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RADIAL BOUNDED SOLUTIONS FOR MODIFIED SCHRÖDINGER EQUATIONS

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Abstract. We study the quasilinear elliptic equation

$$-\operatorname{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) + |u|^{p-2}u = g(x,u) \quad \text{in } \mathbb{R}^N,$$

with $N \geq 2$ and $p > 1$. Here, $A : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a given C^1-Carathéodory function that grows as $|\xi|^p$ with $A_t(x,t,\xi) = \frac{\partial A}{\partial t}(x,t,\xi)$, $a(x,t,\xi) = \nabla_\xi A(x,t,\xi)$ and $g(x,t)$ is a given Carathéodory function on $\mathbb{R}^N \times \mathbb{R}$ which grows as $|\xi|^q$ with $1 < q < p$.

Suitable assumptions on $A(x,t,\xi)$ and $g(x,t)$ set off the variational structure of above problem and its related functional $\mathcal{J}$ is C^1 on the Banach space $X = W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. To overcome the lack of compactness, we assume that the problem has radial symmetry, then we look for critical points of $\mathcal{J}$ restricted to X_r, subspace of the radial functions in X.

Following an approach that exploits the interaction between the intersection norm in X and the norm in $W^{1,p}(\mathbb{R}^N)$, we prove the existence of at least two weak bounded radial solutions, one positive and one negative. For this, we apply a generalized version of the Minimum Principle.

1. Introduction

In this article we look for weak radial bounded solutions for the quasilinear elliptic equation

$$-\operatorname{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) + |u|^{p-2}u = g(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $p > 1$ and $N \geq 2$, $A : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a C^1-Carathéodory function with partial derivatives

$$A_t(x,t,\xi) = \frac{\partial A}{\partial t}(x,t,\xi), \quad a(x,t,\xi) = \Big(\frac{\partial A}{\partial \xi_1}(x,t,\xi), \dots, \frac{\partial A}{\partial \xi_N}(x,t,\xi)\Big)$$

and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a suitable Carathéodory function.

Equation (1.1) generalizes quasilinear equations describing several physical phenomena such as the self-channeling of a high-power ultra short laser, or also some problems which arise in plasma physics, fluid mechanics, mechanics and in the condensed matter theory (see [35] and references therein or also [16] for some model problems).

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If $A(x,t,\xi) = \bar{A}|\xi|^p$ with $\bar{A}$ real constant, (1.1) turns out to be the p-Laplacian equation

$$-\Delta_p u + |u|^{p-2}u = g(x,u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

In the case $p = 2$, equation (1.2) reduces to the following Schrödinger equation

$$-\Delta u + u = g(x,u) \quad \text{in } \mathbb{R}^N$$

which is a central topic in Nonlinear Analysis, see [4, 6, 19, 20, 23, 36, 37]. Many authors studied also (1.2) in the general case $p > 1$, see [3, 5, 27, 30].

We note that (1.2) has a variational structure, but there is a lack of compactness as the problem is settled in the whole Euclidean space $\mathbb{R}^N$ and classical variational tools do not work; thus suitable assumptions on the involved functions are required.

On the other hand, even if the function $A(x,t,\xi)$ has the form $\frac{1}{p}A_1(x,t)|\xi|^p$ but the coefficient $A_1(x,t)$ is not constant, besides the lack of compactness the study of equation (1.1) presents another difficulty: the loss of a direct variational formulation in the space $W^{1,p}(\mathbb{R}^N)$. Let us point out that this problem arises also if we look for solutions verifying homogeneous Dirichlet conditions in a bounded domain Ω. Indeed, the natural action functional

$$J_1(u) = \frac{1}{p}\int_\Omega A_1(x,u)|\nabla u|^p\,dx + \frac{1}{p}\int_\Omega |u|^p\,dx - \int_\Omega G(x,u)\,dx,$$

is not well defined in $W_0^{1,p}(\Omega)$ if $A_1(x,t)$ is unbounded with respect to t. Moreover, even if $A_1(x,t)$ is strictly positive and bounded with respect to t but $\frac{\partial A_1}{\partial t}(x,t) \neq 0$, then J_1 is defined in $W_0^{1,p}(\Omega)$ but it is Gâteaux differentiable only along directions of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Thus, many authors have studied (1.1) by using non-smooth techniques or introducing a suitable change of variable if the term $A(x,t,\xi)$ has a very particular form or giving a "good" definition of critical point either on bounded domains or in unbounded ones, see [1, 2, 7, 8, 17, 18, 21, 22, 28, 29, 35].

More recently, Candela and Palmieri in [10]-[12] considered the functional

$$\mathcal{J}(u) = \int_\Omega A(x,u,\nabla u)\,dx + \frac{1}{p}\int_\Omega |u|^p\,dx - \int_\Omega G(x,u)\,dx,$$

defined on the Banach space $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ equipped with the intersection norm.

Introducing a new weak Cerami-Palais-Smale condition (see Definition 2.1) they state some abstract critical points Theorems. Using this variational approach, the existence of at least one bounded solution of (1.1) in the case $A(x,t,\xi) = \frac{1}{p}A_1(x,t)|\xi|^p$ has been stated when $g(x,t)$ grows as $|t|^q$ with $q > p$ but subcritical and the involved functions are radially symmetric in [14] or the term $|u|^{p-2}u$ is multiplied by a weight $V(x)$ verifying suitable assumptions in [15] (see also [31] and [38] where a generalized (p,q)-Laplacian operator in $\mathbb{R}^N$ is studied).

Always in the presence of a suitable weight $V(x)$, the existence of solutions of equation like to (1.1) has been investigated in [33] (see also [32]) if $A(x,t,\xi)$ is a more general function which grows as $|\xi|^p$ and $g(x,t)$ has a sub-p-linear growth of the type

$$|g(x,t)| \le \eta(x)|t|^{q-1}$$

with η suitable measurable function and $1 < q < p$.

We notice that the results stated in [32, 33] do not cover the case $V(x) = 1$, so they do not apply to the equation (1.1). Therefore, in this paper we want to look for solutions of (1.1) when $A(x,t,\xi)$ and $g(x,t)$, in addition to hypotheses similar to those ones required in [33], are radially symmetric in x. To this aim, in Lemma 4.11 we will state a convergence results in $\mathbb{R}^N$ already proved in bounded domains by Boccardo, Murat and Puel in [7, Lemma 5] (see also [31, Lemma 4.5]).

This article is organized as follows. In Section 2 we introduce a weak Cerami-Palais-Smale condition and the related Minimum Principle (see Proposition 2.2). In Section 3 we give some preliminary assumptions on the functions $A(x,t,\xi)$ and $g(x,t)$ that ensure a variational formulation for the equation (1.1). In Section 4 we consider some further assumptions, then we state our main results (see Theorem 4.5) and we prove some properties of the action functional $\mathcal{J}$ and a convergence result *à la* Boccardo-Murat-Puel in $\mathbb{R}^N$. Finally in Section 5 we prove that $\mathcal{J}$ verifies the weak Cerami-Palais-Smale condition in the subspace X_r of the radial functions of $X = W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and then we state the existence of two nontrivial weak radial bounded solutions, one negative and one positive, thus concluding the proof of Theorem 4.5.

2. Abstract tools

In this section we denote by $(X, \|\cdot\|_X)$ a Banach space with dual space $(X', \|\cdot\|_{X'})$, $(W, \|\cdot\|_W)$ another Banach space such that $X \hookrightarrow W$ continuously, and by $J : X \to \mathbb{R}$ a given C^1 functional.

Nevertheless, to avoid any ambiguity, we will henceforth denote by X the space equipped with its norm $\|\cdot\|_X$, while, if the norm $\|\cdot\|_W$ is involved, we will write it explicitly.

For simplicity, taking $\beta \in \mathbb{R}$, we say that a sequence $(u_n)_n \subset X$ is a *Cerami-Palais-Smale sequence at level* β, briefly $(CPS)_\beta$*-sequence*, if

$$\lim_{n\to+\infty} J(u_n) = \beta \quad \text{and} \quad \lim_{n\to+\infty} \|dJ(u_n)\|_{X'}(1 + \|u_n\|_X) = 0.$$

Moreover, β is a Cerami-Palais-Smale level, briefly (CPS)-level, if there exists a $(CPS)_\beta$-sequence.

The functional J satisfies the classical Cerami-Palais-Smale condition in X at the level β if every $(CPS)_\beta$-sequence converges in X up to subsequences. However, thinking about the setting of our problem, in general a $(CPS)_\beta$-sequence may also exist which is unbounded in $\|\cdot\|_X$ but converges with respect to $\|\cdot\|_W$. Then, we can weaken the Cerami-Palais-Smale condition in an appropriate way according to some ideas developed in previous papers (see, for example, [10]–[12]).

Definition 2.1. The functional J satisfies the *weak Cerami-Palais-Smale condition at level* β ($\beta \in \mathbb{R}$), briefly $(wCPS)_\beta$ condition, if for every $(CPS)_\beta$-sequence $(u_n)_n$, a point $u \in X$ exists such that

(i) $\lim_{n\to+\infty} \|u_n - u\|_W = 0$ (up to subsequences),
(ii) $J(u) = \beta$, $dJ(u) = 0$.

If J satisfies the $(wCPS)_\beta$ condition at each level $\beta \in I$, I real interval, we say that J satisfies the $(wCPS)$ condition in I.

Let us point out that, because of the convergence only in the norm of W, the $(wCPS)_\beta$ condition implies that the set of critical points of J at the β level is compact with respect to $\|\cdot\|_W$, so that we can state a Deformation Lemma and some abstract theorems about critical points (see [12]). In particular, the following Minimum Principle applies (for the proof, see [12, Theorem 1.6]).

Proposition 2.2 (Minimum Principle). *If $J \in C^1(X, \mathbb{R})$ is bounded from below in X and $(wCPS)_\beta$ holds at level $\beta = \inf_X J \in \mathbb{R}$, then J attains its infimum, i.e., $u_0 \in X$ exists such that $J(u_0) = \beta$.*

3. Variational setting and first properties

Here and in the following, let $\mathbb{N} = \{1, 2, \dots\}$ be the set of the strictly positive integers and we denote by $x \cdot y$ the inner product in $\mathbb{R}^N$ and $|\cdot|$ the standard norm on any Euclidean space as the dimension of the considered vector is clear and no ambiguity arises. Furthermore, we denote by:

- $B_R(x) = \{y \in \mathbb{R}^N : |y - x| < R\}$ the open ball in $\mathbb{R}^N$ with center in $x \in \mathbb{R}^N$ and radius $R > 0$;
- $B_R^c = \mathbb{R}^N \setminus B_R(0)$ the complement of the open ball $B_R(0)$ in $\mathbb{R}^N$;
- $\operatorname{meas}(\Omega)$ the usual Lebesgue measure of a measurable set Ω in $\mathbb{R}^N$;
- $L^l(\mathbb{R}^N)$ the Lebesgue space with norm $|u|_l = \left(\int_{\mathbb{R}^N} |u|^l \, dx\right)^{1/l}$ if $1 \le l < +\infty$;
- $L^\infty(\mathbb{R}^N)$ the space of Lebesgue-measurable and essentially bounded functions $u : \mathbb{R}^N \to \mathbb{R}$ with norm
$$|u|_\infty = \operatorname{ess\,sup}_{\mathbb{R}^N} |u|;$$
- $W^{1,p}(\mathbb{R}^N)$ the classical Sobolev space with norm $\|u\|_p = (|\nabla u|_p^p + |u|_p^p)^{\frac{1}{p}}$ if $1 \le p < +\infty$;
- $W_r^{1,p}(\mathbb{R}^N) = \{u \in W^{1,p}(\mathbb{R}^N) : u(x) = u(|x|) \text{ a.e. } x \in \mathbb{R}^N\}$ the subspace of the radial functions of $W^{1,p}(\mathbb{R}^N)$ equipped with the norm $\|\cdot\|_p$ with dual space $(W_r^{1,p}(\mathbb{R}^N))'$.

From the Sobolev Embedding Theorems, for any $l \in [p, p^*]$ with $p^* = \frac{pN}{N-p}$ if $N > p$, or any $l \in [p, +\infty[$ if $p = N$, the Sobolev space $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^l(\mathbb{R}^N)$, i.e., a constant $\sigma_l > 0$ exists such that
$$|u|_l \le \sigma_l \|u\|_p \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N) \tag{3.1}$$
(see, e.g., [9, Corollaries 9.10 and 9.11]). Clearly, it is $\sigma_p = 1$. On the other hand, if $p > N$ then $W^{1,p}(\mathbb{R}^N)$ is continuously imbedded in $L^\infty(\mathbb{R}^N)$ (see, e.g., [9, Theorem 9.12]). Thus, we define
$$X := W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad \|u\|_X = \|u\|_p + |u|_\infty. \tag{3.2}$$

From now on, we assume $1 < p \le N$ as, otherwise, it is $X = W^{1,p}(\mathbb{R}^N)$ and the proofs can be simplified.

