Electronic Journal of Differential Equations, Vol. 2024 (2024), no. 43, pp. 1–25.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2024.43

GROUND STATE SOLUTIONS FOR NONLINEAR SCHRÖDINGER-BOPP-PODOLSKY SYSTEMS WITH NONPERIODIC POTENTIALS

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Abstract. In this article we study the existence of ground-state solutions for the Schrödinger-Bopp-Podolsky equations

$$\begin{gathered} -\Delta u + V(x)u + \phi u = f(x,u) \quad \text{in } \mathbb{R}^3 \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 \quad \text{in } \mathbb{R}^3, \end{gathered}$$

where $V \in C(\mathbb{R}^3,\mathbb{R})$ has different forms on the half spaces, i.e. $V(x) = V_1(x)$ for $x_1 > 0$, and $V(x) = V_2(x)$ for $x_1 < 0$, where $V_1, V_2 \in C(\mathbb{R}^3)$ are periodic in each coordinate. The nonlinearity f is superlinear at infinity with subcritical or critical growth.

1. Introduction

In this article we consider the existence of ground state solutions to Schrödinger-Bopp-Podolsky equations:

$$\begin{gathered} -\Delta u + V(x)u + \phi u = f(x,u) \quad \text{in } \mathbb{R}^3 \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 \quad \text{in } \mathbb{R}^3 \end{gathered} \tag{1.1}$$

where $a > 0$ is the Bopp-Podolsky (BP) parameter. This system, which was first studied in [13], appears when one looks for stationary solutions $u(x)e^{iwt}$ of the Schrödinger equation coupled with the Bopp-Podolsky Lagrangian of the electromagnetic field.

The Bopp-Podolsky theory, developed by Bopp [2], and independently by Podolsky [3], is a second order theory for the electromagnetic field. As the Mie theory [21] and its generalizations given by Born and Infeld [4, 5, 6, 7], it was proposed to deal with the so called infinity problem that appears in the classical Maxwell theory. In fact, by the well-known Gauss law (or Poisson's equation), the electrostatic potential ϕ for a given charge distribution whose density is ρ satisfies the equation

$$-\Delta\phi = \rho \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

2020 *Mathematics Subject Classification.* 35B38, 35A15, 35Q55.
Key words and phrases. Schrödinger-Bopp-Podolsky equation; variational method; Nehari manifold; critical growth.

Submitted April 22, 2024. Published August 12, 2024.

If $\rho = 4\pi\delta_{x_0}$, with $x_0 \in \mathbb{R}^3$, the fundamental solution of (1.2) is $\mathcal{G}(x - x_0)$, where

$$\mathcal{G}(x) = \frac{1}{|x|},$$

and the electrostatic energy is

$$\mathcal{E}_M(\mathcal{G}) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \mathcal{G}|^2 = +\infty.$$

Thus, to overcome this inconvenient new electromagnetic theories appeared. The most important ones are the Born-Infeld theory where equation (1.2) is replaced by

$$-\operatorname{div}\Big(\frac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\Big) = \rho \quad \text{in } \mathbb{R}^3$$

and the Bopp-Podolsky theory where the equation for the electrostatic field is

$$-\Delta\phi + a^2\Delta^2\phi = \rho \quad \text{in } \mathbb{R}^3.$$

In both cases, if $\rho = 4\pi\delta_{x_0}$, their solutions can be written explicitly, and the corresponding energy is finite.

In this article, we focus on the Bopp-Podolsky theory, which then involves the study of the operator $-\Delta + a^2\Delta^2$ whose fundamental solution satisfies

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi\delta_{x_0}$$

and is given by $\mathcal{K}(x - x_0)$, where

$$\mathcal{K}(x) := \frac{1 - e^{-\frac{|x|}{a}}}{|x|}.$$

In particular it presents no singularities at x_0, since

$$\lim_{x\to x_0} \mathcal{K}(x - x_0) = \frac{1}{a}$$

and its energy is

$$\mathcal{E}_{\rm BP}(\mathcal{K}) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\mathcal{K}|^2 \,\mathrm{d}x + \frac{a^2}{2}\int_{\mathbb{R}^3} |\Delta\mathcal{K}|^2 \,\mathrm{d}x < \infty.$$

We refer to [13] for more details.

The most common Schrödinger-Bopp-Podolsky system is

$$\begin{gathered} -\Delta u + V(x)u + K(x)\phi u = f(x,u) \quad \text{in } \mathbb{R}^3 \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 \quad \text{in } \mathbb{R}^3 \end{gathered} \tag{1.3}$$

In recent years, the question of the existence of solutions for (1.3) has been intensively studied by many researchers under a variety of conditions on V, K, f: we refer the reader to the papers [11, 13, 15, 16, 18, 22, 23, 27, 28] and the references therein.

d'Avenia and Siciliano [13] firstly studied the system (1.3) where they assumed $V(x)$ is a positive constant, $K(x) = q^2$ and $f(x,u) = |u|^{p-2}u$ for $p \in (2,6)$. By using a suitable truncation and a useful splitting lemma, they obtained the existence and nonexistence of solutions. In particular, they take two different approaches to overcome the lack of compactness of the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3), 2 < s < 6$: by means of the Splitting Lemma and by looking for solutions in the subspace of radial functions of $H^1(\mathbb{R}^3)$, both of which are only available for the case $p \in (2,6)$.

In [11, 18], the main results extended the existence results in [13] which only dealt with the subcritical case to critical case. In [11], by using some new analytic techniques and new inequalities, Chen and Tang assume $K(x) = 1, f(x, u) = \mu g(u) + u^5$ and prove that system (1.3) admits ground state solutions for all $\mu > 0$ if $p \in (4, 6)$; for all $\mu > \mu_0$ if $p \in (2, 4]$. In [18], Li, Pucci and Tang considered the system when $K(x) = q^2$ and $f(x, u) = \mu|u|^{p-1}u + |u|^4 u$. Under certain assumptions on V, they prove the existence of a nontrivial ground state solution, using the method of the Pohožaev-Nehari manifold, the arguments of Brezis-Nirenberg, the monotonicity trick and a global compactness lemma.

Yang, Chen and Liu [28] assume V is coercive, $K(x) = 1, f(x, u) = \lambda g(u) + |u|^4 u$. By using cut-off functions, the mountain pass theorem and Moser iteration, they prove the existence result without any growth and Ambrosetti-Rabinowitz conditions.

Siciliano and Silva [23] assume V is a positive constant, $K(x) = q^2$ and $f(x, u) = |u|^{p-2}u$ for $p \in (2, 3]$. Different from [13], they apply the fibering approach, and prove the system has no solutions at all for large values of q and has two radial solutions for small q.

