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CARATHÉODORY PERIODIC PERTURBATIONS OF DEGENERATE SYSTEMS

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Abstract. We study the structure of the set of harmonic solutions to T-periodically perturbed coupled differential equations on differentiable manifolds, where the perturbation is allowed to be of Carathéodory-type regularity. Employing degree-theoretic methods, we prove the existence of a noncompact connected set of nontrivial T-periodic solutions that, in a sense, emanates from the set of zeros of the unperturbed vector field. The latter is assumed to be "degenerate": Meaning that, contrary to the usual assumptions on the leading vector field, it is not required to be either trivial nor to have a compact set of zeros. In fact, known results in the "nondegenerate" case can be recovered from our ones. We also provide some illustrating examples of Liénard- and ϕ-Laplacian-type perturbed equations.

1. Introduction and preliminaries

In this article we study the set of harmonic solutions of Carathéodory-type periodic perturbations of autonomous systems on a smooth constraining manifold $\mathcal{M} \subseteq \mathbb{R}^d$. Namely, equations of the form

$$\dot{x} = G(x) + \lambda F(t, x), \quad \lambda \geq 0,$$

where $G\colon \mathcal{M} \to \mathbb{R}^d$ and $F\colon \mathbb{R} \times \mathcal{M} \to \mathbb{R}^d$ are tangent vector fields to $\mathcal{M}$, meaning that $G(\mathfrak{p}) \in T_{\mathfrak{p}}\mathcal{M}$ and $F(t, \mathfrak{p}) \in T_{\mathfrak{p}}\mathcal{M}$ for all $(t, \mathfrak{p}) \in \mathbb{R} \times \mathcal{M}$, and the perturbing term F is T-periodic in t for some given $T > 0$. Here $T_{\mathfrak{p}}\mathcal{M}$ denotes the tangent space to $\mathcal{M}$ at $\mathfrak{p}$. Assuming that G is continuous and F is Carathéodory, we aim to study the structure of the set of the pairs (λ, x), where $x : \mathbb{R} \to \mathcal{M}$ is an absolutely continuous T-periodic function such that $\dot{x}(t) = G(x(t)) + \lambda F(t, x(t))$ for a.e. $t \in \mathbb{R}$. For this purpose we follow a topological approach, based on the concept of *topological degree of a tangent vector field*. This approach was initially pursued by Furi and Pera (see, e.g., [8, 9, 10, 11]) and afterwards applied to other situations. We mention in particular the papers [19, 20] that we wish here, in a sense, to generalize in a unified manner.

In fact, an interesting – and difficult to study – situation presents itself when the autonomous field G is "degenerate" in the sense that its set of zeros is a noncompact submanifold of the constraint. Indeed, the "boundary" cases, that is when $G^{-1}(0)$

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is compact and when $G = 0$, are well understood see, e.g., [11, 20]. The present research is an attempt at filling the "gap" between the results of these two papers. We concentrate on the family of systems where the constraining manifold is of the form $\mathcal{M} = M \times N$, the cartesian product of two smooth boundaryless manifolds $M \subseteq \mathbb{R}^k$ and $N \subseteq \mathbb{R}^s$, and $G\colon M \times N \to \mathbb{R}^k \times \mathbb{R}^s$ is of the form $(0, g)$, i.e., the first component is identically zero and $g\colon M \times N \to \mathbb{R}^s$ is such that $g^{-1}(0)$ compact. To do so we follow the technique introduced in [20], compare also [2] and [15].

More precisely, in this article, we study the following system of coupled equations, depending on the parameter $\lambda \geq 0$, on the product manifold $M \times N$, where $M \subseteq \mathbb{R}^k$ and $N \subseteq \mathbb{R}^s$ are (smooth, boundaryless) differentiable manifolds:

$$\begin{aligned} \dot{x} &= \lambda f\big(t, x, y, \lambda\big), \\ \dot{y} &= g(x, y) + \lambda h\big(t, x, y, \lambda\big). \end{aligned} \tag{1.1}$$

We set our investigation under Carathéodory-type assumptions. Note that a similar problem in the continuous setting has been considered in [2]: however, unlike (1.1), in that paper the first equation is coupled with a periodic perturbation of a particular *nonautomous* differential equation.

It is worth mentioning that our results cannot be deduced from analogous ones in [20], in which the continuity of the involved functions is required. The milder conditions needed here in the Carathéodory setting may allow to study general ϕ-Laplacian-like equations: cp. Example 3.6. The study of periodic solutions for equations involving the ϕ-Laplacian via topological methods has been pursued recently by many authors. We cite, for instance, [1, 3, 4, 6, 7, 18] where the problem is set under Carathéodory assumptions. We will follow this line of investigation in a forthcoming paper.

We will make the following assumptions on the maps f, g, h that appear in (1.1). Hereafter by $T_pM \subseteq \mathbb{R}^k$ we mean the tangent space of M at a point p of M, respectively by $T_qN \subseteq \mathbb{R}^s$ we denote the tangent space of N at $q \in N$. By $L^1_T(\mathbb{R})$ we denote the space of $L^1_{\text{loc}}(\mathbb{R})$, T-periodic maps $\gamma\colon \mathbb{R} \to \mathbb{R}$.

- The map $f\colon \mathbb{R} \times M \times N \times [0, \infty) \to \mathbb{R}^k$ is a Carathéodory, T-periodic vector field tangent to M, meaning that:

(F1) $f(t+T, p, q, \lambda) = f(t, p, q, \lambda) \in T_pM$, for all $(p, q, \lambda) \in M \times N \times [0, \infty)$ and for a.e. $t \in \mathbb{R}$;
(F2) $t \mapsto f(t, p, q, \lambda)$ is measurable, for all $(p, q, \lambda) \in M \times N \times [0, \infty)$;
(F3) $(p, q, \lambda) \mapsto f(t, p, q, \lambda)$ is continuous, for a.e. $t \in \mathbb{R}$;
(F4) for any compact set $K \subseteq M \times N \times [0, \infty)$, there exists a function $\phi_K \in L^1_T(\mathbb{R})$ such that $|f(t, p, q, \lambda)| \leq \phi_K(t)$, for all $(p, q, \lambda) \in K$ and for a.e. $t \in \mathbb{R}$.