Lemma 3.1. *For any $l \ge p$ the Banach space X is continuously embedded in $L^l(\mathbb{R}^N)$, i.e., a constant $\sigma_l > 0$ exists such that*
$$|u|_l \le \sigma_l \|u\|_X \quad \textit{for all } u \in X. \tag{3.3}$$

Proof. If $p = N$ or if $p \le l \le p^*$ the embedding (3.3) follows from (3.1) and (3.2). On the other hand, if $l > p^*$ then, taking any $u \in X$, again (3.2) implies
$$\int_{\mathbb{R}^N} |u|^l \, dx \le |u|_\infty^{l-p} \int_{\mathbb{R}^N} |u|^p \, dx \le |u|_\infty^{l-p} \|u\|_p^p \le \|u\|_X^l,$$
thus (3.3) holds with $\sigma_l = 1$. □

From Lemma 3.1 it follows that if $(u_n)_n \subset X$, $u \in X$ are such that $u_n \to u$ in X, then $u_n \to u$ also in $L^l(\mathbb{R}^N)$ for any $l \ge p$. This result can be weakened as follows.

Lemma 3.2. *If* $(u_n)_n \subset X$, $u \in X$, $M > 0$ *are such that*

$$\|u_n - u\|_p \to 0 \quad \text{as } n \to +\infty, \tag{3.4}$$

$$|u_n|_\infty \leq M \quad \text{for all } n \in \mathbb{N}, \tag{3.5}$$

then $u_n \to u$ *also in* $L^l(\mathbb{R}^N)$ *for all* $l \geq p$.

Proof. Let $1 \leq p < N$ and $l > p^*$ (otherwise, it is a direct consequence of (3.1)). Then, from (3.2), (3.5) and (3.1) we have that

$$\int_{\mathbb{R}^N} |u_n - u|^l \, dx \leq |u_n - u|_\infty^{l-p} \int_{\mathbb{R}^N} |u_n - u|^p \, dx \leq (M + |u|_\infty)^{l-p} \|u_n - u\|_p^p,$$

then (3.4) implies the result. □

From now on, we consider $A : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be such that:

- (A1) A is a C^1-Carathéodory function, i.e., $A(\cdot, t, \xi)$ is measurable for all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $A(x, \cdot, \cdot)$ is C^1 for a.e. $x \in \mathbb{R}^N$;
- (A2) some positive continuous functions $\Phi_i, \phi_i : \mathbb{R} \to \mathbb{R}$, $i \in \{0, 1, 2\}$, exist such that:

$$|A(x, t, \xi)| \leq \Phi_0(t)|t|^p + \phi_0(t)|\xi|^p \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N,$$

$$|A_t(x, t, \xi)| \leq \Phi_1(t)|t|^{p-1} + \phi_1(t)|\xi|^p \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N,$$

$$|a(x, t, \xi)| \leq \Phi_2(t)|t|^{p-1} + \phi_2(t)|\xi|^{p-1} \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N;$$

- (A3) $g(x, t)$ is a Carathéodory function;
- (A4) a function $\eta \in L^{\frac{p}{p-q}}(\mathbb{R}^N)$ exists, with $1 < q < p$, such that

$$0 \leq g(x, t)t \leq \eta(x)|t|^q \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \in \mathbb{R}.$$

Remark 3.3. From (A4) it results that

$$|g(x, t)| \leq \eta(x)|t|^{q-1} \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \in \mathbb{R}.$$

Moreover, (A3) and (A4) imply that $G(x, t) = \int_0^t g(x, s) ds$ is a well defined C^1-Carathéodory function in $\mathbb{R}^N \times \mathbb{R}$ and

$$0 \leq G(x, t) \leq \frac{1}{q}\eta(x)|t|^q \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } t \in \mathbb{R}. \tag{3.6}$$

Remark 3.4. From (A2) it follows that

$$A(x, 0, 0) = A_t(x, 0, 0) = 0 \quad \text{and} \quad a(x, 0, 0) = 0 \text{ for a.e. } x \in \mathbb{R}^N.$$

Moreover, from (A3), (A4) and Remark 3.3 we have that

$$G(x, 0) = g(x, 0) = 0 \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Hence, $u = 0$ is a trivial solution of (1.1).

Proposition 3.5. *Assumptions* (A3) *and* (A4) *imply that*

$$\int_{\mathbb{R}^N} G(x, u) \, dx \in \mathbb{R} \quad \text{for all } u \in X \quad (\text{or better for all } u \in W^{1,p}(\mathbb{R}^N)),$$

$$\int_{\mathbb{R}^N} g(x, u)v \, dx \in \mathbb{R} \quad \text{for all } u, v \in X \quad (\text{or better for all } u, v \in W^{1,p}(\mathbb{R}^N)).$$

Proof. Let $u \in W^{1,p}(\mathbb{R}^N)$. As $\eta \in L^{\frac{p}{p-q}}(\mathbb{R}^N)$ and $|u|^q \in L^{\frac{p}{q}}(\mathbb{R}^N)$, Hölder's inequality with $\frac{p}{p-q}$ and $\frac{p}{q}$ conjugate exponents and (3.6) imply that

$$0 \le \int_{\mathbb{R}^N} G(x,u)dx \le \frac{1}{q}\int_{\mathbb{R}^N} \eta(x)|u|^q\,dx \le \frac{1}{q}|\eta|_{\frac{p}{p-q}}|u|_p^q. \tag{3.7}$$

Moreover, by applying again Hölder's inequality with $\frac{p}{p-q}$, $\frac{p}{q-1}$ and p conjugate exponents, we have

$$\Big|\int_{\mathbb{R}^N} g(x,u)v\,dx\Big| \le \int_{\mathbb{R}^N} |\eta(x)|u|^{q-1}v|\,dx \le |\eta|_{\frac{p}{p-q}}|u|_p^{q-1}|v|_p \tag{3.8}$$

for all $u, v \in W^{1,p}(\mathbb{R}^N)$. □

Remark 3.6. From (A3) and (A4) we have that

$$g(x,u) \in L^{\frac{p}{p-1}}\big(\mathbb{R}^N\big) \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N).$$

Indeed, Hölder's inequality with $\frac{p-1}{p-q}$ and $\frac{p-1}{q-1}$ conjugate exponents implies that

$$\int_{\mathbb{R}^N} |g(x,u)|^{\frac{p}{p-1}}\,dx \le |\eta|_{\frac{p}{p-q}}^{\frac{p}{p-1}}|u|_p^{\frac{p(q-1)}{p-1}}.$$

Let us point out that assumptions (A1) and (A2) imply that $A(x,u,\nabla u) \in L^1(\mathbb{R}^N)$ for any $u \in X$. Therefore, from (3.7) it follows that the functional

$$\mathcal{J}(u) = \int_{\mathbb{R}^N} A(x,u,\nabla u)\,dx + \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\,dx - \int_{\mathbb{R}^N} G(x,u)\,dx \tag{3.9}$$

is well defined for all $u \in X$. Moreover, taking $v \in X$, from (3.8), the Gâteaux differential of functional $\mathcal{J}$ in u along the direction v is given by

$$\begin{aligned}\langle d\mathcal{J}(u), v\rangle = &\int_{\mathbb{R}^N} a(x,u,\nabla u)\cdot\nabla v\,dx + \int_{\mathbb{R}^N} A_t(x,u,\nabla u)v\,dx \\ &+ \int_{\mathbb{R}^N} |u|^{p-2}uv\,dx - \int_{\mathbb{R}^N} g(x,u)v\,dx.\end{aligned} \tag{3.10}$$

Now, we are ready to state the following regularity result.

Proposition 3.7. *Taking $p > 1$, assume that* (A1)—-(A4) *hold. If $(u_n)_n \subset X$, $u \in X$, $M > 0$ are such that* (3.4), (3.5) *hold and*

$$u_n \to u \quad \textit{a.e. in } \mathbb{R}^N \textit{ as } n \to +\infty,$$

then

$$\mathcal{J}(u_n) \to \mathcal{J}(u) \quad \textit{and} \quad \|d\mathcal{J}(u_n) - d\mathcal{J}(u)\|_{X'} \to 0 \quad \textit{as } n \to +\infty.$$

Hence, $\mathcal{J}$ is a C^1 functional on X with Fréchet differential defined as in (3.10).

Proof. It is sufficient to simplify the proof of [33, Prop. 3.10] by observing that the functional $u \in X \mapsto \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\,dx \in \mathbb{R}$ is of class $\mathcal{C}^1$. □

4. Statement of main results

From now on, we assume that in addition to (A1)–(A4), functions $A(x,t,\xi)$ and $g(x,t)$ satisfy the following further conditions:

(A5) there exists a constant $\alpha_0 > 0$ such that

$$A(x,t,\xi) \ge \alpha_0|\xi|^p \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } (t,\xi) \in \mathbb{R}\times\mathbb{R}^N;$$

(A6) there exists a constant η_0 such that
$$A(x,t,\xi) \le \eta_0\ a(x,t,\xi)\cdot\xi \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } (t,\xi)\in\mathbb{R}\times\mathbb{R}^N;$$
(A7) there exists a constant $\alpha_1 > 0$ such that
$$a(x,t,\xi)\cdot\xi + A_t(x,t,\xi)t \ge \alpha_1 a(x,t,\xi)\cdot\xi \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } (t,\xi)\in\mathbb{R}\times\mathbb{R}^N;$$
(A8) there exist constants $\mu > p$ and $\alpha_2 > 0$ such that
$$\mu A(x,t,\xi) - a(x,t,\xi)\cdot\xi - A_t(x,t,\xi)t \ge \alpha_2 A(x,t,\xi) \quad \text{a.e. in } \mathbb{R}^N,$$
for all $(t,\xi)\in\mathbb{R}\times\mathbb{R}^N$;
(A9) for all ξ, $\xi^*\in\mathbb{R}^N$, $\xi\ne\xi^*$, we have
$$[a(x,t,\xi) - a(x,t,\xi^*)]\cdot[\xi-\xi^*] > 0 \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } t\in\mathbb{R};$$
(A10) $A(x,t,\xi) = A(|x|,t,\xi)$ a.e. in $\mathbb{R}^N$, for all $t\in\mathbb{R}$;
(A11) there exist real constants l_1, l_2, η_1, η_2 such that
$$\lim_{t\to 0}\frac{\Phi_1(t)}{|t|^{\eta_1}} = l_1, \quad \lim_{t\to 0}\frac{\Phi_2(t)}{|t|^{\eta_2}} = l_2$$
with Φ_1, Φ_2 as in (A2) and
$$\eta_1 > \frac{p}{N-1}, \quad \eta_2 > \frac{p-1}{N-1}; \tag{4.1}$$
(A12) $g(x,t) = g(|x|,t)$ a.e. in $\mathbb{R}^N$, for all $t\in\mathbb{R}$;
(A13) the function η introduced in (A4) is such that
$$\operatorname{ess\,sup}_{|x|\le 1}\eta(x) \ < \ +\infty;$$
(A14) $\lim_{t\to 0^+}\frac{g(x,t)}{t^{p-1}} = +\infty$ uniformly for a.e. $x\in\mathbb{R}^N, |x|\le 1$.

Example 4.1. The function
$$A(x,t,\xi) = \frac{1}{p}\left(A_1(x) + A_2(x)|t|^\theta\right)|\xi|^p \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } (t,\xi)\in\mathbb{R}\times\mathbb{R}^N$$
with $p>1$ and $\theta>1$, satisfies (A1), (A2), (A5)–(A11) if A_1 and A_2 are two radial functions and there exists a constant $\bar{\alpha_0} > 0$ such that
$$A_1, A_2 \in L^\infty(\mathbb{R}^N), \quad A_1(x)\ge\bar{\alpha_0}, \quad A_1(x)\ge 0 \quad \text{a.e. in } \mathbb{R}^N.$$

We point out some direct consequences of the previous hypotheses.

Remark 4.2. In assumption (A5) we always suppose $\alpha_0\le 1$ while from (A5) and (A6) we suppose $\alpha_1\le 1$ in (A7).

Remark 4.3. From (A7) and (A8) it follows that
$$(\mu-\alpha_2)A(x,t,\xi) \ \ge\ \alpha_1\ a(x,t,\xi)\cdot\xi \quad \text{a.e. in } \mathbb{R}^N, \text{ for all } (t,\xi)\in\mathbb{R}\times\mathbb{R}^N;$$
hence, if also (A5) and (A6) hold, we have $\alpha_2<\mu$. So,
$$A(x,t,\xi)\ge\alpha_3 a(x,t,\xi)\cdot\xi \quad \text{a.e. in } \mathbb{R}^N \text{ for all } (t,\xi)\in\mathbb{R}\times\mathbb{R}^N, \tag{4.2}$$
with $\alpha_3 = \frac{\alpha_1}{\mu-\alpha_2} > 0$. Moreover, from (4.2) and (A8) we have that
$$\mu A(x,t,\xi) - a(x,t,\xi)\cdot\xi - A_t(x,t,\xi)t \ \ge\ \alpha_2\alpha_3\ a(x,t,\xi)\cdot\xi \quad \text{a.e. in } \mathbb{R}^N,$$
for all $(t,\xi)\in\mathbb{R}\times\mathbb{R}^N$.

Remark 4.4. We note that from (A5)–(A8) it follows that

$$-(1-\alpha_1)a(x,t,\xi)\cdot\xi \le A_t(x,t,\xi)t \le (\mu-\alpha_2)A(x,t,\xi) \le (\mu-\alpha_2)\eta_0 a(x,t,\xi)\xi$$

which implies that

$$|A_t(x,t,\xi)t| \le ca(x,t,\xi)\xi \tag{4.3}$$

with $c = \max\{(\mu-\alpha_2)\eta_0, (1-\alpha_1)\}$.

Now, we are able to state our main existence result.

