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# POSITIVE SOLUTION FOR NONLINEAR ELLIPTIC EQUATIONS ON SYMMETRIC DOMAINS

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Abstract. We show the existence of a positive solution for the Schrödinger quasilinear equation with variable exponents above the critical regime. For that matter, we show an embedding into an Orlicz space of functions modeled over radially symmetric domains. Then we use a Galerkin method combined with a fixed-point argument to obtain a solution.

## 1. Introduction

Our aim is to find positive radially symmetric solutions $u(r)$, $r = |x|$, for the equation

$$-\Delta_p u + u^{p-1} = \lambda a(r) u^{q(r)-1} + u^{\theta(r)-1} \quad \text{in } D, \tag{1.1}$$

where the exponents $\theta(r)$ and $q(r)$ are functions satisfying

$$1 < q(r) < p < N, \tag{1.2}$$

$$\theta(r) = \frac{pN}{N-p} + h(r), \tag{1.3}$$

$\lambda > 0$ is a parameter and $a(r), h(r), \theta(r), q(r)$ are positive radially symmetric continuous functions, and $D \subseteq \mathbb{R}^N$, $N \geq 2$, is an open symmetric set centered at the origin (that is, if $x \in D$, then $|x| \in D$). The domain $D$ may be a ball, an annulus, $\mathbb{R}^N$ or $\mathbb{R}^N$ minus a ball. Here $1 < p < N$ and $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the so-called $p$-Laplacian. Equation (1.1) is sometimes called quasilinear Schrödinger equation. Including the case $p = 2$, it was studied in many papers see for instance [1, 2, 4, 7, 8, 9, 10, 11, 12, 13, 17, 18, 19, 21, 23, 24, 28, 31, 33, 42, 45, 44]. Because $\theta(r) > pN/(N-p)$ we say that (1.1) is in the supercritical range in the sense of Sobolev embedding, see Remark 1.5 below.

Throughout this article $h$ is a function with the following properties:

$$h : [0,\infty) \to [0,\infty) \text{ is continuous, and } h(0) = 0; \tag{1.4}$$

$$\text{there are constants } \beta > 2 \text{ and } c > 0 \text{ such that } h(r) \leq c|\log r|^{-\beta} \text{ for } r \text{ near } 0; \tag{1.5}$$

$$\text{there is a constant } c > 0 \text{ such that } h(r) \leq \frac{c}{|1-r|} \text{ for } r \text{ close to } 1. \tag{1.6}$$

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We denote by $W_r^{1,p}(D)$ the closed subspace of $W^{1,p}(D)$ composed by radially symmetric functions on $D$, i.e.,

$$W_r^{1,p}(D) = \{u \in W^{1,p}(D) : u = u(r),\ r = |x|\}, \tag{1.7}$$

endowed with the standard norm

$$\|u\|_{W^{1,p}(D)} = \Big(\int_D (|\nabla u|^p + |u|^p)dx\Big)^{1/p}. \tag{1.8}$$

**Theorem 1.1.** *Let $\theta(r) = p^* + h(r)$, $p^* = pN/(N-p)$, and $h$ satisfy* (1.4)–(1.6)*. Then*

$$\sup\Big\{\int_{\mathbb{R}^N} |u(x)|^{\theta(r)}dx : u \in W_r^{1,p}(\mathbb{R}^N),\ \|u\|_{W^{1,p}(\mathbb{R}^N)} = 1\Big\} < \infty. \tag{1.9}$$

By the conditions on $h(r)$ and the decay of $u(r)$, the integrals are well defined and nonsingular. We denote by $W_{0,r}^{1,p}(D)$ the closed subspace of $W_0^{1,p}(D)$ composed by radially symmetric functions on $D$, i.e.,

$$W_{0,r}^{1,p}(D) = \{u \in W_0^{1,p}(D) : u = u(r),\ r = |x|\}, \tag{1.10}$$

endowed with the norm (1.8). The continuity of the embedding into the Orlicz space reads as follows, see [20, 39] or Section 2 below.

**Corollary 1.2.** *Let $\theta(r) = p^* + h(r)$ with $h \in L_+^\infty(\mathbb{R}^N)$ satisfying* (1.4)–(1.6)*. Then the following embedding is continuous*

$$W_r^{1,p}(\mathbb{R}^N) \hookrightarrow L^{\theta(r)}(\mathbb{R}^N). \tag{1.11}$$

In what follows, we consider an equation more general than (1.1). Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a continuous function, radially symmetric in the first variable and satisfying the growth condition

$$0 \le f(r,t)t \le b_1 t^{\theta(r)} \quad \text{for every } r \in \mathbb{R} \text{ and } t \ge 0, \tag{1.12}$$

where $b_1 > 0$ is a constant, $\theta(r) = p^* + h(r)$, with $h \in L_+^\infty(\mathbb{R}^N)$ satisfying (1.4)–(1.5) (see also (2.3) below), and $p^* = pN/(N-p)$.

Wyhen $D$ is a bounded domain, we deal with the problem

$$\begin{gathered} -\Delta_p u + u^{p-1} = \lambda a(r)u^{q(r)-1} + f(r,u) \quad \text{in } D \\ u > 0 \quad \text{in } D \\ u = 0 \quad \text{on } \partial D. \end{gathered} \tag{1.13}$$

Note that $D$ encompasses balls and an annulus centered at the origin, in which we have the following result.

**Theorem 1.3.** *If $\lambda > 0$ is a constant, $q$ and $a$ are radially symmetric continuous functions, such that $1 < q_- \le q(r) \le q_+ < p$, $q_-, q_+ \in \mathbb{R}$, $a \in L^{\frac{p}{p-q(r)}}(\mathbb{R}^N)$, $a(r) > 0$ in $\mathbb{R}^N$, $r = |x|$, $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying* (1.12)*. Then there exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$ problem* (1.13) *possesses at least one positive radially symmetric solution $u_\lambda \in W_0^{1,p}(D)$. Furthermore, $\|u_\lambda\|_{W_0^{1,p}(D)} \to 0$ as $\lambda \to 0^+$.*

Let $B_R \subseteq \mathbb{R}^N$ be the open ball with radius $R$ centered at the origin. (For $R = 0$ we have $B_0 = \emptyset$.) We study the problem

$$\begin{gathered} -\Delta_p u + u^{p-1} = \lambda a(r) u^{q(r)-1} + f(r,u) \quad \text{in } \mathbb{R}^N \backslash \overline{B_R} \\ u > 0 \quad \text{in } \mathbb{R}^N \backslash \overline{B_R} \\ u = 0 \quad \text{on } \partial(\mathbb{R}^N \backslash \overline{B_R}). \end{gathered} \tag{1.14}$$

Note that $\mathbb{R}^N \backslash \overline{B_R}$ admits the whole space $\mathbb{R}^N$ and exterior domains such as $\mathbb{R}^N \backslash \overline{B_R}$. The next result applies to exterior domains $\mathbb{R}^N \backslash \overline{B_R}$ and the whole space $\mathbb{R}^N$.

**Theorem 1.4.** *If $\lambda > 0$ is a constant, $q$ and $a$ are radially symmetric continuous functions, such that $1 < q_- \le q(r) \le q_+ < p$, $q_-, q_+ \in \mathbb{R}$, $a \in L^{\frac{p}{p-q(r)}}(\mathbb{R}^N)$, $a(r) > 0$ in $\mathbb{R}^N$, $r = |x|$, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (1.12). Then there exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$ problem (1.14) possesses at least one positive radially symmetric solution $u_\lambda \in W_0^{1,p}(\mathbb{R}^N \backslash \overline{B_R})$. Furthermore, $\|u_\lambda\|_{W_0^{1,p}(\mathbb{R}^N \backslash \overline{B_R})} \to 0$ as $\lambda \to 0^+$.*

**Remark 1.5.** The Sobolev embedding with $\theta(r) \le pN/(N-p)$ can be found in [15]. In such cases we say that $f$ in (1.12) or equations (1.13)–(1.14) are subcritical if $\theta(r) < pN/(N-p)$ and critical when $\theta(r) = pN/(N-p)$. The equations studied in this article have the supercritical term $f$ related to $\theta(r) > pN/(N-p)$. About this matter we prove the boundedness of the Sobolev constant in Theorem 1.1 and the embedding stated as Corollary 1.2.

Equation (1.13) and Theorem 1.3 are associated with the existence of a positive solution on domains $D$ which can be balls or an annulus. Other equations in the supercritical range were treated in [7, 23] when $D$ is a ball. Problems on $D$ being an annulus and $f$ with critical growth are studied in [11, 12], non-radial solutions were studied in [16], and uniqueness questions in [17, 18, 42, 45]. With respect to (1.14), when $D$ is an exterior domain $\mathbb{R}^N \backslash \overline{B_R}$ or the whole $\mathbb{R}^N$ we prove Theorem 1.4. The equation with subcritical nonlinearity was treated in the seminal article [13] on a domain like $\mathbb{R}^N \backslash \overline{B_R}$. The critical case was addressed in [2] and on exterior domains without symmetry in [33]. The $p$-Laplacian equation on an exterior domain was studied in [21] by ODE methods, and with a supercritical $f$ in [28] by means of variational methods. Equations with Neumann condition on the inner boundary $\partial B_R$ were investigated in [1, 24, 31, 44] and symmetry issues were analyzed in [36]. The problem in $\mathbb{R}^N$ was studied in [8] for a $p$-Laplacian equation with critical $f$, see also [3, 14]. The supercritical case was treated in [10, 19, 9]. The equations considered in this article also correspond to the concave-convex problems of [4].