For the periodic potential and the nonperiodic potential, Yang, Yuan and Liu [27] study the existence of ground states for a nonlinear Schrödinger-Bopp-Podolsky system with asymptotically periodic potentials:

$$V \in C(\mathbb{R}^3, \mathbb{R}), 0 \le V(x) \le V_\infty(x) \in L^\infty(\mathbb{R}^3), \quad \text{for all } x \in \mathbb{R}^3 \text{ and } V - V_\infty \in \mathcal{F}.$$

Here $\mathcal{F} = \{k(x) : \forall \varepsilon > 0, m(\{x \in B_1(y) : |k(x)| \ge \varepsilon\}) \to 0 \ as \ |y| \to \infty\}$. As a consequence, they also prove existence of ground states for the nonlinear Schrödinger-Bopp-Podolsky system with periodic potentials.

In particular, Cheng and Wang [12] investigated the following Schrödinger-Poisson system with nonperiodic potential and subcritical exponent:

$$\begin{gathered} -\Delta u + V(x)u + \phi u = a(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^3 \\ -\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.4}$$

where $p \in [4, 6), V, a \in C(\mathbb{R}^3, \mathbb{R})$,

$$V(x) = \begin{cases} V_1(x), & x \in \mathbb{R}^3_+, \\ V_2(x), & x \in \mathbb{R}^3_-, \end{cases} \qquad a(x) = \begin{cases} a_1(x), & x \in \mathbb{R}^3_+, \\ a_2(x), & x \in \mathbb{R}^3_-. \end{cases}$$

Here $\mathbb{R}^3_\pm = \{x \in \mathbb{R}^3 : \pm x_1 > 0\}$ and

- (H1) $V_1, V_2, a_1, a_2 \in C(\mathbb{R}^3)$ are T_k periodic in the x_k-direction for $k = 1, 2, 3$ with $T_1 = 1$,
- (H2) essinf $a_i > 0$, for $i = 1, 2$,
- (H3) min $\sigma(-\Delta + V) > 0$.

Borrowing an idea from [14], they got a surface gap soliton ground state by using a variant of Lion concentration compactness lemma and based on the ground state energies of each periodic problem.

Kang, Chen and Tang [17] investigated the following Schrödinger-Poisson system with nonperiodic potential and critical exponent:

$$\begin{gathered} -\Delta u + V(x)u + \phi u = |u|^4 u + \lambda |u|^{p-2}u \quad \text{in } \mathbb{R}^3 \\ -\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.5}$$

where $p \in [4, 6)$, V belongs to $C(\mathbb{R}^3, \mathbb{R})$ and satisfies the following assumptioons:

(A1) $V_0 = \inf_{x\in\mathbb{R}^3} V(x) > 0$, and given V_1 and V_2 periodic functions in each coordinate direction, it is

$$V(x) = \begin{cases} V_1(x), & x \in \mathbb{R}^3_+, \\ V_2(x), & x \in \mathbb{R}^3_-. \end{cases}$$

(A2) $\min \sigma(-\Delta + V) > 0$.

They prove the existence of ground state solutions by splitting lemma and some detailed analysis.

For other papers about periodic and the nonperiodic potential, we refer to [1, 10, 19, 20, 24, 26, 30, 31] and the references therein.

Motivated by the above works, we study a Schrödinger-Bopp-Podolsky system with nonperiodic potentials and subcritial and critical growth. First, we study the case of subcritial growth, i.e.,

$$\begin{gathered} -\Delta u + V(x)u + \phi u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3 \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.6}$$

where $p \in [4, 6)$, V belongs to $C(\mathbb{R}^3, \mathbb{R})$ and satisfies (A1) and (A2). To state our results we need some preliminaries and notation, to used throughout this article.

Let $H^1(\mathbb{R}^3)$ denote the usual Sobolev space with the standard scalar product and squared norm

$$\|u\|^2_{H^1} = \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\,\mathrm{d}x.$$

When the domain of integration is not explicitly written, it is understood to be the whole space. We introduce the subspace of $H^1(\mathbb{R}^3)$,

$$E := \big\{u \in H^1(\mathbb{R}^3) : \int V(x)u^2\,\mathrm{d}x < \infty\big\}$$

which is a Hilbert space and

$$\|u\|^2 := \int (|\nabla u|^2 + V(x)u^2)dx, \quad \forall\, u \in E.$$

Assumption (A2) implies that $\|\cdot\|_{H^1}$ and $\|\cdot\|$ are two equivalent norms on E. Let S_q be the Sobolev embedding constant (see Theorem [25]), then

$$\|u\|_q \le S_q\|u\|, \quad \forall u \in E,\ 2 \le q \le 6. \tag{1.7}$$

Hereafter $\|\cdot\|_q$ is the norm in $L^q(\mathbb{R}^3)$. Let $\mathcal{D}$ be the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{\mathcal{D}}$ induced by the scalar product

$$\langle\phi, \psi\rangle := \int (\nabla\phi\nabla\psi + a^2\Delta\phi\Delta\psi)\,\mathrm{d}x.$$

Then $\mathcal{D}$ is a Hilbert space continuously embedded into $D^{1,2}(\mathbb{R}^3)$ and consequently in $L^6(\mathbb{R}^3)$. Fixed $u \in E$, the Lax-Milgram theorem [29] implies there exists a unique solution in $\mathcal{D}$ of the second equation in (1.6) and is given by

$$\phi_u(x) = \mathcal{K} * u^2 = \int \frac{1 - e^{-\frac{|x-y|}{a}}}{|x-y|} u^2(y)\,\mathrm{d}y. \tag{1.8}$$

Substituting (1.8) into the first equation of (1.6), we have

$$-\Delta u + V(x)u + \phi_u u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3. \tag{1.9}$$

Associated with (1.9) we have the energy functional $\mathcal{I}: E \to \mathbb{R}$ defined by

$$\mathcal{I}(u) = \frac{1}{2}\int (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x + \frac{1}{4}\int \phi_u u^2\,\mathrm{d}x - \frac{1}{p}\int |u|^p\,\mathrm{d}x.$$

Furthermore, one can see that $\mathcal{I}$ is a C^1 functional with the derivative given by

$$\langle \mathcal{I}'(u), v\rangle = \int (\nabla u \nabla v + V(x)uv)\,\mathrm{d}x + \int \phi_u uv\,\mathrm{d}x - \int |u|^{p-2}uv\,\mathrm{d}x, \quad \forall u, v \in E.$$

We define

$$\mathcal{N} := \{u \in E : \langle \mathcal{I}'(u), u\rangle = 0, u \neq 0\}, \tag{1.10}$$

which is the Nehari manifold of $\mathcal{I}$. In this paper, we obtain the existence of ground state solution (1.6) by solving the minimization problem

$$c := \inf_{u\in\mathcal{N}} \mathcal{I}(u). \tag{1.11}$$

By using V_i $(i = 1, 2)$, we consider the auxiliary Schrödinger-Bopp-Podolsky system

$$-\Delta u + V_i(x)u + \phi_u u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3. \tag{1.12}$$

Similarly, we define the working space

$$E_i := \Big\{u \in H^1(\mathbb{R}^3) : \int V_i(x)u^2\,\mathrm{d}x < \infty\Big\},$$

which is a Hilbert space and

$$\|u\|^2_{E_i} := \int_{\mathbb{R}^3} (|\nabla u|^2 + V_i(x)u^2)dx, \quad \forall\ u \in E_i.$$