Theorem 4.5. *Assume that (A1)–(A14) hold, then problem* (1.1) *admits at least two weak nontrivial radial bounded solutions, one negative and one positive.*

We will prove Theorem 4.5 by applying Proposition 2.2 to a suitable restriction of the functional $\mathcal{J}$ introduced in (3.9). To this aim, the following results will be useful.

Proposition 4.6. *Assume that conditions* (A1)–(A5) *hold. Then, there exists positive constants b_1, b_2 such that*

$$\mathcal{J}(u) \ge b_1\|u\|_p^p - b_2\|u\|_p^q \quad \text{for each } u \in X.$$

Hence, functional $\mathcal{J}$ is bounded from below, i.e., there exists a constant $\alpha \in \mathbb{R}$ such that

$$\mathcal{J}(u) \ge \alpha \text{ for any } u \in X, \text{ with } \alpha = \min_{s\ge 0}(b_1 s^p - b_2 s^q).$$

Proof. From (A5) and (3.7) we have

$$\begin{aligned}\mathcal{J}(u) &= \int_{\mathbb{R}^N} A(x,u,\nabla u)dx + \frac{1}{p}\int_{\mathbb{R}^N}|u|^p\,dx - \int_{\mathbb{R}^N} G(x,u)\,dx\\ &\ge \alpha_0\int_{\mathbb{R}^N}|\nabla u|^p\,dx + \frac{1}{p}\int_{\mathbb{R}^N}|u|^p\,dx - \frac{1}{q}|\eta|_{\frac{p}{p-q}}|u|_p^q\\ &\ge b_1\|u\|_p^p - b_2\|u\|_p^q\end{aligned}$$

where $b_1 = \min\{\alpha_0, \frac{1}{p}\}$ and $b_2 = \frac{1}{q}|\eta|_{\frac{p}{p-q}}$. □

Lemma 4.7. *Assume that $g(x,t)$ satisfies conditions* (A3) *and* (A4), *with $1 < q < p$, and consider $(w_n)_n$, $(v_n)_n \subset X$ and $v, w \in X$ such that*

$$\|w_n\|_p \le M_1 \quad \text{for all } n \in \mathbb{N}, \quad w_n \to w \quad \text{a.e. in } \mathbb{R}^N, \tag{4.4}$$

$$\|v_n\|_p \le M_2 \quad \text{for all } n \in \mathbb{N}, \quad v_n \to 0 \quad \text{a.e. in } \mathbb{R}^N, \tag{4.5}$$

for some constants M_1, $M_2 > 0$. Then

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} g(x,w_n)v_n\,dx = 0.$$

Proof. From (4.4), (4.5) and (A3) we have

$$g(x,w_n)v_n \to 0 \quad \text{a.e. in } \mathbb{R}^N. \tag{4.6}$$

Moreover, from (3.8) and by applying again (4.4) and (4.5), it follows that

$$\int_{\mathbb{R}^N}|g(x,w_n)v_n|\,dx \le |\eta|_{\frac{p}{p-q}}|w_n|_p^{q-1}|v_n|_p \le |\eta|_{\frac{p}{p-q}}\|w_n\|_p^{q-1}\|v_n\|_p \le M_1^{q-1}M_2|\eta|_{\frac{p}{p-q}}.$$

As $\eta \in L^{\frac{p}{p-q}}(\mathbb{R}^N)$, for each $\epsilon > 0$ there exists $R > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_R(0)}|g(x,w_n)v_n|\,dx < \epsilon \tag{4.7}$$

for all $n \in \mathbb{N}$. On the other hand, from the absolute continuity of the Lebesgue's integral taking $\epsilon' = \left(\frac{\epsilon}{M_1^{q-1}M_2}\right)^{\frac{p}{p-q}}$ there exists $\delta_\epsilon > 0$ such that

$$\int_A |\eta|^{\frac{p}{p-q}}\,dx \le \epsilon'$$

for all measurable set $A \subset B_R(0)$ with $\mathrm{meas}(A) < \delta_\epsilon$. Thus, it follows that

$$\int_A |g(x,w_n)v_n|\,dx \le \epsilon$$

for all $n \in \mathbb{N}$ and for all measurable set A with $\mathrm{meas}(A) < \delta_\epsilon$. Hence, by Vitali's Convergence Theorem

$$g(x,w_n)v_n \to 0 \quad \text{in } L^1(B_R(0)). \tag{4.8}$$

The conclusion follow from (4.7) and (4.8). □

From now on, to overcome the lack of compactness of the problem we reduce to work in the space of radial functions which is a natural constraint if the problem is radially invariant (see [34]). Thus, in our setting, we consider the space

$$X_r := W_r^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$$

endowed with norm $\|\cdot\|_X$ and we denote by $(X_r', \|\cdot\|_{X_r'})$ its dual space.

Lemma 4.8 (Radial Lemma). *If $N \ge 2$ and $p > 1$, for all $u \in W_r^{1,p}(\mathbb{R}^N)$ it holds*

$$|u(x)| \ \le\ C\frac{\|u\|_p}{|x|^{\frac{N-1}{p}}} \quad \text{a.e. in } \mathbb{R}^N, \tag{4.9}$$

for a suitable constant C depending only on N and p.

For a proof of the above lemma, see [26, Lemma II.1].

Lemma 4.9. *If $p > 1$ then the following compact embeddings hold:*

$$W_r^{1,p}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^l(\mathbb{R}^N) \quad \text{for any } p < l < p^*.$$

The proof of the above lemma is essentially contained in [13, Theorem 3.2] (see also [14, Lemma 4.8]).

Remark 4.10. By assumptions (A10) and (A12), we can be reduced to looking for critical points of the restriction of $\mathcal{J}$ in (3.9) to X_r, which we still denote as $\mathcal{J}$ for simplicity (see [34]).

We recall that Proposition 3.7 implies that functional $\mathcal{J}$ is C^1 on the Banach space X_r, too, if also (A1)–(A4) hold.

Now, we want to extend to $\mathbb{R}^N$ a result stated by Boccardo–Murat–Puel in bounded domains (see [7, Lemma 5]).

Lemma 4.11. *Assume that* (A1), (A2), (A5), (A6), (A9)–(A11) *hold. Let $(u_n)_n \subset X_r, u \in X_r$ be such that*

$$u_n \rightharpoonup u \quad \text{weakly in } W_r^{1,p}(\mathbb{R}^N), \tag{4.10}$$

$$u_n \to u \quad \text{a.e. in } \mathbb{R}^N, \tag{4.11}$$

$$|u_n|_\infty \le M \quad \text{for all } n \in \mathbb{N}, \tag{4.12}$$

$$\int_{\mathbb{R}^N} [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)] \cdot \nabla(u_n - u)dx \to 0. \tag{4.13}$$

Then

$$\int_{\mathbb{R}^N} |\nabla u_n|^p \, dx \to \int_{\mathbb{R}^N} |\nabla u|^p \, dx \quad \textit{as } n \to +\infty. \tag{4.14}$$

Proof. We will use arguments similar to those ones used in bounded domains in [31, Lemma 4.5] (see also [7, Lemma 5]). We will prove that any subsequence of $(u_n)_n$ admits a subsequence satisfying (4.14) and then (4.14) holds for all sequence $(u_n)_n$.

Let f_n be defined by

$$f_n = [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)] \cdot \nabla(u_n - u).$$

From (A9) it follows that $f_n \geq 0$ a.e. in $\mathbb{R}^N$ and from (4.13) we have $f_n \to 0$ in $L^1(\mathbb{R}^N)$.

Thus, from [9, Theorem 4.9] a function $\bar{h} \in L^1(\mathbb{R}^N)$ and a subset Z of $\mathbb{R}^N$ exist such that $\operatorname{meas}(Z) = 0$ and, up to a subsequence,

$$f_n(x) \to 0 \quad \text{and} \quad f_n(x) \leq \bar{h}(x) < \infty \quad \text{for all } x \in \mathbb{R}^N \setminus Z, \text{ for all } n \in \mathbb{N}. \tag{4.15}$$

Moreover, since $u \in X$ and (4.11)–(4.12) hold, we can assume that

$$u_n(x) \to u(x), \quad |u(x)| < +\infty \quad \text{and} \quad |\nabla u(x)| < +\infty, \quad \text{for all } x \in \mathbb{R}^N \setminus Z. \tag{4.16}$$

From (A2) and (A6) we also have

$$\begin{aligned} f_n(x) \geq \frac{\alpha_0}{\eta_0}[|\nabla u_n|^p + |\nabla u|^p] - \Phi_2(u_n)|u_n|^{p-1}|\nabla u| - \phi_2(u_n)|\nabla u_n|^{p-1}|\nabla u| \\ - \Phi_2(u)|u|^{p-1}|\nabla u_n| - \phi_2(u)|\nabla u|^{p-1}|\nabla u_n|. \end{aligned}$$

Since Φ_2, ϕ_2 are continuous functions, by (4.12), (4.15) and (4.16) we find that

$$(\nabla u_n(x))_n \text{ is bounded for all } x \in \mathbb{R}^N \setminus Z.$$

Let $\xi^*(x)$ be a cluster point of $(\nabla u_n(x))_n$. We have $|\xi^*(x)| < \infty$ and, since $f_n(x) \to 0$ and a is a Carathéodory function, it follows that

$$[a(x, u, \xi^*) - a(x, u, \nabla u)] \cdot (\xi^* - \nabla u) = 0,$$

hence (A9) implies that $\nabla u(x) = \xi^*(x)$ for all $x \in \mathbb{R}^N \setminus Z$. From this, we deduce that $\nabla u_n(x)$ converges to $\nabla u(x)$ without passing to subsequence. Hence,

$$\nabla u_n(x) \to \nabla u(x) \quad \text{for all } x \in \mathbb{R}^N \setminus Z. \tag{4.17}$$

Thus, from (A1), (4.16) and (4.17) we have that

$$a(x, u_n(x), \nabla u_n(x)) \to a(x, u(x), \nabla u(x)) \quad \text{for all } x \in \mathbb{R}^N \setminus Z$$

and then

$$a(x, u_n, \nabla u_n) \cdot \nabla u_n \to a(x, u, \nabla u) \cdot \nabla u \quad \text{a.e. in } \mathbb{R}^N. \tag{4.18}$$

Now, from (A5) and (A6) it follows that

$$a(x, u_n, \nabla u_n) \cdot \nabla u_n \geq 0 \quad \text{a.e. in } \mathbb{R}^N. \tag{4.19}$$

From (4.12) and (A2) we obtain that

$$|a(x, u_n, \nabla u_n)| \leq c\big(|\nabla u_n|^{p-1} + |u_n|^{p-1}\big). \tag{4.20}$$

Since (4.10) holds, u_n is bounded in $W^{1,p}(\mathbb{R}^N)$, thus from (4.20) the sequence $(a(x,u_n,\nabla u_n))_n$ is bounded in $(L^{\frac{p}{p-1}}(\mathbb{R}^N))^N$, hence, up to subsequences, it weakly converges to $a(x,u,\nabla u)$ in $(L^{\frac{p}{p-1}}(\mathbb{R}^N))^N$. It follows that

$$\int_{\mathbb{R}^N} a(x,u_n,\nabla u_n)\cdot\nabla u\,dx \to \int_{\mathbb{R}^N} a(x,u,\nabla u)\cdot\nabla u\,dx.$$

In a similar way, we prove that

$$\int_{\mathbb{R}^N} a(x,u_n,\nabla u)\cdot\nabla u\,dx \to \int_{\mathbb{R}^N} a(x,u,\nabla u)\cdot\nabla u\,dx.$$

Now, we prove that

$$\int_{\mathbb{R}^N} a(x,u_n,\nabla u)\cdot\nabla u_n\,dx \to \int_{\mathbb{R}^N} a(x,u,\nabla u)\cdot\nabla u\,dx. \tag{4.21}$$

Clearly, from (A1), (4.11), and (4.17) it follows that

$$a(x,u_n,\nabla u)\cdot\nabla u_n \to a(x,u,\nabla u)\cdot\nabla u \quad \text{a.e. in } \mathbb{R}^N. \tag{4.22}$$

Moreover,

$$\begin{aligned} &\Big|\int_{\mathbb{R}^N} [a(x,u_n,\nabla u)\cdot\nabla u_n - a(x,u,\nabla u)\cdot\nabla u]\,dx\Big| \\ &\le \int_{\mathbb{R}^N} |a(x,u_n,\nabla u)||\nabla u_n|\,dx + \int_{\mathbb{R}^N} a(x,u,\nabla u)\cdot\nabla u\,dx \end{aligned} \tag{4.23}$$

where $a(x,u,\nabla u)\cdot\nabla u \in L^1(\mathbb{R}^N)$ while from (A2), Hölder inequality, (4.10) and (4.12)

$$\Big|\int_{\mathbb{R}^N} a(x,u_n,\nabla u)\cdot\nabla u_n\,dx\Big| \le c(|\nabla u|_p^{p-1} + |(\Phi_2(u_n))^{\frac{1}{p-1}}|u_n||_p^{p-1}. \tag{4.24}$$

We notice that from (A11) we have

$$\lim_{t\to 0}\frac{\Phi_2(t)}{|t|^{\eta_2}} = l_2 \ge 0$$

hence, there exists $\bar\delta > 0$ such that

$$\Phi_2(t) < (l_2+1)|t|^{\eta_2} \quad \text{for all } t\in\mathbb{R}, |t| < \bar\delta.$$