- Analogously the map $h\colon \mathbb{R} \times M \times N \times [0, \infty) \to \mathbb{R}^s$ is a Carathéodory, T-periodic vector field tangent to N, namely:

(H1) $h(t+T, p, q, \lambda) = h(t, p, q, \lambda) \in T_qN$, for all $(p, q, \lambda) \in M \times N \times [0, \infty)$ and for a.e. $t \in \mathbb{R}$;
(H2) $t \mapsto h(t, p, q, \lambda)$ is measurable, for all $(p, q, \lambda) \in M \times N \times [0, \infty)$;
(H3) $(p, q, \lambda) \mapsto h(t, p, q, \lambda)$ is continuous, for a.e. $t \in \mathbb{R}$;
(H4) for any compact set $C \subseteq M \times N \times [0, \infty)$, there exists a function $\psi_C \in L^1_T(\mathbb{R})$ such that $|h(t, p, q, \lambda)| \leq \psi_C(t)$, for all $(p, q, \lambda) \in C$ and for a.e. $t \in \mathbb{R}$.

- The map $g\colon M \times N \to \mathbb{R}^s$ is a continuous, autonomous vector field tangent to N; that is, $g(p, q) \in T_qN$, for all $(p, q) \in M \times N$.

By a *solution* of system (1.1) we mean a function pair $(x, y) \in W^{1,1}_{\rm loc}(M \times N)$ such that the equalities

$$\dot{x}(t) = \lambda f\big(t, x(t), y(t), \lambda\big),$$
$$\dot{y}(t) = g(x(t), y(t)) + \lambda h\big(t, x(t), y(t), \lambda\big)$$

hold for a.e. $t \in \mathbb{R}$.

Since any solution of system (1.1) is (absolutely) continuous, as crucially pointed out in [11, 19], it is convenient to investigate the properties of the T-periodic solutions of (1.1) in the metric space of the continuous functions.

Some further notation is in order. We denote by $C_T(M \times N)$ the set of the $M \times N$-valued, T-periodic, continuous functions with the topology induced by the Banach space $C_T(\mathbb{R}^{k+s})$. We will say that $(\lambda, x, y) \in [0, \infty) \times C_T(M \times N)$ is a *T-triple* for (1.1) if equalities (1.1) hold identically. A T-triple (λ, x, y) is called *trivial* if (x, y) is constant and $\lambda = 0$. Given $(p, q) \in M \times N$, by $\overline{p}$ and $\overline{q}$ we denote the functions constantly equal to p and q, respectively. Thus, a T-triple is trivial if and only if it is of the form $(0, \overline{p}, \overline{q})$ with $(p, q) \in g^{-1}(0)$.

Let $w : M \times N \to \mathbb{R}^k$ be the mean value vector field defined by

$$w(p, q) = \frac{1}{T} \int_0^T f(t, p, q, 0)\, \mathrm{d}t$$

and observe that this is a continuous, autonomous vector field tangent to M; meaning that $w(p, q) \in T_pM$, $\forall (p, q) \in M \times N$

Let now $\nu : M \times N \to \mathbb{R}^{k+s}$ be defined as

$$\nu(p, q) = (w(p, q), \quad g(p, q))\,, \tag{1.2}$$

note that, being w and g vector fields tangent, respectively, to M and N, by definition ν is a vector field tangent to the product manifold $M \times N \subseteq \mathbb{R}^{k+s}$. Let Ω be an open subset of $[0, \infty) \times C_T(M \times N)$. The main result of this paper, Theorem 3.1, establishes a topological condition in terms of the degree of ν in Ω for the existence of a *connected set of nontrivial T-triples* that in a sense "emanates" from the set of zeros of ν in Ω and is not contained in any compact subset of Ω.

Before providing a precise statement and proof we need to recall some basic notions, see Section 2. Section 3 contains our main result, and we close the paper with some illustrating examples, showing the possible shape of the connected "branches" of T-triples in some concrete situation.

2. Topological degree of a tangent vector field

The topological degree of a tangent vector field (sometimes called *rotation* or *characteristic*) plays a crucial role in this paper. Although this is a very well known concept, we summarize here the definitions and properties that are most relevant for our argument.

Assume that $\mathcal{M} \subseteq \mathbb{R}^d$ be a smooth boundaryless manifold. Take a tangent vector field $v\colon \mathcal{M} \to \mathbb{R}^d$ and let $\mathfrak{p} \in v^{-1}(0)$ so, by definition, $v(\mathfrak{p}) \in T_\mathfrak{p}\mathcal{M} \subseteq \mathbb{R}^d$.

When v is C^1 it is known, see e.g. [17], that also the image of the Fréchet derivative $v'(\mathfrak{p})\colon T_\mathfrak{p}\mathcal{M} \to \mathbb{R}^d$ of v at $\mathfrak{p}$ is contained into $T_\mathfrak{p}\mathcal{M}$. That is, when $\mathfrak{p}$ is a zero of v, $v'(\mathfrak{p})$ is an endomorphism of $T_\mathfrak{p}\mathcal{M}$. In particular, the determinant $\det v'(\mathfrak{p})$ is well defined. When $\mathfrak{p} \in v^{-1}(0)$ and $\det v'(\mathfrak{p}) \neq 0$ we say that $\mathfrak{p}$ is a *nondegenerate zero* of v. In this case we define its *index* $i(v, \mathfrak{p})$ as $\operatorname{sign}\big(\det v'(\mathfrak{p})\big)$.