By (A2), we have also $\min \sigma(\Delta + V_i) > 0$ for $i = 1, 2$. Then, we can deduce that $\|\cdot\|, \|\cdot\|_{E_i}$ are equivalent to $\|\cdot\|_{H^1}$. Hence, $\|\cdot\|$ is equivalent to $\|\cdot\|_{E_i}$, where $i = 1, 2$. In addition, the corresponding energy functional $\mathcal{I}_i : E_i \to \mathbb{R}$ is defined by

$$\mathcal{I}_i(u) = \frac{1}{2}\int (|\nabla u|^2 + V_i(x)u^2)\,\mathrm{d}x + \frac{1}{4}\int \phi_u u^2\,\mathrm{d}x - \frac{1}{p}\int |u|^p\,\mathrm{d}x.$$

It is a C^1 functional with the derivative given by

$$\langle \mathcal{I}_i'(u), v\rangle = \int (\nabla u \nabla v + V_i(x)uv)\,\mathrm{d}x + \int \phi_u uv\,\mathrm{d}x - \int |u|^{p-2}uv\,\mathrm{d}x, \quad \forall u, v \in E_i.$$

The minimisation problem on the Nehari manifolds is

$$\mathcal{N}_i := \{u \in E_i : \langle \mathcal{I}_i'(u), u\rangle = 0, u \neq 0\} \quad \text{and} \quad c_i := \inf_{u\in\mathcal{N}_i} \mathcal{I}_i(u). \tag{1.13}$$

Now, we summarize our first results as follows.

Theorem 1.1. *Suppose* (A1) *and* (A2) *hold and* $p \in [4, 6)$. *If* $c < \min\{c_1, c_2\}$, *then* (1.6) *has a positive ground state solution* $\overline{u}_0$ *with* $\mathcal{I}(\overline{u}_0) = c$.

A sufficient condition that guarantees $c < \min\{c_1, c_2\}$ is given in the next result.

Theorem 1.2. *Suppose* (A1) *and* (A2) *hold and* $p \in [4, 6)$. *Let* w_i *be a positive ground state solution to* (1.12) *for* $i = 1, 2$ *and assume that either*

$$c_1 \le c_2, \quad \int_{\mathbb{R}^3_-} (V_2 - V_1)w_1^2\,\mathrm{d}x < 0,$$

or

$$c_2 \le c_1, \quad \int_{\mathbb{R}^3_+} (V_1 - V_2)w_2^2\,\mathrm{d}x < 0.$$

Then, $c < \min\{c_1, c_2\}$ *and thus* (1.6) *has a positive ground state solution.*

Secondly, we study the case of critial growth.

$$\begin{gathered} -\Delta u + V(x)u + \phi u = |u|^4 u + \lambda |u|^{p-2} u \quad \text{in } \mathbb{R}^3 \\ -\Delta\phi + a^2\Delta^2\phi = 4\pi u^2 \quad \text{in } \mathbb{R}^3, \end{gathered} \tag{1.14}$$

where $p \in [4,6), \lambda > 0, V(x) \in C(\mathbb{R}^3, \mathbb{R})$ and satisfies (A1), (A2). As before we define the main objects.

Similar to the case of subcritial growth, we define the energy functional

$$\mathcal{J}(u) = \frac{1}{2}\int (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x + \frac{1}{4}\int \phi_u u^2\,\mathrm{d}x - \frac{1}{6}\int |u|^6\,\mathrm{d}x - \frac{\lambda}{p}\int |u|^p\,\mathrm{d}x,$$

and for all $u, v \in E$, its derivative satisfies

$$\langle \mathcal{J}'(u), v\rangle = \int (\nabla u\nabla v + V(x)uv)\,\mathrm{d}x + \int \phi_u uv\,\mathrm{d}x - \int |u|^4 uv\,\mathrm{d}x - \lambda\int |u|^{p-2}uv\,\mathrm{d}x.$$

We define the Nehari manifold of $\mathcal{J}$ and the minimization problem

$$\mathcal{M} := \{u \in E : \langle \mathcal{J}'(u), u\rangle = 0, u \neq 0\} \quad \text{and} \quad m := \inf_{u\in\mathcal{M}} \mathcal{J}(u). \tag{1.15}$$

By using V_i, we consider the auxiliary Schrödinger-Bopp-Podolsky system

$$-\Delta u + V_i(x)u + \phi_u u = |u|^4 u + \lambda|u|^{p-2}u \quad \text{in } \mathbb{R}^3. \tag{1.16}$$

For all $u, v \in E_i$, we have

$$\begin{gathered} \mathcal{J}_i(u) = \frac{1}{2}\int (|\nabla u|^2 + V_i(x)u^2)\,\mathrm{d}x + \frac{1}{4}\int \phi_u u^2\,\mathrm{d}x - \frac{1}{6}\int |u|^6\,\mathrm{d}x - \frac{\lambda}{p}\int |u|^p\,\mathrm{d}x, \\ \langle \mathcal{J}_i'(u), v\rangle = \int (\nabla u\nabla v + V_i(x)uv)\,\mathrm{d}x + \int \phi_u uv\,\mathrm{d}x - \int |u|^4 uv\,\mathrm{d}x - \lambda\int |u|^{p-2}uv\,\mathrm{d}x. \end{gathered}$$

Let the Nehari manifold of $\mathcal{J}_i$ be

$$\mathcal{M}_i := \{u \in E_i : \langle \mathcal{J}_i'(u), u\rangle = 0, u \neq 0\} \quad \text{and} \quad m_i := \inf_{u\in\mathcal{M}_i} \mathcal{J}_i(u). \tag{1.17}$$

We have our second result.

Theorem 1.3. *Suppose* (A1) *and* (A2) *hold,* $m < \min\{m_1, m_2\}$ *and either* $p \in (4,6), \lambda > 0$ *or* $p = 4, \lambda > 0$ *sufficiently large. Then* (1.14) *has a positive ground state solution* $\hat{u}_0$ *with* $\mathcal{J}(\hat{u}_0) = m$.

Next we give a condition that guarantees $m < \min\{m_1, m_2\}$.

Theorem 1.4. *Suppose* (A1) *and* (A2) *hold and* $p \in [4,6)$. *Let* v_i *be a positive ground state solution to* (1.16) *for* $i = 1, 2$ *and either*

$$m_1 \leq m_2, \quad \int_{\mathbb{R}^3_-} (V_2 - V_1)v_1^2\,\mathrm{d}x < 0,$$

or

$$m_2 \leq m_1, \quad \int_{\mathbb{R}^3_+} (V_1 - V_2)v_2^2\,\mathrm{d}x < 0.$$

Then, $m < \min\{m_1, m_2\}$ *and thus* (1.14) *has a positive ground state solution.*

Notation. We use following notation along this article.