Therefore, taking $\bar M = \sup_n \|u_n\|_p$ and $\bar R$ such that $\frac{C\bar M}{\bar R^{\frac{N-1}{p}}} < \bar\delta$, using (4.9) in Radial Lemma it holds

$$|u_n(x)| \le \frac{C\bar M}{|x|^{\frac{N-1}{p}}} \le \frac{C\bar M}{\bar R^{\frac{N-1}{p}}} < \bar\delta \quad \text{for all } x\in\mathbb{R}^N, |x| > \bar R$$

and therefore, using again Radial Lemma a constant $\bar C > 0$ exists such that for $|x| > \bar R$,

$$(\Phi_2(u_n))^{\frac{p}{p-1}}|u_n|^p \le (l_2+1)^{\frac{p}{p-1}}|u_n|^{\frac{\eta_2 p}{p-1}}|u_n|^p \le \frac{\bar C}{|x|^{(N-1)(\frac{\eta_2}{p-1}+1)}} \in L^1(B^c_{\bar R}) \tag{4.25}$$

since from (4.1) and simple calculations it follows that $(N-1)(\frac{\eta_2}{p-1}+1) > N$. Thus, from (4.23)–(4.25) for each $\epsilon > 0$ there exists $R > \bar R$ such that

$$\Big|\int_{B^c_R} [a(x,u_n,\nabla u)\cdot\nabla u_n - a(x,u,\nabla u)\cdot\nabla u]\,dx\Big| \le \epsilon. \tag{4.26}$$

On the other hand, from (4.24) and (4.12), since $u_n \to u$ in $L^p(B_R(0))$ for each $\epsilon > 0$,

$$\begin{aligned} &\Big|\int_{B_R(0)} [a(x,u_n,\nabla u)\cdot\nabla u_n - a(x,u,\nabla u)\cdot\nabla u]\,dx\Big| \\ &\le c(|\nabla u|_{p,B_R(0)}^{p-1} + |u|_{p,B_R(0)}^{p-1}). \end{aligned} \tag{4.27}$$

From the absolute continuity of the Lebesgue integral, there exists $\delta_\epsilon > 0$ such that

$$\Big|\int_A [a(x,u_n,\nabla u)\cdot\nabla u_n - a(x,u,\nabla u)\cdot\nabla u]\,dx\Big| < \epsilon \tag{4.28}$$

for all measurable set $A \subset B_R(0)$ with $\mathrm{meas}(A) < \delta_\epsilon$. Hence, from (4.22) Vitali's Theorem holds and

$$\int_{B_R(0)} a(x,u_n,\nabla u)\cdot\nabla u_n\,dx \to \int_{B_R(0)} a(x,u,\nabla u)\cdot\nabla u\,dx. \tag{4.29}$$

Finally, (4.21) follows from (4.26) and (4.29). Hence, from (4.13) we finally find that

$$\int_{\mathbb{R}^N} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx \to \int_{\mathbb{R}^N} a(x,u,\nabla u)\cdot\nabla u\,dx. \tag{4.30}$$

Now, we set

$$y_n = a(x,u_n,\nabla u_n)\cdot\nabla u_n \quad \text{and} \quad y = a(x,u,\nabla u)\cdot\nabla u.$$

So, from (4.19), (4.18), (A2) and (4.30) we obtain that

$$y_n \ge 0, \quad y_n \to y \quad \text{a.e. in } \mathbb{R}^N, \quad y \in L^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} y_n\,dx \to \int_{\mathbb{R}^N} y\,dx.$$

From Brezis-Lieb's Lemma [9] it results

$$a(x,u_n,\nabla u_n)\cdot\nabla u_n \to a(x,u,\nabla u)\cdot\nabla u \quad \text{in } L^1(\mathbb{R}^N),$$

hence, using again [9, Theorem 4.9] a function $H \in L^1(\mathbb{R}^N)$ exists such that

$$a(x,u_n,\nabla u_n)\cdot\nabla u_n \le H(x) \quad \text{a.e. in } \mathbb{R}^N. \tag{4.31}$$

Moreover, from (A5), (A6) and (4.31) we have that

$$\frac{\alpha_0}{\eta_0}\,(|\nabla u_n|^p) \le a(x,u_n,\nabla u_n)\cdot\nabla u_n \le H(x),$$

thus, (4.14) follows from (4.17) and Lebesgue's Convergence Theorem. □

5. Proof of the main result

The aim of this section is to prove that $\mathcal{J}$ satisfies the $(wCPS)_\beta$-condition in X_r and then to apply Proposition 2.2 to the functional $\mathcal{J}$ on X_r. To prove the weak Cerami-Palais-Smale condition, we need some preliminary lemmas.

Firstly, let us point out that, while if $p > N$ the two norms $\|\cdot\|_X$ and $\|\cdot\|_p$ are equivalent, if $p \le N$ sufficient conditions are required for the boundedness of a $W^{1,p}$-function. Even if we are working in $W_r^{1,p}(\mathbb{R}^N)$, we need a condition for functions u in $W^{1,p}(\Omega)$, Ω bounded, as in the following result.

Lemma 5.1. *Let Ω be an open bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$, consider p, r so that $1 < p \le r < p^*$, $p \le N$, and take $v \in W^{1,p}(\Omega)$. If $\gamma > 0$ and $k_0 \in \mathbb{N}$ exist such that*

$$k_0 \ge \operatorname{ess\,sup}_{\partial\Omega} v(x),$$

$$\int_{\Omega_k^+} |\nabla v|^p \, dx \le \gamma\Big(k^r \ \operatorname{meas}(\Omega_k^+) + \int_{\Omega_k^+} |v|^r \, dx\Big) \quad \text{for all } k \ge k_0,$$

with $\Omega_k^+ = \{x \in \Omega : v(x) > k\}$, then $\operatorname{ess\,sup}_\Omega v$ is bounded from above by a positive constant which can be chosen so that it depends only on meas(Ω), *N, p, r, γ, k_0, $|v|_{p^*}$ ($|v|_l$ for some $l > r$ if $p^* = +\infty$). Vice versa, if*

$$-k_0 \le \operatorname{ess\,inf}_{\partial\Omega} v(x)$$

and

$$\int_{\Omega_k^-} |\nabla v|^p \, dx \le \gamma\Big(k^r \ \operatorname{meas}(\Omega_k^-) + \int_{\Omega_k^-} |v|^r \, dx\Big) \quad \text{for all } k \ge k_0$$

holds with $\Omega_k^- = \{x \in \Omega : v(x) < -k\}$, then $\operatorname{ess\,sup}_\Omega(-v)$ is bounded from above by a positive constant which can be chosen so that it depends only on meas(Ω), *N, p, r, γ, k_0, $|v|_{p^*}$ ($|v|_l$ for some $l > r$ if $p^* = +\infty$).*

The proof follows from [24, Theorem II.5.1] but reasoning as in [11, Lemma 4.5].

By applying Lemma 5.1, we will prove that the weak limit in $W_r^{1,p}(\mathbb{R}^N)$ of a $(CPS)_\beta$-sequence has to be bounded in $\mathbb{R}^N$. For simplicity, in the following proofs, when a sequence $(u_n)_n$ is involved, we use the notation $(\varepsilon_n)_n$ for any infinitesimal sequence depending only on $(u_n)_n$ while $(\varepsilon_{k,n})_n$ for any infinitesimal sequence depending not only on $(u_n)_n$ but also on some fixed integer k. Moreover, c denotes any strictly positive constant independent of n which can change from line to line.

Proposition 5.2. *Let $1 < q < p$ and assume that* (A1)–(A7), (A10), (A12), (A13) *hold. Then, taking any $\beta \in \mathbb{R}$ and a $(CPS)_\beta$-sequence $(u_n)_n \subset X_r$, it follows that $(u_n)_n$ is bounded in $W_r^{1,p}(\mathbb{R}^N)$ and a constant $\beta_0 > 0$ exists such that*

$$|u_n(x)| \le \beta_0 \quad \text{for a.e. } x \in \mathbb{R}^N \text{ with } |x| \ge 1 \text{ and for all } n \in \mathbb{N}. \tag{5.1}$$

Moreover, there exists $u \in X_r$ such that, up to subsequences,

$$u_n \rightharpoonup u \quad \text{weakly in } W_r^{1,p}(\mathbb{R}^N), \tag{5.2}$$

$$u_n \to u \quad \text{strongly in } L^l(\mathbb{R}^N) \text{ for each } l \in]p, p^*[, \tag{5.3}$$

$$u_n \to u \quad \text{a.e. in } \mathbb{R}^N, \tag{5.4}$$

as $n \to +\infty$.

Proof. Let $\beta \in \mathbb{R}$ be fixed and consider a sequence $(u_n)_n \subset X_r$ such that

$$\mathcal{J}(u_n) \to \beta \quad \text{and} \quad \|d\mathcal{J}(u_n)\|_{X_r'}(1 + \|u_n\|_{X_r}) \to 0 \quad \text{as } n \to +\infty. \tag{5.5}$$

From Proposition 4.6, as $q < p$, $(u_n)_n$ is bounded in $W_r^{1,p}(\mathbb{R}^N)$ and therefore Lemma 4.8 implies the uniform estimate (5.1). Furthermore, $u \in W_r^{1,p}(\mathbb{R}^N)$ exists such that (5.2)–(5.4) hold, up to subsequences.

Now, we have just to prove that $u \in L^\infty(\mathbb{R}^N)$. Clearly, (5.1) and (5.4) imply

$$\operatorname{ess\,sup}_{|x| \ge 1} |u(x)| \le \beta_0 < +\infty. \tag{5.6}$$

Then, it is sufficient to prove that

$$\operatorname{ess\,sup}_{|x| \le 1} |u(x)| < +\infty. \tag{5.7}$$

Arguing by contradiction, let us assume that either

$$\operatorname{ess\,sup}_{|x|\le 1} u(x) = +\infty \tag{5.8}$$

or

$$\operatorname{ess\,sup}_{|x|\le 1}(-u(x)) = +\infty. \tag{5.9}$$

If, for example, (5.8) holds then, for any fixed $k \in \mathbb{N}$, $k > \beta_0$ we have that

$$\operatorname{meas}(B_k^+) > 0 \quad \text{with } B_k^+ = \{x \in B_1(0) : u(x) > k\}. \tag{5.10}$$

We note that the choice of k and (5.6) imply that

$$B_k^+ = \{x \in \mathbb{R}^N : u(x) > k\}. \tag{5.11}$$

Moreover, if we set

$$B_{k,n}^+ = \{x \in B_1(0) : u_n(x) > k\}, \quad n \in \mathbb{N},$$

the choice of k and (5.1) imply that

$$B_{k,n}^+ = \{x \in \mathbb{R}^N : u_n(x) > k\} \quad \text{for all } n \in \mathbb{N}. \tag{5.12}$$

Now, consider the new function $R_k^+ : t \in \mathbb{R} \to R_k^+ t \in \mathbb{R}$ such that

$$R_k^+ t = \begin{cases} 0 & \text{if } t \le k \\ t - k & \text{if } t > k\,. \end{cases}$$

By definition and (5.11), respectively (5.12), it results

$$R_k^+ u(x) = \begin{cases} 0 & \text{if } x \notin B_k^+ \\ u(x) - k & \text{if } x \in B_k^+, \end{cases} \quad R_k^+ u_n(x) = \begin{cases} 0 & \text{if } x \notin B_{k,n}^+ \\ u_n(x) - k & \text{if } x \in B_{k,n}^+\,. \end{cases} \tag{5.13}$$

Clearly, (5.1), (5.6) and $k > \beta_0$ imply

$$R_k^+ u \in W_0^{1,p}(B_1(0)) \quad \text{and} \quad R_k^+ u_n \in W_0^{1,p}(B_1(0)) \quad \text{for all } n \in N. \tag{5.14}$$

From (5.2) it follows that $R_k^+ u_n \rightharpoonup R_k^+ u$ weakly in $W_r^{1,p}(\mathbb{R}^N)$, then, from (5.14), in $W_0^{1,p}(B_1(0))$. As $W_0^{1,p}(B_1(0)) \hookrightarrow\hookrightarrow L^l(B_1(0))$ for any $1 \le l < p^*$, then

$$\lim_{n\to+\infty} \int_{B_1(0)} |R_k^+ u_n|^l \,dx = \int_{B_1(0)} |R_k^+ u|^l \,dx \quad \text{for } 1 \le l < p^*. \tag{5.15}$$

Moreover, from (5.3) we have $u_n \to u$ strongly in $L^l(B_1(0))$ for any $l \in]p, p^*[$ and then

$$\lim_{n\to+\infty} \int_{B_1(0)} |u_n|^l \,dx = \int_{B_1(0)} |u|^l \,dx \quad \text{for } 1 \le l < p^*. \tag{5.16}$$

Thus, by the weak lower semi-continuity of the norm $\|\cdot\|_p$, we have that

$$\int_{\mathbb{R}^N} |\nabla R_k^+ u|^p \,dx + \int_{\mathbb{R}^N} |R_k^+ u|^p \,dx \le \liminf_{n\to+\infty} \Big(\int_{\mathbb{R}^N} |\nabla R_k^+ u_n|^p \,dx + \int_{\mathbb{R}^N} |R_k^+ u_n|^p \,dx \Big),$$

i.e., from (5.13)–(5.15) we have

$$\begin{aligned} \int_{B_k^+} |\nabla u|^p \,dx + \int_{B_1(0)} |R_k^+ u|^p \,dx &\le \liminf_{n\to+\infty} \Big(\int_{B_{k,n}^+} |\nabla u_n|^p \,dx + \int_{B_1(0)} |R_k^+ u_n|^p \,dx \Big) \\ &= \liminf_{n\to+\infty} \int_{B_{k,n}^+} |\nabla u_n|^p \,dx + \int_{B_1(0)} |R_k^+ u|^p \,dx. \end{aligned}$$

