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NONEXISTENCE RESULTS FOR FRACTIONAL DIFFERENTIAL INEQUALITIES

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Abstract. We prove nonexistence of global solution of fractional differential inequalities of the form $D^\alpha u(t) \geq \lambda t^\beta |u(t)|^p$ when $p > 1$ for each of the Riemann-Liouville and Caputo fractional derivatives. This is motivated by work of Laskri and Tatar (Comput. Math. Appl. (2010)) and Shan and Lv (Filomat (2024)). The result of Laskri-Tatar was claimed to be false by Zhang, Liu, Wu and Cui (J. Funct. Spaces (2017)) with a correction and counter-example. We show that the counter-example and the claims are not accurate. We use a different method to that of Laskri and Tatar, our result supports the one of Laskri and Tatar. We also improve on the result in Shan and Lv paper by considering a more general problem and giving a more precise conclusion.

1. Introduction

Some years ago Laskri and Tatar [15] proved that global solutions of an inequality for the Riemann-Liouville (R-L) fractional derivative do not exist. They study the inequality

$$D^\alpha u(t) \geq t^\beta |u(t)|^p,\ t > 0,\ \text{where } p > 1, 0 < \alpha < 1 \text{ and } \beta > -\alpha, \tag{1.1}$$

with initial condition (IC) $I^{1-\alpha}u(0) = b \geq 0$, where $I^{1-\alpha}u$ is the R-L fractional integral and $D^\alpha u = D(I^{1-\alpha}u)$ denotes the R-L fractional derivative; detailed definitions are given later in the paper. The authors considered solutions belonging to a space they denoted $L^\alpha := \{u \in L^1 : D^\alpha u \in L^1\}$. Their result is the following.

Theorem 1.1 ([15, Theorem 1]). *Assume that $\beta + \alpha > 0$ and $1 < p < \frac{1+\beta}{1-\alpha}$. Then, problem* (1.1) *does not admit global nontrivial solutions when $b \geq 0$.*

They give an example where for $p \geq \frac{1+\beta}{1-\alpha}$ non-zero solutions exist for all t, so $p_0 := \frac{1+\beta}{1-\alpha}$ is a critical exponent. We will show that for $p \geq p_0$ and $b > 0$ there is no solution in the space we use for solutions, while for $p < \frac{1+\beta}{1-\alpha}$ any solution must fail to exist at or before some finite value T_1 explicitly determined by the data and parameters. The reason for these differences is that we have $b > 0$, the example has $b = 0$. We expect that for $1 < p < \frac{1+\beta}{1-\alpha}$ solutions will become unbounded at some point $T_2 \leq T_1$ (blow-up) but this requires knowledge of what occurs at the

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endpoint of a maximal interval of existence and we do not know of any such result for R-L differential equations.

Zhang, Liu, Wu, and Cui [25] claim that the proof in [15] has a flaw and that the result is not correct. They give a 'counter-example' and a 'correct version' of Theorem 1.1. Unfortunately the correction is wrong, and the counter-example has an error, which invalidates their claims. We explain this fully in Section 4.

One of the problems studied by Shan and Lv [18] is the initial value problem (IVP) for the Caputo fractional derivative

$$D_*^\alpha u(t) = u^p(t),\ t > 0,\ \text{with IC } u(0) = u_0 > 0. \tag{1.2}$$

Shan-Lv assert that when $1 < p \le 1/(1-\alpha)$ every solution of (1.2) blows-up (becomes unbounded) at some T^* and the same holds when $p > 1/(1-\alpha)$ and u_0 has a sufficiently large positive lower bound. They do not fully prove this, they prove nonexistence but do not prove blow-up nor do they mention any continuation result that would give blow-up.

For this Caputo derivative equation there is a continuation result of Eloe and Masthay [6, Theorem 2.4] and of Wu and Liu [24, Theorem 4.1] which asserts that when f is continuous, a solution u of

$$D_*^\alpha u(t) = f(t, u(t)),\ t > 0,\ \text{with IC } u(0) = u_0,$$

exists on a maximal interval of existence $[0, T_0)$ and, if T_0 is finite, $u(t) \to \infty$ as $t \to T_0^-$. This could be used to justify the blow-up claims of Shan-Lv.

We improve the Shan-Lv result by considering the more general inequality as in (1.1) with the possibly singular term t^β. We also prove, by a simple method, an improved result, namely that, *for any initial value* $u_0 > 0$, when $p > 1$ global solutions do not exist. Since we have a more general case of an inequality and a singular term, we do not know how to prove that this is blow-up.

There are a number of generalizations of the results in [15], for example the papers [7, 8, 9] but they use different methods and have little relevance to this paper so we do not discuss them.

Some comments on existence theorems. There are existence results for Caputo fractional differential equations of the form

$$D_*^\alpha u(t) = t^{-\eta} f(t, u(t)),\ t > 0,\ u(0) = u_0,$$

where $0 < \eta < \alpha$ and also for a somewhat more general non-negative function f. For local existence, existence on some possibly short interval, continuity of f is sufficient, while for global existence it is usually supposed that $|f(t,u)| \le C_1 + C_2|u|^p$ where $p \le 1$, and a Gronwall or Bihari type inequality is used to get suitable a priori bounds, see for example [14, 20, 21, 22]. Our results show that for global existence $p \le 1$ cannot be improved to have $p > 1$.

For the R-L fractional derivative case

$$D^\alpha u(t) = f(t, u(t)),\ t > 0,\ \lim_{t\to 0+} t^{1-\alpha} u(t) = u^0, \tag{1.3}$$

when $|f(t,u)| \le k(t) + l(t)|u|$, under a variety of conditions on k, l, which allow singularities in the t variable, Zhu [26, 27, 28] has proved various global existence theorems.

In the papers [1, 2, 3], Becker-Burton-Purnaras give interesting results concerning existence theory for R-L fractional differential equations. With a sign condition, opposite to the sign we have, global L^1 solutions are possible for $p > 1$. For example, in [1, Example 4.12], $D^{1/2}u(t) = -\frac{\sqrt{\pi}}{2}t^{3/4}u^{3/2}$ with the initial condition $\lim_{t\to 0+} t^{1/2}u(t) = 1$ is shown to have an explicit global solution $u(t) = \frac{1}{\sqrt{t(1+t)}}$, $t > 0$.

A result closely related to the problem we study gives local existence.

Theorem 1.2 ([2, Theorem 3.1]). *Let $0 < \alpha < 1$, $\beta > -1$, and $p \geq 0$ satisfy $\beta - p + \alpha(1+p) > 0$. Suppose that $f : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous. Suppose there are nonnegative constants K_1, K_2 such that $|f(t,u)| \leq K_1 + K_2 t^\beta |u|^p$ for $u \in \mathbb{R}$ and $0 < t < T_0$, where $T_0 \in (0,\infty]$. Then, for $u^0 \neq 0$, the problem*

$$D^\alpha u(t) = f(t,u(t)),\ t > 0,\ \lim_{t\to 0+} t^{1-\alpha}u(t) = u^0,$$

has a solution in $C_{\alpha-1}[0,T]$ for some $T \in (0,T_0)$. Here $u \in C_{\alpha-1}$ means that u is continuous on $(0,T]$ and $\lim_{t\to 0+} t^{1-\alpha}u(t)$ exists.

