)$, $j = 1, 2, 3$ be meromorphic functions in $\mathbb{C}^n$ such that f_1 is not a constant, and $f_1 + f_2 + f_3 = 1$, and such that*

$$\sum_{j=1}^{3}\{N_2(r, \frac{1}{f_j}) + 2\overline{N}(r, f_j)\} < \lambda T(r, f_1) + O(\log^+ T(r, f_1)),$$

for all r outside possibly a set with finite logarithmic measure, where $\lambda(< 1)$ is a possible number. Then either $f_2 \equiv 1$ or $f_3 \equiv 1$.

Here, $N_2(r, \frac{1}{f})$ is the counting function of the zeros of f in $|z| \leq r$, where the simple zero is counted once and the multiple zero is counted twice.

3. Proof of Theorem 2.1

Suppose that f is a transcendental entire solution with finite order of equation (2.1). Denote

$$\mathcal{D}_1(f) = \frac{1}{\sqrt{2}}(m + n), \quad \mathcal{D}_2(f) = \frac{1}{\sqrt{2}}(m - n), \tag{3.1}$$

where m, n are entire functions in $\mathbb{C}^2$. Thus, equation (2.1) can be rewritten as

$$(1 + \alpha)m^2 + (1 - \alpha)n^2 = 1. \tag{3.2}$$

If $\sqrt{1+\alpha}m$ is not a transcendental entire function, by (3.2), then $\sqrt{1-\alpha}n$ is not a transcendental entire function, which implies that f is not a transcendental entire function, a contradiction with the assumption that f is a transcendental entire function.

Hence, $\sqrt{1+\alpha}m$ and $\sqrt{1-\alpha}n$ are transcendental functions. We can rewrite equation (3.2) as

$$(\sqrt{1+\alpha}m + i\sqrt{1-\alpha}n)(\sqrt{1+\alpha}m - i\sqrt{1-\alpha}n) = 1. \tag{3.3}$$

Noting that m, n are transcendental entire functions with finite order, we have that $\sqrt{1+\alpha}m + i\sqrt{1-\alpha}n$ and $\sqrt{1+\alpha}m - i\sqrt{1-\alpha}n$ have no zeros and poles. Therefore, by Lemmas 2.9 and 2.10, there exists a nonconstant polynomial $p(z)$ in $\mathbb{C}^2$ such that

$$\sqrt{1+\alpha}m + i\sqrt{1-\alpha}n = e^{p(z)}, \quad \sqrt{1+\alpha}m - i\sqrt{1-\alpha}n = e^{-p(z)}. \tag{3.4}$$

In view of (3.1) and (3.4), it follows that

$$\begin{aligned}\mathcal{D}_1(f) &= \lambda_1 f(z+c) + \lambda_2 \frac{\partial f}{\partial z_1} \\ &= \frac{\sqrt{2}}{4}[(\frac{1}{\sqrt{1+\alpha}} + \frac{1}{i\sqrt{1-\alpha}})e^{p(z)} + (\frac{1}{\sqrt{1+\alpha}} - \frac{1}{i\sqrt{1-\alpha}})e^{-p(z)}] \\ &= \frac{\sqrt{2}}{4}(A_1 e^{p(z)} + A_2 e^{-p(z)})\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\mathcal{D}_2(f) &= \lambda_3 f(z+c) + \lambda_4 \frac{\partial f}{\partial z_1} \\ &= \frac{\sqrt{2}}{4}[(\frac{1}{\sqrt{1+\alpha}} - \frac{1}{i\sqrt{1-\alpha}})e^{p(z)} + (\frac{1}{\sqrt{1+\alpha}} + \frac{1}{i\sqrt{1-\alpha}})e^{-p(z)}] \\ &= \frac{\sqrt{2}}{4}(A_2 e^{p(z)} + A_1 e^{-p(z)}).\end{aligned} \tag{3.6}$$