Let us now briefly look at the construction of the degree. Let $v\colon \mathcal{M} \to \mathbb{R}^d$ be a continuous tangent vector field on $\mathcal{M}$, and $V \subseteq \mathcal{M}$ be an open subset such that $v^{-1}(0) \cap V$ is compact. In this case we say that v is *admissible* (for the degree) in V or that the pair (v, V) is admissible. We associate to the admissible pair (v, V) the integer $\deg(v, V)$ which, so to speak, counts algebraically the number of zeros of v in V (see e.g. [14, 17] and references therein). To give a precise meaning to the last sentence consider the case when v is smooth and $v^{-1}(0) \cap V$ consists of a finite number of nondegenerate zeros, in this situation we define $\deg(v, V)$ as the sum of the indices of these zeros. That is

$$\deg(v, V) = \sum_{\mathfrak{p} \in v^{-1}(0) \cap V} i(v, \mathfrak{p}) = \sum_{p \in v^{-1}(0) \cap V} \operatorname{sign}\big(\det v'(\mathfrak{p})\big). \tag{2.1}$$

In the general case when v is admissible in V the degree is defined by taking a sufficiently close smooth approximation of v with finitely many nondegenerate zeros in V (see e.g. [12]).

The degree of a tangent vector field enjoys many of the properties of the classical Brouwer degree such as solution, excision, additivity, homotopy invariance, normalization etc. Indeed, When $\mathcal{M} = \mathbb{R}^d$, W is a bounded open neighborhood of $w^{-1}(0) \cap V$ whose closure is contained in V, $\deg(v, W)$ is equal to the Brouwer degree (see, e.g. the classical [14, 16, 17] or the more recent [5]) $\deg(v, W, 0)$, of v at 0. We do not list these properties as they are easily found in any of the above references. We only mention that the properties of normalization, additivity and homotopy invariance can be used as axioms to uniquely determine to the notion of degree of a tangent vector field (see [13]).

Remark 2.1. By the Poincaré-Hopf Theorem, when $\mathcal{M}$ is a compact manifold, $\deg(v, \mathcal{M})$ coincides with the Euler-Poincaré characteristic of $\mathcal{M}$, so it is independent of v. Observe, in particular, that when $\mathcal{M} = \{\mathfrak{p}\}$ is a singleton, one has

$$\deg(\mathbf{0}, \mathcal{M}) = 1 \tag{2.2}$$

where $\mathbf{0}$ denotes the zero vector field.

Remark 2.2. Let $U_1 \subseteq M$ and $U_2 \subseteq N$ be open and let $U = U_1 \times U_2$. Consider the tangent vector field on $M \times N$ given by $v(p, q) = \big(v^1(p), v^2(q)\big)$ for $(p, q) \in M \times N$, where $v^1\colon M \to \mathbb{R}^k$ and $v^2\colon N \to \mathbb{R}^s$ are tangent vector fields. As a consequence of (2.1) and the construction of degree outlined above, one can prove that when (v^1, U_1) and (v^2, U_2) are admissible for the degree of tangent vector fields, then so is (v, U) and we have

$$\deg(v, U) = \deg(v^1, U_1)\, \deg(v^2, U_2). \tag{2.3}$$

3. Main result

In the sequel, given an open subset Ω of $[0, \infty) \times C_T(M \times N)$, we let

$$\Omega_{M\times N} = \big\{(p, q) \in M \times N : (0, \overline{p}, \overline{q}) \in \Omega\big\}.$$

We are now in a position to state and prove our main result.

Theorem 3.1. *Let f, g and h be as in system* (1.1)*, let ν be as in* (1.2)*, and let Ω be an open subset of* $[0, \infty) \times C_T(M \times N)$*. Assume that* $\deg\big(\nu, \Omega_{M\times N}\big)$ *is well-defined and nonzero. Then there exists a connected set Γ of nontrivial T-triples in*

Ω *of* (1.1) *whose closure in* $[0,\infty)\times C_T(M\times N)$ *intersects*

$$\big\{(0,\overline{p},\overline{q})\in[0,\infty)\times C_T(M\times N):(p,q)\in\nu^{-1}(0)\cap\Omega_{M\times N}\big\}$$

and is not contained in any compact subset of Ω*. In particular, if* $M\times N$ *is closed in* $\mathbb{R}^{k+s}$ *and* $\Omega=[0,\infty)\times C_T(M\times N)$*, then* Γ *is unbounded.*

Remark 3.2. By formulas (2.2) and (2.3), by taking $M=\{p\}$ and $N=\{q\}$ and comparing the notions of T-triple with that of T-pairs in [19] and [11], respectively, one can use Theorem 3.1 to retrieve the main results of these papers. Namely, theorem [19, Th. 3.1] and [11, Th. 2.2] that separately ensure the existence of a connected set of nontrivial T-triples as in Theorem 3.1 for equations of the form

$$\dot{x}=\lambda f(t,x),\quad \text{or}\quad \dot{y}=g(y)+\lambda h(t,y),$$

respectively: That is, for equations equivalent to system (1.1) when N, resp. M, is a singleton. Such results, although similar, are not directly comparable. In particular, the former is not a consequence of the latter since the vector field $(p,q)\mapsto\big(\mathbf{0},g(q)\big)$ is not admissible for the degree, unless M is compact.

The proof of Theorem 3.1 is rather elaborate and requires some preliminary steps.