The outline of this article is as follows. Section 2 presents some preliminary results about extension of Sobolev functions, Brouwer theorem, comparison principles and function spaces with variable exponents. Sometimes in the paper we pass from the domain $D$ to the whole $\mathbb{R}^N$ in order to stress that the estimates are independent of $D$ and since we are keeping in mind the use of Corollary 1.2. In Section 3 we prove of Theorem 1.1 and Corollary 1.2 by means of integral estimates which are nonsingular and well defined because of the conditions on $h(r)$ and the polynomial decay of $u(r)$. In Section 4 we present the Lipschitz approximate functions of $f$ that will be useful for solving the approximated equation in Section 5 with the aid of a Galerkin type scheme. In doing so we use a Schauder basis. In the limit the solution of those approximate equations tend to a solution of the original equation

(1.13) on a ball. We apply the Palais principle to show that the solution is radially symmetric. We prove Theorems 1.3 and 1.4 in Sections 6 and 7, respectively. Obtaining a positive solution in $\mathbb{R}^N$ is done by means of a diagonal argument that allows us to let $R$ approach $\infty$.

## 2. Preliminaries

From now on, when a function defined in $D$ is radially symmetric, for convenience, we will use the same notation to represent the function on $x$ or $r = |x|$.

2.1. **Extension.** Let $u \in W_0^{1,p}(D)$. In what follows, we denote by $\tilde{u}$ the canonical extension of $u$ by 0 outside $D$, that is,

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in D, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus D. \end{cases} \tag{2.1}$$

It is well known that $u \in W_0^{1,p}(D)$ implies $\tilde{u} \in W^{1,p}(\mathbb{R}^N)$ (see e.g. [15, Proposition 9.18]).

2.2. **Brouwer Theorem.** The following lemma is a generalization of the classical Brouwer fixed point theorem found in Kesavan [32], the proof is performed in [6]. We denote $\langle \cdot, \cdot \rangle^{1/2}$ the Euclidean inner product with its induced norm and $|\cdot|_m$ is any other norm.

**Lemma 2.1.** *Let $F : (\mathbb{R}^m, |\cdot|_m) \to (\mathbb{R}^m, |\cdot|_m)$ be a continuous function such that $\langle F(\xi), \xi \rangle \geq 0$ for every $\xi \in \mathbb{R}^m$ with $|\xi|_m = \vartheta$ for some $\vartheta > 0$, and $|\cdot|_m$ is any norm. Then, there exists $z_0$ in the closed ball $\overline{B}_\vartheta^m(0) := \{z \in \mathbb{R}^m; |z|_m \leq \vartheta\}$ such that $F(z_0) = 0$.*

2.3. **Comparison principle.** Assume that $D$ is a bounded domain in $\mathbb{R}^N$ with $C^2$ boundary $\partial D$. Next we present a subtle adaptations of the results achieved in [26, Theorems 3 and 5] for a non-autonomous function.

We present a couple of comparison principles for a subsolution and for a supersolution of the problem

$$\begin{gathered} -\Delta_p u + |u|^{p-2} u = g(x, u) \quad \text{in } D \\ u = 0 \quad \text{on } \partial D, \end{gathered} \tag{2.2}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function.

We say that $u_1 \in W^{1,p}(D)$ is a subsolution of (2.2) if $u_1 \leq 0$ on $\partial D$ and

$$\int_D (|\nabla u_1|^{p-2} \nabla u_1 \nabla \varphi + |u_1|^{p-2} u_1 \varphi) dx \leq \int_D g(x, u_1) \varphi dx$$

for all $\varphi \in W_0^{1,p}(D)$ with $\varphi \geq 0$ in $D$ provided the integral $\int_D g(x, u_1) \varphi dx$ exists. We say that $u_2 \in W^{1,p}(D)$ is a supersolution of (2.2) if the reversed inequalities are satisfied with $u_2$ in place of $u_1$ for all $\varphi \in W_0^{1,p}(D)$ with $\varphi \geq 0$ in $D$.

The next comparison results are particular cases of the ones achieved in [27, Theorems 3 and 5].

**Proposition 2.2.** *Let $g : D \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $g(x,t)/t^{p-1}$ is decreasing for $t > 0$. Assume that $u_1$ and $u_2$ are a positive subsolution and a positive supersolution of problem (2.2), respectively. If $u_2(x) > u_1(x) = 0$ for all $x \in \partial D$, $u_i \in C^{1,\alpha}(\overline{D})$ with some $\alpha \in (0,1)$, $\Delta_p u_i \in L^\infty(D)$, for $i, j = 1, 2$, then $u_2 \geq u_1$ in $D$.*

Whenever $u_1$ and $u_2$ satisfy the homogeneous Dirichlet boundary condition we can state the following result.

**Proposition 2.3.** *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $g(x,t)/t^{p-1}$ is decreasing for $t > 0$. Assume that $u_1, u_2 \in C_0^{1,\alpha}(\overline{D})$, with some $\alpha \in (0,1)$, are a positive subsolution and a positive supersolution of problem (2.2), respectively. If $\Delta_p u_i \in L^\infty(D)$, for $i, j = 1, 2$, $u_1/u_2 \in L^\infty(D)$ and $u_2/u_1 \in L^\infty(D)$, then $u_2 \geq u_1$ in $D$.*

2.4. **Function spaces with variable exponents.** In the sequel we display some results of the Lebesgue spaces with variable exponents (see [20, 25, 39] for more details). Let $D$ be an open domain in $\mathbb{R}^N$, $1 < p < N$ and $p^* = pN/(N-p)$. Define

$$L^\infty_+(D) = \{y : y \in L^\infty(D), \inf_{x \in D} y(x) > 1\}. \tag{2.3}$$

For each $y \in L^\infty_+(D)$, we define

$$y_- = y_-(D) = \inf_{x \in D} y(x), \; y_+ = y_+(D) = \sup_{x \in D} y(x).$$

For $y \in L^\infty_+(D)$, the space

$$L^{y(x)} = \big\{u : \text{ is real measurable}, \int_D |u(x)|^{y(x)} dx \leq \infty\big\} \tag{2.4}$$

is equipped with a Banach norm

$$\|u\|_{y(x)} = \inf\big\{\sigma > 0 : \int_D |\frac{u(x)}{\sigma}|^{y(x)} dx \leq 1\big\}.$$

And $L^{y(x)}$ is called Orlicz space (see also Musielak-Orlicz spaces in [20, 39]) and $\|\cdot\|_{y(x)}$ is the Luxemburg norm.

**Proposition 2.4.** *Let $u \in L^{y(x)}(D)$ and $\|u\|_{y(x)} = \lambda$.*

$$\text{If } \lambda \geq 1, \text{ then } \lambda^{y_-} \leq \int_D |u(x)|^{y(x)} dx \leq \lambda^{y_+}.$$
$$\text{If } \lambda \leq 1, \text{ then } \lambda^{y_+} \leq \int_D |u(x)|^{y(x)} dx \leq \lambda^{y_-}.$$

**Proposition 2.5.** *The conjugate space of $L^{y(x)}(D)$ is $L^{y_o(x)}(D)$, where $1/y(x) + 1/y_o(x) = 1$. Furthermore, for $u \in L^{y(x)}(D)$, $v \in L^{y_o(x)}(D)$, we have the inequality*

$$\big|\int_D u(x)v(x)dx\big| \leq 2\|u\|_{y(x)}\|v\|_{y_o(x)}.$$

**Proposition 2.6.** *Let $D$ be an open bounded domain in $\mathbb{R}^N$ with the cone property. If $y \in L^\infty_+(D)$ satisfies*

$$y(x) \leq y_+ < p^* \text{ for all } x \in D,$$

*then the following embedding is compact*

$$W^{1,p}(D) \hookrightarrow L^{y(x)}(D).$$

## 3. Proof of Theorem 1.1

Let $u:\mathbb{R}^N\to\mathbb{R}$ be a radial function and $v:[0,\infty)\to\mathbb{R}$ such that $u(x)=v(r)$ for all $x\in\mathbb{R}^N$. For convenience, we will keep the same notation $u$ for both cases, i.e. $u(x)=v(r)=u(r)$. It is well-known that (see e.g. [29])

$$\int_{\mathbb{R}^N} u(x)dx=\omega_N\int_0^\infty u(r)r^{N-1}dr,\quad \text{where } \omega_N=\frac{2\pi^{N/2}}{\Gamma(N/2)}.$$

Thus, if $u\in W^{1,p}_{0,r}(\mathbb{R}^N)$ we can write

$$\|u\|_{W^{1,p}(\mathbb{R}^N)}=\Big(\omega_N\int_0^\infty(|Du'|^p+|u|^p)r^{N-1}dr\Big)^{1/2}. \tag{3.1}$$

Let $C_0^\infty(\mathbb{R}^N)$ be the space of infinitely differentiable functions with compact support. We denote by $C_{0,r}^\infty(\mathbb{R}^N)$ the subspace of $C_0^\infty(\mathbb{R}^N)$ of radially symmetric functions.