Noting that $D = \lambda_1\lambda_4 - \lambda_2\lambda_3 \neq 0$, and solving the system consisting of (3.5) and (3.6), we deduce that

$$f(z+c) = \frac{\sqrt{2}}{4D}[(\lambda_4 A_1 - \lambda_2 A_2)e^{p(z)} + (\lambda_4 A_2 - \lambda_2 A_1)e^{-p(z)}], \tag{3.7}$$

$$\frac{\partial f}{\partial z_1} = \frac{\sqrt{2}}{4D}[(\lambda_1 A_2 - \lambda_3 A_1)e^{p(z)} + (\lambda_1 A_1 - \lambda_3 A_2)e^{-p(z)}]. \tag{3.8}$$

Then (3.7) and (3.8) yield

$$\omega_1 e^{p(z+c)} + \omega_2 e^{-p(z+c)} = \omega_3 \frac{\partial p}{\partial z_1} e^{p(z)} - \omega_4 \frac{\partial p}{\partial z_1} e^{-p(z)}, \tag{3.9}$$

where

$$\begin{gathered}\omega_1 = \lambda_1 A_2 - \lambda_3 A_1,\ \omega_2 = \lambda_1 A_1 - \lambda_3 A_2, \\ \omega_3 = \lambda_4 A_1 - \lambda_2 A_2,\ \omega_4 = \lambda_4 A_2 - \lambda_2 A_1.\end{gathered}$$

If $\omega_2 = 0$, then $\omega_1 \neq 0$. Otherwise, $A_1^2 = A_2^2$, a contradiction. If $\omega_3 \frac{\partial p}{\partial z_1} = 0$, then either $\frac{\partial p}{\partial z_1} = 0$ or $\omega_3 = 0$. If $\frac{\partial p}{\partial z_1} = 0$, then from (3.9), it follows that $\omega_1 e^{p(z+c)} = 0$, which implies $\omega_1 = 0$, a contradiction. If $\omega_3 = 0$, then $\omega_4 \neq 0$. Otherwise, $A_1^2 = A_2^2$, a contradiction. Thus, (3.9) yieldS

$$e^{p(z+c)+p(z)} = -\frac{\omega_4}{\omega_1}\frac{\partial p}{\partial z_1}.$$

Since $p(z)$ is a nonconstant polynomial, $p(z+c)+p(z)$ can not be a constant. Hence, the above equation implies a contradiction that the left-hand side is transcendental but the right-hand side is not transcendental. Thus, it follows that $\omega_3 \neq 0$. Then, $\omega_4 \neq 0$. Otherwise, $\omega_2 = \omega_4$ deduces a contradiction that $D = 0$. By combining this with (3.9), yields

$$\omega_1 e^{p(z+c)+p(z)} = \omega_3 \frac{\partial p}{\partial z_1} e^{2p(z)} - \omega_4 \frac{\partial p}{\partial z_1}. \tag{3.10}$$

Noting that $N(r, \frac{1}{e^{p(z+c)+p(z)}}) = 0$, $N(r, e^{p(z+c)+p(z)}) = 0$ and $N(r, \frac{1}{e^{2p(z)}}) = 0$, by the Nevanlinna second main theorem in several complex variables, and in view of

(3.10), we conclude that

$$\begin{aligned}T(r, e^{p(z+c)+p(z)}) &\leq N(r, \frac{1}{e^{p(z+c)+p(z)}}) + N(r, \frac{1}{e^{p(z+c)+p(z)} - \chi}) \\ &\quad + N(r, e^{p(z+c)+p(z)}) + S(r, e^{p(z+c)+p(z)}) \\ &\leq N(r, \frac{1}{e^{p(z+c)+p(z)} - \chi}) + S(r, e^{p(z+c)+p(z)}) \\ &= N(r, \frac{1}{\frac{\omega_3}{\omega_1}\frac{\partial p}{\partial z_1} e^{2p(z)}}) + S(r, e^{p(z+c)+p(z)}) = S(r, e^{p(z+c)+p(z)}),\end{aligned}$$

where $\chi = -\frac{\omega_4}{\omega_1}\frac{\partial p}{\partial z_1}$. This is a contradiction. We conclude that $\omega_2 \neq 0$.