We introduce some further notation. A triple $(\lambda,p,q)\in[0,\infty)\times M\times N$ is called a *starting point* (of T-periodic solutions) of (1.1) if the Cauchy problem

$$\begin{gathered}\dot{x}=\lambda f\big(t,x,y,\lambda\big),\\ \dot{y}=g(x,y)+\lambda h\big(t,x,y,\lambda\big),\\ x(0)=p,\\ y(0)=q,\end{gathered}\tag{3.1}$$

has a T-periodic solution. A starting point is *trivial* when $\lambda=0$ and $g(p,q)=0$. Roughly speaking, the concept of starting point is the finite-dimensional counterpart of that of T-triple. The set of all the starting points of (1.1) will be denoted by $\mathcal{S}$. The investigation of this set is a crucial step towards the proof of our main result. To this end, it is convenient to place ourselves under uniqueness conditions. Thus, we assume provisionally that g is C^1 and that the following Lipschitz-like conditions on the maps f and h hold in addition to the (F1)–(F4) and (H1)–(H4):

(F5) For any compact set $K\subseteq M\times N\times[0,\infty)$, there exists $\gamma_K\in L^1_T(\mathbb{R})$ such that

$$|f(t,p_1,q_1,\lambda_1)-f(t,p_2,q_2,\lambda_2)|\le\gamma_K(t)\left(|p_1-p_2|+|q_1-q_2|+|\lambda_1-\lambda_2|\right),$$

for all $(p_1,q_1,\lambda_1),(p_2,q_2,\lambda_2)\in K$ and for a.e. $t\in\mathbb{R}$.

(H5) For any compact set $C\subseteq M\times N\times[0,\infty)$, there exists $\eta_C\in L^1_T(\mathbb{R})$ such that

$$|h(t,p_1,q_1,\lambda_1)-h(t,p_2,q_2,\lambda_2)|\le\eta_C(t)\left(|p_1-p_2|+|q_1-q_2|+|\lambda_1-\lambda_2|\right),$$

for all $(p_1,q_1,\lambda_1),(p_2,q_2,\lambda_2)\in C$ and for a.e. $t\in\mathbb{R}$.

We point out that properties (F5) and (H5) are, in a sense, generic. This is crucial to develop our argument in a Carathéodory setting. We explicitly state this observation about f; an analogous remark holds for h as well.

Remark 3.3. Assume that f satisfies (F1)–(F4). Then, there exists a sequence $\{f_n\}$ of equi-Carathéodory, T-periodic vector fields tangent to M that can be assumed to satisfy (F5) such that if $p_n \to p_0$ then, for all $(q, \lambda) \in N \times [0, \infty)$ and for a.e. $t \in \mathbb{R}$,

$$f_n(t, p_n, q, \lambda) \to f(t, p_0, q, \lambda).$$

Namely (cf. [11, 19]):

$$f_n(t, p, q, \lambda) = \pi_p\Big(\int_M \varphi_n(p, u) f(t, u, q, \lambda) \mathrm{d}u\Big),$$

where $\pi_p : \mathbb{R}^k \to T_pM$ is the orthogonal projection and $\varphi_n : M \times M \to \mathbb{R}$ is a smooth function such that:

(1) $\varphi_n(p, u) \geq 0$ for all $(p, u) \in M \times M$;
(2) $\varphi_n(p, u) = 0$ whenever $|p - u| > 1/n$;
(3) $\int_M \varphi_n(p, u) \mathrm{d}u = 1$ for any $p \in M$.

Note that, when g is of class C^1 and under the assumptions (F1)–(F5) and (H1)–(H5), by continuous dependence, the set

$$D = \{(\lambda, p, q) \in [0, \infty) \times M \times N : \text{ the solution of (3.1) is defined on } [0, T]\}$$

is open. Note that the set $\mathcal{S}$ of all the starting points of (1.1) is a closed subset of D, even if it could be not so in $[0, \infty) \times M \times N$; therefore, $\mathcal{S}$ is locally compact.

The next intermediate result, roughly speaking, requires more regularity of the involved vector fields compared to our main Theorem 3.1. In particular, we assume that g is C^1, and that f and h satisfy (F5) and (H5), respectively. Such extra assumptions will be removed in the main theorem via an approximation procedure. The proof can be carried out following closely [20, Theorem 4.5], and therefore we omit it.

Theorem 3.4. *Let f, g, h be as in* (1.1) *and let ν be as in* (1.2)*. Assume that g is of class C^1 and that conditions* (F1)–(F5) *and* (H1)–(H5) *hold. Let U be an open subset of D such that $\nu^{-1}(0) \cap U_0$ is compact. If $\deg\big(\nu, U_0\big) \neq 0$, then the set of the nontrivial starting points in U admits a connected subset whose closure in U meets $\{0\} \times (\nu^{-1}(0) \cap U_0)$ and is not compact.*

The next lemma is a Whyburn-type topological result which is crucial in the proof of Theorem 3.1.

Lemma 3.5 ([10])**.** *Let Y_0 be a compact subset of a locally compact metric space Y. Assume that every compact subset of Y containing Y_0 has nonempty boundary. Then $Y \backslash Y_0$ contains a connected set whose closure in Y is non-compact and intersects Y_0.*

We are now in the position to prove our main result.

Proof of Theorem 3.1. Let X be the set of T-triples of (1.1). One can prove that X is closed in $[0, \infty) \times C_T(M \times N)$.

Let Ω be as in the statement. Let us prove that Ω contains a connected set Γ of nontrivial T-triples, whose closure in $X \cap \Omega$ intersects

$$\big\{(0, \overline{p}, \overline{q}) \in [0, \infty) \times C_T(M \times N) : (p, q) \in \nu^{-1}(0) \cap \Omega_{M \times N}\big\}$$

and is not compact.