**Lemma 3.1.** *Let $u\in W^{1,p}_{0,r}(\mathbb{R}^N)$. Then*

$$|u(r)|\le C\min\big\{\frac{1}{r^{(N-p)/p}},\frac{1}{r^{(N-1)/p}}\big\}\|u\|_{W^{1,p}(\mathbb{R}^N)}. \tag{3.2}$$

*Proof.* If $\phi\in C_{0,r}^\infty(\mathbb{R}^N)$, then

$$\begin{aligned}
|\phi(r)| &\le \Big|\int_r^\infty \phi'(s)ds\Big|\\
&\le \int_r^\infty \big|\phi'(s)s^{(N-1)/p}\frac{1}{s^{(N-1)/p}}ds\big|\\
&\le \Big(\int_r^\infty|\phi'(s)|^p s^{N-1}ds\Big)^{1/p}\Big(\int_r^\infty\frac{1}{s^{\frac{N-1}{p-1}}}ds\Big)^{\frac{p-1}{p}}\\
&\le \Big(\int_r^\infty(|\phi'(s)|^p+|\phi(s)|^p)s^{N-1}ds\Big)^{1/p}\Big(\int_r^\infty\frac{1}{s^{\frac{N-1}{p-1}}}ds\Big)^{\frac{p-1}{p}}\\
&\le \frac{1}{\omega_N^{1/p}}\Big(\frac{p-1}{N-p}\Big)^{\frac{p-1}{p}}\frac{1}{r^{(N-p)/p}}\|\phi\|_{W^{1,p}(\mathbb{R}^N)}.
\end{aligned} \tag{3.3}$$

On the other hand,

$$\begin{aligned}
(\phi(r))^p &= -p\int_r^\infty \phi'(s)\phi(s)^{p-1}ds\\
&\le p\int_r^\infty|\phi'(s)||\phi(s)|^{p-1}s^{N-1}\frac{1}{s^{N-1}}ds\\
&\le \frac{p}{r^{N-1}}\int_r^\infty\big(\frac{|\phi'(s)|^p}{p}+\frac{|\phi(s)|^p}{p/(p-1)}\big)s^{N-1}ds\\
&\le \frac{1}{\omega_N}\max\{1,p-1\}\frac{1}{r^{N-1}}\|\phi\|^p_{W^{1,p}(\mathbb{R}^N)}.
\end{aligned}$$

Thus, we arrive at

$$|\phi(r)|\le C\frac{1}{r^{(N-1)/p}}\|\phi\|_{W^{1,p}(\mathbb{R}^N)}. \tag{3.4}$$

Since $C_{0,r}^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}_{0,r}(\mathbb{R}^N)$, we obtain (3.2). □

*Proof of Theorem 1.1.* Let $u \in W^{1,p}_{0,r}(\mathbb{R}^N)$ with $\|u\|_{W^{1,p}(\mathbb{R}^N)} = 1$. Let $\rho \in (0,1)$ to be chosen later on, we can write

$$\begin{aligned}
&\frac{1}{w_p}\int_{\mathbb{R}^N} |u(r)|^{p^*+h(r)}dx\\
&= \underbrace{\int_0^\rho |u(r)|^{p^*+h(r)} r^{N-1}dr}_{I} + \underbrace{\int_\rho^1 |u(r)|^{p^*+h(r)} r^{N-1}dr}_{J} + \underbrace{\int_1^\infty |u(r)|^{p^*+h(r)} r^{N-1}dr}_{K}.
\end{aligned}$$

We estimate $I, J, K$ in 3 steps. In the first two steps we use ideas similar to those in [23, Theorem 2.1].

**Step 1.** Estimate of $I$.

$$\begin{aligned}
I &= \int_0^\rho |u(r)|^{p^*+h(r)} r^{N-1}dr\\
&\le \int_0^\rho |u(r)|^{p^*}(|u(r)|^{h(r)} - 1) r^{N-1}dr + \int_0^1 |u(r)|^{p^*} r^{N-1}dr\\
&\le \int_0^\rho |u(r)|^{p^*}(|u(r)|^{h(r)} - 1) r^{N-1}dr + C_{N,p},
\end{aligned}$$

where $C_{N,p}$ is the constant in the Sobolev constant embedding

$$W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N).$$

By Lemma (3.1), we have

$$\begin{aligned}
\int_0^\rho |u(r)|^{p^*}(|u(r)|^{h(r)} - 1) r^{N-1}dr &\le \overline{C}\int_0^\rho \frac{1}{r^{(N-p)p^*/p}}\Big(\frac{1}{r^{(N-p)h(r)/p}} - 1\Big) r^{N-1}dr\\
&= \overline{C}\int_0^\rho \frac{1}{r}\Big(\exp[\frac{-(N-p)h(r)}{p}\log r] - 1\Big)dr\\
&\le \overline{C}d\frac{N-p}{p}\int_0^\rho h(r)\frac{|\log r|}{r}dr < \infty,
\end{aligned}$$

whence the above inequality follows from (1.5).

**Step 2.** Estimate of $J$. By (1.6) and Lemma (3.1), we obtain

$$\begin{aligned}
J &= \int_\rho^1 |u(r)|^{p^*+h(r)} r^{N-1}dr\\
&\le \overline{C}\int_\rho^1 \frac{1}{r^{\frac{(N-p)}{p}(p^*+h(r))}} r^{N-1}dr\\
&= \overline{C}\int_\rho^1 \frac{1}{r^{1+\frac{N-p}{p}h(r)}}dr\\
&= -\overline{C}\int_{1-\rho}^0 \frac{1}{(1-s)^{1+\frac{N-p}{p}h(1-s)}}ds\\
&\le \overline{C}\int_0^{1-\rho} \frac{1}{(1-s)^{1+\frac{N-p}{p}\frac{c}{s}}}ds\\
&= \overline{C}\int_0^{1-\rho} \frac{1}{\exp{(1+\frac{N-p}{p}\frac{c}{s})}\log(1-s)}ds < \infty.
\end{aligned}$$

**Step 3.** Estimate of $K$. The third integral $K$ can be bounded with the aid of (1.4) and Lemma 3.1. Indeed

$$\begin{aligned}
K &= \int_1^\infty |u(r)|^{p^*+h(r)} r^{N-1} dr \\
&\leq \overline{C} \int_1^\infty r^{-\frac{N-1}{p}(p^*+h(r))+(N-1)} dr \\
&\leq \overline{C} \int_1^\infty r^{-\frac{N-1}{p}p^*+(N-1)} dr \\
&= \overline{C} \int_1^\infty r^{-p(N-1)/(N-p)} dr < \infty.
\end{aligned} \tag{3.5}$$

From the previous steps, we infer (1.9). The proof of Theorem 1.1 is complete. □

*Proof of Corollary 1.2.* Notice that $\theta(r) = p^* + h(r)$ belongs to $L^\infty_+(\mathbb{R}^N)$ since $\theta(r) \geq p^* > 1$. Define the space

$$L^{p^*+h(r)} = \big\{u : \mathbb{R}^N \to \mathbb{R} : \text{real measurable}, \int_{\mathbb{R}^N} |u(r)|^{p^*+h(r)} dx \leq \infty \big\}$$

equipped with the norm

$$\|u\|_{p^*+h(r)} = \inf \big\{\sigma > 0 : \int_{\mathbb{R}^N} |\frac{u(r)}{\sigma}|^{p^*+h(r)} dx \leq 1 \big\}.$$

Taking that $u \in W^{1,p}_r(\mathbb{R}^N)$ with $\|u\|_{W^{1,p}_r(\mathbb{R}^N)} = 1$. Theorem 1.1 yields that there exists $C$ such that

$$\int_{\mathbb{R}^N} |u(r)|^{p^*+h(r)} dx \leq C < \infty,$$

where $C$ does not depend on $u$. By Proposition 2.4, we obtain

$$\|u\|_{L^{p^*+h(r)}} \leq C_0 = \max\{C^{p-}, C^{p+}\}.$$

Thus,

$$\|u\|_{L^{p*+h(r)}} \leq C_0 \|u\|_{W^{1,p}_r(\mathbb{R}^N)},$$

for every $u \in W^{1,p}_r(\mathbb{R}^N)$. □

## 4. Approximate functions

The continuous function $f$ satisfying (1.12) can be approximated by Lipschitz functions $f_k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ defined by

$$f_k(r,s) = \begin{cases}
-k[G(r,-k-\frac{1}{k}) - G(r,-k)], & \text{if } s \leq -k \\
-k[G(r,s-\frac{1}{k}) - G(r,s)], & \text{if } -k \leq s \leq -\frac{1}{k} \\
k^2 s[G(r,-\frac{2}{k}) - G(r,-\frac{1}{k})], & \text{if } -\frac{1}{k} \leq s \leq 0 \\
k^2 s[G(r,\frac{2}{k}) - G(r,\frac{1}{k})], & \text{if } 0 \leq s \leq \frac{1}{k} \\
k[G(r,s+\frac{1}{k}) - G(r,s)], & \text{if } \frac{1}{k} \leq s \leq k \\
k[G(r,k+\frac{1}{k}) - G(r,k)], & \text{if } s \geq k,
\end{cases} \tag{4.1}$$

where $G(r,s) = \int_0^s f(r,\zeta)d\zeta$, $G_s = f$ and $G(r,0)$=0.

The following approximation result was proved in [41] and it uses the explicit expression of the sequence (4.1).

**Lemma 4.1.** *Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $sf(r,s) \geq 0$ for every $s \in \mathbb{R}$. Then there exists a sequence $f_k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ of continuous functions satisfying*

(i) *$sf_k(r,s) \geq 0$ for every $s \in \mathbb{R}$;*

(ii) *for each $k \in \mathbb{N}$ there is a continuous function $c_k(r)$ such that*

$$|f_k(r,\xi) - f_k(r,\eta)| \leq c_k(r)|\xi - \eta|$$

*for every $\xi, \eta \in \mathbb{R}$;*

(iii) *$f_k$ converges uniformly to $f$ in bounded sets.*

The following result gives the growth behavior to the sequence of functions $c_k(\cdot)$, and the proof can be found in [7, Proposition 5].

**Lemma 4.2.** *One can choose the sequence of Lipschitz constants $c_k(\cdot)$, defined in Lemma 4.1, satisfying the following estimates*

$$c_k(r) \leq Ck \sup_t \big\{|f(r,t)|; t \in [-k - \frac{1}{k}, k + \frac{1}{k}]\big\}, \quad \forall r \in [0,R], \tag{4.2}$$

*where the constant $C$ does not depend on neither $r$ nor $k$.*

A similar version of the next lemma was presented in [7, Lemma 2]. However, since here we can deal with unbounded domains, a slight different statement is needed and deserves a proof.