Similarly, we have $\omega_1 \neq 0$, $\omega_3 \neq 0$, $\omega_4 \neq 0$. Thus, we rewrite (3.9) in the form

$$\frac{\omega_3}{\omega_2}\frac{\partial p}{\partial z_1} e^{p(z+c)+p(z)} - \frac{\omega_4}{\omega_2}\frac{\partial p}{\partial z_1} e^{p(z+c)-p(z)} - \frac{\omega_1}{\omega_2} e^{2p(z+c)} = 1. \tag{3.11}$$

In view of $\frac{\omega_1}{\omega_2} e^{2p(z+c)} \neq 0$ and $e^{p(z+c)+p(z)}$ is nonconstant, by Lemma 2.3,

$$\frac{\omega_4}{\omega_2}\frac{\partial p}{\partial z_1} e^{p(z+c)-p(z)} \equiv -1. \tag{3.12}$$

Thus, it follows from (3.11) that

$$\frac{\omega_3}{\omega_1}\frac{\partial p}{\partial z_1} e^{p(z)-p(z+c)} \equiv 1. \tag{3.13}$$

Since $p(z)$ is a polynomial, (3.12) (or (3.13)) implies $p(z+c) - p(z) = \zeta$, where ζ is a constant in $\mathbb{C}$. Thus, it follows that $p(z) = L(z) + H(z) + B$, where $L(z) = \alpha_1 z_1 + \alpha_2 z_2, \alpha_1, \alpha_2 \in \mathbb{C}, H(z) := H(s), H(s)$ is a polynomial in $s = c_2 z_1 - c_1 z_2, c_1, c_2 \in \mathbb{C}$, and $B \in \mathbb{C}$. Next, we prove that $H(z) \equiv 0$. (3.12) implies

$$\frac{\omega_4}{\omega_2}\alpha_1 + \frac{\omega_4}{\omega_2} c_2 \frac{dH}{ds} \equiv -e^{-\zeta},$$

which also means that $\deg_s H \leq 1$. Thus, the form of $L(z) + H(z) + B$ is still the linear form of $\alpha_1 z_1 + \alpha_2 z_2 + B, \alpha_1, \alpha_2, B \in \mathbb{C}$, which means that $H(z) \equiv 0$. Hence, it follows that $p(z) = L(z) + B = \alpha_1 z_1 + \alpha_2 z_2 + B, \alpha_1, \alpha_2, B \in \mathbb{C}$. By substituting this into (3.12) and (3.13), we deduce that

$$-\frac{\omega_4}{\omega_2}\alpha_1 e^{L(c)} \equiv 1, \quad \frac{\omega_3}{\omega_1}\alpha_1 e^{-L(c)} \equiv 1, \tag{3.14}$$

which implies that

$$\alpha_1^2 = -\frac{\omega_1\omega_2}{\omega_3\omega_4}, \quad e^{2L(c)} = -\frac{\omega_2\omega_3}{\omega_1\omega_4}. \tag{3.15}$$

By applying (3.14) to (3.7), we have

$$\begin{aligned}f(z) &= \frac{\sqrt{2}}{4D}[\omega_3 e^{L(z)+B-L(c)} + \omega_4 e^{-L(z)-B+L(c)}] \\ &= \frac{\sqrt{2}}{4D}[\frac{\omega_1}{\alpha_1} e^{L(z)+B} - \frac{\omega_2}{\alpha_1} e^{-L(z)-B}].\end{aligned}$$

The proof of Theorem 2.1 is complete.