Assume first that g is of class C^1 and that conditions (F1)–(F5) and (H1)–(H5) hold. Let us denote by $\mathcal{S}$ the set of all the starting points of (1.1), and by $\widehat{\mathcal{S}}$ the set of the starting points (λ, p, q) such that the corresponding T-triple (λ, x, y), where (x, y) is a solution of (3.1), is contained in Ω. Note that $\widehat{\mathcal{S}}$ is an open subset of $\mathcal{S}$; thus, we can find an open subset U of D such that $\mathcal{S} \cap U = \widehat{\mathcal{S}}$. By construction, we have

$$\nu^{-1}(0) \cap \Omega_{M\times N} = \nu^{-1}(0) \cap \widehat{\mathcal{S}}_0 = \nu^{-1}(0) \cap U_0;$$

and thus, by definition of degree, $\deg\big(\nu, U_0\big) = \deg\big(\nu, \Omega_{M\times N}\big) \neq 0$. Therefore, Theorem 3.4 applies, yielding the existence of a connected set $\Sigma \subseteq U$, made up of nontrivial starting points of (1.1), whose closure in U is not compact and meets $\{0\} \times (\nu^{-1}(0) \cap U_0)$.

Let now

$$\mathfrak{h} : X \to \mathcal{S}$$

be the map that assigns to any T-triple (λ, x, y) the starting point $(\lambda, x(0), y(0))$. Observe that $\mathfrak{h}$ is continuous, onto and, by the assumptions on f, g, h, it is also one to one. Moreover, by the continuous dependence on initial data, we get the continuity of the inverse $\mathfrak{h}^{-1} : \mathcal{S} \to X$. Therefore $\mathfrak{h}$ maps $X \cap \Omega$ homeomorphically onto $S \cap U$, and the trivial T-triples correspond to the trivial starting points under this homeomorphism. This implies that $\Upsilon := \mathfrak{h}^{-1}(\Sigma)$ satisfies the requirements.

Let us now remove the additional assumptions on f, g and h. Let $Y = X \cap \Omega$ and

$$Y_0 = \big\{(0, \overline{p}, \overline{q}) \in [0, \infty) \times C_T(M \times N) : (p, q) \in \nu^{-1}(0) \cap \Omega_{M\times N}\big\}.$$

Then, to prove our result, it is sufficient to apply Lemma 3.5.

Assume, by contradiction, that the pair (Y, Y_0) does not satisfy the hypothesis of Lemma 3.5. That is, there exists a compact subset K of Y containing Y_0 whose boundary, in Y, is empty. Then, there exists an open subset W of Ω, with closure $\overline{W}$ contained in Ω and such that $W \cap Y = K$, $\partial W \cap Y = \emptyset$. Observe that being K compact and $[0, \infty) \times M \times N$ locally compact, we can choose W in such a way that the set

$$\{(\lambda, x(t), y(t)) \in [0, \infty) \times M \times N : (\lambda, x, y) \in W, t \in [0, T]\}$$

is contained in a compact subset $\widetilde{K}$ of $[0, \infty) \times M \times N$. This implies that W is bounded with complete closure in Ω, and $W_{M\times N}$ is a relatively compact subset of $\Omega_{M\times N}$. Hence, in particular, ν is nonzero on the boundary of $W_{M\times N}$ (relative to $M \times N$), and the same is true for its components, w and g.

Let us now approximate, respectively, g by a a sequence $\{g_n\}$ of smooth (autonomous) vector fields tangent to N, uniformly converging to g on compact subsets of $M \times N$, and f by a sequence $\{f_n\}$ of equi-Carathéodory, T-periodic vector fields tangent to M, as in Remark 3.3. For each $n \in \mathbb{N}$, let

$$w_n(p, q) = \frac{1}{T} \int_0^T f_n(t, p, q, 0)\, \mathrm{d}t$$

be the mean value vector field, tangent to M, associated to f_n; by construction, the sequence $\{w_n(p, q)\}$ converges uniformly to $w(p, q)$ on compact subsets of $M \times N$.

Now, let

$$\nu_n(p, q) = (w_n(p, q), g_n(p, q)).$$

Note that, since the zeros of ν in $\Omega_{M\times N}$ lie in a compact subset of $W_{M\times N}$, for n large enough, the homotopy $H(s, p, q) = s\nu_n(p, q) + (1 - s)\nu(p, q)$, $s \in [0, 1]$, is admissible for the degree in $W_{M\times N}$. Thus, $\deg\big(\nu_n, W_{M\times N}\big)$ is well-defined and, by the homotopy invariance property of the degree, it coincides with $\deg\big(\nu, W_{M\times N}\big)$. Hence, by excision,

$$\deg\big(\nu_n, \Omega_{M\times N}\big) = \deg\big(\nu_n, W_{M\times N}\big) = \deg\big(\nu, W_{M\times N}\big) = \deg\big(\nu, \Omega_{M\times N}\big) \neq 0.$$

Therefore, for n sufficiently large, the first part of the proof can be applied to system

$$\begin{aligned} \dot{x} &= \lambda f_n\big(t, x, y, \lambda\big), \\ \dot{y} &= g_n(x, y) + \lambda h_n\big(t, x, y, \lambda\big) \end{aligned} \tag{3.2}$$

where, again, $\{h_n\}$ a sequence of equi-Carathéodory, T-periodic vector fields tangent to N, as in Remark 3.3.