**Lemma 4.3.** *Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (1.12) for every $s \in \mathbb{R}$. Then the sequence $(f_k)$ of Lemma 4.1 satisfies*

(i) *for all $k \in \mathbb{N}$, $0 \leq sf_k(r,s) \leq K_1|s|^{\theta(r)}$ for every $|s| \geq 1/k$;*

(ii) *for all $k \in \mathbb{N}$, $0 \leq sf_k(r,s) \leq K_1 \frac{1}{k^{p_- -1}}|s|$ for every $|s| \leq 1/k$,*

*where $C_1$ is a positive constant independent of $k$.*

*Proof.* In this proof the constant $b_1$ is the one of (1.12). According to the definition (2.3),

$$p_- = \inf_{r \geq 0} \theta(r) \quad \textit{and} \quad p_+ = \sup_{r \geq 0} \theta(r).$$

**Step 1.** Suppose that $-k \leq s \leq -1/k$. By the mean value theorem, there exists $\eta \in (s - \frac{1}{k}, s)$ such that

$$f_k(r,s) = -k[G(r, s - \frac{1}{k}) - G(r,s)] = -kG_s(r,\eta)(s - \frac{1}{k} - s) = f(r,\eta),$$
$$sf_k(r,s) = sf(r,\eta).$$

Since $s - \frac{1}{k} < \eta < s < 0$ and $f(r,\eta) < 0$, we have $sf(r,\eta) \leq \eta f(r,\eta)$. Therefore,

$$\begin{aligned}
0 \leq sf_k(r,s) &\leq \eta f(r,\eta) \\
&\leq b_1|\eta|^{\theta(r)} \leq b_1|s - \frac{1}{k}|^{\theta(r)} \\
&\leq b_1(|s| + \frac{1}{k})^{\theta(r)} \\
&\leq b_1(2|s|)^{\theta(r)} \\
&\leq b_1 2^{p_+}|s|^{\theta(r)}.
\end{aligned}$$

**Step 2.** Assume $\frac{1}{k} \le s \le k$. By the mean value theorem, there exists $\eta \in (s, s+\frac{1}{k})$ such that

$$f_k(r,s) = k[G(r,s+\frac{1}{k}) - G(r,s)] = kG_s(r,\eta)(s+\frac{1}{k} - s) = f(r,\eta),$$
$$sf_k(r,s) = sf(r,\eta).$$

Since $0 < s < \eta < s + \frac{1}{k}$ and $f(r,\eta) > 0$, we have $sf(r,\eta) \le \eta f(r,\eta)$. Therefore,

$$0 \le sf_k(r,s) \le \eta f(r,\eta) \le b_1|\eta|^{\theta(r)} \le b_1|s+\frac{1}{k}|^{\theta(r)} \le b_1 2^{p_+}|s|^{\theta(r)}.$$

**Step 3.** Suppose that $|s| \ge k$, then

$$f_k(r,s) = \begin{cases} -k[G(r,-k-\frac{1}{k}) - G(r,-k)], & \text{if } s \le -k \\ k[G(r,k+\frac{1}{k}) - G(r,k)], & \text{if } s \ge k. \end{cases} \tag{4.3}$$

If $s \le -k$, again by the mean value theorem, there exists $\eta \in (-k-\frac{1}{k}, -k)$ such that

$$f_k(r,s) = k[G(r,-k-\frac{1}{k}) - G(r,-k)] = -kG_s(r,\eta)(-k-\frac{1}{k} - (-k)) = f(r,\eta),$$
$$sf_k(r,s) = sf(r,\eta).$$

Since $-k - \frac{1}{k} < \eta < -k < 0$ and $k < |\eta| < k + \frac{1}{k}$, we conclude that

$$\begin{aligned} 0 \le sf_k(r,s) &= \frac{s}{\eta}\eta f(r,\eta) \\ &\le \frac{|s|}{|\eta|} b_1|\eta|^{\theta(r)} = b_1|s||\eta|^{\theta(r)-1} \\ &\le b_1|s|(k+\frac{1}{k})^{\theta(r)-1} \\ &\le b_1|s|(|s|+\frac{1}{k})^{\theta(r)-1} \\ &\le b_1|s|(2|s|)^{\theta(r)-1} \\ &\le b_1 2^{p_+}|s|^{\theta(r)}. \end{aligned} \tag{4.4}$$

If $s \ge k$, by the mean value theorem, there exists $\eta \in (k, k+\frac{1}{k})$ such that

$$f_k(r,s) = k[G(r,k+\frac{1}{k}) - G(r,k)] = kG_s(r,\eta)(k+\frac{1}{k} - k) = f(r,\eta).$$

By using similar computations as for (4.4) one has

$$0 \le sf_k(r,s) = sf(r,\eta) = \frac{s}{\eta}\eta f(r,\eta) \le \frac{|s|}{|\eta|} b_1|\eta|^{\theta(r)} \le b_1 2^{p_+}|s|^{\theta(r)}.$$

**Step 4.** Assume $-\frac{1}{k} \le s \le \frac{1}{k}$. Then

$$f_k(r,s) = \begin{cases} k^2 s[G(r,-2/) - G(r,-1/k)], & \text{if } -1/k \le s \le 0 \\ k^2 s[G(r,2/k) - G(r,1/k)], & \text{if } 0 \le s \le 1/k. \end{cases}$$

If $-1/k \le s \le 0$, by the mean value theorem, there exists $\eta \in (-2/k, -1/k)$ such that

$$f_k(r,s) = k^2 s[G(r,-\frac{2}{k}) - G(r,-\frac{1}{k})] = k^2 sG_s(r,\eta)(-\frac{2}{k} - (-\frac{1}{k})) = -ksf(r,\eta).$$

Therefore

$$
\begin{aligned}
0 \le s f_k(r,s) &= -k s^2 f(r,\eta) = -k \frac{s^2}{\eta} \eta f(r,\eta) \\
&\le k \frac{s^2}{|\eta|} \eta f(r,\eta) \le b_1 k |s|^2 |\eta|^{\theta(r)-1} \\
&\le b_1 k |s|^2 (\frac{2}{k})^{\theta(r)-1} \\
&\le b_1 \frac{2^{\theta(r)-1}}{k^{p_- -2}} |s|^2 \\
&\le b_1 2^{p_+} \frac{1}{k^{p_- -1}} |s|.
\end{aligned} \tag{4.5}
$$

If $0 \le s \le 1/k$, by the mean value theorem, there exists $\eta \in (1/k, 2/k)$ such that

$$
f_k(r,s) = k^2 s[G(r,\frac{2}{k}) - G(r,\frac{1}{k})] = k^2 s G_s(r,\eta)(\frac{2}{k} - \frac{1}{k}) = k s f(r,\eta).
$$

Using similar computations as for (4.5) one obtains

$$
0 \le s f_k(r,s) = k s^2 f(r,\eta) = k \frac{s^2}{|\eta|} \eta f(r,\eta) \le b_1 2^{p_+} \frac{1}{k^{p_- -1}} |s|.
$$

The proof of the lemma follows by taking $K_1 = b_1 2^{p_+}$, where $b_1$ is given in (1.12). □

## 5. Approximate equation

Let $D \subseteq \mathbb{R}^N$ be an open, bounded and symmetric set centered at the origin. We say that $u \in W_0^{1,p}(D)$ is a solution of (1.13) if $u(x) > 0$ in $D$ and

$$
\int_D |\nabla u|^{p-2} \nabla u \nabla \phi dx + \int_D |u|^{p-2} u \phi dx = \lambda \int_D a(x) |u|^{q(r)-2} u \phi dx + \int_D f(r,u) \phi dx,
$$

for all $\phi \in W_0^{1,p}(D)$.

We will employ the following auxiliary problem in our reasoning later.

$$
\begin{gathered}
-\Delta_p u + |u|^{p-2} u = \lambda a(x) |u|^{q(r)-2} u + f_n(r,u) + \frac{\varphi}{n} \quad \text{in } D \\
u > 0 \quad \text{in } D \\
u(x) = 0 \quad \text{on } \partial D,
\end{gathered} \tag{5.1}
$$

with $n > 0$ a integer number, $\varphi > 0$ is a fixed function such that $\varphi \in L^\infty(\mathbb{R}^N) \cap L^{p'}(\mathbb{R}^N)$ and $f_n$ is given by Lemmas 4.1 and 4.3.

**Lemma 5.1.** *There exists $\lambda^* > 0$ and $n^* \in \mathbb{N}$ such that (5.1) has a solution $u_n \in C_0^1(\overline{D})$ such that $\partial u_n / \partial \nu < 0$ on $\partial D$ for every $\lambda \in (0, \lambda^*)$ and $n \ge n^*$. Furthermore,*

$$
\|u_n\|_{W^{1,p}(D)} \le \vartheta,
$$

*where $\vartheta$ does not depend on $n$.*

*Proof.* Let $\mathcal{B} = \{w_1, w_2, \dots, w_n, \dots\}$ be a Schauder basis (see [15] for details) for the Banach space $(W_{0,r}^{1,p}(D), \|\cdot\|_{W^{1,p}(D)})$. For each positive integer $m$, let

$$
W_m = [w_1, w_2, \dots, w_m]
$$

be the $m$-dimensional subspace of $W^{1,p}_{0,r}(D)$ generated by $\{w_1, w_2, \dots, w_m\}$ with norm induced from $W^{1,p}_{0,r}(D)$. Let $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$, notice that

$$|\xi|_m := \big\| \sum_{j=1}^{m} \xi_j w_j \big\|_{W^{1,p}(D)} \tag{5.2}$$

defines a norm in $\mathbb{R}^m$ (see [5] for the details).