Note that $\beta - p + \alpha(1+p) > 0$ is equivalent to $p < \frac{\alpha+\beta}{1-\alpha}$ so it is implicit that $\alpha + \beta > 0$. Thus local existence is possible in this case.

Our Theorem 4.2 has a similar but larger critical value of p. We show that when $1 < p < \frac{1+\beta}{1-\alpha}$ solutions can exist only on a finite interval $[0,T)$, where $T \leq T_1$ for some explicitly determined T_1, while for $p \geq \frac{1+\beta}{1-\alpha}$ existence in the space $C_{\alpha-1}[0,T]$ is impossible for any $T > 0$.

In the paper [2], for the problem

$$D^\alpha u(t) = u^n(t),\ t > 0,\ \lim_{t\to 0+} t^{1-\alpha}u(t) = u^0, \tag{1.4}$$

with $n \in \mathbb{N}$, the authors showed that whether or not a solution of the initial value problem (1.4) exists on an interval $(0,T]$, for some $T > 0$, depends on the value of α. One of their results is the following.

Theorem 1.3 ([2, Theorem 3.11]). *Let $0 < \alpha < 1$, $n \in \mathbb{N}$ and $u^0 \neq 0$. The initial value problem (1.4) has a solution if and only if $\alpha > \frac{n-1}{n}$. Moreover, the solution is unique.*

For the problem (1.4) with n replaced by p, a special case of our Theorem 4.2 shows that for $u^0 > 0$ there does not exist a nontrivial solution in the space $C_{\alpha-1}[0,T]$ if $p \geq \frac{1}{1-\alpha}$, that is $\alpha \leq \frac{p-1}{p}$, which partially extends Theorem 1.3.

Lan [11, 12, 13] has studied more general fractional problems that include the Caputo and R-L fractional equations as special cases, and he has proved equivalences between fractional differential and integral equations.

2. Preliminaries

We consider real valued functions defined on an arbitrary finite interval $[0,T]$, which is, by a simple change of variable, equivalent to any finite interval.

In this paper all functions are supposed to be measurable, all integrals are Lebesgue integrals and $L^1[0,T]$ denotes the usual space of Lebesgue integrable functions; we will often simply write L^1.

The space of functions that are continuous on $[0,T]$ is denoted by $C[0,T]$, or simply C, and is endowed with the supremum norm $\|u\|_\infty := \max_{t\in[0,T]} |u(t)|$, $C^1 = C^1[0,T]$ will denote the space of continuously differentiable functions.

When studying fractional integrals and derivatives, functions such as $t^{\alpha-1}$ arise where typically $0 < \alpha < 1$. This leads to consideration of a weighted space of functions that are continuous except at $t = 0$ and have an integrable singularity at $t = 0$. For $\gamma > -1$ we define the space denoted $C_\gamma = C_\gamma[0, T]$ by

$$C_\gamma[0, T] := \{u \in C(0, T] \text{ such that } \lim_{t\to 0+} t^{-\gamma} u(t) \text{ exists}\}. \tag{2.1}$$

Then $u \in C_\gamma$ if and only if $u(t) = t^\gamma U(t)$ for some function $U \in C[0, T]$ and we define $\|u\|_\gamma := \|U\|_\infty$. The spaces of functions with a singularity at $t = 0$ are $C_{-\gamma}$ where $\gamma > 0$. For $0 < \gamma < 1$ we have $C_{-\gamma} \subset L^1$.

We also use the space of absolutely continuous functions which is denoted AC. The space AC is the appropriate space for the fundamental theorem of the calculus for Lebesgue integrals. In fact, we have the following equivalence.

$$\begin{gathered} u \in AC[0, T] \text{ if and only if } u' \in L^1[0, T],\ u'(t) \text{ exists for almost every} \\ \text{(a.e.) } t \in [0, T] \text{ and } u(t) - u(0) = \int_0^t u'(s)\, ds \text{ for all } t \in [0, T]. \end{gathered} \tag{2.2}$$

The Gamma and Beta functions frequently occur in fractional problems. The Gamma function is, for $\alpha > 0$, given by

$$\Gamma(\alpha) := \int_0^\infty s^{\alpha-1} \exp(-s)\, ds. \tag{2.3}$$

The Gamma function has the property $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ for $\alpha > 0$. The Beta function is defined for $\alpha > 0, \beta > 0$ by

$$B(\alpha, \beta) := \int_0^1 (1 - s)^{\alpha-1} s^{\beta-1}\, ds. \tag{2.4}$$

These are well defined Lebesgue integrals and it is well known that $B(\alpha, \beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$. We will also use the following property which is proved by the simple substitution $s = t\sigma$.

Lemma 2.1. *Let $t > 0$ and $\alpha > 0$, $\beta > 0$. Then we have*

$$\int_0^t (t - s)^{\alpha-1} s^{\beta-1}\, ds = t^{\alpha+\beta-1} B(\alpha, \beta). \tag{2.5}$$

These properties will be used without further mention.

We will use the so-called Riemann-Liouville (R-L) fractional integral. Using this we will consider the two most often used fractional derivatives, the R-L and the Caputo versions. The R-L fractional integral is defined for L^1 functions as follows.

Definition 2.2. The Riemann-Liouville (R-L) fractional integral of order $\alpha > 0$ of a function $u \in L^1[0, T]$ is defined for a.e. t by

$$I^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} u(s)\, ds. \tag{2.6}$$

The integral $I^\alpha u$ is the convolution of the L^1 functions h, u where $h(t) = t^{\alpha-1}/\Gamma(\alpha)$, so by the well known results on convolutions $I^\alpha u$ is defined as an L^1 function, in particular $I^\alpha u(t)$ is defined and finite for a.e. t. If $\alpha = 1$ this is the usual integration operator which we denote I. We define $I^\alpha u(0) := \lim_{t\to 0+} I^\alpha u(t)$ *if this limit exists*, otherwise it is not defined. Detailed discussion of these operators can be found in the texts [5, 10, 17], a survey of some important results is given in the free to access paper [21]. One useful result is the semigroup property as follows, see for example [5, Theorem 2.2], [17, (2.21)], [21, Lemma 2.4].

Lemma 2.3 (Semigroup property)**.** *Let $\alpha, \beta > 0$ and $u \in L^1[0,T]$. Then $I^\alpha I^\beta(u) = I^{\alpha+\beta}(u)$ as L^1 functions, thus, $I^\alpha I^\beta(u)(t) = I^{\alpha+\beta}(u)(t)$ for a.e. $t \in [0,T]$, in fact for every t for which $I^{\alpha+\beta}(|u|)(t)$ exists. If u is continuous this holds for all $t \in [0,T]$. If $u \in L^1$ and $\alpha + \beta \geq 1$ equality again holds for all $t \in [0,T]$.*

In this paper we only consider fractional derivatives of order $0 < \alpha < 1$. Let D denote the usual differentiation operator, $Du = u'$. The Riemann-Liouville (R-L) fractional derivative of order $\alpha \in (0,1)$ is informally defined by $D^\alpha u(t) = D(I^{1-\alpha}u)(t)$.