Let X_n denote the set of T-triples of (3.2). By the above argument, there exists a connected set Γ_n of nontrivial T-triples whose closure in Ω is noncompact and meets

$$\big\{(0, \overline{p}, \overline{q}) \in [0, \infty) \times C_T(M \times N) : (p, q) \in \nu_n^{-1}(0) \cap \Omega_{M\times N}\big\}.$$

Since W is bounded with complete closure, any Γ_n must intersect the complement of W in Ω, which implies the existence of a triple $(\lambda_n, x_n, y_n) \in \partial W \cap \Gamma_n$. Therefore, since for any $n \in \mathbb{N}$ and $t \in \mathbb{R}$ we have $(\lambda_n, x_n(t), y_n(t)) \in \widetilde{K}$, the compactness of K implies the existence of a pair of functions γ, η in $L^1_T(\mathbb{R})$ such that

$$\begin{gathered} |\dot{x}_n(t)| = |\lambda_n f_n\big(t, x_n(t), y_n(t), \lambda_n\big)| \le \gamma(t), \\ |\dot{y}_n(t)| = |g_n(x_n(t), y_n(t)) + \lambda_n h_n\big(t, x_n(t), y_n(t), \lambda_n\big)| \le \eta(t) \end{gathered}$$

for all $n \in \mathbb{N}$ and a.a. $t \in \mathbb{R}$. Consequently, the sequences $\{x_n\}$ and $\{y_n\}$ are equicontinuous, so that, by Ascoli's theorem we may assume that $(x_n, y_n) \to (x_0, y_0)$ in $C_T(M \times N)$ and, without loss of generality, $\lambda_n \to \lambda_0$, so that $(\lambda_0, x_0, y_0) \in \partial W$. Moreover, by the assumptions on the sequences $\{f_n\}$, $\{g_n\}$, $\{h_n\}$ we have, for a.a. $t \in [0, T]$,

$$\begin{gathered} g_n(x_n(t), y_n(t)) \to g(x_0(t), y_0(t)), \\ f_n\big(t, x_n(t), y_n(t), \lambda_n\big) \to f\big(t, x_0(t), y_0(t), \lambda_0\big), \\ h_n\big(t, x_n(t), y_n(t), \lambda_n\big) \to h\big(t, x_0(t), y_0(t), \lambda_0\big). \end{gathered}$$

Therefore, by the dominated convergence theorem, for a.a. $t \in [0, T]$,

$$\begin{gathered} \dot{x}_0(t) = x_0(0) + \lambda_0 \int_0^t f\big(s, x_0(s), y_0(s), \lambda_0\big)\, \mathrm{d}s, \\ \dot{y}_0(t) = y_0(0) + \int_0^t \big[g(x_0(s), y_0(s)) + \lambda_0 h\big(s, x_0(s), y_0(s), \lambda_0\big)\big]\, \mathrm{d}s \end{gathered}$$

In other words, (x_0, y_0) is a T-periodic solution of the system

$$\begin{aligned} \dot{x} &= \lambda_0 f\big(t, x, y, \lambda_0\big), \\ \dot{y} &= g(x, y) + \lambda_0 h\big(t, x, y, \lambda_0\big). \end{aligned}$$

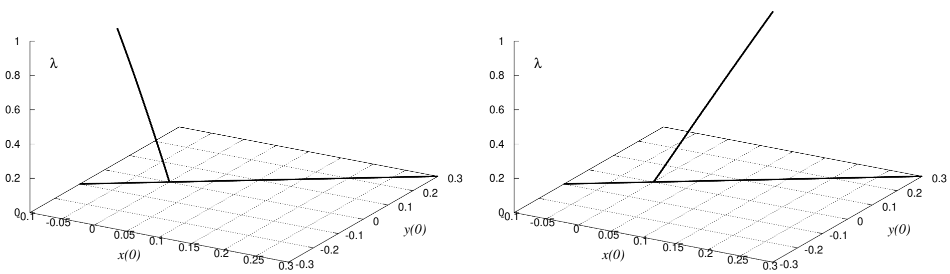
Thus, (λ_0, x_0, y_0) is a T-triple of (1.1).

Now, if $\lambda_0 > 0$, then $(\lambda_0, x_0, y_0) \in Y$. Otherwise, if $\lambda_0 = 0$, an argument similar to the one used in proving the necessary condition for bifurcation (see [11, Theorem 2.1]) shows that $(0, \overline{p}, \overline{q}) \in Y_0$. Therefore, in any case, $(\lambda_0, x_0, y_0) \in \partial W \cap Y$, which is a contradiction. Consequently, a straightforward application of Lemma 3.5 to the pair (Y, Y_0) implies the first part of our assertion.

To complete the proof, assume $M \times N$ closed in $\mathbb{R}^{k+s}$ and take $\Omega = [0,\infty) \times C_T(M \times N)$. Then, there exists a connected set Γ of nontrivial T-triples of (1.1) whose closure is not compact and meets $\{0\} \times \nu^{-1}(0)$. We need to prove that Γ is unbounded. Assume the contrary.

Note that, as a consequence of the Ascoli–Arzelà Theorem, when $M \times N$ closed in $\mathbb{R}^{k+s}$ any bounded closed set of T-triples is compact. Thus, in this case, the closure of Γ in $[0,\infty) \times C_T(M \times N)$ is compact. This is a contradiction, and the assertion follows. □

We close this article with two illustrating examples. In the first one we deal with a Liénard-type equation while in the second we study a ϕ-laplacian like equation.



(a) $\theta(t,\lambda) = 2\pi\cos(2\pi t)$ (b) $\theta(t,\lambda) = 2\pi\sin(2\pi t)$

FIGURE 1. Starting points of (3.4) with $\phi = 1$, $\psi(t,y,\lambda) = \big(\frac{1}{2} + \sin(2\pi t)\big)y$, $a = b = 0$, for two different choices of θ. One easily checks that those contained in the xy-plane are trivial and so correspond to trivial T-triples