Hence,

$$\int_{B_k^+} |\nabla u|^p \, dx \le \liminf_{n\to+\infty} \int_{B_{k,n}^+} |\nabla u_n|^p \, dx. \tag{5.17}$$

On the other hand, since $\|R_k^+ u_n\|_X \le \|u_n\|_X$ holds, it follows that

$$|\langle d\mathcal{J}(u_n), R_k^+ u_n\rangle| \le \|d\mathcal{J}(u_n)\|_{X_r'} \|u_n\|_X \,.$$

Then (5.5) and (5.10) imply that $n_k \in \mathbb{N}$ exists such that

$$\langle d\mathcal{J}(u_n), R_k^+ u_n\rangle < \operatorname{meas}(B_k^+) \quad \text{for all } n \ge n_k. \tag{5.18}$$

Let us point out that, since $\alpha_1 \le 1$, assumptions (A5)–(A7) imply that

$$\begin{aligned}
\langle d\mathcal{J}(u_n), R_k^+ u_n\rangle &= \int_{B_{k,n}^+} a(x,u_n,\nabla u_n)\cdot \nabla u_n \, dx + \int_{B_{k,n}^+} A_t(x,u_n,\nabla u_n)(u_n-k)\, dx \\
&\quad + \int_{B_{k,n}^+} |u_n|^{p-2}u_n(u_n-k)\, dx - \int_{B_{k,n}^+} g(x,u_n) R_k^+ u_n \, dx \\
&= \int_{B_{k,n}^+} \big(1-\frac{k}{u_n}\big)[a(x,u_n,\nabla u_n)\cdot\nabla u_n + A_t(x,u_n,\nabla u_n)u_n]\, dx \\
&\quad + \int_{B_{k,n}^+} \frac{k}{u_n} a(x,u_n,\nabla u_n)\cdot \nabla u_n\, dx + \int_{B_{k,n}^+} |u_n|^{p-2}u_n(u_n-k)\, dx \\
&\quad - \int_{B_{k,n}^+} g(x,u_n)R_k^+ u_n\, dx \\
&\ge \alpha_1 \int_{B_{k,n}^+} a(x,u_n,\nabla u_n)\cdot\nabla u_n\, dx - \int_{B_{k,n}^+} g(x,u_n)R_k^+ u_n\, dx.
\end{aligned}$$

Hence, from the previous inequalities, (A5) and (A6) it follows that

$$\frac{\alpha_0\alpha_1}{\eta_0} \int_{B_{k,n}^+} |\nabla u_n|^p \, dx \le \langle d\mathcal{J}(u_n), R_k^+ u_n\rangle + \int_{B_{k,n}^+} g(x,u_n)R_k^+ u_n \, dx. \tag{5.19}$$

Now, from (5.14), (5.15) and (A4) we obtain

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} g(x,u_n)R_k^+ u_n \, dx = \int_{\mathbb{R}^N} g(x,u)R_k^+ u \, dx. \tag{5.20}$$

Thus, from (5.17)–(5.20) and (A13) we obtain that

$$\begin{aligned}
\int_{B_k^+} |\nabla u|^p \, dx &\le c\Big(\operatorname{meas}(B_k^+) + \int_{B_k^+} g(x,u)R_k^+ u\, dx\Big) \\
&\le c\ \operatorname{meas}(B_k^+) + c\int_{B_k^+} \eta(x)|u|^q \, dx \\
&\le \bar{c}\Big(\operatorname{meas}(B_k^+) + \int_{B_k^+} |u|^p\Big)
\end{aligned}$$

with $\bar{c} = \max\{c, \operatorname{ess\,sup}_{|x|\le 1} \eta(x)\}$ since

$$\int_{B_k^+} \eta(x)|u|^q \, dx \le \int_{B_k^+} \eta(x)|u|^p \, dx \le \operatorname{ess\,sup}_{|x|\le 1} \eta(x) \int_{B_k^+} |u|^p \, dx$$

as $q < p$ and $u(x) > 1$ for all $x \in B_k^+$.

Thus, we obtain

$$\int_{B_k^+} |\nabla u|^p \, dx \le \bar{c}\Big(\operatorname{meas}(B_k^+) + \int_{B_k^+} |u|^p \Big).$$

As this inequality holds for all $k > \beta_0$, Lemma 5.1 implies that (5.8) is not true. Thus, (5.9) must hold. In this case, fixing any $k \in \mathbb{N}$, $k > \beta_0$, we have

$$\operatorname{meas}(B_k^-) > 0, \quad \text{with } B_k^- = \{x \in B_1(0) : u(x) < -k\},$$

and we can consider $R_k^- : t \in \mathbb{R} \to R_k^- t \in \mathbb{R}$ such that

$$R_k^- t = \begin{cases} 0 & \text{if } t \ge -k \\ t + k & \text{if } t < -k\,. \end{cases}$$

Thus, reasoning as above, but replacing R_k^+ with R_k^-, and applying again Lemma 5.1 we prove that (5.9) cannot hold. Hence, (5.7) has to be true. □

We are ready to prove the $(wCPS)$ condition in $\mathbb{R}$ by adapting the arguments developed in [10, Proposition 3.4], also in [11, Proposition 4.6], to our setting in the whole space $\mathbb{R}^N$.

Proposition 5.3. *If* $1 < q < p$ *and* (A1)–(A13) *hold, then functional* $\mathcal{J}$ *satisfies the weak Cerami-Palais-Smale condition in* X_r *at each level* $\beta \in \mathbb{R}$.

Proof. Let $\beta \in \mathbb{R}$ be fixed and consider a sequence $(u_n)_n \subset X_r$ verifying (5.5). By Proposition 5.2, the uniform estimate (5.1) holds and there exists $u \in X_r$ such that, up to subsequences, (5.2)–(5.4) are satisfied.

We need to prove the following three steps:

(1) Define $T_k : \mathbb{R} \to \mathbb{R}$ such that

$$T_k t = \begin{cases} t & \text{if } |t| \le k \\ k\frac{t}{|t|} & \text{if } |t| > k, \end{cases} \tag{5.21}$$

with $k \ge \max\{|u|_\infty, \beta_0\}$. Then, as $n \to +\infty$, we have

$$\mathcal{J}(T_k u_n) \to \beta, \tag{5.22}$$

$$\|d\mathcal{J}(T_k u_n)\|_{X_r'} \to 0; \tag{5.23}$$

(2) $\|u_n - u\|_p \to 0$ if $n \to +\infty$, as

$$\|T_k u_n - u\|_p \to 0 \quad \text{as } n \to +\infty; \tag{5.24}$$

(3) $\mathcal{J}(u) = \beta$ and $d\mathcal{J}(u) = 0$.

Step 1. Taking any $k > \max\{|u|_\infty, \beta_0\}$, if we set

$$B_{k,n} = \{x \in B_1(0) : |u_n(x)| > k\}, \quad n \in \mathbb{N}, \tag{5.25}$$

the choice of k and (5.1) imply that

$$B_{k,n} = \{x \in \mathbb{R}^N : |u_n(x)| > k\} \quad \text{for all } n \in \mathbb{N}. \tag{5.26}$$

Then, from (5.21) and (5.26) we have that

$$T_k u_n(x) = \begin{cases} u_n(x) & \text{for a.e. } x \notin B_{k,n} \\ k\frac{u_n(x)}{|u_n(x)|} & \text{for } x \in B_{k,n} \end{cases} \tag{5.27}$$

and

$$|T_k u_n|_\infty \le k, \quad \|T_k u_n\|_p \le \|u_n\|_p \quad \text{for each } n \in \mathbb{N}.$$

Defining $R_k : \mathbb{R} \to \mathbb{R}$ such that

$$R_k t = t - T_k t = \begin{cases} 0 & \text{if } |t| \le k \\ t - k\frac{t}{|t|} & \text{if } |t| > k, \end{cases}$$

from (5.26) it results that

$$R_k u_n(x) = \begin{cases} 0 & \text{for a.e. } x \notin B_{k,n} \\ u_n(x) - k\frac{u_n(x)}{|u_n(x)|} & \text{for } x \in B_{k,n}; \end{cases} \tag{5.28}$$

hence, (5.25) and (5.28) imply that

$$R_k u_n \in W_0^{1,p}(B_1(0)) \quad \text{for all } n \in N. \tag{5.29}$$

Since $k > |u|_\infty$, we deduce that

$$T_k u(x) = u(x) \quad \text{and} \quad R_k u(x) = 0 \quad \text{for a.e. } x \in \mathbb{R}^N;$$

thus, from (5.2) it follows that $R_k u_n \rightharpoonup 0$ weakly in $W_r^{1,p}(\mathbb{R}^N)$, and, from (5.29), in $W_0^{1,p}(B_1(0))$. From the compact embedding of $W_0^{1,p}(B_1(0))$ in $L^l(B_1(0))$ for $1 \le l < p^*$, we have that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |R_k u_n|^l \, dx = 0 \quad \text{for } 1 \le l < p^*. \tag{5.30}$$

Now, arguing as in the proof of (5.19) but replacing $R_k^+ u_n$ with $R_k u_n$ we obtain

$$\begin{aligned} \frac{\alpha_0\alpha_1}{\eta_0} \int_{B_{k,n}} |\nabla u_n|^p \, dx &\le \alpha_1 \int_{B_{k,n}} a(x, u_n, \nabla u_n) \cdot \nabla u_n \, dx \\ &\le \langle d\mathcal{J}(u_n), R_k u_n \rangle + \int_{B_{k,n}} g(x, u_n) R_k u_n \, dx. \end{aligned} \tag{5.31}$$

We note that (5.5) and $\|R_k u_n\|_X \le \|u_n\|_X$ imply that

$$\lim_{n\to+\infty} |\langle d\mathcal{J}(u_n), R_k u_n \rangle| = 0; \tag{5.32}$$

while the boundedness of the sequences $(\|u_n\|_p)_n$ and $(\|R_k u_n\|_p)_n$, (5.4), (5.6), (5.28), and Lemma 4.7 imply that

$$\lim_{n\to+\infty} \int_{B_{k,n}} g(x, u_n) R_k u_n \, dx = 0. \tag{5.33}$$

From (5.31)–(5.33) we obtain that

$$\lim_{n\to+\infty} \int_{B_{k,n}} |\nabla u_n|^p \, dx = 0, \tag{5.34}$$

$$\lim_{n\to+\infty} \int_{B_{k,n}} a(x, u_n, \nabla u_n) \cdot \nabla u_n \, dx = 0. \tag{5.35}$$

Hence, from (5.28), (5.30), and (5.34) it follows that

$$\lim_{n\to+\infty} \|R_k u_n\|_p = 0. \tag{5.36}$$

Moreover, from (5.4), (5.25), and $k > |u|_\infty$ we obtain

$$\lim_{n\to+\infty} \text{meas}(B_{k,n}) = 0, \tag{5.37}$$

which together (5.16) implies

$$\lim_{n\to+\infty}\int_{B_{k,n}}|u_n|^l\,dx=0\quad\text{for } 1\le l<p^*. \tag{5.38}$$

From (3.9) and (5.27) we have

$$\begin{aligned}
&\mathcal{J}(T_ku_n)\\
&=\int_{\mathbb{R}^N\setminus B_{k,n}}A(x,u_n,\nabla u_n)\,dx+\int_{B_{k,n}}A\left(x,k\frac{u_n}{|u_n|},0\right)\,dx\\
&\quad+\frac{1}{p}\int_{\mathbb{R}^N\setminus B_{k,n}}|u_n|^p\,dx+\frac{1}{p}\int_{B_{k,n}}k^p\,dx-\int_{\mathbb{R}^N}G(x,T_ku_n)\,dx\\
&=\mathcal{J}(u_n)-\int_{B_{k,n}}A(x,u_n,\nabla u_n)\,dx+\int_{B_{k,n}}A\left(x,k\frac{u_n}{|u_n|},0\right)\,dx\\
&\quad-\frac{1}{p}\int_{B_{k,n}}|u_n|^p\,dx+\frac{1}{p}\int_{B_{k,n}}k^p\,dx-\int_{\mathbb{R}^N}(G(x,T_ku_n)-G(x,u_n))\,dx.
\end{aligned}\tag{5.39}$$

From (A5), (A6) and (5.35) we have

$$\int_{B_{k,n}}A(x,u_n,\nabla u_n)\,dx\le\eta_0\int_{B_{k,n}}a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx\to 0, \tag{5.40}$$

while (A2), (5.4), (5.37), and (5.38) imply

$$\begin{aligned}
\int_{B_{k,n}}A\Big(x,k\frac{u_n}{|u_n|},0\Big)\,dx&\le\int_{B_{k,n}}\Phi_0\Big(k\frac{u_n}{|u_n|}\Big)k^p\,dx\\
&\le\Big(\max_{|t|\le k}\Phi_0(t)\Big)k^p\,\text{meas}\,B_{k,n}\to 0
\end{aligned}\tag{5.41}$$

and

$$-\frac{1}{p}\int_{B_{k,n}}|u_n|^p\,dx+\frac{1}{p}\int_{B_{k,n}}k^p\,dx\to 0. \tag{5.42}$$

Furthermore, from (5.27), we have

$$\int_{\mathbb{R}^N}(G(x,T_ku_n)-G(x,u_n))\,dx=\int_{B_{k,n}}\Big(G\Big(x,k\frac{u_n}{|u_n|}\Big)-G(x,u_n)\Big)\,dx\to 0 \tag{5.43}$$

since (3.7), (5.37), and (5.38) imply that

$$\int_{B_{k,n}}G\Big(x,k\frac{u_n}{|u_n|}\Big)\,dx\le\frac{1}{q}|\eta|_{\frac{p}{p-q}}k^q(\text{meas}(B_{k,n}))^{\frac{q}{p}}\to 0$$

and

$$\int_{B_{k,n}}G(x,u_n)\,dx\le\frac{1}{q}|\eta|_{\frac{p}{p-q}}\Big(\int_{B_{k,n}}|u_n|^q\,dx\Big)^{\frac{q}{p}}\to 0.$$

Then, (5.22) follows from (5.5) and (5.39)–(5.43).