- $C, \bar{C}$ and C_i ($i = 1, 2, \dots$) denote positive constants which may change from line to line.
- $\to$ and $\rightharpoonup$ denote strong and weak convergence in the related function spaces, respectively.
- $\not\to$ and $\not\rightharpoonup$ denote not strong and weak convergence in the related function spaces, respectively.
- $B_R(x_0)$ denotes the ball centered at $x_0 \in \mathbb{R}^3$ with radius R.
- $p' = \frac{p}{p-1}$ is the conjugate exponent of p, E^{-1} denotes the dual space of E.
- $o_n(1)$ denotes a vanishing sequence in the specified space.
- S is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$.

2. Preliminary results

In this section, we give some properties of ϕ_u, which will be used later.

Lemma 2.1 ([13, Lemma 3.4])**.** *For every $u \in H^1(\mathbb{R}^3)$ we have:*

(i) *for every $y \in \mathbb{R}^3$, $\phi_{u(\cdot+y)} = \phi_u(\cdot + y)$;*
(ii) $\phi_u \geq 0$*;*
(iii) *for every $s \in (3, +\infty], \phi_u \in L^s(\mathbb{R}^3) \cap C^0(\mathbb{R}^3)$;*
(iv) *for every $s \in (\frac{3}{2}, +\infty], \nabla\phi_u \in L^s(\mathbb{R}^3) \cap C^0(\mathbb{R}^3)$;*
(v) $\phi_u \in \mathcal{D}$*;*
(vi) $\|\phi_u\|_6 \leq C\|u\|^2$*;*
(vii) *ϕ_u is the unique minimizer in $\mathcal{D}$ of the functional*
$$E(\phi) = \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{a^2}{2}\|\Delta\phi\|_2^2 - \int \phi u \,\mathrm{d}x, \quad \forall \phi \in \mathcal{D};$$
(viii) $\int \phi_u u^2 dx \leq S^2\|u\|_{12/5}^4$*;*
(ix) *if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $\mathcal{D}$.*

Let us define the function $\Psi : H^1(\mathbb{R}^3) \to \mathbb{R}$ by
$$\Psi(u) = \int \phi_u u^2 \,\mathrm{d}x.$$
It is clear that for all fixed $u \in H^1(\mathbb{R}^3)$, we have $\Psi(u(\cdot + y)) = \Psi(u)$ for all $y \in \mathbb{R}^3$ and that Ψ is weakly lower semi-continuous in $H^1(\mathbb{R}^3)$. The next lemma shows that the functional Ψ and its derivative Ψ' have the Brezis-Lieb splitting property, which is similar to the well-known Brézis-Lieb lemma.

Lemma 2.2 ([18, Lemma 2.5])**.** *If $u_n \rightharpoonup u$ in E and $u_n \to u$ a.e. in $\mathbb{R}^3$, then*

(i) $\Psi(u_n - u) = \Psi(u_n) - \Psi(u) + o_n(1)$.
(ii) $\Psi'(u_n - u) = \Psi'(u_n) - \Psi'(u) + o_n(1)$ *in* E^{-1}.

Lemma 2.3 (Brézis-Lieb lemma [8])**.** *If $u_n \rightharpoonup u$ in E, then:*

(i) $\|u_n - u\|^2 = \|u_n\|^2 - \|u\|^2 + o_n(1)$.
(ii) $\|u_n - u\|_s^s = \|u_n\|_s^s - \|u\|_s^s + o_n(1)$*, where* $s \in (2, 6]$.
(iii) $\|u_n - u\|^{s-2}(u_n - u) = \|u_n\|^{s-2}u_n - \|u\|^{s-2}u + o_n(1)$ *in* E^{-1}.

3. Subcritical case

In this section, we give the proof of Theorems 1.1 and 1.2. First, we give some properties of $\mathcal{N}$ defined in (1.10).

Lemma 3.1. *Suppose that* (A1), (A2) *are satisfied, then we have:*
- (i) *for any $u \in E \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$. Moreover, $\mathcal{I}(t_u u) = \max_{t>0} \mathcal{I}(tu)$;*
- (ii) *$\mathcal{N}$ is a natural constraint for the functional $\mathcal{I}$, i.e., critical points of $\mathcal{I}$ on $\mathcal{N}$ are critical points of $\mathcal{I}$ on E;*
- (iii) *the functional $\mathcal{I}$ is bounded away from zero on $\mathcal{N}$, i.e., $c = \inf_{u\in\mathcal{N}} \mathcal{I}(u) > 0$.*

Proof. (i) We first consider the case $p > 4$. In fact, for all $t > 0$, $tu \in \mathcal{N}$ is equivalent to
$$t^2\|u\|^2 + t^4 \int \phi_u u^2 \, dx = t^p \int |u|^p \, dx.$$
Set $a_1 = \|u\|^2 > 0$, $a_2 = \int \phi_u u^2 \, dx > 0$, $a_3 = \int |u|^p \, dx > 0$. Then we obtain $a_1 t^2 + a_2 t^4 = a_3 t^p$. Let
$$g(t) = a_1 t^2 + a_2 t^4 - a_3 t^p.$$
Since $p > 4$, then $g(t) \to -\infty$ as $t \to \infty$ and $g(t) > 0$ as $t \to 0$. So there exists a solution $t = t_u > 0$ such that $g(t) = 0$, i.e., $t_u u \in \mathcal{N}$. Furthermore, since $\mathcal{I}'(tu) = g(t)$, we deduce that $\mathcal{I}(t_u u) = \max_{t>0} \mathcal{I}(tu)$. It remains to show the uniqueness of t_u. In fact,
$$\begin{aligned} g'(t) &= 2a_1 t + 4a_2 t^3 - pa_3 t^{p-1}, \\ g''(t) &= 2a_1 + 12a_2 t^2 - p(p-1)a_3 t^{p-2}, \\ g'''(t) &= 24a_2 t - p(p-1)(p-2)a_3 t^{p-3}, \end{aligned}$$
By a direct calculation, we obtain that $g'''(t) = 0$ has a unique solution t_u''', and $g'''(t) > 0$ with $0 < t < t_u'''$, $g'''(t) < 0$ with $t > t_u'''$. So $g''(t) = 0$ has a unique solution t_u'', and $g''(t) > 0$ with $0 < t < t_u''$, $g''(t) < 0$ with $t > t_u''$. By iterating this procedure, we obtain the uniqueness of t_u. Next, we consider the case $p = 4$. Define
$$A = \{u \in E \setminus \{0\} : \int \phi_u u^2 \, dx < \int u^4 \, dx\}.$$
We show that the set A is non-empty. In fact, take $u_0 \in C_0^\infty(\mathbb{R}^3, [0,1])$ satisfying $u_0 = 1$ in $B_{R_0}(0)$ and $u_0 = 0$ for $\mathbb{R}^3 \setminus B_{2R_0}(0)$, where R_0 is a positive constant to be determined. Then by Lemma 2.1, on one hand,
$$\int \phi_{u_0} u_0^2 \, dx \le S^2 \Big(\int_{B_{2R_0}(0)} |u_0|^{12/5} \, dx \Big)^{5/3} \le S^2 \frac{4}{3}\pi (2R_0)^5. \tag{3.1}$$
On the other hand,
$$\int u_0^4 \, dx \ge \int_{B_{R_0}(0)} u_0^4 \, dx = \frac{4}{3}\pi R_0^3, \tag{3.2}$$
and a suitable choice of R_0 permits to have $S^2(2R_0)^5 < R_0^3$, implying that $u_0 \in A$. Let
$$h(t) = \mathcal{I}(tu) = \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\Big(\int \phi_u u^2 \, dx - \int u^4 \, dx\Big).$$
We take $u \in A$, it is easy to verify that $h(t) > 0$ for t sufficient small and $h(t) < 0$ for t sufficient large. Then similar to the case of $p \in (4,6)$, it is not difficult to

verify that there exists a unique $t_u > 0$ such that $h'(t_u) = 0$, i.e., $t_u u \in \mathcal{N}$ and $\mathcal{I}(t_u u) = \max_{t>0} \mathcal{I}(tu)$.