For $D\, I^{1-\alpha}u(t)$ to be defined at a point t, it is necessary that $I^{1-\alpha}u$ should be differentiable at t which requires some extra condition, which we now discuss.

It is useful to know when the fractional derivative and fractional integral are inverse operations, that is when a fractional differential equation (FDE) with an initial condition is equivalent to an integral equation.

One frequently used, but imprecise statement, is as follows. If $0 < \alpha < 1$, then u satisfies $D^\alpha u = f$ and $I^{1-\alpha}(0) = c/\Gamma(\alpha)$ if and only if $u(t) = I^\alpha f(t) + ct^{\alpha-1}$.

If $u \in L^1$ then $I^{1-\alpha}u \in L^1$ but need not be differentiable. Assuming additionally that $I^{1-\alpha}u$ is differentiable almost everywhere then $D^\alpha u(t) = f(t)$ can be satisfied for a.e. t, but it is not equivalent to an integral equation, it is necessary to always have $I^{1-\alpha}u \in AC$. This was noted long ago in the monograph [17], see [17, Definition 2.4] and the related comments in the 'Notes to §2.6'. It was recalled in [21]. Therefore a suitable precise definition is as follows.

Definition 2.4. For $\alpha \in (0,1)$ and $u \in L^1$ the R-L fractional derivative $D^\alpha u$ is defined when $I^{1-\alpha}u \in AC$ as an L^1 function by

$$D^\alpha u(t) := D\, I^{1-\alpha}u(t), \text{ a.e. } t \in [0,T]. \tag{2.7}$$

Then we do have an equivalence which is stated below in Proposition 3.1.

The Caputo differential operator, or Caputo fractional derivative, is usually used for *continuous functions* u and is defined via the R-L derivative, as in the texts [5, Definition 3.2], [10, (2.4.1)].

Definition 2.5. For $\alpha \in (0,1)$, $u \in C$ and $I^{1-\alpha}(u - u(0)) \in AC$ the Caputo derivative $D_*^\alpha u$ is defined by

$$D_*^\alpha u := D^\alpha(u - u(0)). \tag{2.8}$$

This defines $D_*^\alpha u$ as an L^1 function, so $D_*^\alpha u(t)$ is defined and finite for a.e. t.

The Caputo derivative of a constant is 0 but the R-L derivative is not, $D^\alpha c = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha}$ for $t > 0$.

There is another commonly used definition of Caputo derivative namely:

Definition 2.6. For $0 < \alpha < 1$, the Caputo derivative $D_C^\alpha u$ is defined for $u \in AC$ as an L^1 function by

$$D_C^\alpha u(t) := I^{1-\alpha}u'(t), \text{ for a.e. } t. \tag{2.9}$$

We will not use the definition $D_C^\alpha u$ because it has the severe disadvantage that for u continuous the 'equivalence' between the fractional initial value problem (IVP) $D^\alpha u(t) = f(t),\ u(0) = u_0$ and the Volterra integral equation $u(t) = u_0 + (I^\alpha f)(t)$ is not valid. I^α maps $C[0,T]$ into $C[0,T]$ but not (all of) $C[0,T]$ into $AC[0,T]$; examples are in Cichon-Salem [4, Counter-Example 1], and Webb [21, addendum]. A detailed discussion is given in [14]. When $u \in AC$, $D_*^\alpha u = D_C^\alpha u$, so often there is no reason to use D_C^α.

3. Riemann-Liouville equivalences

For $0 < \alpha < 1$ an initial value problem for the R-L fractional differential equation (FDE) $D^\alpha u = f$, $I^{1-\alpha}u(0) = \lim_{t\to 0+} I^\alpha u(t) = c_0$, with $f \in L^1$ can be studied in the space L^1. An equivalence with an integral equation is given by the following result, for example [10, Lemma 2.5(b)], [17, Theorem 2.4] and [21, Proposition 6.1].

Proposition 3.1. *Let $f \in L^1[0,T]$ and $c_0 \in \mathbb{R}$. Then a function $u \in L^1$ such that $I^{1-\alpha}u \in AC$ satisfies $D^\alpha u(t) = f(t)$ a.e. and $I^{1-\alpha}u(0) = c_0$ if and only $u \in L^1$ satisfies the Volterra integral equation*

$$u(t) = c_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \ a.e. \ t \in [0,T]. \tag{3.1}$$

We illustrate part of the argument needed for the proof of Proposition 3.1 by showing a positivity result.

Lemma 3.2. *For $0 < \alpha < 1$ suppose that $u \in L^1$ and $I^{1-\alpha}u \in AC[0,T]$ and that $D^\alpha u = f$ where $f \in L^1$ and $f(t) \ge 0$ a.e. on $[0,T]$. Then $I^{1-\alpha}u(0) = c_0 \ge 0$ implies that $u(t) \ge c_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for a.e. $t \in (0,T]$. If, in addition, $u \in C_{\alpha-1}$ then $t^{1-\alpha}u(t) \ge \frac{c_0}{\Gamma(\alpha)}$ for all $t \in (0,T]$.*

Proof. $D^\alpha u = f$ means that $D(I^{1-\alpha}u) = f$. Since $I^{1-\alpha}u \in AC[0,T]$ and $I^{1-\alpha}u(0) = c_0$, this can be integrated to give $I^{1-\alpha}u(t) = c_0 + If(t)$, for all $t \in [0,T]$. Applying I^α and using the semigroup property gives $Iu = c_0 \frac{t^\alpha}{\Gamma(\alpha+1)} + I(I^\alpha f)$. Since all terms are AC, the derivatives exist a.e., which gives $u(t) = c_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + (I^\alpha f)(t)$ for a.e. t, and proves the result since $(I^\alpha f)(t) \ge 0$ for a.e. t. The last part follows since then both sides are continuous functions of $t \in (0,T]$. □

The problem (3.1) can also be studied in the smaller space $C_{\alpha-1} = C_{-(1-\alpha)}$ when the initial condition $\lim_{t\to 0+} I^{1-\alpha}u(t) = c_0$ is replaced by $\lim_{t\to 0+} t^{1-\alpha}u(t) = c_0/\Gamma(\alpha)$.

The two limits are related as follows.

Lemma 3.3. *Let $0 < \alpha < 1$ and suppose that $u \in L^1$. Then*

$$\lim_{t\to 0+} u(t)t^{1-\alpha} = u^0 \ \text{implies that} \ I^{1-\alpha}u(0) = \lim_{t\to 0+} I^{1-\alpha}u(t) = u^0\Gamma(\alpha).$$

The result is proved for example in [21, Lemma 6.3], a longer proof is given in [1, Theorem 6.1], also the more general case when $\alpha \in \mathbb{C}$ with $0 < \mathrm{Re}(\alpha) < 1$ is proved in [10, Lemma 3.2, page 151].