To prove (5.23), we take $v\in X_r$ such that $\|v\|_X=1$; hence, $|v|_\infty\le 1$, $\|v\|_W\le 1$. From (3.10) and (5.27) we have

$$\begin{aligned}
&\langle d\mathcal{J}(T_ku_n),v\rangle\\
&=\int_{\mathbb{R}^N}a(x,T_ku_n,\nabla T_ku_n)\cdot\nabla v\,dx+\int_{\mathbb{R}^N}A_t(x,T_ku_n,\nabla T_ku_n)v\,dx\\
&\quad+\int_{\mathbb{R}^N}|T_ku_n|^{p-2}T_ku_nv\,dx-\int_{\mathbb{R}^N}g(x,T_ku_n)v\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N\setminus B_{k,n}} a(x,u_n,\nabla u_n)\cdot\nabla v\,dx + \int_{B_{k,n}} a\Big(x,k\frac{u_n}{|u_n|},0\Big)\cdot\nabla v \\
&\quad + \int_{\mathbb{R}^N\setminus B_{k,n}} A_t(x,u_n,\nabla u_n)v\,dx + \int_{B_{k,n}} A_t\Big(x,k\frac{u_n}{|u_n|},0\Big)v\,dx \\
&\quad + \int_{\mathbb{R}^N\setminus B_{k,n}} |u_n|^{p-2}u_n v\,dx + \int_{B_{k,n}} k^{p-1}\frac{u_n}{|u_n|}v\,dx - \int_{\mathbb{R}^N} g(x,T_k u_n)v\,dx \\
&= \langle d\mathcal{J}(u_n),v\rangle - \int_{B_{k,n}} a(x,u_n,\nabla u_n)\cdot\nabla v\,dx - \int_{B_{k,n}} A_t(x,u_n,\nabla u_n)v\,dx \\
&\quad - \int_{B_{k,n}} |u_n|^{p-2}u_n v\,dx + \int_{B_{k,n}} (g(x,u_n)-g(x,T_k u_n))v\,dx + \epsilon_n,
\end{aligned}
$$

since (A2), (5.37), Hölder inequality and $|\nabla v|_p \le 1, |v|_\infty \le 1$ imply that

$$
\begin{aligned}
\Big|\int_{B_{k,n}} a\Big(x,k\frac{u_n}{|u_n|},0\Big)\cdot\nabla v\,dx\Big| &\le \int_{B_{k,n}} \Phi_2\Big(k\frac{u_n}{|u_n|}\Big)k^{p-1}|\nabla v|dx \\
&\le \Big(\max_{|t|\le k}\Phi_2(t)\Big)\Big(\int_{B_{k,n}} k^p\,dx\Big)^{\frac{p-1}{p}} \to 0,
\end{aligned} \tag{5.44}
$$

$$
\begin{aligned}
\Big|\int_{B_{k,n}} A_t\Big(x,k\frac{u_n}{|u_n|},0\Big)v\,dx\Big| &\le \int_{B_{k,n}} \Phi_1\Big(k\frac{u_n}{|u_n|}\Big)k^{p-1}dx \\
&\le \Big(\max_{|t|\le k}\Phi_1(t)\Big)k^{p-1}\operatorname{meas}(B_{k,n}) \to 0,
\end{aligned} \tag{5.45}
$$

$$
\Big|\int_{B_{k,n}} k^{p-1}\frac{u_n}{|u_n|}v\,dx\Big| \le k^{p-1}\operatorname{meas}(B_{k,n}) \to\ 0, \tag{5.46}
$$

where all the limits hold uniformly with respect to v.

Furthermore, from (4.3) and (5.35) we have that

$$
\lim_{n\to+\infty}\int_{B_{k,n}} |A_t(x,u_n,\nabla u_n)u_n|dx = 0,
$$

and then, since $1 \le k \le |u_n|$ on $B_{k,n}$ and $|v|_\infty \le 1$, we obtain

$$
\begin{aligned}
\Big|\int_{B_{k,n}} A_t(x,u_n,\nabla u_n)v\,dx\Big| &\le \int_{B_{k,n}} |A_t(x,u_n,\nabla u_n)|dx \\
&\le \int_{B_{k,n}} |A_t(x,u_n,\nabla u_n)||u_n|\,dx \to 0
\end{aligned} \tag{5.47}
$$

uniformly with respect to v, while from (5.38), Hölder inequality and $|v|_p \le 1$ we have

$$
\Big|\int_{B_{k,n}} |u_n|^{p-2}u_n v\,dx\Big| \le \Big(\int_{B_{k,n}} |u_n|^p\,dx\Big)^{\frac{p-1}{p}} \to 0.
$$

Moreover, from (3.8), (5.37), (5.38), and $|v|_p \le 1$ it results

$$
\Big|\int_{B_{k,n}} g(x,u_n)v\,dx\Big| \le |\eta|_{\frac{p}{p-q}}\Big(\int_{B_{k,n}} |u_n|^p\,dx\Big)^{\frac{q-1}{p}} \to 0
$$

uniformly with respect to v, and

$$
\Big|\int_{B_{k,n}} g(x,T_k u_n)v\,dx\Big| \le |\eta|_{\frac{p}{p-q}}\Big(\int_{B_{k,n}} |T_k u_n|^p\,dx\Big)^{\frac{q-1}{p}} \to 0
$$

uniformly with respect to v. Thus, summing, from (5.5) we obtain

$$|\langle d\mathcal{J}(T_k u_n), v\rangle| \le \varepsilon_{k,n} + \Big|\int_{B_{k,n}} a(x,u_n,\nabla u_n)\cdot\nabla v\,dx\Big|. \tag{5.48}$$

Now, to estimate the last integral in (5.48), following the notation introduced in the proof of Proposition 5.2, let us consider the set $B^+_{k,n}$ and the test function

$$\varphi^+_{k,n} = vR^+_k u_n.$$

By definition, we have $\|\varphi^+_{k,n}\|_X \le 2\|u_n\|_X$; thus, (5.5) implies

$$\|d\mathcal{J}(u_n)\|_{X'_r}\|\varphi^+_{k,n}\|_X \le \varepsilon_n.$$

From definition (5.13) and direct computations we note that

$$\begin{aligned}
\langle d\mathcal{J}(u_n),\varphi^+_{k,n}\rangle &= \int_{B^+_{k,n}} a(x,u_n,\nabla u_n)R^+_k u_n\cdot\nabla v\,dx + \int_{B^+_{k,n}} a(x,u_n,\nabla u_n)\cdot v\nabla u_n\,dx\\
&\quad + \int_{B^+_{k,n}} A_t(x,u_n,\nabla u_n)vR^+_k u_n\,dx + \int_{B^+_{k,n}} |u_n|^{p-2}u_n vR^+_k u_n\,dx\\
&\quad - \int_{B^+_{k,n}} g(x,u_n)vR^+_k u_n\,dx\,,
\end{aligned}$$

where, since $B^+_{k,n}\subset B_{k,n}$, from (5.37) we have

$$\lim_{n\to+\infty}\operatorname{meas}(B^+_{k,n}) = 0,$$

while $|v|_\infty \le 1$, (5.35), (5.47), (5.38), and (3.8) imply

$$\begin{aligned}
\Big|\int_{B^+_{k,n}} a(x,u_n,\nabla u_n)\cdot v\nabla u_n\,dx\Big| &\le \int_{B^+_{k,n}} a(x,u_n,\nabla u_n)\cdot\nabla u_n\,dx \to\ 0,\\
\Big|\int_{B^+_{k,n}} A_t(x,u_n,\nabla u_n)vR^+_k u_n dx\Big| &\le \int_{B^+_{k,n}} |A_t(x,u_n,\nabla u_n)|(u_n-k)dx\\
&\le \int_{B^+_{k,n}} |A_t(x,u_n,\nabla u_n)|u_n\,dx \to\ 0,\\
\Big|\int_{B^+_{k,n}} |u_n|^{p-2}u_n vR^+_k u_n\,dx\Big| &\le \int_{B^+_{k,n}} |u_n|^p\,dx \to 0,\\
\Big|\int_{B^+_{k,n}} g(x,u_n)vR^+_k u_n\,dx\Big| &\le \int_{B^+_{k,n}} |g(x,u_n)||u_n|\,dx\\
&\le |\eta|_{\frac{p}{p-q}}\Big(\int_{B^+_{k,n}} |u_n|^p\Big)^{\frac{q-1}{p}} \to 0
\end{aligned}$$

uniformly with respect to v. From the previous estimates it follows that

$$\lim_{n\to+\infty}\int_{B^+_{k,n}} a(x,u_n,\nabla u_n)R^+_k u_n\cdot\nabla v\,dx = 0 \tag{5.49}$$

Now, if we fix $k > \max\{|u|_\infty,\beta_0\}+1$, all the previous computations hold also for $k-1$ and then in particular, (5.34), (5.38), and (5.49) become

$$\lim_{n\to+\infty}\int_{B_{k-1,n}} |\nabla u_n|^p\,dx = 0,\quad \lim_{n\to+\infty}\int_{B_{k-1,n}} |u_n|^p\,dx = 0, \tag{5.50}$$

$$\lim_{n\to+\infty}\int_{B^+_{k-1,n}} a(x,u_n,\nabla u_n)R^+_{k-1}u_n\cdot\nabla v\,dx=0. \tag{5.51}$$

From (5.51) since $B^+_{k,n}\subset B^+_{k-1,n}$, we have

$$\begin{aligned}
\epsilon_{k,n} &= \int_{B^+_{k-1,n}} a(x,u_n,\nabla u_n)R^+_{k-1}u_n\cdot\nabla v\,dx\\
&= \int_{B^+_{k,n}} a(x,u_n,\nabla u_n)R^+_{k-1}u_n\cdot\nabla v\,dx\\
&\quad + \int_{B^+_{k-1,n}\setminus B^+_{k,n}} a(x,u_n,\nabla u_n)R^+_{k-1}u_n\cdot\nabla v\,dx\\
&= \int_{B^+_{k,n}} a(x,u_n,\nabla u_n)R^+_{k}u_n\cdot\nabla v\,dx + \int_{B^+_{k,n}} a(x,u_n,\nabla u_n)\cdot\nabla v\,dx\\
&\quad + \int_{B^+_{k-1,n}\setminus B^+_{k,n}} a(x,u_n,\nabla u_n)R^+_{k-1}u_n\cdot\nabla v\,dx
\end{aligned}$$

where (A2), (5.13), the properties of $B^+_{k-1,n}\setminus B^+_{k,n}$, Hölder inequality, $|\nabla v|_p\le 1$, and (5.50) imply

$$\begin{aligned}
&\Big|\int_{B^+_{k-1,n}\setminus B^+_{k,n}} a(x,u_n,\nabla u_n)R^+_{k-1}u_n\cdot\nabla v\,dx\Big|\\
&\le k\int_{B^+_{k-1,n}\setminus B^+_{k,n}} |a(x,u_n,\nabla u_n)||\nabla v|\,dx\\
&\le k\max_{|t|\le k}\Phi_2(t)\int_{B^+_{k-1,n}\setminus B^+_{k,n}} |u_n|^{p-1}|\nabla v|\,dx\\
&\quad + k\max_{|t|\le k}\phi_2(t)\int_{B^+_{k-1,n}\setminus B^+_{k,n}} |\nabla u_n|^{p-1}|\nabla v|\,dx\\
&\le k\max_{|t|\le k}\Phi_2(t)\Big(\int_{B^+_{k-1,n}\setminus B^+_{k,n}} |u_n|^{p}\,dx\Big)^{\frac{p-1}{p}}\\
&\quad + k\max_{|t|\le k}\phi_2(t)\Big(\int_{B^+_{k-1,n}\setminus B^+_{k,n}} |\nabla u_n|^{p}\,dx\Big)^{\frac{p-1}{p}}\to 0.
\end{aligned}$$

The above arguments imply

$$\Big|\int_{B^+_{k,n}} a(x,u_n,\nabla u_n)\cdot\nabla v\,dx\Big|\le\varepsilon_{k,n}. \tag{5.52}$$

Similar arguments apply also if we consider $B^-_{k,n}$ and the test functions

$$\varphi^-_{k,n}=vR^-_k u_n,\quad \varphi^-_{k-1,n}=vR^-_{k-1}u_n;$$

hence, we have

$$\Big|\int_{B^-_{k,n}} a(x,u_n,\nabla u_n)\cdot\nabla v\,dx\Big|\le\varepsilon_{k,n}. \tag{5.53}$$

Thus, (5.23) follows from (5.48), (5.52) and (5.53) as all $\varepsilon_{k,n}$ are independent of v.