Using the above notation, we can identify the spaces $(W_m, \|\cdot\|_{W^{1,p}(D)})$ and $(\mathbb{R}^m, |\cdot|_m)$ by the isometric linear transformation

$$u = \sum_{j=1}^{m} \xi_j w_j \in W_m \mapsto \xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m. \tag{5.3}$$

Define the function $F : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$F(\xi) = (F_1(\xi), F_2(\xi), \dots, F_m(\xi)),$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}^m$,

$$\begin{aligned} F_j(\xi) = &\int_D |\nabla u|^{p-2} \nabla u \nabla w_j dx + \int_D |u|^{p-2} u w_j dx - \lambda \int_D a(x)(u_+)^{q(r)-1} w_j dx \\ &- \int_D f_n(r, u_+) w_j dx - \frac{1}{n} \int_D \varphi w_j dx, \end{aligned}$$

$j = 1, 2, \dots, m$, and $u = \sum_{i=1}^{m} \xi_i w_i \in W_m$. Therefore,

$$\begin{aligned} \langle F(\xi), \xi \rangle = &\int_D |\nabla u|^p dx + \int_D |u|^p dx - \lambda \int_D a(x)(u_+)^{q(r)} dx \\ &- \int_D f_n(r, u_+) u_+ dx - \frac{1}{n} \int_D \varphi u dx, \end{aligned} \tag{5.4}$$

where $u_+ = \max\{u, 0\}$, $u_- = u_+ - u$.

For a given $u \in W_m$ we define

$$D_n^+ = \{x \in D : |u(r)| \ge \frac{1}{n}\}, \quad D_n^- = \{x \in D : |u(r)| < \frac{1}{n}\}.$$

Thus, we can write (5.4) as

$$\langle F(\xi), \xi \rangle = \langle F(\xi), \xi \rangle_P + \langle F(\xi), \xi \rangle_N,$$

where

$$\begin{aligned} \langle F(\xi), \xi \rangle_P = &\int_{D_n^+} |\nabla u|^p dx + \int_{D_n^+} |u|^p dx - \lambda \int_{D_n^+} a(x)(u_+)^{q(r)} dx \\ &- \int_{D_n^+} f_n(r, u_+) u_+ dx - \frac{1}{n} \int_{D_n^+} \varphi u dx \end{aligned}$$

and

$$\begin{aligned} \langle F(\xi), \xi \rangle_N = &\int_{D_n^-} |\nabla u|^p dx + \int_{D_n^-} |u|^p dx - \lambda \int_{D_n^-} a(x)(u_+)^{q(r)} dx \\ &- \int_{D_n^-} f_n(r, u_+) u_+ dx - \frac{1}{n} \int_{D_n^-} \varphi u dx. \end{aligned}$$

Now we consider

$$|\xi|_m = \|u\|_{W_0^{1,p}(D)} = \vartheta \tag{5.5}$$

for some $0 < \vartheta \le 1$ to be chose later. **Step 1.** Since the embedding $W^{1,p}(\mathbb{R}^N) \subset$

$L^\tau(\mathbb{R}^N)$ is continuous for all $p \le \tau \le p^*$ (see [15, Corollary 9.11]), we have

$$\int_{D_n^+} |a(r)|(u_+)^{q(r)} dx \le C\|a\|_{L^{p/(p-q(x))}(\mathbb{R}^N)} \|\tilde{u}\|^{q_-}_{W^{1,p}(\mathbb{R}^N)} \le \frac{C_1}{2}\|\tilde{u}\|^{q_-}_{W^{1,p}(\mathbb{R}^N)}. \quad (5.6)$$

By Lemma 4.3 and Corollary 1.2, we obtain

$$\int_{D_n^+} f_n(r,u_+)u_+ dx \le K_1 \int_{D_n^+} |u_+|^{\theta(r)} dx \le C_2\|\tilde{u}\|^{p_-}_{W^{1,p}(\mathbb{R}^N)}. \quad (5.7)$$

Since $\varphi \in L^{p'}(\mathbb{R}^N)$, we have

$$\int_{D_n^+} \varphi u dx \le \|\varphi\|_{L^{p'}(\mathbb{R}^N)} \|\tilde{u}\|_{L^p(\mathbb{R}^N)} \le \frac{C_3}{2}\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^N)}. \quad (5.8)$$

It follows from (5.6), (5.7) and (5.8) that

$$\begin{aligned} \langle F(\xi),\xi\rangle_P \ge \int_{D_n^+} |\nabla u|^p dx + \int_{D_n^+} |u|^p dx - \lambda\frac{C_1}{2}\|\tilde{u}\|^{q_-}_{W^{1,p}(\mathbb{R}^N)} \\ - C_2\|\tilde{u}\|^{p_-}_{W^{1,p}(\mathbb{R}^N)} - \frac{C_3}{2n}\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^N)}. \end{aligned} \quad (5.9)$$

**Step 2.** In a similarly way, we obtain

$$\int_{D_n^-} |a(x)|(u_+)^q dx \le \frac{C_1}{2}\|\tilde{u}\|^{q_-}_{W^{1,p}(\mathbb{R}^N)}. \quad (5.10)$$

By Lemma 4.3, we obtain

$$\begin{aligned} \int_{D_n^-} f_n(r,u_+)u_+ dx &\le K_1\frac{1}{n^{p_- -1}} \int_{D_n^-} |u_+| dx \\ &\le K_1\frac{1}{n^{p_- -1}} \Big(\int_{D_n^-} dx\Big)^{\frac{1}{p'}} \Big(\int_{\mathbb{R}^N} |\tilde{u}|^p dx\Big)^{1/p} \\ &\le K_1|D|^{1/p'}\frac{1}{n^{p_- -1}}\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^N)}. \end{aligned} \quad (5.11)$$

Also we have

$$\int_{D_n^-} \varphi u dx \le \frac{C_3}{2}\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^N)}. \quad (5.12)$$

It follows from (5.10), (5.11) and (5.12) that

$$\begin{aligned} \langle F(\xi),\xi\rangle_N \ge \int_{D_n^-} |\nabla u|^p dx + \int_{D_n^-} |u|^p dx - \lambda\frac{C_1}{2}\|\tilde{u}\|^{q_-}_{W^{1,p}(\mathbb{R}^N)} \\ - K_1|D|^{1/p'}\frac{1}{n^{p_- -1}}\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^N)} - \frac{C_3}{2n}\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^N)}. \end{aligned} \quad (5.13)$$

Using that $\|u\|^p_{W^{1,p}(D)} = \|\tilde{u}\|^p_{W^{1,p}(\mathbb{R}^N)} = \|\nabla\tilde{u}\|^p_{L^N(\mathbb{R}^N)} + \|\tilde{u}\|^p_{L^N(\mathbb{R}^N)}$, inequalities (5.9) and (5.13) imply

$$\begin{aligned} \langle F(\xi),\xi\rangle \ge \|\tilde{u}\|^p_{W^{1,p}(\mathbb{R}^N)} - \lambda C_1\|\tilde{u}\|^{q_-}_{W^{1,p}(\mathbb{R}^N)} - C_2\|\tilde{u}\|^{p_-}_{W^{1,p}(\mathbb{R}^N)} \\ - \Big(K_1|D|^{1/p'}\frac{1}{n^{p_- -1}} + \frac{C_3}{n}\Big)\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^N)}. \end{aligned} \quad (5.14)$$

Since $|\xi|_m = \|\tilde{u}\|_{W^{1,p}(\mathbb{R}^N)} = \vartheta$, we have

$$\langle F(\xi),\xi\rangle \ge \vartheta^p - \lambda C_1\vartheta^{q_-} - C_2\vartheta^{p_-} - \Big(\frac{K_1|D|^{1/p'}}{n^{p_- -1}} + \frac{C_3}{n}\Big)\vartheta.$$

If $\vartheta$ is such that

$$\vartheta \leq \frac{1}{(2C_2)^{\frac{1}{p_- - p}}},$$

then $\vartheta^p - C_2\vartheta^{p_-} \geq \frac{\vartheta^p}{2}$. Thus, choosing

$$\vartheta := \min\Big\{\frac{1}{(2C_2)^{\frac{1}{p_- - p}}}, 1\Big\}, \tag{5.15}$$

we obtain

$$\langle F(\xi), \xi\rangle \geq \frac{\vartheta^p}{2} - \lambda C_1 \vartheta^{q_-} - \Big(\frac{K_1|D|^{1/p'}}{n^{p_- -1}} + \frac{C_3}{n}\Big)\vartheta.$$

We define $\varsigma := \frac{\vartheta^p}{2} - \lambda C_1\vartheta^{q_-}$. If we choose

$$\lambda^* := \frac{\vartheta^{p-q_-}}{4C_1} > 0,$$

then $\varsigma > \frac{\vartheta^p}{4}$ for all $0 < \lambda < \lambda^*$. We choose $n^* \in \mathbb{N}$ such that

$$\Big(\frac{K_1|D|^{1/p'}}{n^{p_- -1}} + \frac{C_3}{n}\Big)\vartheta < \frac{\varsigma}{2}$$

for every $n \geq n^*$. Notice that $n^*$ depends on the domain $D$. Since $\xi \in \mathbb{R}^m$ is such that $|\xi|_m = \vartheta$, then for $\lambda < \lambda^*$ and $n \geq n^*$ we obtain

$$\langle F(\xi), \xi\rangle \geq \frac{\varsigma}{2} > 0. \tag{5.16}$$

Since $f_n$ is a Lipschitz function (for each $n \in \mathbb{N}$), it easy to see that $F : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function. Thus, for each $\lambda < \lambda^*$ and $n > n^*$ fixed, Lemma 2.1 ensure the existence of $y \in \mathbb{R}^m$ with $|y|_m \leq \vartheta$ and such that $F(y) = 0$. In other words, there exists $u_m \in W_m$ satisfying

$$\|u_m\|_{W^{1,p}(D)} \leq \vartheta, \tag{5.17}$$

and such that

$$\begin{aligned} &\int_D |\nabla u_m|^{p-2}\nabla u_m \nabla w dx + \int_D |u_m|^{p-2}u_m w\, dx \\ &= \lambda \int_D a(x)(u_{m+})^{q(x)-1} w dx + \int_D f_n(r, u_{m+}) w dx + \frac{1}{n}\int_D \varphi w\, dx, \end{aligned} \tag{5.18}$$

for all $w \in W_m$.

**Remark 5.2.** It is important to mention that $\vartheta$, given in (5.15), does not depend on the domain $D$, $m$ nor $n$. For this matter, Corollary 1.2 plays an important role.