The converse of this result is false.

Example 3.4. Let $0 < \gamma \le \alpha < 1$ and let $\mathbb{Q}$ denote the rational numbers. Let

$$u(t) := \begin{cases} t^{\gamma-1}, & \text{for } t \in (0,T] \cap \mathbb{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $u \in L^1[0,T]$, $\lim_{t\to 0+} I^{1-\alpha}u(t) = 0$ but $\lim_{t\to 0+} t^{1-\alpha}u(t)$ does not exist.

Proof. Since $u(t) = 0$ a.e. on $[0,T]$, $(I^{1-\alpha}u)(t) = 0$ for every $t > 0$ so $\lim_{t\to 0+} I^{1-\alpha}u(t) = 0$. Also we have

$$t^{1-\alpha}u(t) := \begin{cases} t^{\gamma-\alpha}, & \text{for } t \in (0,T] \cap \mathbb{Q}, \\ 0, & \text{otherwise,} \end{cases}$$

and for $\alpha \ge \gamma$, $\lim_{t\to 0+} t^{1-\alpha}u(t)$ does not exist. □

The following Proposition [1, Theorem 6.2] proves the equivalence between the FDE and the Volterra integral equation in the space $C_{\alpha-1}$.

Proposition 3.5. *Let $0 < \alpha < 1$, and let f be continuous on $(0,T] \times J$ where $J \subset \mathbb{R}$ is an unbounded interval. If $u : (0,T] \to J$ is continuous, $u \in L^1[0,T]$ and $t \mapsto f(t,u(t))$ belongs to $L^1[0,T]$, then u satisfies the initial value problem,*

$$D^\alpha u(t) = f(t,u(t)), \quad t \in (0,T], \ \lim_{t\to 0+} t^{1-\alpha}u(t) = u^0, \tag{3.2}$$

if and only if it satisfies the Volterra integral equation

$$u(t) = u^0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s))\,ds, \ t \in (0,T]. \tag{3.3}$$

The difference between these equivalence results is that Proposition 3.5 assumes that $D^\alpha u(t)$ exists for every $t \in (0,T]$ and functions that are continuous on $(0,T]$ are considered, as opposed to supposing that functions are in L^1 and $D^\alpha u(t)$ exists only a.e. in Proposition 3.1. The conditions in Proposition 3.5 imply that $I^{1-\alpha}u \in AC$ as shown in [21, Remark 6.6].

4. Non-existence for R-L inequalities

Henceforth for non-existence results we consider $p > 1$ since there are theorems of existence on an arbitrary interval $[0,T]$ when $p \le 1$, for example Zhu [26, 27] for the R-L case, and [14, 23] for the Caputo case.

In the paper [15] Laskri and Tatar study the inequality

$$D^\alpha u(t) \ge t^\beta |u(t)|^p, \ t > 0, \text{ where } p > 1, \text{ with IC } I^{1-\alpha}u(0) = b. \tag{4.1}$$

They consider $u \in L^\alpha$ where $L^\alpha := \{u \in L^1 : D^\alpha u \in L^1\}$. Their result is as follows.

Theorem 4.1 ([15, Theorem 1])**.** *Assume that $\beta + \alpha > 0$ and $1 < p < \frac{1+\beta}{1-\alpha}$. Then, problem* (4.1) *does not admit global nontrivial solutions when $b \ge 0$.*

From Lemma 3.2 we see that for $b > 0$ any solution must be positive a.e.. When $b = 0$ there is the trivial solution $u = 0$ but nontrivial solutions are possible, see Remark 4.5 below. Laskri and Tatar use a test function method due to Mitidieri and Pokhozhaev [16].

Zhang, Liu, Wu, and Cui [25] claim that the proof in [15] has some flaws and that the result is not correct. They give a 'counter-example' and a 'correct version' of Theorem 4.1. Unfortunately both the correction and the counter-example are wrong. We now explain these points.

Laskri-Tatar [15] consider a nonincreasing $C^1[0,\infty)$ test function $\varphi \ge 0$ such that, for some $\tau > 0$,

$$\varphi(t) = \begin{cases} 1, & \text{if } t \le \tau/2, \\ 0, & \text{if } t \ge \tau. \end{cases}$$

Then for a supposed positive solution u of (4.1), the integral $\int_0^\tau \varphi'(t) I^{1-\alpha}u(t)\,dt$ is estimated from above. The integrand in this integral is non-positive and is negative on an interval. The mistake in [15] is that absolute value signs have been omitted but, in fact, it is the absolute value of the integral that is estimated. When the absolute value signs are added in [15] the flaw claimed in [25] disappears.

The paper [25] claims, that for $1 < p < \frac{1+\beta}{1-\alpha}$ and $b \ge 0$, based on their 'counter-examples', the problem has infinitely many global nontrivial positive solutions. Also it is claimed that the problem does not have any global nontrivial *negative* solutions. The last point actually follows immediately from Lemma 3.2, not from a correction of the proof of [15].

The 'counter-example' claimed in [25] is for the case $\alpha = 1/2$, $\beta = -1/6$, $p = 3/2$. It is stated that, for $c = \Gamma(3/2)$,

$$u_1(t) = \begin{cases} c^2 t^{1/2}, & \text{if } t \le 1, \\ c^2 t^{-4/5}, & \text{if } t > 1, \end{cases}$$

is a global solution of (4.1) with $b = 0$.

For $t \le 1$, $D^{1/2}u_1(t) = D(I^{1/2}u_1)(t) = c^3$ and the inequality (4.1) holds. For $t > 1$ the authors give $D^{1/2}u_1(t) = c^3 t^{-13/10}$. This is not correct. It seems that they apply a known formula, but that formula is for a function equal to a single power of t for all $t > 0$. For $h(t) = c^2 t^{-4/5}$ for $t > 0$, the formula gives $D^{1/2}h(t) = c^2 c_1 t^{-13/10}$ where the constant c_1 is negative, so it is not equal to $c^3 t^{-13/10}$. The correct calculation of $D^{1/2}u_1(t) = D(I^{1/2}u_1)(t)$ for $t > 1$ starts as follows.

$$\begin{aligned} I^{1/2}u_1(t) &= \frac{1}{\Gamma(1/2)} \int_0^t (t-s)^{-1/2} u_1(s)\, ds \\ &= c^2 \frac{1}{\Gamma(1/2)} \Big(\int_0^1 (t-s)^{-1/2} s^{1/2}\, ds + \int_1^t (t-s)^{-1/2} s^{-4/5}\, ds \Big). \end{aligned}$$

The first integral is a decreasing function of t so it contributes a negative amount to the fractional derivative, the other term is not known in terms of elementary functions and its monotonicity properties are not easy to prove, but a Maple calculation and graph suggest that it first increases then decreases, so it would not be a counter-example.