Example 3.6. Consider the Liénard-type perturbed differential equation

$$\ddot{y} + \phi(y)\dot{y} + \lambda\big(\psi(t,y,\lambda) + \theta(t,\lambda)\big) = 0, \quad \lambda \geq 0, \tag{3.3}$$

where $\phi\colon \mathbb{R} \to \mathbb{R}$ is continuous and $\psi\colon \mathbb{R} \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ and $\theta\colon \mathbb{R} \times [0,\infty) \to \mathbb{R}$ satisfy Carathéodory conditions and are T-periodic in t, for $T > 0$ given. We also assume that θ has zero average on $[0,T]$ for all λ. We can rewrite (3.3) in the Liénard plane as follows:

$$\begin{gathered} \dot{x} = -\lambda\psi(t,y,\lambda), \\ \dot{y} = x - \Phi(y) - \lambda\Theta(t,\lambda), \end{gathered} \tag{3.4}$$

where Φ and Θ can be taken, respectively, as $\Phi(y) = \int_a^y \phi(s)\,\mathrm{d}s$ and $\Theta(t,\lambda) = \int_b^t \theta(\tau,\lambda)\,\mathrm{d}\tau$ with $a, b \in \mathbb{R}$ arbitrary constants. Notice that, since θ has zero average, Θ is T-periodic. Let $\Omega = [0,\infty) \times C_T(\mathbb{R} \times \mathbb{R})$ and

$$\nu(p,q) = \Big(\frac{1}{T}\int_0^T \psi(t,q,0)\,\mathrm{d}t\,,\, p - \Phi(q)\Big).$$

According to our notation, we have $\Omega_{\mathbb{R}\times\mathbb{R}} = \mathbb{R} \times \mathbb{R}$. When $\deg(\nu,\mathbb{R}^2) \neq 0$, Theorem 3.1 yields an unbounded connected set Γ of nontrivial T-triples whose closure intersects the set

$$\mathfrak{T} := \{(0,\overline{p},\overline{q}) \in [0,\infty) \times C_T(\mathbb{R} \times \mathbb{R}) : \nu(p,q) = 0\}.$$

It is not difficult to prove that the choice of the constants a and b do not affect the degree of ν. In fact, b has no influence on ν at all, whereas changing the value of a only induces a translation of the set of zeros along the p-axis because the zeros of the map $q \mapsto \int_0^T \psi(t,q,0)\,\mathrm{d}t$ remain unchanged.

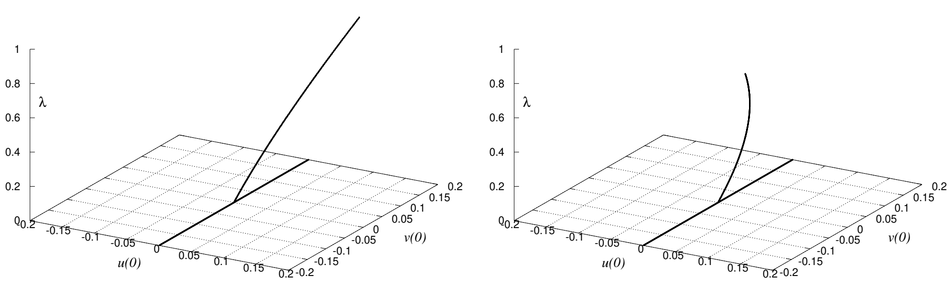
Let $\varpi\colon [0,\infty)\times C_T(\mathbb{R}\times\mathbb{R}) \to [0,\infty)\times C_T(\mathbb{R})$ the projection given by $\varpi(\lambda,x,y) = (\lambda,y)$ and let $\Upsilon = \varpi(\Gamma)$. Clearly Υ is a connected set, consisting of pairs $(\lambda,y) \in [0,\infty)\times C_T(\mathbb{R})$ with y a solution of (3.3). Notice also that if $(0,y)\in\Upsilon$ then y is not constant. To check the latter assertion we proceed by contradiction: Assume y constant and let $(0,x,y)\in\Gamma$ be any T-triple such that $\varpi(0,x,y) = (0,y)$ and observe that when $\lambda = 0$, (3.4) implies that x is constant as well. Thus $(0,x,y)$ is a trivial T-triple, a contradiction.

Let $\{(\lambda_n,x_n,y_n)\}_{n\in\mathbb{N}} \subseteq \Gamma$ be a sequence converging to $(0,\overline{p},\overline{q})$ with $(\overline{p},\overline{q}) \in \nu^{-1}(0)$. (such a sequence exists because the closure $\overline{\Gamma}$ of Γ intersects $\mathfrak{T}$). Thus q satisfies $\int_0^T \psi(t,q,0)\,\mathrm{d}t = 0$. The sequence $\{(\lambda_n,y_n)\}_{n\in\mathbb{N}} \subseteq \Upsilon$ converges to $(0,\overline{q})$ which, hence, is contained in the closure $\overline{\Upsilon}$ of Υ in $[0,\infty)\times C_T(\mathbb{R})$. In other words, $\overline{\Upsilon}$ intersects the set of pairs $(0,\overline{q})$ such that $\int_0^T \psi(t,q,0)\,\mathrm{d}t = 0$.

In conclusion, when $\deg(\nu,\mathbb{R}^2)\neq 0$, there exists an unbounded connected set Υ of pairs (λ,y), with y a T-periodic solution of (3.3) that is not constant for $\lambda = 0$, whose closure in $[0,\infty)\times C_T(\mathbb{R})$ intersects the set

$$\big\{(0,\overline{q}) \in [0,\infty)\times C_T(\mathbb{R}) : \textstyle\int_0^T \psi(t,q,0)\,\mathrm{d}t = 0\big\}.$$

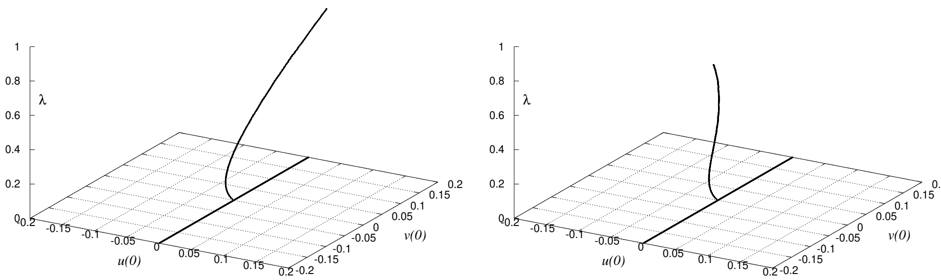
It is worth mentioning that the perturbing term θ, independent of y, does not enter in the definition of ν. As a consequence, the mere existence of the set Υ does not depend on the choice of θ or, to put it differently, it is impossible to destroy Υ by selecting a suitable θ. Taking different functions θ, though, may actually change Υ. To illustrate this fact, consider figure 1 where it is represented a portion of the set of starting points of (3.4) with $\phi = 1$, $\psi(t,y,\lambda) = \big(\frac{1}{2} + \sin(2\pi t)\big)y$, $a = b = 0$, and two different choices of θ.