4. Proof of Theorem 2.6

Suppose that f is a transcendental entire solution with finite order of equation (2.2). Denote

$$\mathcal{D}_1(f)=\frac{1}{\sqrt{2}}(u+v),\quad \mathcal{D}_3(f)=\frac{1}{\sqrt{2}}(u-v), \tag{4.1}$$

where u,v are entire functions in $\mathbb{C}^2$. Thus, equation (2.2) can be rewritten as

$$(1+\alpha)u^2+(1-\alpha)v^2=1. \tag{4.2}$$

As in Theorem 2.1, we discuss the folowing two cases.

Case 1. $\sqrt{1+\alpha}u$ is a constant. We denote

$$\sqrt{1+\alpha}u=\gamma_1,\quad \gamma_1\in\mathbb{C}. \tag{4.3}$$

In view of equation (4.2), it follows that $\sqrt{1-\alpha}v$ is also a constant. We denote

$$\sqrt{1-\alpha}v=\gamma_2,\ \gamma_2\in\mathbb{C}. \tag{4.4}$$

This leads to $\gamma_1^2+\gamma_2^2=1$. Thus, we deduce from (4.1) that

$$\mathcal{D}_1(f)=\lambda_1 f(z+c)+\lambda_2\frac{\partial f}{\partial z_1}=\frac{1}{\sqrt{2}}(\frac{\gamma_1}{\sqrt{1+\alpha}}+\frac{\gamma_2}{\sqrt{1-\alpha}}), \tag{4.5}$$

$$\mathcal{D}_3(f)=\lambda_3 f(z+c)+\lambda_4\frac{\partial f}{\partial z_2}=\frac{1}{\sqrt{2}}(\frac{\gamma_1}{\sqrt{1+\alpha}}-\frac{\gamma_2}{\sqrt{1-\alpha}}). \tag{4.6}$$

In view of (4.5) and (4.6), we have

$$\lambda_2\lambda_3\frac{\partial f}{\partial z_1}-\lambda_1\lambda_4\frac{\partial f}{\partial z_2}=\frac{1}{\sqrt{2}}\big[\lambda_3\big(\frac{\gamma_1}{\sqrt{1+\alpha}}+\frac{\gamma_2}{\sqrt{1-\alpha}}\big)-\lambda_1\big(\frac{\gamma_1}{\sqrt{1+\alpha}}-\frac{\gamma_2}{\sqrt{1-\alpha}}\big)\big]. \tag{4.7}$$

By differentiating both two sides of (4.5) and (4.6) for the variables z_2 and z_1 respectively, and by combining with the fact $\frac{\partial^2 f}{\partial z_1\partial z_2}=\frac{\partial^2 f}{\partial z_2\partial z_1}$, we deduce that

$$\lambda_2\lambda_3\frac{\partial f(z+c)}{\partial z_1}-\lambda_1\lambda_4\frac{\partial f(z+c)}{\partial z_2}=0, \tag{4.8}$$

which also implies by (4.7) that

$$\lambda_3\big(\frac{\gamma_1}{\sqrt{1+\alpha}}+\frac{\gamma_2}{\sqrt{1-\alpha}}\big)=\lambda_1\big(\frac{\gamma_1}{\sqrt{1+\alpha}}-\frac{\gamma_2}{\sqrt{1-\alpha}}\big). \tag{4.9}$$

Thus, from (4.9) and $\gamma_1^2+\gamma_2^2=1$ it follows that

$$\gamma_1=\pm\frac{\sqrt{1+\alpha}(\lambda_1+\lambda_3)}{\sqrt{2\lambda_1^2+2\lambda_3^2+4\alpha\lambda_1\lambda_3}},\quad \gamma_2=\pm\frac{\sqrt{1-\alpha}(\lambda_1-\lambda_3)}{\sqrt{2\lambda_1^2+2\lambda_3^2+4\alpha\lambda_1\lambda_3}}.$$