However, the authors of [25] claim that, by "the proof of the above example", a more general case also holds, namely for

$$u_2(t) = \begin{cases} k t^{\alpha}, & \text{if } t \le 1, \\ k t^{-\mu - (\alpha+\beta)/(p-1)}, & \text{if } t > 1, \end{cases}$$

where $p > 1$ and $k = \Gamma(\alpha)^{1/(p-1)}$, $\mu \ge 1$, $\alpha + \beta > 0$, it is claimed that u_2 is a global solution of the inequality (4.1). For this case we can prove this is false by giving a counter-example that can be readily checked. Let $\alpha = 1/2$, $\beta = 1$, $\mu = 1$ and $p = 2$. The corresponding fractional integral is as follows. For $t \le 1$,

$$I^{1/2}u_2(t) = \frac{1}{\Gamma(1/2)} \int_0^t (t-s)^{-1/2} k s^{1/2}\, ds = kt\Gamma(3/2).$$

The fractional derivative $D^{1/2}u_2(t) = D(I^{1/2}u_2)(t) = k\Gamma(3/2)$ is constant for $t \le 1$. For $t > 1$ we have

$$\begin{aligned} I^{1/2}u_2(t) &= \frac{1}{\Gamma(1/2)} \int_0^t (t-s)^{-1/2} u_2(s)\, ds \\ &= \frac{1}{\Gamma(1/2)} \Big(\int_0^1 (t-s)^{-1/2} k s^{1/2}\, ds + \int_1^t (t-s)^{-1/2} k s^{-5/2}\, ds \Big). \end{aligned}$$

The second integral can be evaluated and we have, for $t > 1$,

$$\int_1^t (t-s)^{-1/2} k s^{-5/2}\, ds = k \frac{2\sqrt{t-1}(t-2)}{3t^2}.$$

This function of t first decreases for $t \in (1, t_1]$ where $t_1 = 4 - 2\sqrt{2} \approx 1.17157$, then increases to a maximum at $t_2 = 4+2\sqrt{2} \approx 6.82843$ and then decreases again. Since the first part of $I^{1/2}u_2(t)$ is a decreasing function of t we see that the fractional derivative is certainly negative except possibly for t in part of the interval (t_1, t_2), so can never be a global solution of (4.1) since the right side is always non-negative.

For the case $b = 0$ it is possible to have explicit solutions of the corresponding R-L equation $D^\alpha u(t) = t^\beta |u(t)|^p$ when $p > 1$ for some values of α, β, see the details in Example 4.6 below. The values used in the above discussion do not fit the example.

We will prove non-existence results by a different method to that used by Laskri-Tatar [15] which supports their conclusion. We do not believe the space L^α is an adequate space to consider the inequality (4.1). For, in the proof of [15, Theorem 1], the integration by parts requires $I^{1-\alpha} \in AC$, thus a better definition of L^α is $L^\alpha := \{u \in L^1 : I^{1-\alpha}u \in AC\}$. We will use a smaller space and have $u \in C_{\alpha-1}$ so that $u \in L^1$ and is continuous on $(0, T]$.

For $0 < \alpha < 1$, $\alpha + \beta > 0$ and $p > 1$ we will study the following problem.

$$D^\alpha u(t) \geq t^\beta |u(t)|^p, \ t > 0, \ \text{with IC} \lim_{t\to 0+} t^{1-\alpha}u(t) = u^0. \tag{4.2}$$

By a *solution* u of (4.2) on an interval $[0, T]$ we will mean that $u \in C_{\alpha-1}[0, T]$, $I^{1-\alpha}u \in AC[0, T]$, $D^\alpha u$ is continuous on $(0, T]$, $t^\beta |u(t)|^p \in L^1[0, T]$, the inequality is satisfied for all $t \in (0, T]$ and the IC is satisfied. A *global solution* would be a solution $u(t)$ which exists for all finite $t > 0$. We will prove that no global solution exists, a precise statement is given in the following Theorem. Firstly we prove a result for an equation.

Theorem 4.2. *For $p > 1$ and $u^0 > 0$ let $f \in L^1$ with f continuous on $(0, T]$ and $f(t) \geq 0$ for $t > 0$. Then for $\lambda > 0$ and $\beta + \alpha > 0$, the equation $D^\alpha u(t) = \lambda t^\beta |u(t)|^p + f(t)$, $t > 0$ with $\lim_{t\to 0} t^{1-\alpha}u(t) = u^0$ does not have a global solution $u \in C_{\alpha-1}$. In fact, for $p \geq \frac{1+\beta}{1-\alpha}$ there is no solution in the space $C_{\alpha-1}[0, T]$ for any $T > 0$, while for $1 < p < \frac{1+\beta}{1-\alpha}$ there exists T_1, explicitly determined by the given data and parameters, such that a solution can exist on an interval $[0, T]$ only if $T < T_1$.*

Note that β can be negative. The hypothesis $\beta + \alpha > 0$ ensures that $\frac{1+\beta}{1-\alpha} > 1$.

Proof. From Lemma 3.2 we note that, if a solution u exists, then $t^{1-\alpha}u(t) \geq u^0 > 0$ for all t in its interval of existence, so $u(t) > 0$ for $t > 0$ in its interval of existence. Since $D^\alpha u \in L^1$, it is necessary that $t^\beta u^p(t) \in L^1$, otherwise no solution can exist. However, when $t^{1-\alpha}u(t) = U(t) \geq u^0 > 0$, with U continuous on $[0, T]$, $t^\beta u^p(t) = t^{\beta - p(1-\alpha)}U^p(t)$ is in $L^1[0, T]$ for some $T > 0$ if and only if $\beta - p(1-\alpha) > -1$, that is $p < \frac{1+\beta}{1-\alpha}$, so no solution can exist when $p \geq \frac{1+\beta}{1-\alpha}$. We recover this again below but for now we suppose a solution exists on an interval $[0, T]$ with $T > 0$. Let $v = cu$ where $c^{p-1} = \lambda$, then $v \in C_{\alpha-1}$ satisfies

$$D^\alpha v(t) = t^\beta v^p(t) + cf(t), \ t \in (0, T], \ \text{with IC} \lim_{t\to 0} t^{1-\alpha}v(t) = c\, u^0.$$

By Proposition 3.5, v satisfies

$$v(t) = c\, u^0 t^{\alpha-1} + I^\alpha(s^\beta v^p(s))(t) + cI^\alpha f(t), \ t \in (0, T].$$

Then $w(t) = t^{1-\alpha}v(t)$ is continuous for $t \in [0,T]$. Discarding the last non-negative term, we have w satisfies the inequality

$$\begin{aligned}
w(t) &\geq c\,u^0 + t^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^\beta v^p(s)\,ds\\
&= c\,u^0 + \frac{1}{\Gamma(\alpha)}\int_0^t (1-s/t)^{\alpha-1}s^{\beta-p(1-\alpha)}w^p(s)\,ds\\
&\geq c\,u^0 + \frac{1}{\Gamma(\alpha)}\int_0^t s^{\beta-p(1-\alpha)}w^p(s)\,ds,
\end{aligned}$$

where we used $(1-s/t)^{\alpha-1} \geq 1$. Note that $w(t) \geq c\,u^0$ for all $t \in [0,T]$. If $\beta - p(1-\alpha) \leq -1$, that is, $p \geq \frac{1+\beta}{1-\alpha}$, then

$$\int_0^t s^{\beta-p(1-\alpha)}w^p(s)\,ds \geq \int_0^t s^{\beta-p(1-\alpha)}(c\,u^0)^p\,ds,$$

and the last integral does not exist for any $t > 0$, so there is no solution in the space $C_{\alpha-1}[0,T]$ for any $T > 0$. This can be thought of as instantaneous blow-up, or blow-up at 0.