(ii) For each $u \in \mathcal{N}$, from (1.7) and (1.10),

$$0 = \|u\|^2 + \int \phi_u u^2 \,\mathrm{d}x - \int |u|^p \,\mathrm{d}x \geq \|u\|^2 - \int |u|^p \,\mathrm{d}x \geq \|u\|^2 - S_p^p \|u\|^p.$$

Since $p \in [4, 6)$, this implies that

$$\|u\| \geq (\frac{1}{S_p^p})^{\frac{1}{p-2}} > 0. \tag{3.3}$$

Define $\mathcal{A}(u) := \langle \mathcal{I}'(u), u\rangle$, by a direct computation,

$$\begin{aligned} \langle \mathcal{A}'(u), u\rangle &= 2\|u\|^2 + 4\int \phi_u u^2 \,\mathrm{d}x - p\int |u|^p \,\mathrm{d}x \\ &= (2-p)\|u\|^2 + (3-p)\int \phi_u u^2 \,\mathrm{d}x \\ &\leq (2-p)\|u\|^2 \\ &< (2-p)\Big(\frac{1}{S_p^p}\Big)^{\frac{1}{p-2}} < 0. \end{aligned} \tag{3.4}$$

Then there exists $\mu \in \mathbb{R}$ such that $\mathcal{I}'(u) = \mu \mathcal{A}'(u)$. Therefore

$$0 = \langle \mathcal{I}'(u), u\rangle = \mu \langle \mathcal{A}'(u), u\rangle,$$

which implies $\mu = 0$ by (3.4) and then $\mathcal{I}'(u) = \mu \mathcal{A}'(u) = 0$.

(iii) For any $u \in \mathcal{N}$, we can deduce from $p \in [4, 6)$ and (3.3) that

$$\begin{aligned} \mathcal{I}(u) &= (\frac{1}{2} - \frac{1}{p})\|u\|^2 + (\frac{1}{4} - \frac{1}{p})\int \phi_u u^2 \,\mathrm{d}x \\ &\geq (\frac{1}{2} - \frac{1}{p})\|u\|^2 \geq (\frac{1}{2} - \frac{1}{p})(\frac{1}{S_p^p})^{\frac{2}{p-2}} > 0. \end{aligned}$$

This completes the proof. □

Corollary 3.2. *For any $u \in E_i \setminus \{0\}$, there exists a unique $t_{ui} > 0$ such that $t_{ui} u \in \mathcal{N}_i$. Moreover, $\mathcal{I}_i(t_{ui} u) = \max_{t>0} \mathcal{I}_i(tu)$, where $i = 1, 2$.*

Lemma 3.3. *Suppose that* (A1), (A2) *are satisfied, then there exists a $(PS)_c$ sequence $\{u_n\} \subset \mathcal{N}$, namely such that $\mathcal{I}(u_n) \to c$ and $\mathcal{I}'(u_n) \to 0$ as $n \to +\infty$. Moreover, the sequence $\{u_n\}$ is bounded and bounded away from zero on $\mathcal{N}$.*

Proof. By the Ekeland Variational Principle, there exists a sequence $\{u_n\} \subset \mathcal{N}$ such that

$$\mathcal{I}(u_n) \to c \quad \text{and} \quad \mathcal{I}'|_{\mathcal{N}}(u_n) \to 0 \quad \text{as} \quad n \to +\infty.$$

Since $u_n \in \mathcal{N}$, we obtain

$$0 = \langle \mathcal{I}'(u_n), u_n\rangle = \|u_n\|^2 + \int \phi_{u_n} u_n^2 \,\mathrm{d}x - \int |u_n|^p \,\mathrm{d}x,$$

and by a direct calculation, for $p \in [4, 6)$,

$$\mathcal{I}(u_n) = \mathcal{I}(u_n) - \frac{1}{4}\langle \mathcal{I}'(u_n), u_n\rangle = \frac{1}{4}\|u_n\|^2 + (\frac{1}{4} - \frac{1}{p})\int |u_n|^p \,\mathrm{d}x \geq \frac{1}{4}\|u_n\|^2.$$

Then, it follows from $\mathcal{I}(u_n) \to c$ as $n \to +\infty$ that $\{u_n\}$ is bounded. From the definition of $\mathcal{A}$ in the proof of Lemma 3.1, we have as $n \to +\infty$,

$$o_n(1) = \nabla|_{\mathcal{N}} \mathcal{I}(u_n) = \mathcal{I}'(u_n) + \mu_n \mathcal{A}'(u_n) \tag{3.5}$$

for some $\mu_n \in \mathbb{R}$. Taking the scalar product with u_n (which is bounded), we obtain that

$$o_n(1) = \langle \mathcal{I}'(u_n), u_n\rangle + \mu_n \langle \mathcal{A}'(u_n), u_n\rangle = \mu_n \langle \mathcal{A}'(u_n), u_n\rangle. \tag{3.6}$$

Since $u_n \in \mathcal{N}$ and $p \in [4, 6)$, we obtain, as in equation (3.4)

$$\langle \mathcal{A}'(u_n), u_n\rangle < (2-p)\Big(\frac{1}{S_p^p}\Big)^{\frac{1}{p-2}} < 0,$$

and (3.6) gives $\mu_n \to 0$.

It follows from (3.5) and (3.6) that $\mu_n \to 0$ and $\mathcal{I}'(u_n) \to 0$ in E^{-1} as $n \to +\infty$. Moreover, by (3.3), $\{u_n\}_{n\in\mathbb{N}}$ is bounded and bounded away from zero. □

Since $\{u_n\}$ is bounded in E, passing to a subsequence, there exists $\overline{u} \in E$ such that as $n \to \infty$,

$$\begin{gathered} u_n \rightharpoonup \overline{u} \quad \text{in } E, \\ u_n \to \overline{u} \quad \text{in } L^r_{\rm loc}(\mathbb{R}^3), r \in [1, 6), \\ u_n(x) \to \overline{u}(x) \quad \text{a.e. on } \mathbb{R}^3. \end{gathered} \tag{3.7}$$

Here is some preparatory work that will be used later.