(a) $h(t,\lambda) = \cos(2\pi t) + \lambda\sin(2\pi t)$ (b) $h(t,\lambda) = \cos(2\pi t) - \lambda\sin(2\pi t)$

FIGURE 2. Starting points of (3.6) with $\phi(s) = s|s| + 2s$, $f(t,s,r,\lambda) = \sin(2\pi t) + s$, for two different choices of h. One easily checks that those contained in the xy-plane are trivial and so correspond to trivial T-triples.

Example 3.7. Let $\phi\colon \mathbb{R}\to\mathbb{R}$ be C^1 with continuous inverse, and let $f\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[0,\infty)\to\mathbb{R}$ be a map that satisfy Carathéodory assumptions and is T-periodic



(a) $h(t,\lambda) = \cos(2\pi t) + \lambda \sin(2\pi t)$ (b) $h(t,\lambda) = \cos(2\pi t) - \lambda \sin(2\pi t)$

FIGURE 3. Starting points of (3.6) with $\phi(s) = -20s$, $f(t,s,r,\lambda) = \sin(2\pi t) + s$, for two different choices of h. Again, one easily checks that those contained in the xy-plane are trivial and so correspond to trivial T-triples.

in t. Consider the ϕ-Laplacian-like equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[\phi\big(\dot v(t) + \lambda h(t,\lambda)\big)] = \lambda f\big(t, v(t), \dot v(t), \lambda\big), \quad \lambda \geq 0, \tag{3.5}$$

where h is a T-periodic C^1 function. Setting $u(t) = \phi\big(\dot v(t) + \lambda h(t,\lambda)\big)$ or, equivalently, $\dot v(t) = \phi^{-1}\big(u(t)\big) - \lambda h(t,\lambda)$, we can rewrite (3.5) as the system

$$\begin{gathered} \dot u(t) = \lambda f\Big(t, v(t), \phi^{-1}\big(u(t)\big) - \lambda h(t,\lambda), \lambda\Big), \\ \dot v(t) = \phi^{-1}\big(u(t)\big) - \lambda h(t,\lambda). \end{gathered} \tag{3.6}$$

Let $\Omega = [0,\infty) \times C_T(\mathbb{R}\times\mathbb{R})$ so that $\Omega_{\mathbb{R}\times\mathbb{R}} = \mathbb{R}\times\mathbb{R}$, and

$$\nu(p,q) = \Big(\frac{1}{T}\int_0^T f\big(t, q, \phi^{-1}(p), 0\big)\,\mathrm{d}t\,,\ \phi^{-1}(p)\Big).$$

By Theorem 3.1, when $\deg(\nu, \mathbb{R}^2) \neq 0$, there exists an unbounded connected set Γ of nontrivial T-triples whose closure intersects the set

$$\mathfrak{T} := \{(0,\overline{p},\overline{q}) \in [0,\infty) \times C_T(\mathbb{R}\times\mathbb{R}) : \nu(p,q) = 0\}.$$

Assuming that $\deg(\nu, \mathbb{R}\times\mathbb{R}) \neq 0$, similar considerations to example 3.6 show the existence of an unbounded connected set $\Upsilon \subseteq [0,\infty) \times C_T(\mathbb{R})$ of pairs (λ, v), where v is a T-periodic solution of (3.5) that is not constant when $\lambda = 0$, such that the closure $\overline{\Upsilon}$ of Υ in $[0,\infty) \times C_T(\mathbb{R})$ intersects the set

$$\big\{(0,\overline{q}) \in [0,\infty) \times C_T(\mathbb{R}) : \textstyle\int_0^T f(t,q,0,0)\,\mathrm{d}t = 0\big\}.$$

As in Example 3.6, observe that the set Υ depends on the perturbing term h although no choice of h can make Υ disappear. Figure 2 represents a portion of the set of starting points of (3.5) with $\phi(s) = s|s| + 2s$, $f(t,s,r,\lambda) = \sin(2\pi t) + s$, and two different choices of h.

Notice in passing that, ϕ being an isomorphism, by (2.1) and using the indicated construction of the degree, it is not difficult to prove that

$$\big|\deg(\nu, \mathbb{R}\times\mathbb{R})\big| = |\deg(w, \mathbb{R})|, \tag{3.7}$$

where w denotes the map $q \mapsto \frac{1}{T}\int_0^T f\big(t,q,0,0\big)\,\mathrm{d}t$. A complete proof of this formula however, would take us too far from the scope of the paper, so we omit it.

Combining formula (3.7) with the above arguments, one one sees that what is really necessary for the existence of Υ is the condition $\deg(w, \mathbb{R}) \neq 0$ instead of $\deg(\nu, \mathbb{R} \times \mathbb{R}) \neq 0$. This shows that, although Υ may depend on the diffeomorphism ϕ, its mere existence is not affected by the actual choice of this map. An investigation of this phenomenon will be pursued elsewhere; here we only show an example with $\phi(s) = -20s$, see figure 3, illustrating how figure 2 is altered by a different choice of ϕ.

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