For $\beta - p(1-\alpha) > -1$, that is, for $1 < p < \frac{1+\beta}{1-\alpha}$, let $\gamma = p(1-\alpha) - \beta$, then $\gamma < 1$. By Theorem 1.2, local solutions can certainly exist in some cases. We suppose that u and hence also w exist on an interval $(0,T]$ with $T > 0$. Thus we have $w(t) \geq c\,u^0 + \frac{1}{\Gamma(\alpha)}\int_0^t s^{-\gamma}w^p(s)\,ds$ for all $t \in [0,T]$ since terms are continuous. Let $g(t) := c\,u^0 + \frac{1}{\Gamma(\alpha)}\int_0^t s^{-\gamma}w^p(s)\,ds$, then $g \in AC$, g is strictly increasing, $g(t) \geq g(0) = c\,u^0 > 0$, and $g'(t) = t^{-\gamma}w^p(t) \geq \frac{1}{\Gamma(\alpha)}t^{-\gamma}g^p(t)$. Since $g^p \in AC$ is positive, integrating $\frac{g'}{g^p} \geq \frac{1}{\Gamma(\alpha)}t^{-\gamma}$ gives

$$\frac{g^{1-p}(t) - g^{1-p}(0)}{1-p} \geq \frac{1}{\Gamma(\alpha)}\frac{t^{1-\gamma}}{1-\gamma},$$

that is

$$g^{1-p}(t) \leq (cu^0)^{1-p} - (p-1)\frac{1}{\Gamma(\alpha)}\frac{t^{1-\gamma}}{1-\gamma}, \text{ for all } t \in [0,T].$$

Let $T_1 = \big[\frac{(1-\gamma)\Gamma(\alpha)}{(p-1)(cu^0)^{p-1}}\big]^{\frac{1}{1-\gamma}}$. Then we have $(cu^0)^{1-p} - (p-1)\frac{1}{\Gamma(\alpha)}\frac{T_1^{1-\gamma}}{1-\gamma} \leq 0$, but since $g(t)$ is positive on its interval of existence, $g^{1-p}(t)$ cannot exist at $t = T_1$. Thus w, and hence u, can exist on an interval $[0,T]$ only for $T < T_1$. □

Remark 4.3. We expect that the solution u exists on an interval $(0,T_2)$ for some $T_2 \leq T_1$ and blows up at T_2 but we do not know a proof of this. If by some means we knew that $T_2 = T_1$ then we would have blow-up. Laskri-Tatar [15] prove non-existence of a global solution but do not have any estimate of the interval of existence $(0,T_2)$.

Theorem 4.4. *Let $p > 1$ and $\lambda > 0$. The fractional inequality $D^\alpha u(t) \geq \lambda t^\beta u^p(t)$ for $t > 0$, together with $\lim_{t\to 0} t^{1-\alpha}u(t) = u^0 > 0$, does not have a global solution $u \in C_{\alpha-1}$. In fact, for $p \geq \frac{1+\beta}{1-\alpha}$ there is no solution in $C_{\alpha-1}[0,T]$ for any $T > 0$, while for $p < \frac{1+\beta}{1-\alpha}$ any solution can only exist on an interval $[0,T]$ for $T < T_1$, with T_1 as given in Theorem 4.2.*

Proof. Let $f(t) = D^\alpha u(t) - \lambda t^\beta u^p(t)$. For $p \geq \frac{1+\beta}{1-\alpha}$ and $u \in C_{\alpha-1}$ the term $\lambda t^\beta u^p(t)$ is not integrable and no solution exists. Otherwise we can apply Theorem 4.2 noting that T_1 does not depend on f. □

Remark 4.5. Laskri-Tatar [15] state that $p_0 = \frac{1+\beta}{1-\alpha}$ is a critical exponent, arguing that no global solution exists for $1 < p < p_0$, while for $p \geq p_0$ a global solution exists by quoting an explicit example given in [10, Example 3.3]. This appears to contradict our result in Theorem 4.2, but the real reason is that the example has $u^0 = 0$ while we have $u^0 > 0$. The case $p \geq p_0$ is not discussed in [15] when $u^0 > 0$.

We now give extra details of this example since it is only stated in [15]. They cite the monograph [10, Example 3.3], but there is a missing minus sign in the constant term in the formula in [10, page 177]) which misprint is copied in [15].

Example 4.6. Consider the equation

$$D^\alpha u = \lambda t^\beta |u(t)|^p,\ t > 0,\ \text{with}\ \lim_{t\to 0} t^{1-\alpha}u(t) = 0 \text{ and } \lambda > 0,\ p > 0. \tag{4.3}$$

The zero function is a solution. We show that a positive solution of the form $u(t) = ct^r$, $c > 0$ can exist. Let $p_0 = \frac{1+\beta}{1-\alpha}$, where $0 < \alpha < 1$, but β can be of either sign. We have by direct calculation,

$$I^{1-\alpha}u = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} cs^r\,ds = ct^{1-\alpha+r}\frac{\Gamma(1+r)}{\Gamma(2+r-\alpha)},$$

where we impose $1 + r - \alpha > 0$ for this to be AC and to have a nonzero derivative. Therefore $D^\alpha u(t) = c\frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}$ for $1 + r - \alpha > 0$. Hence, for $u = ct^r$ to be a solution, we require that

$$r - \alpha = \beta + pr \text{ and } c^{p-1} = \frac{\Gamma(1+r)}{\lambda\Gamma(1+r-\alpha)}. \tag{4.4}$$

The value $p = 1$ is possible only if $\alpha + \beta = 0$ and λ, r satisfy $\frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} = \lambda$.
If $\alpha + \beta = 0$ and $p \neq 1$ then $r = 0$ which gives a constant solution with $c^{p-1} = 1/(\lambda\Gamma(1-\alpha))$.

We now suppose that $\alpha + \beta \neq 0$. For $p > 1$, we have $r = \frac{-(\alpha+\beta)}{p-1}$. It is readily verified that $1 - \alpha + r > 0$ if $p(1-\alpha) > 1 + \beta$, that is $p > p_0$. Also it then follows that $u \in C_{\alpha-1}[0, T]$ and $\lambda t^\beta u^p \in L^1[0, T]$ for every $T > 0$, thus u is a solution when $p > 1$ and $p > p_0$. Note that $p_0 > 1$ if and only if $\alpha + \beta > 0$. Therefore if $\alpha + \beta > 0$ we must have $p > p_0 > 1$, whereas if $\alpha + \beta < 0$ any $p > 1$ is allowed. When $\alpha + \beta < 0$, we have $r > 0$ and u is continuous; this also gives an explicit solution for the Caputo derivative case when $u(0) = 0$.
For $0 < p < 1$, we have $r = \frac{\alpha+\beta}{1-p}$ and $1 + r - \alpha > 0$ requires $p < p_0$. If $p_0 \geq 1$, that is $\alpha + \beta > 0$, any $p < 1$ is allowed, while if $p_0 \leq 1$, that is $\alpha + \beta \leq 0$, it must be $0 < p < p_0 < 1$.