Lemma 3.4 ([13, Lemma B.5]). *Let $\{y_n\} \subset \mathbb{R}^3, v \in H^1(\mathbb{R}^3), \{v_n\} \subset H^1(\mathbb{R}^3)$ be bounded.*

(i) *If $|y_n| \to +\infty$, then $v(\cdot + y_n) \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$.*
(ii) *If $\{y_n\}$ is bounded, then, up to a subsequence,*

$$v_n \not\rightharpoonup 0 \quad \text{in } H^1(\mathbb{R}^3) \Longrightarrow v_n(\cdot + y_n) \not\rightharpoonup 0 \quad \text{in } H^1(\mathbb{R}^3)$$

Now, we give the splitting lemma of a $(PS)_c$ sequence of $\mathcal{I}$, which plays a crucial role for the subsequent discussion.

Lemma 3.5 (Splitting Lemma). *Let $\{u_n\} \subset E$ be a bounded $(PS)_c$ sequence of $\mathcal{I}$ at level $c > 0$ and assume that $u_n \rightharpoonup \overline{u}$ in E. Then, passing to a subsequence, either u_n strongly converges to $\overline{u}$, or setting $k \in \mathbb{N} \cup \{0\}$, there are sequences $\{u^i\}_{i=1}^k \subset E$ and $y_n^i \in T_1\mathbb{Z} \times T_2\mathbb{Z} \times T_3\mathbb{Z} \subset \mathbb{R}^3$ with $1 \le i \le k$ such that*

(i) *$(-1)^{\varepsilon_i - 1}(y_n^i)_1 \to +\infty$ and $|(y_n^i)_1 - (y_n^j)_1| \to +\infty$ for $1 \le i \ne j \le k$, as $n \to +\infty$, where $(y_n^i)_\gamma$ denotes the γ-th component of y_n^i, $1 \le \gamma \le 3$.*
(ii) *$u_n \to \overline{u} + \sum_{i=1}^k u^i(\cdot - y_n^i)$ in E.*
(iii) *$\mathcal{I}(u_n) = \mathcal{I}(\overline{u}) + \sum_{i=1}^k \mathcal{I}_{\varepsilon_i}(u^i) + o_n(1)$.*
(iv) *$\mathcal{I}'_{\varepsilon_i}(u^i) = 0$ and $u^i \ne 0$ with $1 \le i \le k$.*

where $\varepsilon_i = \{1, 2\}$.

Proof. Let us divide the proof in various steps.

Step 1: Let $z_n^1 = u_n - \overline{u}$. We have two possibilities:
If $z_n^1 \to 0$ in E, then the first alternative follows and the proof is concluded.
If $z_n^1 \not\to 0$ in E. Let $\varepsilon_i = \{1, 2\}$ and

$$\mathbb{R}^3_{(-1)^{\varepsilon_i}} = \begin{cases} \mathbb{R}^3_-, & \varepsilon_i = 1, \\ \mathbb{R}^3_+, & \varepsilon_i = 2. \end{cases} \tag{3.8}$$

Since $z_n^1 \rightharpoonup 0$ in E, it follows from Lemma 2.2 and Lemma 2.3, for any $\varphi \in C_0^\infty(E,\mathbb{R})$, as $n\to\infty$,

$$
\begin{aligned}
\langle \mathcal{I}'(u_n),\varphi\rangle &= \int (\nabla u_n\nabla\varphi + V(x)u_n\varphi)\,\mathrm{d}x + \int \phi_{u_n}u_n\varphi\,\mathrm{d}x - \int |u_n|^{p-2}u_n\varphi\,\mathrm{d}x\\
&= \int (\nabla \overline{u}\nabla\varphi + V(x)\overline{u}\varphi)\,\mathrm{d}x + \int (\nabla z_n^1\nabla\varphi + V(x)z_n^1\varphi)\,\mathrm{d}x + \int \phi_{\overline{u}}\overline{u}\varphi\,\mathrm{d}x\\
&\quad + \int \phi_{z_n^1}z_n^1\varphi\,\mathrm{d}x - \int |\overline{u}|^{p-2}\overline{u}\varphi\,\mathrm{d}x - \int |z_n^1|^{p-2}z_n^1\varphi\,\mathrm{d}x + o_n(1)\\
&= \langle \mathcal{I}'(\overline{u}),\varphi\rangle + \langle \mathcal{I}'(z_n^1),\varphi\rangle + o_n(1)\\
&= \langle \mathcal{I}'(\overline{u}),\varphi\rangle + \int (\nabla z_n^1\nabla\varphi + V_1 z_n^1\varphi)\,\mathrm{d}x + \int_{\mathbb{R}^3_-} V_2 z_n^1\varphi\,\mathrm{d}x\\
&\quad + \int \phi_{z_n^1}z_n^1\varphi\,\mathrm{d}x - \int |z_n^1|^{p-2}z_n^1\varphi\,\mathrm{d}x + o_n(1)\\
&= \langle \mathcal{I}'(\overline{u}),\varphi\rangle + \langle \mathcal{I}'_{\varepsilon_1}(z_n^1),\varphi\rangle + \int_{\mathbb{R}^3_{(-1)^{\varepsilon_1}}} (-1)^{\varepsilon_1}(V_1-V_2)z_n^1\varphi\,\mathrm{d}x + o_n(1)\\
&= \langle \mathcal{I}'(\overline{u}),\varphi\rangle + \langle \mathcal{I}'_{\varepsilon_1}(z_n^1),\varphi\rangle + o_n(1).
\end{aligned}
$$

Which together with $\mathcal{I}'(u_n)\to 0$ and $\mathcal{I}'(\overline{u})=0$ imply that

$$\langle \mathcal{I}'_{\varepsilon_1}(z_n^1),\varphi\rangle \to 0, \quad \text{as } n\to\infty. \tag{3.9}$$

In addition, for any $\varphi\in C_0^\infty(E,\mathbb{R})$, by Lemmas 2.2 and 2.3, as $n\to\infty$,

$$\langle \mathcal{I}'(u_n),\varphi\rangle = \langle \mathcal{I}'(\overline{u}),\varphi\rangle + \langle \mathcal{I}'(z_n^1),\varphi\rangle + o_n(1). \tag{3.10}$$

Since $\mathcal{I}'(u_n)\to 0$ as $n\to\infty$, From (3.10) and $\mathcal{I}'(\overline{u})=0$ it follows that

$$\langle \mathcal{I}'(z_n^1),\varphi\rangle \to 0.$$

Setting $\varphi = z_n^1$, we obtain $\langle \mathcal{I}'(z_n^1), z_n^1\rangle \to 0$, i.e.,

$$\|z_n^1\|^2 + \int \phi_{z_n^1}(z_n^1)^2\,\mathrm{d}x - \int |z_n^1|^p\,\mathrm{d}x \to 0. \tag{3.11}$$

Let

$$\delta_1 := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)} |z_n^1|^2\,\mathrm{d}x.$$