Step 2. We note that (5.2)–(5.4) imply that, if $n \to +\infty$,

$$\begin{gathered}T_k u_n \rightharpoonup u \quad \text{weakly in } W_r^{1,p}(\mathbb{R}^N),\\ T_k u_n \to u \quad \text{strongly in } L^l(\mathbb{R}^N) \text{ for each } l \in]p, p^*[,\\ T_k u_n \to u \quad \text{a.e. in } \mathbb{R}^N.\end{gathered}$$

Now, arguing as in [1], let us consider the real map

$$\psi : t \in \mathbb{R} \mapsto \psi(t) = t\mathrm{e}^{\bar{\eta}t^2} \in \mathbb{R},$$

where $\bar{\eta} > (\frac{\beta}{2\alpha})^2$ will be fixed once α, $\beta > 0$ are chosen in a suitable way later. By definition,

$$\alpha\psi'(t) - \beta|\psi(t)| > \frac{\alpha}{2} \quad \text{for all } t \in \mathbb{R}. \tag{5.54}$$

If we define $v_{k,n} = T_k u_n - u$, since $k > |u|_\infty$, we have that $|v_{k,n}|_\infty \le 2k$ for all $n \in \mathbb{N}$. Therefore,

$$|\psi(v_{k,n})| \le \psi(2k), \quad 0 < \psi'(v_{k,n}) \le \psi'(2k) \quad \text{a.e. in } \mathbb{R}^N \text{ for all } n \in \mathbb{N}, \tag{5.55}$$

$$\psi(v_{k,n}) \to 0, \quad \psi'(v_{k,n}) \to 1 \quad \text{a.e. in } \mathbb{R}^N \text{ as } n \to +\infty. \tag{5.56}$$

Furthermore, we note that

$$|\psi(v_{k,n})| \le |v_{k,n}|\mathrm{e}^{4k^2\bar{\eta}} \quad \text{a.e. in } \mathbb{R}^N \text{ for all } n \in \mathbb{N},$$

thus, direct computations imply that $(\|\psi(v_{k,n})\|_X)_n$ is bounded, and so from (5.56), up to subsequences, we have

$$\psi(v_{k,n}) \rightharpoonup 0 \quad \text{weakly in } W_r^{1,p}(\mathbb{R}^N), \tag{5.57}$$

while from (5.23) it follows that

$$\langle d\mathcal{J}(T_k u_n), \psi(v_{k,n})\rangle \to 0 \quad \text{as } n \to +\infty,$$

where

$$\begin{aligned}&\langle d\mathcal{J}(T_k u_n), \psi(v_{k,n})\rangle\\ &= \int_{\mathbb{R}^N\setminus B_{k,n}} a(x, u_n, \nabla u_n)\cdot\nabla\psi(v_{k,n})\,dx + \int_{B_{k,n}} a\Big(x, k\frac{u_n}{|u_n|}, 0\Big)\cdot\nabla\psi(v_{k,n})\,dx\\ &\quad + \int_{\mathbb{R}^N\setminus B_{k,n}} A_t(x, u_n, \nabla u_n)\psi(v_{k,n})\,dx + \int_{B_{k,n}} A_t\Big(x, k\frac{u_n}{|u_n|}, 0\Big)\psi(v_{k,n})\,dx\\ &\quad + \int_{\mathbb{R}^N\setminus B_{k,n}} |u_n|^{p-2}u_n\psi(v_{k,n})\,dx + \int_{B_{k,n}} k^{p-1}\frac{u_n}{|u_n|}\psi(v_{k,n})\,dx\\ &\quad - \int_{\mathbb{R}^N} g(x, T_k u_n)\psi(v_{k,n})\,dx.\end{aligned}$$

Since $(\|\psi(v_{k,n})\|_X)_n$ is bounded, arguing as in (5.44)–(5.46) it follows that

$$\begin{gathered}\lim_{n\to+\infty}\int_{B_{k,n}} a\Big(x, k\frac{u_n}{|u_n|}, 0\Big)\cdot\nabla\psi(v_{k,n})\,dx = 0,\\ \lim_{n\to+\infty}\int_{B_{k,n}} A_t\Big(x, k\frac{u_n}{|u_n|}, 0\Big)\psi(v_{k,n})\,dx = 0,\\ \lim_{n\to+\infty}\int_{B_{k,n}} k^{p-1}\frac{u_n}{|u_n|}\psi(v_{k,n})\,dx = 0.\end{gathered}$$

Furthermore, from Lemma 4.7 with $w_n = T_k u_n$ and $v_n = \psi(v_{k,n})$, we have

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} g(x, T_k u_n)\psi(v_{k,n})\, dx = 0.$$

Hence, summing, the previous relations imply

$$\begin{aligned}\varepsilon_{k,n} = &\int_{\mathbb{R}^N\setminus B_{k,n}} a(x, u_n, \nabla u_n)\psi'(v_{k,n}) \cdot \nabla v_{k,n}\, dx \\ &+ \int_{\mathbb{R}^N\setminus B_{k,n}} A_t(x, u_n, \nabla u_n)\psi(v_{k,n})\, dx \\ &+ \int_{\mathbb{R}^N\setminus B_{k,n}} |u_n|^{p-2} u_n \psi(v_{k,n})\, dx.\end{aligned} \tag{5.58}$$

We note that from (A2),

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N\setminus B_{k,n}} A_t(x, u_n, \nabla u_n)\psi(v_{k,n})\, dx\Big| \\ &\le \int_{\mathbb{R}^N\setminus B_{k,n}} \Big(\Phi_1(u_n)|u_n|^{p-1} + \max_{|t|\le k} \phi_1(t)|\nabla u_n|^p\Big)|\psi(v_{k,n})| dx.\end{aligned} \tag{5.59}$$

We prove that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N\setminus B_{k,n}} \Phi_1(u_n)|u_n|^{p-1}|\psi(v_{k,n})|\, dx = 0. \tag{5.60}$$

In fact, since the sequence $(u_n)_n$ is bounded in $W^{1,p}_r(\mathbb{R}^N)$, there exists a constant $\tilde{M} > 0$ such that

$$\|u_n\|_p \le \tilde{M}, \quad \|u_n - u\|_p \le \tilde{M} \quad \text{for all } n \in \mathbb{N}.$$

Moreover, from assumption (A11),

$$\lim_{t\to 0} \frac{\Phi_1(t)}{|t|^{\eta_1}} = l_1 \quad \text{with } l_1 \ge 0,$$

hence, there exists $\delta_1 > 0$ such that

$$\Phi_1(t) < (l_1 + 1)|t|^{\eta_1} \quad \text{for all } t \in \mathbb{R}, |t| < \delta_1. \tag{5.61}$$

Now, fixing $\epsilon > 0$, as from (4.1) it follows that $(\eta_1 + p)\frac{N-1}{p} > N$, then there exists R_ϵ such that

$$\frac{C\tilde{M}}{R_\epsilon^{\frac{N-1}{p}}} < \delta_1, \tag{5.62}$$

$$(l_1 + 1)(C\tilde{M})^{p+\eta_1} \mathrm{e}^{\bar{\eta}\frac{C^2\tilde{M}^2}{R_\epsilon^{2\frac{N-1}{p}}}} \int_{B^c_{R_\epsilon}} \frac{1}{|x|^{(\eta_1+p)\frac{N-1}{p}}} dx < \epsilon \tag{5.63}$$

where C is the constant introduced in (4.9). From (4.9) and (5.62), it follows that

$$|u_n(x)| \le C\frac{\tilde{M}}{|x|^{\frac{N-1}{p}}} \le C\frac{\tilde{M}}{R_\epsilon^{\frac{N-1}{p}}} < \delta_1 \quad \text{a.e. } x \in \mathbb{R}^N \text{ with } |x| > R_\epsilon;$$

hence, (5.61), (4.9), and (5.63) imply

$$\int_{(\mathbb{R}^N\setminus B_{k,n})\cap B^c_{R_\epsilon}} \Phi_1(u_n)|u_n|^{p-1}|\psi(v_{k,n})|\, dx$$

$$\begin{aligned}&\le \int_{(\mathbb{R}^N\setminus B_{k,n})\cap B^c_{R_\epsilon}} (l_1+1)|u_n|^{\eta_1+p-1}|u_n-u|\mathrm{e}^{\bar\eta\|u_n-u\|^2_W}\,dx\\
&\le (l_1+1)(C\tilde M)^{\eta_1+p}\mathrm{e}^{\bar\eta\frac{C^2\tilde M^2}{R_\epsilon{}^{2\frac{N-1}{p}}}}\int_{B^c_{R_\epsilon}}\frac{1}{|x|^{(\eta_1+p)\frac{N-1}{p}}}dx<\epsilon\end{aligned}$$

while from Hölder's inequality

$$\begin{aligned}&\int_{(\mathbb{R}^N\setminus B_{k,n})\cap B_{R_\epsilon}} \Phi_1(u_n)|u_n|^{p-1}|\psi(v_{k,n})|\,dx\\
&\le \big(\max_{|t|\le k}\Phi_1(t)\big)|u_n|_p^{p-1}\Big(\int_{B_{R_\epsilon}}|\psi(v_{k,n})|^p\,dx\Big)^{1/p}\to 0\end{aligned}$$

since (5.57) implies that $\psi(v_{k,n})\to 0$ in $L^p_{loc}(\mathbb{R}^N)$. Then, (5.60) holds and from (A5) and (A6) it follows that

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus B_{k,n}}|\nabla u_n|^p|\psi(v_{k,n})|\,dx\\
&\le \frac{\eta_0}{\alpha_0}\int_{\mathbb{R}^N\setminus B_{k,n}} a(x,u_n,\nabla u_n)\cdot\nabla u_n|\psi(v_{k,n})|\,dx\\
&= \frac{\eta_0}{\alpha_0}\int_{\mathbb{R}^N\setminus B_{k,n}} a(x,u_n,\nabla u_n)\cdot\nabla v_{k,n}|\psi(v_{k,n})|\,dx\\
&\quad+\frac{\eta_0}{\alpha_0}\int_{\mathbb{R}^N\setminus B_{k,n}} a(x,u_n,\nabla u_n)\cdot\nabla u|\psi(v_{k,n})|dx,\end{aligned}\tag{5.64}$$

where the boundedness of $(u_n)_n$ in $W^{1,p}_r(\mathbb{R}^N)$, (A2), Hölder's inequality, (5.56) and the Lebesgue Dominated Convergence Theorem imply that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N\setminus B_{k,n}} a(x,u_n,\nabla u_n)\cdot\nabla u|\psi(v_{k,n})|dx\Big|\\
&\le \int_{\mathbb{R}^N\setminus B_{k,n}}\Phi_2(u_n)|u_n|^{p-1}|\nabla u||\psi(v_{k,n})|dx\\
&\quad+\int_{\mathbb{R}^N\setminus B_{k,n}}\phi_2(u_n)|\nabla u_n|^{p-1}|\nabla u||\psi(v_{k,n})|dx\\
&\le \Big(\max_{|t|\le k}\Phi_2(t)\Big)|u_n|_p^{p-1}\Big(\int_{\mathbb{R}^N\setminus B_{k,n}}|\nabla u|^p|\psi(v_{k,n})|^pdx\Big)^{1/p}\\
&\quad+\Big(\max_{|t|\le k}\phi_2(t)\Big)|\nabla u_n|_p^{p-1}\Big(\int_{\mathbb{R}^N\setminus B_{k,n}}|\nabla u|^p|\psi(v_{k,n}|^p)\,dx\Big)^{1/p}\to 0.\end{aligned}\tag{5.65}$$

From (5.58)–(5.60), (5.64), (5.65), (A5) and (A6) we obtain

$$\begin{aligned}\epsilon_{k,n}\ge&\int_{\mathbb{R}^N\setminus B_{k,n}} a(x,u_n,\nabla u_n)\psi'(v_{k,n})\cdot\nabla v_{k,n}dx\\
&-\frac{\eta_0}{\alpha_0}\max_{|t|\le k}\phi_1(t)\int_{\mathbb{R}^N\setminus B_{k,n}} a(x,u_n,\nabla u_n)\cdot\nabla v_{k,n}|\psi(v_{k,n})|\,dx\\
&+\int_{\mathbb{R}^N\setminus B_{k,n}}|u_n|^{p-2}u_n\psi(v_{k,n})\,dx.\end{aligned}$$