The characteristic equations of (4.8) are

$$\frac{dz_1}{dt}=\lambda_2\lambda_3,\quad \frac{dz_2}{dt}=-\lambda_1\lambda_4,\quad \frac{df}{dt}=0.$$

By using the initial conditions $z_1=0, z_2=u$, and $f=f(0,u):=\phi(u)$ with a parameter u, we obtain the following parametric representation for the solutions of the characteristic equations:

$$z_1=\lambda_2\lambda_3 t,\quad z_2=-\lambda_1\lambda_4 t+u,\quad f(t,u)=\int_0^t 0dt+\phi(u)=\phi(u),$$

where $\phi(u)$ is a transcendental entire function with finite order in $u = z_2 + \frac{\lambda_1\lambda_4}{\lambda_2\lambda_3} z_1$. Then, by combining this with $t = \frac{1}{\lambda_2\lambda_3} z_1$ and $u = z_2 + \frac{\lambda_1\lambda_4}{\lambda_2\lambda_3} z_1$, the solution of equation (4.8) has the form

$$f(z_1, z_2) = \phi(z_2 + \frac{\lambda_1\lambda_4}{\lambda_2\lambda_3} z_1). \tag{4.10}$$

By substituting (4.10) into (4.5), we obtain that

$$\phi(u + u_0) + \frac{\lambda_4}{\lambda_3}\phi'(u) = \pm\frac{1}{\sqrt{\lambda_1^2 + \lambda_3^2 + 2\alpha\lambda_1\lambda_3}}, \tag{4.11}$$

where $u_0 = c_2 + \frac{\lambda_1\lambda_4}{\lambda_2\lambda_3} c_1$.

Case 2. $\sqrt{1+\alpha}u$ is not a constant. Then we can rewrite equation (4.2) as

$$(\sqrt{1+\alpha}u + i\sqrt{1-\alpha}v)(\sqrt{1+\alpha}u - i\sqrt{1-\alpha}v) = 1. \tag{4.12}$$

Noting that u, v are transcendental entire functions with finite order, we have that $\sqrt{1+\alpha}u + i\sqrt{1-\alpha}v$ and $\sqrt{1+\alpha}u - i\sqrt{1-\alpha}v$ have no zeros and poles. Thus, by Lemmas 2.9 and 2.10, there exists a nonconstant polynomial $q(z)$ in $\mathbb{C}^2$ such that

$$\sqrt{1+\alpha}u + i\sqrt{1-\alpha}v = e^{q(z)}, \quad \sqrt{1+\alpha}u - i\sqrt{1-\alpha}v = e^{-q(z)}. \tag{4.13}$$

In view of (4.1) and (4.13), similar to Theorem 2.1, it follows that

$$\mathcal{D}_1(f) = \lambda_1 f(z+c) + \lambda_2\frac{\partial f}{\partial z_1} = \frac{\sqrt{2}}{4}(A_1e^{q(z)} + A_2e^{-q(z)}), \tag{4.14}$$

$$\mathcal{D}_3(f) = \lambda_3 f(z+c) + \lambda_4\frac{\partial f}{\partial z_2} = \frac{\sqrt{2}}{4}(A_2e^{q(z)} + A_1e^{-q(z)}), \tag{4.15}$$

which means

$$\lambda_2\lambda_3\frac{\partial f}{\partial z_1} - \lambda_1\lambda_4\frac{\partial f}{\partial z_2} = \frac{\sqrt{2}}{4}[(\lambda_3A_1 - \lambda_1A_2)e^{q(z)} + (\lambda_3A_2 - \lambda_1A_1)e^{-q(z)}]. \tag{4.16}$$