We claim that $\delta_1\neq 0$. If not, by Lion's lemma, $z_n^1\to 0$ in $L^r(\mathbb{R}^3)$ for $r\in(2,6)$. By Hölder's inequality, we obtain that

$$\int \phi_{z_n^1}(z_n^1)^2\,\mathrm{d}x \le \Big(\int |\phi_{z_n^1}|^6\,\mathrm{d}x\Big)^{1/6}\Big(\int |z_n^1|^{12/5}\,\mathrm{d}x\Big)^{5/6} \le CS\|z_n^1\|_{12/5}^2 \to 0.$$

From (3.11) it follows that $\|z_n^1\|^2\to 0$, which contradicts $z_n^1 \not\to 0$ in E. Then, there exists $y_n^1\in T_1\mathbb{Z}\times T_2\mathbb{Z}\times T_3\mathbb{Z}\subset\mathbb{R}^3$ such that

$$\int_{B_1(y_n^1)} |z_n^1|^2 dx \ge \frac{\delta_1}{2}. \tag{3.12}$$

Let $\xi_n^1 := (0,(y_n^1)_2,(y_n^1)_3)$, $\sigma_n^1 := ((y_n^1)_1,0,0)$, and $w_n^1 = z_n^1(\cdot+\xi_n^1)$. Clearly, $\|w_n^1\| = \|z_n^1\|$ and $w_n^1 \rightharpoonup 0$ in E but not strongly. Therefore, by (3.12), we obtain

$$\int_{B_1(\sigma_n^1)} |w_n^1|^2 dx \ge \frac{\delta_1}{2}. \tag{3.13}$$

It is easy to check that $|(y_n^1)_1| = |\sigma_n^1| \to +\infty$, that is,

$$(-1)^{\varepsilon_1 - 1}(y_n^1)_1 \to +\infty. \tag{3.14}$$

Considering the sequence $\{w_n^1(\cdot + \sigma_n^1)\}$, which is bounded in E_{ε_1}, there exists $u^1 \in E_{\varepsilon_1}$ satisfying

$$\begin{gathered}
w_n^1(\cdot + \sigma_n^1) \rightharpoonup u^1 \quad \text{in } E_{\varepsilon_1},\\
w_n^1(\cdot + \sigma_n^1) \to u^1 \quad \text{in } L^r_{\rm loc}(\mathbb{R}^3),\\
w_n^1(x + \sigma_n^1) \to u^1(x) \text{ a.e. in } \mathbb{R}^3.
\end{gathered}$$

From (3.13), we obtain $u^1 \neq 0$. From (3.9), for any $\varphi \in C_0^\infty(E, \mathbb{R})$, we obtain

$$\langle \mathcal{I}'_{\varepsilon_1}(w_n^1(\cdot + \sigma_n^1)), \varphi\rangle = \langle \mathcal{I}'_{\varepsilon_1}(z_n^1(\cdot + y_n^1)), \varphi\rangle = \langle \mathcal{I}'_{\varepsilon_1}(z_n^1), \varphi(\cdot - y_n^1)\rangle \to 0.$$

Hence, we have $\mathcal{I}'_{\varepsilon_1}(w_n^1(\cdot + \sigma_n^1)) \to 0$ since $C_0^\infty(E, \mathbb{R})$ is dence in E, it follows that

$$\mathcal{I}'_{\varepsilon_1}(u^1) = 0. \tag{3.15}$$

Step 2: Let $z_n^2 = z_n^1 - u^1(x - y_n^1)$. Then, $z_n^2 \rightharpoonup 0$ in E because the norm of E is equivalent to the norm of E_{ε_1}. In addition, from Lemmas 2.2 and 2.3, by the simple calculation, as $n \to \infty$,

$$\begin{aligned}
\mathcal{I}(u_n) &= \frac{1}{2}\|u_n\|^2 + \frac{1}{4}\int \phi_{u_n} u_n^2 \,dx - \frac{1}{p}\|u_n\|_p^p\\
&= \frac{1}{2}(\|z_n^1\|^2 + \|\overline{u}\|^2) + \frac{1}{4}\Big(\int \phi_{\overline{u}}\overline{u}^2 \,dx + \int \phi_{z_n^1}(z_n^1)^2 \,dx\Big)\\
&\quad - \frac{1}{p}(\|z_n^1\|_p^p + \|\overline{u}\|_p^p) + o_n(1)\\
&= \mathcal{I}(z_n^1) + \mathcal{I}(\overline{u}) + o_n(1)\\
&= \mathcal{I}(z_n^2) + \mathcal{I}(u^1) + \mathcal{I}(\overline{u}) + o_n(1)\\
&= \mathcal{I}(z_n^2) + \mathcal{I}_{\varepsilon_1}(u^1) + \int_{\mathbb{R}^3_{(-1)^{\varepsilon_1}}} (-1)^{\varepsilon_1}(V_1 - V_2)u^1 \,dx + \mathcal{I}(\overline{u}) + o_n(1)\\
&= \mathcal{I}(z_n^2) + \mathcal{I}_{\varepsilon_1}(u^1) + \mathcal{I}(\overline{u}) + o_n(1).
\end{aligned} \tag{3.16}$$

We have two possibilities:

If $z_n^2 \to 0$ in E, i.e., $z_n^1 - u^1(x - y_n^1) = u_n - \overline{u} - u^1(x - y_n^1) \to 0$, i.e.,

$$u_n \to \overline{u} + u^1(x - y_n^1). \tag{3.17}$$

From (3.16), we obtain

$$\mathcal{I}(u_n) = \mathcal{I}(\overline{u}) + \mathcal{I}_{\varepsilon_1}(u^1) + o_n(1). \tag{3.18}$$

Then the Lemma is proved for $k = 1$. It follows from (3.14), (3.17), (3.18), and (3.15).

If $z_n^2 \not\to 0$ in E. From Lemmas 2.2 and 2.3, we obtain that for any $\varphi \in C_0^\infty(E,\mathbb{R})$, as $n\to\infty$,

$$\begin{aligned}\langle \mathcal{I}'(z_n^1),\varphi\rangle &= \langle \mathcal{I}'(z_n^2),\varphi\rangle + \langle \mathcal{I}'(u^1),\varphi(x+y_n^1)\rangle + o_n(1)\\ &= \langle \mathcal{I}'(z_n^2),\varphi\rangle + \langle \mathcal{I}'_{\varepsilon_1}(u^1),\varphi(x+y_n^1)\rangle\\ &\quad + \int_{\mathbb{R}^3_{(-1)^{\varepsilon_1}}} (-1)^{\varepsilon_1}(V_1-V_2)u^1\varphi(x+y_n^1)dx + o_n(1)\\ &= \langle \mathcal{I}'(z_n^2),\varphi\rangle + \langle \mathcal{I}'_{\varepsilon_1}(u^1),\varphi(x+y_n^1)\rangle + o_n(1).\end{aligned} \tag{3.19}$$