Remark 4.7. Our Theorem 4.2 shows that, when $\alpha + \beta > 0$, p_0 is critical for $u^0 > 0$ because when $1 < p < p_0$ solutions can exist only for $0 < t \leq T < T_1$ for an explicit T_1, but solutions do not exist in the space $C_{\alpha-1}[0, T]$ for any $T > 0$ when $p \geq p_0 > 1$. Thus $u^0 = 0$ is an exceptional case.

5. Blow-up for Caputo derivative inequalities

For $0 < \alpha < 1$, $\alpha + \beta \geq 0$, $\lambda > 0$ and $p > 1$, we will investigate continuous functions u that satisfy the inequality

$$D_*^\alpha u(t) \geq \lambda t^\beta |u(t)|^p \text{ with } u(0) = u_0 > 0. \tag{5.1}$$

Our aim is to prove nonexistence of nontrivial global solutions. Of course, for $u_0 = 0$ the trivial solution $u = 0$ exists for all t. Since the Caputo derivative is defined in terms of the R-L derivative it should be no surprise that a similar result to Theorem 4.2 holds. However there are some differences. In the Caputo case it is supposed that u is continuous but $D_*^\alpha u$ need not be continuous.

By a *solution* u of the problem (5.1) on an interval $[0, T]$ we will mean that $u \in C[0, T]$, $I^{1-\alpha} \in AC[0, T]$, and $t^\beta |u(t)|^p \in L^1[0, T]$, the inequality is satisfied for $t \in (0, T]$ and the IC is satisfied. By a *global solution* we mean $u(t)$ is a solution for all $t > 0$.

We first give a result which will prove useful. It can be deduced from more general known results, for example [11, Theorem 3.2], [14, Lemma 4], [21, Theorem 5.1]. For completeness we give the simple proof.

Lemma 5.1. *Let $0 < \alpha < 1$, and for $f \in L^1$ suppose that $u \in C[0, T]$ and $I^{1-\alpha}(u - u_0) \in AC[0, T]$ satisfies*

$$D_*^\alpha u(t) = f(t), \text{ for a.e. } t > 0, \text{ with IC } u(0) = u_0, \tag{5.2}$$

Then u satisfies the Volterra integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s)\, ds, \text{ a.e. } t \in [0, T]. \tag{5.3}$$

Proof. Since u is continuous, for $M > 0$ there exists $\delta > 0$ such that $|u(s) - u_0| < M$ for $0 \leq s < \delta$. Then we have, for $0 < t < \delta$,

$$|I^{1-\alpha}(u - u_0)(t)| \leq \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} |u(s) - u_0|\, ds \leq \frac{M}{\Gamma(2 - \alpha)} t^{1-\alpha},$$

thus $I^{1-\alpha}(u - u_0)(0) = 0$. By definition, $D_*^\alpha u = f$ means that $D(I^{1-\alpha}(u - u_0)) = f$. Since $I^{1-\alpha}(u - u_0)) \in AC[0, T]$, by integration and the above calculation we obtain $I^{1-\alpha}(u - u_0))(t) = If(t)$ for all t. Then, applying I^α and using the semigroup property gives $I(u - u_0)(t) = I(I^\alpha f)(t)$. The functions on both sides of this equation are absolutely continuous, so are differentiable almost everywhere, and we get $u(t) - u_0 = I^\alpha f(t)$ for a.e. t. □

Remark 5.2. The converse needs more condition since, for $f \in L^1$, we only have $I^\alpha f \in L^1$, and $u(t) - u_0 = I^\alpha f(t)$ for a.e. t does not imply $u(0) = u_0$. There is an equivalence when f is continuous as is proved in Diethelm [5, Lemma 6.2], and there are equivalences under some conditions on f weaker than continuity, for example [11, Theorem 3.2], [14, Lemma 4] and [21, Theorem 4.6].

Shan-Lv [18] studied the Caputo IVP

$$D_*^\alpha u(t) = u^p(t), \ t > 0, \text{ with IC } u(0) = u_0. \tag{5.4}$$

This is a special case of the problem we study with $\beta = 0$. They asserted that if $u_0 > 0$ and $1 < p \leq 1/(1 - \alpha)$ then every solution of (5.4) blows-up in finite time, [18, Theorem 2.2]. They also asserted that when $p > 1/(1 - \alpha)$ and u_0 has an explicit positive lower bound, then solutions blow-up in finite time. In fact they proved non-existence of global solutions but did not prove that this is blow-up.

We improve their result by considering the more general inequality with the possibly singular term t^β. We prove that, for *for every* $p > 1$ and *any initial value* $u_0 > 0$, solutions can only exist on a finite interval $[0, T]$ with an explicit upper bound $T < T_1$.

We start with the IVP, for $\alpha + \beta > 0$, $f \in L^1$, $f(t) \geq 0$ for $t > 0$.

$$D_*^\alpha u(t) = \lambda t^\beta u^p(t) + f(t) \text{ with } u(0) = u_0 > 0. \tag{5.5}$$

A solution $u \in C[0, T]$ of (5.5) satisfies

$$u(t) = u_0 + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (s^\beta u^p(s) + f(s))\, ds, \ \textit{a.e.}\ t \in [0, T]. \tag{5.6}$$

Since p is a real number, it is implicit that solutions of (5.5) are positive. In fact, $u_0 > 0$ and continuity imply that a solution $u(t)$ will be positive for small $t > 0$ and then from (5.6) it follows that $u(t) > u_0$ for $t > 0$ on its interval of existence.

We first consider the case $1 < p \leq \frac{1+\beta}{1-\alpha}$.

Theorem 5.3. *Let $f \in L^1$ be non-negative and let $\lambda > 0$ and $\alpha + \beta > 0$. Consider the problem*

$$D_*^\alpha u(t) = \lambda t^\beta u^p(t) + f(t), \ \textit{a.e.}\ t > 0, \ \textit{with IC}\ u(0) = u_0. \tag{5.7}$$

For any $u_0 > 0$ and for $1 < p \leq \frac{1+\beta}{1-\alpha}$ there does not exist a global solution u. More precisely, there exists $T_1 > 0$, explicitly determined by the parameters of the problem, such that a solution u can only exist on an interval $[0, T]$ where $T < T_1$.