Since $W_m \subset W^{1,p}_{0,r}(D)$ for all $m \in \mathbb{N}$, and $\vartheta$ does not depend on $m$, then $(u_m)_{m\in\mathbb{N}}$ is a bounded sequence in $W^{1,p}_0(D)$. Therefore, for some subsequence, there exists $u \in W^{1,p}_0(D)$ such that

$$u_m \rightharpoonup u \quad \text{weakly in } W^{1,p}_{0,r}(D), \tag{5.19}$$

$$u_m \to u \quad \text{in } L^s(D),\ p \leq s < p^*, \tag{5.20}$$

$$u_m \to u, \quad \text{a.e. in }, D. \tag{5.21}$$

Thus,

$$\|u\|_{W^{1,p}(D)} \leq \liminf_{m\to\infty} \|u_m\|_{W^{1,p}(D)} \leq \vartheta. \tag{5.22}$$

Now we claim that

$$u_m \to u \quad \text{in } W^{1,p}_{0,r}(D). \tag{5.23}$$

Indeed, using that $\mathcal{B} = \{w_1, w_2, \dots, w_n, \dots\}$ is a Schauder basis of $W^{1,p}_{0,r}(D)$, for every $u \in W^{1,p}_{0,r}(D)$ there exists a unique sequence $(\alpha_n)_{n\geq 1}$ in $\mathbb{R}$ such that $u = \sum_{j=1}^{\infty} \alpha_j w_j$, so that

$$\psi_m := \sum_{j=1}^{m} \alpha_j w_j \to u \quad \text{in } W^{1,p}_{0,r}(D) \text{ as } m \to \infty. \tag{5.24}$$

Using $w = (u_m - \psi_m) \in W_m$ as test function in (5.18), we obtain

$$\begin{aligned} &\int_D |\nabla u_m|^{p-2} \nabla u_m \nabla (u_m - \psi_m) dx + \int_D |u_m|^{p-2} u_m (u_m - \psi_m) dx \\ &= \lambda \int_D a(x)(u_{m+})^{q(x)-1}(u_m - \psi_m) dx + \int_D f_n(r, u_{m+})(u_m - \psi_m) dx \\ &\quad + \frac{1}{n} \int_D \varphi (u_m - \psi_m) dx. \end{aligned} \tag{5.25}$$

It is easy to see that

$$\begin{aligned} &\int_D \Big( |u_m|^{p-1} + |\lambda a(x)(u_{m+})^{q(x)-1}| + \frac{1}{n}|\varphi| \Big) |u_m - \psi_m| dx \\ &\leq \Big( \|u_n\|^{p-1}_{L^p(D)} + \lambda \|a\|_{L^{p/(p-q(x))}(D)} \|u_n\|^{q_- -1}_{L^p(D)} + \frac{1}{n} \|\varphi\|_{L^{p'}(D)} \Big) \|u_m - \psi_m\|_{L^p(D)}. \end{aligned} \tag{5.26}$$

By Lemma 4.3, one has

$$\int_D [f_n(r, u_{m+})]^{\frac{\theta(r)}{\theta(r)-1}} dx = \int_{D_n^+} [f_n(r, u_{m+})]^{\frac{\theta(r)}{\theta(r)-1}} dx + \int_{D_n^-} [f_n(r, u_{m+})]^{\frac{\theta(r)}{\theta(r)-1}} dx,$$

$$\begin{aligned} \int_{D_n^+} [f_n(r, u_{m+})]^{\frac{\theta(r)}{\theta(r)-1}} dx &\leq (\max\{K_1, 1\})^{p^*} \int_{D_n^+} |u_{m+}|^{\theta(r)} dx \\ &\leq (\max\{K_1, 1\})^{p^*} \int_D |u_{m+}|^{\theta(r)} dx \\ &\leq K_2 \|u_m\|_{W^{1,p}(D)} \end{aligned}$$

$$\begin{aligned} \int_{D_n^-} [f_n(r, u_{m+})]^{\frac{\theta(r)}{\theta(r)-1}} dx &\leq (\max\{K_1, 1\})^{p^*} \frac{1}{n^{p^*-2}} \int_{D_n^-} |u_{m+}|^{\theta(r)} dx \\ &\leq (\max\{K_1, 1\})^{p^*} |D|. \end{aligned}$$

Since $\|u_m\|_{W^{1,p}(D)} \leq \vartheta$, by the estimates above, we obtain

$$\int_D [f_n(r, u_{m+})]^{\frac{\theta(r)}{\theta(r)-1}} dx \leq C, \tag{5.27}$$

where $C$ does not depend on $m$. Hence $f_n(r, u_{m+})$ is bounded in $L^{\frac{\theta(r)}{\theta(r)-1}}(D)$. Applying Corollary 1.2, (5.24) and (5.27) we conclude that

$$\lim_{m\to\infty} \int_D f_n(r, u_{m+})(u_m - \psi_m) dx = 0. \tag{5.28}$$

Notice that

$$\begin{aligned}\int_D f_n(r,u_{m+})(u_m-\psi_m)dx &\le \tilde{C}\|u_m-\psi_m\|_{\theta(r)}\\ &\le \tilde{\tilde{C}}\|u_m-\psi_m\|_{W^{1,p}(D)} \to 0 \quad \text{as } m\to\infty.\end{aligned} \tag{5.29}$$

By (5.17) and (5.24), we obtain

$$\lim_{m\to\infty}\int_D |\nabla u_m|^{p-2}\nabla u_m \nabla(u-\psi_m)dx = 0. \tag{5.30}$$

By (5.25), (5.26), (5.28) and (5.30), we obtain

$$\lim_{m\to\infty}\int_D |\nabla u_m|^{p-2}\nabla u_m \nabla(u_m-u)dx = 0. \tag{5.31}$$

It is sufficient to apply the $(S_+)-$ property of $-\Delta_p$ (see, e.g., [38, Proposition 3.5.]) to obtain (5.23).

For every $m\ge k$ we obtain

$$\begin{aligned}&\int_D |\nabla u_m|^{p-2}\nabla u_m\nabla w_k dx + \int_D |u_m|^{p-2}u_m w_k dx\\ &= \lambda\int_D a(x)(u_{m+})^{q(x)-1}w_k dx + \int_D f_n(r,u_{m+})w_k dx + \frac{1}{n}\int_D \varphi w_k dx,\end{aligned} \tag{5.32}$$

for all $w_k\in W_k$.

It follows from (5.23) that

$$\begin{aligned}&\int_D |\nabla u|^{p-2}\nabla u\nabla w_k dx + \int_D |u|^{p-2}u w_k dx\\ &= \lambda\int_D a(x)(u_+)^{q(x)-1}w_k dx + \int_D f_n(r,u_+)w_k dx + \frac{1}{n}\int_D \varphi w_k dx,\end{aligned} \tag{5.33}$$

for all $w_k\in W_k$. Since $[W_k]_{k\in\mathbb{N}}$ is dense in $W^{1,p}_{0,r}(D)$ we conclude that

$$\begin{aligned}&\int_D |\nabla u|^{p-2}\nabla u\nabla w dx + \int_D |u|^{p-2}u w\, dx\\ &= \lambda\int_D a(x)(u_+)^{q(x)-1}w dx + \int_D f_n(r,u_+)w\, dx + \frac{1}{n}\int_D \varphi w\, dx,\end{aligned} \tag{5.34}$$

for all $w\in W^{1,p}_{0,r}(D)$.

Before concluding, we will check that $u$ satisfy (5.34) for all $w\in W^{1,p}_0(D)$. Indeed, we will use a symmetric critical principle of Palais [40] in Banach spaces developed in [22].

Let $O(N)$ be the subgroup of isometries $g: W^{1,p}_0(D)\to W^{1,p}_0(D)$ corresponding to all rotations, that is, $O(N)$ is the orthogonal group of dimension $N$. The subspace of $W^{1,p}_0(D)$ consisting of radially symmetric functions, $O(N)$-invariant, is given by $W^{1,p}_{0,r}(D)=\{u\in W^{1,p}_0(D): g(u)=u, \text{ for all } g\in O(N)\}$, see (1.10).

Let $u\in W^{1,p}_{0,r}(D)$ satisfying (5.34). Define $\Phi(u)\in W^{1,p}_0(D)^*$ (the dual space) by

$$\begin{aligned}(\Phi(u),w) &= \int_D |\nabla u|^{p-2}\nabla u\nabla w dx + \int_D |u|^{p-2}u w\, dx\\ &\quad - \lambda\int_D a(x)(u_+)^{q(x)-1}w dx - \int_D f_n(r,u_+)w dx - \frac{1}{n}\int_D \varphi w\, dx.\end{aligned} \tag{5.35}$$

By (5.34), we have that

$$(\Phi(u), w) = 0, \quad w \in W^{1,p}_{0,r}(D),$$

and $\Phi(u)$ is invariant under the action of $O(N)$. By [22], we can infer that $\Phi(u) \equiv 0$. In other words, $u$ satisfy (5.34) for all $w \in W^{1,p}_0(D)$, i.e.

$$\begin{aligned} &\int_D |\nabla u|^{p-2}\nabla u \nabla w dx + \int_D |u|^{p-2}uw\, dx \\ &= \lambda \int_D a(x)(u_+)^{q(x)-1} w dx + \int_D f_n(r, u_+) w dx + \frac{1}{n}\int_D \varphi w\, dx, \end{aligned} \tag{5.36}$$

for all $w \in W^{1,p}_0(D)$.