By differentiating both two sides of (4.14) and (4.15) for the variables z_2 and z_1 respectively, and by combining with the fact $\frac{\partial^2 f}{\partial z_1\partial z_2} = \frac{\partial^2 f}{\partial z_2\partial z_1}$, we conclude that

$$\begin{aligned} &\lambda_2\lambda_3\frac{\partial f(z+c)}{\partial z_1} - \lambda_1\lambda_4\frac{\partial f(z+c)}{\partial z_2} \\ &= \frac{\sqrt{2}}{4}[(\lambda_2A_2\frac{\partial q}{\partial z_1} - \lambda_4A_1\frac{\partial q}{\partial z_2})e^q + (\lambda_4A_2\frac{\partial q}{\partial z_2} - \lambda_2A_1\frac{\partial q}{\partial z_1})e^{-q}]. \end{aligned} \tag{4.17}$$

It follows from (4.16) and (4.17) that

$$\upsilon_1e^{q(z+c)} + \upsilon_2e^{-q(z+c)} = \upsilon_3e^{q(z)} + \upsilon_4e^{-q(z)}. \tag{4.18}$$

where

$$\upsilon_1 = \lambda_3A_1 - \lambda_1A_2, \quad \upsilon_2 = \lambda_3A_2 - \lambda_1A_1,$$
$$\upsilon_3 = \lambda_2A_2\frac{\partial q}{\partial z_1} - \lambda_4A_1\frac{\partial q}{\partial z_2}, \quad \upsilon_4 = \lambda_4A_2\frac{\partial q}{\partial z_2} - \lambda_2A_1\frac{\partial q}{\partial z_1}.$$

If $\upsilon_2 = 0$, then $\upsilon_1 \neq 0$. Otherwise, $A_1^2 = A_2^2$, a contradiction. If $\upsilon_3 = 0$, then $\upsilon_4 \neq 0$. Otherwise, (4.18) leads to $\upsilon_1e^{q(z+c)} = 0$, a contradiction. Thus, it follows that $\upsilon_3 \neq 0$. Similarly, $\upsilon_4 \neq 0$. By combining this with (4.18), we have

$$e^{q(z+c)+q(z)} = \frac{\upsilon_4}{\upsilon_1}.$$

Since $q(z)$ is a nonconstant polynomial, $q(z+c)+q(z)$ can not be a constant. Hence, the above equation implies a contradiction that the left-hand side is transcendental but the right-hand side is not transcendental. Therefore, by (4.18), we obtain that

$$\upsilon_1 e^{q(z+c)+q(z)} = \upsilon_3 e^{2q(z)} + \upsilon_4. \tag{4.19}$$

Similar to Case 2 in Theorem 2.1, by the Nevanlinna second main theorem in several complex variables, and in view of (4.19), we conclude that it is a contradiction.

We conclude that $\upsilon_2 \neq 0$. Similarly, we have $\upsilon_1 \neq 0$. Thus, (4.18) leads to

$$\frac{\upsilon_3}{\upsilon_2} e^{q(z+c)+q(z)} + \frac{\upsilon_4}{\upsilon_2} e^{q(z+c)-q(z)} - \frac{\upsilon_1}{\upsilon_2} e^{2q(z+c)} = 1, \tag{4.20}$$

If $\upsilon_3 = 0$, then $\upsilon_4 \neq 0$. Otherwise, (4.20) leads to $-\frac{\upsilon_1}{\upsilon_2} e^{2q(z+c)} = 1$, a contradiction. Then (4.20) becomes

$$e^{2q(z+c)} = \frac{\upsilon_4}{\upsilon_1} e^{q(z+c)-q(z)} - \frac{\upsilon_2}{\upsilon_1}. \tag{4.21}$$