Proof. Suppose a solution u exists on an interval $[0, T]$ where $T > 1$; otherwise we can take any $T_1 > 1$. Define $c > 0$ by $c^{p-1} = \lambda$. Then $v = cu$ is a solution of

$$D_*^\alpha v(t) = t^\beta v^p(t) + cf(t), \text{ a.e. } t \in (0, T), \text{with IC } v(0) = v_0 = cu_0 > 0,$$

By Lemma 5.1, for a.e. $t \in (0, T]$, v satisfies the equation

$$\begin{aligned} v(t) &= v_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^\beta v^p(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} cf(s)\, ds, \\ &= v_0 + \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \int_0^t (1 - s/t)^{\alpha-1} s^\beta v^p(s)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} cf(s)\, ds. \end{aligned} \tag{5.8}$$

The last term can be discarded and we obtain

$$t^{1-\alpha} v(t) \geq t^{1-\alpha} v_0 + \frac{1}{\Gamma(\alpha)} \int_0^t s^{\beta - p(1-\alpha)} (s^{1-\alpha} v(s))^p\, ds, \text{ a.e. } t.$$

Let $w(t) := t^{1-\alpha} v(t)$. Then w is continuous and satisfies

$$w(t) \geq t^{1-\alpha} v_0 + \frac{1}{\Gamma(\alpha)} \int_0^t s^{\beta - p(1-\alpha)} w^p(s)\, ds. \tag{5.9}$$

For $t \in [1, T]$ we have

$$w(t) \geq v_0 + \frac{1}{\Gamma(\alpha)} \int_1^t s^{\beta - p(1-\alpha)} w^p(s)\, ds. \tag{5.10}$$

Let $\gamma = p(1-\alpha) - \beta$, then $1 < p \le \frac{1+\beta}{1-\alpha}$ implies that $\gamma \le 1$ and (5.10) can be written

$$w(t) \ge v_0 + \frac{1}{\Gamma(\alpha)} \int_1^t s^{-\gamma} w^p(s)\, ds.$$

Let $g(t) = v_0 + \frac{1}{\Gamma(\alpha)} \int_1^t s^{-\gamma} w^p(s)\, ds$ for $t \in [1, T]$. Now terms are continuous so $g \in AC$, $g(1) = v_0$, $g(t) \ge v_0 > 0$ for all $t \ge 1$ and

$$g'(t) = \frac{1}{\Gamma(\alpha)} t^{-\gamma} w^p(t) \ge \frac{1}{\Gamma(\alpha)} t^{-\gamma} g^p(t), \text{ so } \frac{g'}{g^p} \ge \frac{t^{-\gamma}}{\Gamma(\alpha)},\ t \in [1, T].$$

We can integrate from 1 to $t \le T$ to obtain

$$\begin{gathered} g^{1-p}(t) \le v_0^{1-p} - \frac{(p-1)}{\Gamma(\alpha)} \frac{(t^{1-\gamma} - 1)}{1-\gamma},\ \text{for } \gamma < 1, \\ g^{1-p}(t) \le v_0^{1-p} - \frac{(p-1)}{\Gamma(\alpha)} \ln t,\ \text{for } \gamma = 1. \end{gathered} \tag{5.11}$$

It is clear that there exists $T_1 > 1$ (it can be written explicitly) such that the terms on the right become zero, hence $g^{1-p}(T_1)$ does not exist, thus u can only exist on an interval $[0, T]$ with $T < T_1$. □

Corollary 5.4. *Let $1 < p \le \frac{1+\beta}{1-\alpha}$, $\lambda > 0$ and $\alpha + \beta > 0$. The problem $D_*^\alpha u(t) \ge \lambda t^\beta u^p(t)$, $t > 0$, with $u(0) = u_0 > 0$, does not have a global solution.*

Proof. The proof of Theorem 5.3 applies since for $f(t) = D_*^\alpha u(t) - \lambda t^\beta u^p(t)$, $f \in L^1$ and $f(t) \ge 0$ for $t > 0$. □

We now can deal with the case $p > \frac{1+\beta}{1-\alpha}$ very simply and it gives the following result.

Theorem 5.5. *Let $\lambda > 0$ and $\alpha + \beta > 0$. For any $p > 1$ and any $u_0 > 0$ there does not exist a global solution u of the problem*

$$D_*^\alpha u(t) \ge \lambda t^\beta u^p(t),\ t > 0, \text{with } u(0) = u_0 > 0. \tag{5.12}$$

Proof. A solution u of (5.12) will satisfy $u(t) \ge u_0 > 0$ on its interval of existence. For $1 < p \le \frac{1+\beta}{1-\alpha}$ the result is shown in Corollary 5.4, so suppose that $p > \frac{1+\beta}{1-\alpha}$. Write $p = p_0 + p_1$ where $p_0 = \frac{1+\beta}{1-\alpha}$. Then $u^p = u^{p_1} u^o \ge u_0^{p_1} u^{p_0}$ and from (5.12) we obtain

$$D_*^\alpha u(t) \ge (u_0^{p_1} \lambda) t^\beta u^{p_0}(t),\ \text{with IC } u(0) = u_0 > 0.$$

By Corollary 5.4, u can only exist on some interval $[0, T]$ where $T < \hat{T}_1$ and $\hat{T}_1$ is determined by the parameters of the problem. □

Remark 5.6. When $\alpha + \beta < 0$, the opposite sign to the one we have considered, and $p > 1$, there exists a continuous solution of the form ct^r with $c > 0$ for the Caputo problem with initial data 0

$$D_*^\alpha u(t) = \lambda t^\beta u^p(t),\ u(0) = u_0 = 0.$$

The calculation is given above in Example 4.6. The solution is ct^r for $r = \frac{-\alpha-\beta}{p-1}$. This nontrivial continuous solution exists if $\alpha + \beta < 0$ for any $p > 1$.

Remark 5.7. In the paper by Shan and Lv [18], who have the special case $\beta = 0$, at the corresponding stage (5.8) of our proof of Theorem 5.3, the authors use the inequality $(t-s)^{\alpha-1} \geq (t+1)^{\alpha-1}$ together with comparison principles. We discuss the more general result using a different inequality at that point and simple comparisons. The special case of $\beta = 0$ in Theorem 5.3 gives a similar conclusion to [18, Theorem 2.2 (1)]. Theorem 5.5 improves [18, Theorem 2.2 (2)], which uses an inequality that requires the initial condition u_0 to be bounded below by a sufficiently large explicit constant. They write that the solution blows-up but do not mention any continuation theorem that would give the proof of this. By using Wu and Liu [24, Theorem 4.1], or Eloe-Masthay [6, Theorem 2.4], the blow-up at some $T_2 < T^*$ can be justified for the case of an equation.

Systems of Caputo inequalities are studied in [19]. A nonexistence result for the inequality $D_C^\alpha u(t) \geq \lambda t^\beta u^p$ for $\beta \geq 0$ and $p > 1$ is given in [19, Proposition 3.1], using a test function and capacity method. The given condition is $\beta + 1 \geq \beta p'$ where $1/p + 1/p' = 1$. This is equivalent to $\beta < p - 1$ so it cannot be a sharp estimate; perhaps there is a typo. Blow-up is claimed but it seems to need further explanation for fractional inequalities.

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