Furthermore, $u \geq 0$ in $D$. In fact, since $u_- \in W^{1,p}_0(D)$ then from (5.36) we obtain

$$\begin{aligned} -\|u_-\|^p_{W^{1,p}(D)} &= \int_D |\nabla u|^{p-2}\nabla u \nabla u_- dx + \int_D |u|^{p-2}uu_- dx \\ &= \lambda \int_D a(x)(u_+)^{q(x)-1} u_- dx + \int_D f_n(r, u_+)u_- dx + \frac{1}{n}\int_D \varphi u_- dx \geq 0. \end{aligned}$$

Then $u_- \equiv 0$ a.e. in $D$, whence $u \geq 0$ a.e. in $D$. Moreover, $u \not\equiv 0$ is valid due to $\frac{\varphi}{n} > 0$ in $D$. Applying the strong maximum principle [43] we obtain $u > 0$ in $D$ and $\partial u/\partial\nu < 0$ on $\partial D$. By Lemma 4.1 and [34, Theorem 7.1] we conclude that $u \in L^\infty(D)$. Thus, [35, Theorem 1] ensure the regularity up to the boundary $u \in C^{1,\beta}(\overline{D})$, for some $\beta \in (0, 1)$. Therefore, we conclude that proof of the lemma taking $u_n = u$. □

## 6. Proof of Theorem 1.3

For the proof of Theorem 1.3 we need the following result.

**Lemma 6.1.** *For each constant $b > 0$, the problem*

$$\begin{gathered} -\Delta_p u + u^{p-1} = b u^{q(r)-1} \quad \text{in } D \\ u > 0 \quad \text{in } D \\ u = 0 \quad \text{on } \partial D, \end{gathered} \tag{6.1}$$

*where $D$ is a bounded domain in $\mathbb{R}^N$ with $C^2$ boundary $\partial D$, admits a solution $u_0 \in C^{1,\beta}(\overline{D})$ such that $\partial u_0/\partial\nu < 0$ on $\partial D$.*

*Proof.* This result is more or less standard, but we will sketch the proof. Given a constant $b > 0$, we define the functional $I : W^{1,p}_0(D) \to \mathbb{R}$ by

$$I(u) = \frac{1}{p}\int_D |\nabla u|^p dx + \frac{1}{p}\int_D |u|^p dx - b\int_D \frac{1}{q(r)}(u^+)^{q(r)} dx \quad \text{for all } u \in W^{1,p}_0(D),$$

where $u^+ = \max\{0, u\}$. Notice that $I$ is of class $C^1$. Using the Sobolev embedding, Proposition 2.4 and Proposition 2.6, we have the estimate

$$I(u) \geq \frac{1}{p}\|u\|^p_{W^{1,p}(D)} - c(\|u\|^{q_+}_{W^{1,p}(D)} + \|u\|^{q_-}_{W^{1,p}(D)}) \quad \text{for all } u \in W^{1,p}_0(D),$$

with a constant $c > 0$. Since $p > q_+ \geq q_- > 1$, $I$ is bounded from below and coercive. Considering that the first two terms in the expression of $I$ are convex and continuous on $W^{1,p}_0(D)$ and the embedding of $W^{1,p}_0(D)$ into $L^{q(r)}(D)$ is compact, we infer that $I$ is sequentially weakly lower semi-continuous. Therefore, there exists $u_0 \in W_0^{1,p}(D)$ such that

$$I(u_0) = \inf_{u \in W_0^{1,p}(D)} I(u)$$

(see, e.g., [37, Theorems 1.1, 1.2]). Hence $u_0$ is a critical point of $I$ that reads as

$$\int_D |\nabla u_0|^{p-2}\nabla u_0 \nabla v dx + \int_D |u_0|^{p-2}u_0 v dx = b\int_D (u_0^+)^{q(r)-1} v dx \tag{6.2}$$

for all $v \in W_0^{1,p}(D)$. Since the variable exponent $q(r)$ is subcritical, we obtain with standard bootstrap arguments that $u_0 \in C^{1,\beta}(\overline{D})$.

It remains to justify that $u_0 > 0$. Inserting $v = -u_0^- = -\max\{0, -u_0\}$ in (6.2) leads to $u_0^- = 0$, so $u_0 \geq 0$ in $D$. We observe that the condition $1 < q_- \leq q_+ < p$ ensures that $I(tu) < 0$ provided $u \neq 0$ and $t > 0$ is sufficiently small, which implies that $u_0 \neq 0$. Finally, we can verify that the strong maximum principle applies in the case of equation (6.2). We conclude that $u_0 > 0$ in $D$, so $u_0$ is a (weak) solution of problem (6.1). Applying the Hopf boundary point lemma we obtain $\partial u_0/\partial \nu < 0$ on $\partial D$. □

*Proof of Theorem 1.3.* First we show that (1.13) has a positive solution. For each $n \in \mathbb{N}$ we know, by Lemma 5.1, that equation (5.1) has a (weak) solution $u_n \in W_{0,r}^{1,p}(D) \cap C^{1,\beta}(\overline{D})$, thus

$$\begin{aligned}
&\int_D |\nabla u_n|^{p-2}\nabla u_n \nabla w dx + \int_D |u_n|^{p-2}u_n w\, dx \\
&= \lambda \int_D a(x)(u_n)^{q(x)-1} w dx + \int_D f_n(r, u_n) w dx + \frac{1}{n}\int_D \varphi w\, dx,
\end{aligned} \tag{6.3}$$

for all $w \in W_0^{1,p}(D)$.

By (5.22) we have

$$\|u_n\|_{W^{1,p}(D)} \leq \vartheta \leq 1, \quad \forall n \in \mathbb{N}, \tag{6.4}$$

and $\vartheta$ does not depend on $n$ (indeed, see Remark 5.2). Thus, for a subsequence again relabeled as $(u_n)$, there exists $u \in W_0^{1,p}(D)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(D) \text{ as } n \to \infty. \tag{6.5}$$

Since $u_n \to u$ a.e. in $D$, by the uniform convergence of Lemma 4.1 (ii) we have

$$f_n(\cdot, u_n(\cdot)) \to f(\cdot, u(\cdot)) \quad \text{a.e. in } D. \tag{6.6}$$

By Lemma 4.3, one has

$$\int_D [f_n(r, u_n)]^{\frac{\theta(r)}{\theta(r)-1}} dx = \int_{D_n^+} [f_n(r, u_n)]^{\frac{\theta(r)}{\theta(r)-1}} dx + \int_{D_n^-} [f_n(r, u_n)]^{\frac{\theta(r)}{\theta(r)-1}} dx,$$

$$\begin{aligned}
\int_{D_n^+} [f_n(r, u_n)]^{\frac{\theta(r)}{\theta(r)-1}} dx &\leq (\max\{K_1, 1\})^{p^*} \int_{D_n^+} |u_n|^{\theta(r)} dx \\
&\leq (\max\{K_1, 1\})^{p^*} \int_D |u_n|^{\theta(r)} dx \\
&\leq K_2 \|u_n\|_{W^{1,p}(D)}^{p^*},
\end{aligned}$$

$$\begin{aligned}
\int_{D_n^-} [f_n(r, u_n)]^{\frac{\theta(r)}{\theta(r)-1}} dx &\leq (\max\{K_1, 1\})^{p^*} \frac{1}{n^{p^*-2}} \int_{D_n^-} |u_n|^{\theta(r)} dx \\
&\leq (\max\{K_1, 1\})^{p^*} |D|.
\end{aligned}$$

Since $\|u_n\|_{W^{1,p}(D)} \leq \vartheta$, by the estimates before, we obtain

$$\int_D [f_n(r,u_n)]^{\frac{\theta(r)}{\theta(r)-1}} dx \leq C,$$

where $C$ does not depend on $n$. Hence $f_n(r,u_n)$ is bounded in $L^{\frac{\theta(r)}{\theta(r)-1}}(D)$ and by similar arguments like [30, Theorem 13.44] leads to

$$f_n(r,u_n) \rightharpoonup f(r,u) \quad \text{weakly in } L^{\frac{\theta(r)}{\theta(r)-1}}(D). \tag{6.7}$$

By (6.5), (6.7) and Proposition 2.6, we can pass to the limit in (6.3) to obtain

$$\int_D |\nabla u|^{p-2}\nabla u \nabla w dx + \int_D |u_n|^{p-2} uw\, dx = \lambda \int_D a(x)(u)^{q(x)-1} w dx + \int_D f(r,u) w dx, \tag{6.8}$$

for all $w \in W_0^{1,p}(D)$. Thus, $u$ is a solution of (1.13).

We need to prove that the limit function $u$ does not vanish. For this matter, fix a positive constant $\lambda \in (0,\lambda^*)$ such that

$$\lambda^* = \frac{\vartheta^{2-q_-}}{4C_1} \tag{6.9}$$

was given in Lemma 5.1. Since $a > 0$ is a continuous function, define

$$a_\tau = \inf_D a(r).$$

Then, according to Lemma 6.1, there exists a positive solution $u_{\lambda,R}$ of

$$\begin{gathered} -\Delta_p u + u^{p-1} = \lambda a_\tau u^{q(r)-1} \quad \text{in } D \\ u > 0 \quad \text{in } D \\ u = 0 \quad \text{on } \partial D. \end{gathered}$$

Let $u_n$ be a positive solution of problem (5.1) obtained by Lemma 5.1. We observe that $u_\lambda/u_n, u_n/u_\lambda \in L^\infty(D)$ because $u_{\lambda,R}$ and $u_n$ are positive functions belonging to $C_0^{1,\beta}(\overline{D})$ and satisfying $\partial u_n/\partial\nu < 0$, $\partial u_{\lambda,R}/\partial\nu < 0$ on $\partial D$. Hence we are able to apply Proposition 2.3 with $u_1 = u_{\lambda,R}$, $u_2 = u_n$, $g(r,t) = \lambda a_\tau t^{q(r)}$. Notice that

$$\lambda t^{q(r)} + f_n(r,t) + \frac{\phi}{n} \geq \lambda a_\tau t^{q(r)} = g(r,t).$$

Hence, $u_1$ and $u_2$ are a positive subsolution and a positive supersolution of problem (6.1), respectively. In this way, by Proposition 2.3 we see that $u_n \geq u_{\lambda,R} > 0$ in $D$ for every $n \geq 1$. Therefore, in the limit as $n \to \infty$ we obtain that $u \geq u_{\lambda,R}$ a.e. in $D$. Thus, by letting to the limit, we conclude that $u$ is a positive solution of problem (1.13).