Noting that $N(r, \frac{1}{e^{2q(z+c)}}) = 0$, $N(r, e^{2q(z+c)})) = 0$ and

$$N(r, \frac{1}{\frac{\upsilon_4}{\upsilon_1} e^{q(z+c)-q(z)}}) = 0,$$

by the Nevanlinna second main theorem in several complex variables, and in view of (4.21), we conclude that

$$\begin{aligned}
&T(r, e^{2q(z+c)}) \\
&\leq N(r, \frac{1}{e^{2q(z+c)}}) + N(r, \frac{1}{\frac{\upsilon_2}{\upsilon_1} + e^{2q(z+c)}}) + N(r, e^{2q(z+c)}) + S(r, e^{2q(z+c)}) \\
&\leq N(r, \frac{1}{\frac{\upsilon_4}{\upsilon_1} e^{q(z+c)-q(z)}}) + S(r, e^{2q(z+c)}) = S(r, e^{2q(z+c)}),
\end{aligned}$$

which leads to a contradiction. Hence $\upsilon_3 \neq 0$. Similarly, we have $\upsilon_4 \neq 0$. By Lemma 2.11, and the fact that $e^{q(z+c)+q(z)}$ is nonconstant yields

$$\frac{\upsilon_4}{\upsilon_2} e^{q(z+c)-q(z)} \equiv 1. \tag{4.22}$$

Thus, in view of (4.20),

$$\frac{\upsilon_3}{\upsilon_1} e^{q(z)-q(z+c)} \equiv 1. \tag{4.23}$$

Since $q(z)$ is a polynomial, (4.22) (or (4.23)) implies $q(z+c) - q(z) = \eta$, where η is a constant in $\mathbb{C}$. Thus, it follows that $q(z) = L(z) + H(z) + B$, where $L(z) = \alpha_1 z_1 + \alpha_2 z_2$, $\alpha_1, \alpha_2 \in \mathbb{C}$, $H(z) := H(s)$, $H(s)$ is a polynomial in $s = c_2 z_1 - c_1 z_2$, $c_1, c_2 \in \mathbb{C}$, and $B \in \mathbb{C}$. Next, we prove that $H(z) \equiv 0$. In view of (4.22) and (4.23), we have

$$\frac{1}{\upsilon_2}[\sigma_1 - (c_1\lambda_4 A_2 + c_2\lambda_2 A_1)\frac{dH}{ds}] = A^{-1}, \tag{4.24}$$

$$\frac{1}{\upsilon_1}[\sigma_2 + (c_2\lambda_2 A_2 + c_1\lambda_4 A_1)\frac{dH}{ds}] = A, \tag{4.25}$$

where

$$A = e^{\eta} = e^{L(c)}, \quad \sigma_1 = \lambda_4 A_2 \alpha_2 - \lambda_2 A_1 \alpha_1, \quad \sigma_2 = \lambda_2 A_2 \alpha_1 - \lambda_4 A_1 \alpha_2.$$

This implies that both $(c_1\lambda_4 A_2 + c_2\lambda_2 A_1)\frac{dH}{ds}$ and $(c_2\lambda_2 A_2 + c_1\lambda_4 A_1)\frac{dH}{ds}$ are constants. By combining this with the fact $c_1\lambda_4 \neq \pm c_2\lambda_2$, it follows that $\frac{dH}{ds}$ is a constant; that is, $\deg_s H \leq 1$. Thus, the form of $L(z) + H(z) + B$ is still the linear form of $\alpha_1 z_1 + \alpha_2 z_2 + B, \alpha_1, \alpha_2, B \in \mathbb{C}$, which means that $H(z) \equiv 0$. It follows that $q(z) = L(z) + B = \alpha_1 z_1 + \alpha_2 z_2 + B, \alpha_1, \alpha_2, B \in \mathbb{C}$. Thus, we can deduce from (4.24) and (4.25) that

$$\frac{\sigma_1}{\upsilon_2} e^{L(c)} = 1, \ \frac{\sigma_2}{\upsilon_1} e^{-L(c)} = 1,$$

that is,

$$\frac{\sigma_1 \sigma_2}{\upsilon_2 \upsilon_1} = 1, \quad e^{L(c)} = \frac{\upsilon_2}{\sigma_1} = \frac{\sigma_2}{\upsilon_1}. \tag{4.26}$$