Thus, setting

$$h_{k,n}(x) = \psi'(v_{k,n}) - \frac{\eta_0}{\alpha_0} \max_{|t|\le k} \phi_1(t)|\psi(v_{k,n})|,$$

and choosing, in the definition of ψ, constants $\alpha = 1$ and $\beta = \frac{\eta_0}{\alpha_0} \max_{|t|\le k} \phi_1(t)$, from (5.54) it results

$$h_{k,n}(x) > \frac{1}{2} \quad \text{a.e. in } \mathbb{R}^N. \tag{5.66}$$

Therefore,

$$\begin{aligned}
\varepsilon_{k,n} \ge & \int_{\mathbb{R}^N\setminus B_{k,n}} h_{k,n} a(x,u_n,\nabla u_n)\cdot \nabla v_{k,n}\,dx \\
&+ \int_{\mathbb{R}^N\setminus B_{k,n}} |u_n|^{p-2}u_n \psi(v_{k,n})\,dx \\
=& \int_{\mathbb{R}^N\setminus B_{k,n}} a(x,u,\nabla u)\cdot \nabla v_{k,n}\,dx \\
&+ \int_{\mathbb{R}^N\setminus B_{k,n}} h_{k,n} \left(a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)\right)\cdot \nabla v_{k,n}\,dx \\
&+ \int_{\mathbb{R}^N\setminus B_{k,n}} \left(h_{k,n} a(x,u_n,\nabla u) - a(x,u,\nabla u)\right)\cdot \nabla v_{k,n}\,dx \\
&+ \int_{\mathbb{R}^N\setminus B_{k,n}} (|u_n|^{p-2}u_n - |u|^{p-2}u)\psi(v_{k,n})\,dx \\
&+ \int_{\mathbb{R}^N\setminus B_{k,n}} |u|^{p-2}u\psi(v_{k,n})\,dx,
\end{aligned} \tag{5.67}$$

where (5.2), respectively (5.57) imply that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N\setminus B_{k,n}} a(x,u,\nabla u)\cdot \nabla v_{k,n}\,dx = 0, \quad \lim_{n\to+\infty} \int_{\mathbb{R}^N\setminus B_{k,n}} |u|^{p-2}u\psi(v_{k,n})\,dx = 0.$$

Now, we want to prove that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N\setminus B_{k,n}} \Big(h_{k,n} a(x,u_n,\nabla u) - a(x,u,\nabla u)\Big)\cdot \nabla v_{k,n}\,dx = 0. \tag{5.68}$$

Indeed, recalling that $(\nabla v_{k,n})_n$ is bounded in $L^p(\mathbb{R}^N)$, arguing as in the proof of (4.26), from (A11) for all $\epsilon > 0$ there exists $R_\epsilon > 0$ such that

$$\int_{(\mathbb{R}^N\setminus B_{k,n})\cap B^c_{R_\epsilon}} |h_{k,n} a(x,u_n,\nabla u) - a(x,u,\nabla u)|^{\frac{p}{p-1}}\,dx < \epsilon \tag{5.69}$$

where $(\mathbb{R}^N \setminus B_{k,n}) \cap B^c_{R_\epsilon} = B^c_{R_\epsilon}(0)$. On the other hand, we note that (A1), (5.4) and (5.56) infer that

$$h_{k,n} a(x,u_n,\nabla u) - a(x,u,\nabla u) \to 0 \quad \text{a.e. in} \mathbb{R}^N,$$

while from Hölder's inequality it follows that

$$\begin{aligned}
&|\int_{\mathbb{R}^N\setminus B_{k,n}} \Big(h_{k,n} a(x,u_n,\nabla u) - a(x,u,\nabla u)\Big)\cdot \nabla v_{k,n}\,dx| \\
&\le \Big(\int_{\mathbb{R}^N\setminus B_{k,n}} |h_{k,n} a(x,u_n,\nabla u) - a(x,u,\nabla u)|^{\frac{p}{p-1}}\,dx\Big)^{\frac{p-1}{p}} |\nabla v_{k,n}|_p.
\end{aligned} \tag{5.70}$$

From (5.55) and (A2) we have that for each $x \in (\mathbb{R}^N \setminus B_{k,n})$,

$$\begin{aligned} &|h_{k,n}a(x,u_n,\nabla u) - a(x,u,\nabla u)|^{\frac{p}{p-1}} \\ &\le \Big(\psi'(2k)\Big(\Phi_2(u_n)|u_n|^{p-1} + (\max_{|t|\le k}\phi_2(t))|\nabla u|^{p-1}\Big) + |a(x,u,\nabla u)|\Big)^{\frac{p}{p-1}} \\ &\le c(1+|\nabla u|^p), \end{aligned} \tag{5.71}$$

hence, the Lebesgue Dominated Convergence Theorem implies that

$$\lim_{n\to+\infty}\int_{(\mathbb{R}^N\setminus B_{k,n})\cap B_{R_\epsilon}(0)} |h_{k,n}a(x,u_n,\nabla u) - a(x,u,\nabla u)|^{\frac{p}{p-1}}\,dx = 0. \tag{5.72}$$

Thus, from (5.66) and (5.67), by using the previous estimate, the strong convexity of the power function with exponent $p>1$, (A9) and $e^{\bar\eta v_{k,n}^2} \ge 1$ we obtain

$$\begin{aligned} \varepsilon_{k,n} \ge \frac{1}{2}\int_{\mathbb{R}^N\setminus B_{k,n}} &\Big(a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)\Big)\cdot\nabla(u_n-u)\,dx \\ &+ \int_{\mathbb{R}^N\setminus B_{k,n}} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)\,dx. \end{aligned}$$

Using again (A9) and the strong convexity of the power function with exponent $p>1$ we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_{k,n}} \Big(a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)\Big)\cdot\nabla(u_n-u)\,dx = 0, \tag{5.73}$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_{k,n}} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u)\,dx = 0 \tag{5.74}$$

Next we prove that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_{k,n}} |u_n-u|^p\,dx = 0. \tag{5.75}$$

In fact, if $p \ge 2$,

$$|u_n-u|^p \le (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n-u) \quad \text{a.e. } x\in\mathbb{R}^N, \text{ for all } n\in\mathbb{N}; \tag{5.76}$$

thus, (5.76) implies (5.75). On the other hand, if $p\in(1,2)$, it is $\frac{p}{p-1} > p$; thus, as $(T_k u_n)_n$ is bounded in $W^{1,p}(\mathbb{R}^N)$ and $|T_k u_n| \le k$ a.e. $x\in\mathbb{R}^N\setminus B_{k,n}$, for all $n\in\mathbb{N}$, it follows that $(T_k u_n)_n$ is bounded in $L^\ell(\mathbb{R}^N)$ for any $\ell \ge p$, and in particular is bounded in $L^{\frac{p}{p-1}}(\mathbb{R}^N)$. Passing to a subsequence, $T_k u_n \rightharpoonup u$ in $L^{\frac{p}{p-1}}(\mathbb{R}^N)$, hence

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} |T_k u_n|^{p-2}T_k u_n u\,dx = \int_{\mathbb{R}^N}|u|^p\,dx,$$

which implies, together (5.37), that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_{k,n}} |u_n|^{p-2}u_n u\,dx = \int_{\mathbb{R}^N}|u|^p\,dx. \tag{5.77}$$

Moreover, since $(T_k u_n)_n$ is bounded in $L^p(\mathbb{R}^N)$ and $u \in L^{\frac{p}{p-1}}(\mathbb{R}^N)$, up to subsequences, we have that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} |u|^{p-2}u T_k u_n\,dx = \int_{\mathbb{R}^N}|u|^p\,dx,$$

i.e., using again (5.37),

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_{k,n}}|u|^{p-2}uu_n\,dx=\int_{\mathbb{R}^N}|u|^p\,dx. \tag{5.78}$$

Hence, from (5.74), (5.77), (5.78), (5.37), and (5.38) we obtain

$$\begin{aligned}
0&=\lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_{k,n}}\big(|u_n|^p+|u|^p-|u_n|^{p-2}u_nu-|u|^{p-2}uu_n\big)\,dx\\
&=\lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_{k,n}}|u_n|^p\,dx+\int_{\mathbb{R}^N}|u|^p\,dx\\
&\quad-\lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_{k,n}}|u_n|^{p-2}u_nu\,dx-\lim_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_{k,n}}|u|^{p-2}uu_n\,dx\\
&=\lim_{n\to+\infty}\int_{\mathbb{R}^N}|u_n|^p\,dx-\int_{\mathbb{R}^N}|u|^p\,dx,
\end{aligned}$$

i.e.,

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|u_n|^p\,dx=\int_{\mathbb{R}^N}|u|^p\,dx.$$

Thus, by applying Brezis-Lieb's Lemma (see [9]), condition (5.75) follows, also in the case $1<p<2$.

In each case, $T_ku_n\to u$ in $L^p(\mathbb{R}^N)$. Finally, as $T_ku_n\to u$ a.e. in $\mathbb{R}^N$ and $|T_ku_n|_\infty\le k$ for all $n\in\mathbb{N}$, from (5.73), we can apply Lemma 4.11 to the sequence $(T_ku_n)_n$ obtaining that $\nabla T_ku_n\to\nabla u$ in $L^p(\mathbb{R}^N)$. Thus, (5.24) follows.

Step 3. The proof follows from (5.24), Proposition 3.7, (5.22) and (5.23). □

Proof of Theorem 4.5. The functional $\mathcal{J}$ is bounded from below in X (see Proposition 4.6) and satisfies condition $(wCPS)$ in $\mathbb{R}$ (see Proposition 5.3), thus, from Proposition 2.2, $\mathcal{J}$ admits a minimum point u^* in X. Clearly, it is

$$\mathcal{J}(u^*)=\min_{u\in X}\mathcal{J}(u)\le\mathcal{J}(0)=0.$$

Now, we prove that u^* is not trivial since $\mathcal{J}(u^*)<0$. To this aim, we consider $\varphi_1\in W_0^{1,p}(B_1(0))$ the unique eigenfunction associated to the first eigenvalue λ_1 of $-\Delta_p$ in $B_1(0)$ (see [25]). It results

$$\begin{gathered}
\varphi_1>0 \text{ a.e. in } B_1(0),\quad \varphi_1\in L^\infty(B_1(0)),\\
\int_{B_1(0)}|\varphi_1|^p\,dx=1,\quad \int_{B_1(0)}|\nabla\varphi_1|^p\,dx=\lambda_1.
\end{gathered}$$

We denote again by φ_1 its null extension to $\mathbb{R}^N\setminus B_1(0)$.

Let us remark that φ_1 is radial since by the Pólya-Szegö inequality we have

$$\lambda_1=|\nabla\varphi_1|_p^p\ge|\nabla\varphi_1^\star|_p^p,$$

where $\varphi_1^\star$ is the Schwartz rearrangement of φ_1.

Taking $\tau\in(0,1)$, from (A2) we have

$$\begin{aligned}
\mathcal{J}(\tau\varphi_1)&=\int_{\mathbb{R}^N}A(x,\tau\varphi_1,\nabla(\tau\varphi_1))dx+\frac{1}{p}\int_{\mathbb{R}^N}|\tau\varphi_1|^p\,dx-\int_{\mathbb{R}^N}G(x,\tau\varphi_1)dx\\
&\le\int_{B_1(0)}\left(\Phi_0(\tau\varphi_1(x))|\tau\varphi_1(x)|^p+\phi_0(\tau\varphi_1(x))|\nabla(\tau\varphi_1(x))|^p\right)dx
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\tau^p}{p}\int_{B_1(0)}|\varphi_1|^p\,dx-\int_\Omega G(x,\tau\varphi_1)dx\\
&\le c_1\tau^p-\int_{B_1(0)}G(x,\tau\varphi_1)dx,
\end{aligned}
$$

where $c_1=\max_{0\le t\le|\varphi_1|_\infty}\Phi_0(t)+\lambda_1\max_{0\le t\le|\varphi_1|_\infty}\phi_0(t)+\frac{1}{p}$.

Now, from (A14) there exists a constant $\delta>0$ such that for each $s\in[0,\delta]$ and for a.e. $x\in B_1(0)$ it is $G(x,s)>2c_1s^p$. Then, for any $\tau>0$ small sufficient, in particular $0<\tau<\frac{\delta}{|\varphi_1|_\infty}$, it results

$$
\mathcal{J}(\tau\varphi_1)\le c_1\tau^p-2c_1\tau^p<0.
$$

Finally, let us prove that $\mathcal{J}$ has at least two solutions, one negative and one positive. For this, let us denote by $u_+=\max\{0,u\}$ and $u_-=\max\{0,-u\}$, the positive and the negative part of u, respectively, so that $u=u_+-u_-$.

If we replace $g(x,u)$ by $g_+(x,u):=g(x,u_+)$, all the previous statements still hold true for the functional $\mathcal{J}_+$ obtained by replacing G with G_+, defined as $G_+(x,t)=\int_0^t g_+(x,s)ds$. In particular, $\mathcal{J}_+$ has a nontrivial critical point u. Hence, from (A5) and (A6) we find that

$$
\begin{aligned}
0=\langle d\mathcal{J}_+(u),-u_-\rangle&=\int_{\mathbb{R}^N}a(x,-u_-,\nabla(-u_-))\nabla(-u_-)\,dx\\
&\quad+\int_{\mathbb{R}^N}A_t(x,-u_-,\nabla(-u_-))(-u_-)dx+\int_{\mathbb{R}^N}|u_-|^pdx-\int_{\mathbb{R}^N}g_+(x,u)u_-dx\\
&\ge\frac{\alpha_0\alpha_1}{\eta_0}\int_{\mathbb{R}^N}|\nabla u_-|^p\,dx+\int_{\mathbb{R}^N}|u_-|^p\,dx\\
&\ge\frac{\alpha_0\alpha_1}{\eta_0}\|u_-\|_W.
\end{aligned}
$$

Hence, $u_-=0$ a.e. in $\mathbb{R}^N$, and u is a positive critical point of $\mathcal{J}$.

Similarly, replacing $g(x,u)$ with $g(x,-u_-)$, we find a negative solution of (1.1). □

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