The solution we just found is being written as $u_\lambda$ with explicit dependence on $\lambda$. We will deduce that $\|u_\lambda\|_{W^{1,p}(D)} \to 0$ as $\lambda \to 0$. Fix the pair $(\lambda,u_\lambda)$, where $\lambda \in (0,\lambda^*)$ and $u_\lambda$ is the corresponding solution of problem (1.13),given by the previous steps. Using $w = u_\lambda$ as a test function in (6.8) and recalling (2.1), we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\nabla \tilde{u}_\lambda|^p dx + \int_{\mathbb{R}^N} \tilde{u}_\lambda^p dx &= \int_D |\nabla u_\lambda|^p dx + \int_D u_\lambda^p dx \\
&= \lambda \int_D a(r) u_\lambda^{q(r)} dx + \int_D f(r, u_\lambda) u_\lambda dx \\
&= \lambda \int_{\mathbb{R}^N} a(r) \tilde{u}_\lambda^{q(r)} dx + \int_{\mathbb{R}^N} f(r, \tilde{u}_\lambda) \tilde{u}_\lambda dx \\
&\le \lambda C_1 \|\tilde{u}_\lambda\|_{W^{1,p}(\mathbb{R}^N)}^{q_-} + C_2 \|\tilde{u}_\lambda\|_{W^{1,p}(\mathbb{R}^N)}^{p_-},
\end{aligned} \tag{6.10}
$$

where $C_1$, $C_2$ are given in (5.6), (5.7), respectively. Since $\tilde{u}_\lambda \neq 0$, from (6.10), we have the estimate

$$
\|\tilde{u}_\lambda\|_{W^{1,p}(\mathbb{R}^N)}^{p-q_-} (1 - C_2 \|\tilde{u}_\lambda\|_{W^{1,p}(\mathbb{R}^N)}^{p_- -2}) \le \lambda C_1. \tag{6.11}
$$

Combining (5.22), (6.4), we obtain

$$
\|\tilde{u}_\lambda\|_{W^{1,p}(\mathbb{R}^N)}^{p_- -2} \le \frac{1}{2C_2}.
$$

Thus,

$$
\|u_\lambda\|_{W^{1,p}(D)} = \|\tilde{u}_\lambda\|_{W^{1,p}(\mathbb{R}^N)} \le (2\lambda C_1)^{1/(p-q_-)}. \tag{6.12}
$$

We conclude that $\|u_\lambda\|_{W^{1,p}(\mathbb{R}^N)} \to 0$ as $\lambda \to 0$. The proof of the theorem is complete. □

## 7. Proof of Theorem 1.4

Let $R > 0$. In what follows, we denote $B_n = B_n(0)$ the open ball centered at the origin and of radius $n \in \mathbb{N}$, for some $n > R$. The space $W^{1,p}(B_n)$ is equipped with the norm

$$
\|u\|_{2,n} = \Big( \int_{B_n} \big( |\nabla u|^p + |u|^p \big) dx \Big)^{1/p}.
$$

*Proof of Theorem 1.4.* Applying Theorem 1.3 with $D = B_n \backslash \overline{B_R}$ (and $n > R$), we obtain a positive solution $u_n \in W_0^{1,p}(B_n) \cap C^{1,\beta}(\overline{B_n})$ of the problem

$$
\begin{gathered}
-\Delta_p u + u^{p-1} = \lambda a(r) u^{q(r)-1} + f(r, u) \quad \text{in } B_n \\
u > 0 \quad \text{in } B_n \\
u = 0 \quad \text{on } \partial B_n,
\end{gathered} \tag{7.1}
$$

Again, (5.22) and (6.12) show the uniform boundedness of the sequence $(u_n)_{n>R}$ in $W_0^{1,p}(B_n)$, that is, defining

$$
\tilde{\vartheta} = \min \big\{ \vartheta, (2\lambda C_1)^{1/(p-q_-)} \big\}
$$

we obtain

$$
\|u_n\|_{2,n} \le \tilde{\vartheta} \quad \text{for all } n \in \mathbb{N}. \tag{7.2}
$$

Fix $m \in \mathbb{N}$. If $n \ge m > R$, by (7.2) we have

$$
\|u_n\|_{2,m} \le \|u_n\|_{2,n} \le \tilde{\vartheta}. \tag{7.3}
$$

Therefore, for a subsequence if necessary, there exists $u \in W^{1,p}(B_m)$ such that

$$
\begin{gathered}
u_n \rightharpoonup u \quad \text{in } W^{1,p}(B_m), \\
u_n \to u \quad \text{for a.e. } x \in B_m, \\
u_n \to u \quad \text{in } L^p(B_m),
\end{gathered}
$$

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } L^{\theta(r)/(\theta(r)-1)}(B_m), \\ u_n \rightharpoonup u \quad \text{in } L^{q(r)-1}(B_m), \text{ as } n \to \infty. \end{gathered}$$

Recalling that $u_n > 0$ in $B_m$, by the above convergences, we infer that $u$ is a nonnegative solution of the problem

$$-\Delta_p u + u^{p-1} = \lambda a(r) u^{q(r)-1} + f(r,u) \quad \text{in } B_m, \quad u \geq 0 texton \partial B_m.$$

By a diagonal argument we obtain a subsequence of $(\tilde{u}_n)$ and a function $u \in W^{1,p}(\mathbb{R}^N)$ such that

$$\begin{gathered} \tilde{u}_n \rightharpoonup u \quad \text{in } W^{1,p}(\mathbb{R}^N), \\ \tilde{u}_n \to u \quad \text{for a.e. } x \in \mathbb{R}^N, \\ \tilde{u}_n \rightharpoonup u \quad \text{in } L^{\theta(r)/(\theta(r)-1)}(\mathbb{R}^N), \\ \tilde{u}_n \rightharpoonup u \quad \text{in } L^{q(r)-1}(\mathbb{R}^N) \text{ as } n \to \infty. \end{gathered} \tag{7.4}$$

Indeed, fix $\varphi \in C_0^\infty(\mathbb{R}^N)$, let $m \in \mathbb{N}$ such that $\operatorname{supp}(\varphi) \subset B_m$. For $n$ large enough, we have

$$\begin{gathered} \int_{\mathbb{R}^N} |\nabla \tilde{u}_n|^{p-2} \nabla \tilde{u}_n \nabla \varphi dx = \int_{B_m} |\nabla \tilde{u}_n|^{p-2} \nabla \tilde{u}_n \nabla \varphi dx \\ \to \int_{B_m} |\nabla u|^{p-2} \nabla u \nabla \varphi dx = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi dx, \\ \int_{\mathbb{R}^N} |\tilde{u}_n|^{p-2} \tilde{u}_n \varphi dx = \int_{B_m} |\tilde{u}_n|^{p-2} \tilde{u}_n \varphi dx \\ \to \int_{B_m} |u|^{p-2} u \varphi dx = \int_{\mathbb{R}^N} |u|^{p-2} u \varphi dx, \\ \int_{\mathbb{R}^N} a(x) (\tilde{u}_n)^{q(x)-1} \varphi dx = \int_{B_m} a(x) (\tilde{u}_n)^{q(x)-1} \varphi dx \\ \to \int_{B_m} a(x) u^{q(x)-1} \varphi dx = \int_{\mathbb{R}^N} a(x) u^{q(x)-1} \varphi dx, \\ \int_{\mathbb{R}^N} f(x, \tilde{u}_n) \varphi dx = \int_{B_m} f(x, \tilde{u}_n) \varphi dx \\ \to \int_{B_m} f(x,u) \varphi dx = \int_{\mathbb{R}^N} f(x,u) \varphi dx. \end{gathered}$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, these convergence properties ensure that $u$ is a weak solution of problem (1.14).

The next step is to show that the limit function $u$ does not vanish in $\mathbb{R}^N$. Fix $\lambda \in (0, \lambda^*)$, with $\lambda^*$ satisfying (6.9). Lemma 6.1 provides a solution $u_{\lambda,m}$ of the problem

$$\begin{gathered} -\Delta_p u + u^{p-1} = \lambda a_\tau u^{q(r)-1} \quad \text{in } B_m \\ u > 0 \quad \text{in } B_m \\ u = 0 \quad \text{on } \partial B_m. \end{gathered}$$

Since $\lambda a(r) t^{q(r)-1} + f(r,t) \geq \lambda t^{q(r)-1}$ for all $x \in \mathbb{R}^N$ and $t > 0$, we can apply Proposition 2.2 to the functions $u_{\lambda,m}$ and $\tilde{u}_n$ with $n \geq m$, in place of $u_1 = u_{\lambda,m}$

and $u_2 = \tilde{u}_n$, respectively, which renders $\tilde{u}_n \geq u_{\lambda,m}$ in $B_m$ for every $n \geq m$. This enables us to deduce that $u_\lambda := u \geq u_{\lambda,m}$ in $B_m$, so

$$u_\lambda(x) > 0 \quad \text{a.e. } x \in \mathbb{R}^N,$$

since $m$ was arbitrarily chosen.

Furthermore, since $\tilde{u}_n$ converges weakly to $u$ in $W^{1,p}(\mathbb{R}^N)$, we obtain

$$\|\nabla \tilde{u}_n\|_{L^p(\mathbb{R}^N)} + \|\tilde{u}_n\|_{L^p(\mathbb{R}^N)} = \|\tilde{u}_n\|_{2,n} \leq \tilde{\vartheta}.$$

By means of (7.2), according to the iteration process, we can check that $u_\lambda \in W^{1,p}(\mathbb{R}^N)$, and $\|u_\lambda\|_{W^{1,p}(\mathbb{R}^N)} \to 0$ as $\lambda \to 0$, see also (6.10), (6.11), (6.12). $\square$

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