On the other hand, in view of (4.16), we can deduce that

$$\begin{aligned} &\lambda_2 \lambda_3 \frac{\partial f(z)}{\partial z_1} - \lambda_1 \lambda_4 \frac{\partial f(z)}{\partial z_2} \\ &= \frac{\sqrt{2}}{4} [\upsilon_1 e^{\alpha_1 z_1 + \alpha_2 z_2 + B} + \upsilon_2 e^{-(\alpha_1 z_1 + \alpha_2 z_2 + B)}]. \end{aligned} \tag{4.27}$$

The characteristic equations of (4.27) are

$$\frac{dz_1}{dt} = \lambda_2 \lambda_3, \quad \frac{dz_2}{dt} = -\lambda_1 \lambda_4,$$

$$\frac{df}{dt} = \frac{\sqrt{2}}{4} [\upsilon_1 e^{\alpha_1 z_1 + \alpha_2 z_2 + B} + \upsilon_2 e^{-(\alpha_1 z_1 + \alpha_2 z_2 + B)}].$$

By using the initial comditions $z_1 = 0, z_2 = u$, and $f = f(0, u) := \varphi_0(u)$ with a parameter u, we obtain the following parametric representation for the solutions of the characteristic equations: $z_1 = \lambda_2 \lambda_3 t$, $z_2 = -\lambda_1 \lambda_4 t + u$,

$$f(t, u) = \frac{\sqrt{2}}{4} \int_0^t [\upsilon_1 e^{\alpha_1 z_1 + \alpha_2 z_2 + B} + \upsilon_2 e^{-(\alpha_1 z_1 + \alpha_2 z_2 + B)}] dt + \varphi(u),$$

where $\varphi(u)$ is an entire function with finite order in u such that

$$\varphi(u) = \begin{cases} \varphi_0(u), & \lambda_2 \lambda_3 \alpha_1 - \lambda_1 \lambda_4 \alpha_2 = 0; \\ \varphi_0(u) - \frac{\sqrt{2}}{4} (\frac{\upsilon_1 e^{\alpha_2 u + B}}{\lambda_2 \lambda_3 \alpha_1 - \lambda_1 \lambda_4 \alpha_2} - \frac{\upsilon_2 e^{-(\alpha_2 u + B)}}{\lambda_2 \lambda_3 \alpha_1 - \lambda_1 \lambda_4 \alpha_2}), & \lambda_2 \lambda_3 \alpha_1 - \lambda_1 \lambda_4 \alpha_2 \neq 0. \end{cases}$$

If $\lambda_2 \lambda_3 \alpha_1 - \lambda_1 \lambda_4 \alpha_2 = 0$, then

$$f(z_1, z_2) = \frac{\sqrt{2}}{4 \lambda_2 \lambda_3} z_1 [\upsilon_1 e^{\alpha_2 u + B} + \upsilon_2 e^{-\alpha_2 u - B}] + \varphi_0(u).$$

If $\lambda_2 \lambda_3 \alpha_1 - \lambda_1 \lambda_4 \alpha_2 \neq 0$, then

$$f(z_1, z_2) = \frac{\sqrt{2}}{4} [\frac{\upsilon_1}{\lambda_2 \lambda_3 \alpha_1 - \lambda_1 \lambda_4 \alpha_2} e^{L(z) + B} - \frac{\upsilon_2}{\lambda_2 \lambda_3 \alpha_1 - \lambda_1 \lambda_4 \alpha_2} e^{-L(z) - B}] + \varphi(u).$$

By substituting this expression into (4.14) and combining it with (4.26), we can deduce that $\varphi(u)$ satisfies

$$\varphi(u + u_0) + \frac{\lambda_4}{\lambda_3} \varphi'(u) = 0. \tag{4.28}$$

The proof of Theorem 2.6 is complete.

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