It follows from $\langle \mathcal{I}'(z_n^1),\varphi\rangle \to 0$ and $\mathcal{I}'_{\varepsilon_1}(u^1)=0$ that $\langle \mathcal{I}'(z_n^2),\varphi\rangle \to 0$. Setting $\varphi = z_n^2$, we obtain $\langle \mathcal{I}'(z_n^2),z_n^2\rangle \to 0$, i.e.,

$$\|z_n^2\|^2 + \int \phi_{z_n^2}(z_n^2)^2dx - \int |z_n^2|^p dx \to 0.$$

Let

$$\delta_2 := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}|z_n^2|^2dx.$$

Simlar to $\delta_1\neq 0$, we obtain $\delta_2 \neq 0$. Then, there exists $y_n^2 \in T_1\mathbb{Z}\times T_2\mathbb{Z}\times T_3\mathbb{Z}\subset\mathbb{R}^3$ such that

$$\int_{B_1(y_n^2)}|z_n^2|^2dx \geq \frac{\delta_2}{2}.$$

Let $\xi_n^2 := (0,(y_n^2)_2,(y_n^2)_3), \sigma_n^2 := ((y_n^2)_1,0,0)$, and $w_n^2 = z_n^2(x+\xi_n^2)$. Clearly, $\|w_n^2\| = \|z_n^2\|$ and $w_n^2 \rightharpoonup 0$ in E. Therefore,

$$\int_{B_1(\sigma_n^2)}|w_n^2|^2dx \geq \frac{\delta_2}{2}. \tag{3.20}$$

It is easy to check that as $n\to\infty$, $|(y_n^2)_1| = |\sigma_n^2| \to +\infty$; that is,

$$(-1)^{\varepsilon_2-1}(y_n^2)_1 \to +\infty. \tag{3.21}$$

Then $w_n^2(\cdot+\sigma_n^2) \not\rightharpoonup 0$ in E. In addition, we claim that

$$|(y_n^2)_1 - (y_n^1)_1| \to +\infty. \tag{3.22}$$

To see this, first observe that

$$\begin{aligned}&w_n^2(\cdot+\sigma_n^2+(y_n^2)_1-(y_n^1)_1)\\ &= z_n^2(\cdot+\sigma_n^2+(y_n^2)_1-(y_n^1)_1+\xi_n^2)\\ &= z_n^1(\cdot+\sigma_n^2+(y_n^2)_1-(y_n^1)_1+\xi_n^2) - u^1(\cdot+\sigma_n^2+(y_n^2)_1-(y_n^1)_1+\xi_n^2-y_n^1)\\ &= w_n^1(\cdot+\sigma_n^2+(y_n^2)_1-(y_n^1)_1+\xi_n^2-\xi_n^1) - u^1(\cdot+\sigma_n^2+(y_n^2)_1-(y_n^1)_1+\xi_n^2-y_n^1)\\ &= w_n^1(\cdot+\sigma_n^2+y_n^2-y_n^1) - u^1(\cdot+\sigma_n^2+y_n^2-y_n^1-\sigma_n^1)\end{aligned}$$

From Lemma 3.4 (ii), since $w_n^2(\cdot+\sigma_n^2)\not\rightharpoonup 0$ in E, if it were $|(y_n^2)_1-(y_n^1)_1| \not\to +\infty$, we obtain

$$w_n^2(\cdot+\sigma_n^2+(y_n^2)_1-(y_n^1)_1)\not\rightharpoonup 0.$$

On the other hand, since $w_n^1(\cdot+\sigma_n^1)\rightharpoonup u^1$, we obtain

$$w_n^1(\cdot+\sigma_n^2+y_n^2-y_n^1) - u^1(\cdot+\sigma_n^2+y_n^2-y_n^1-\sigma_n^1) \rightharpoonup 0,$$

which is a contradiction. So we obtain (3.22). Moreover, for any $\varphi \in C_0^\infty(E,\mathbb{R})$, we obtain as $n \to \infty$,

$$\begin{aligned}\langle \mathcal{I}'_{\varepsilon_2}(z_n^1), \varphi\rangle &= \langle \mathcal{I}'_{\varepsilon_2}(z_n^2), \varphi\rangle + \langle \mathcal{I}'_{\varepsilon_2}(u^1), \varphi(x+y_n^1)\rangle \\ &= \langle \mathcal{I}'_{\varepsilon_2}(z_n^2), \varphi\rangle + \langle \mathcal{I}'_{\varepsilon_1}(u^1), \varphi(x+y_n^1)\rangle \\ &\quad + \int (V_{\varepsilon_2} - V_{\varepsilon_1}) u^1 \varphi(x+y_n^1) dx + o_n(1) \\ &= \langle \mathcal{I}'_{\varepsilon_2}(z_n^2), \varphi\rangle + \langle \mathcal{I}'_{\varepsilon_1}(u^1), \varphi(x+y_n^1)\rangle + o_n(1).\end{aligned} \tag{3.23}$$

By $\langle \mathcal{I}'_{\varepsilon_2}(z_n^1), \varphi\rangle \to 0$ and $\mathcal{I}'_{\varepsilon_1}(u^1) = 0$, we obtain $\mathcal{I}'_{\varepsilon_2}(z_n^2) \to 0$ since $C_0^\infty(E,\mathbb{R})$ is dense in E. Considering the sequence $\{w_n^2(\cdot + \sigma_n^2)\}$, which is bounded in E_{ε_2}, there exists $u^2 \in E_{\varepsilon_2}$ satisfying

$$\begin{gathered} w_n^2(\cdot + \sigma_n^2) \rightharpoonup u^2 \quad \text{in } E_{\varepsilon_2}, \\ w_n^2(\cdot + \sigma_n^2) \to u^2 \quad \text{in } L^r_{\rm loc}(\mathbb{R}^3), \\ w_n^2(x + \sigma_n^2) \to u^2(x) \text{ a.e. on } \mathbb{R}^3. \end{gathered}$$

We can see that $u^2 \neq 0$. For any $\varphi \in C_0^\infty(E,\mathbb{R})$, we obtain

$$\begin{aligned}\langle \mathcal{I}'_{\varepsilon_2}(u^2), \varphi\rangle &= \langle \mathcal{I}'_{\varepsilon_2}(w_n^2(\cdot + \sigma_n^2)), \varphi\rangle + o_n(1) \\ &= \langle \mathcal{I}'_{\varepsilon_2}(z_n^2(\cdot + y_n^2)), \varphi\rangle + o_n(1) \\ &= \langle \mathcal{I}'_{\varepsilon_2}(z_n^2), \varphi(\cdot - y_n^2)\rangle + o_n(1) = o_n(1).\end{aligned}$$

Hence, since $C_0^\infty(E,\mathbb{R})$ is dense in E, we have

$$\mathcal{I}'_{\varepsilon_2}(u^2) = 0. \tag{3.24}$$

Step 3: Let $z_n^3 = z_n^2 - u^2(x - y_n^2)$. From (3.16), we have

$$\begin{aligned}\mathcal{I}(u_n) &= \mathcal{I}(z_n^2) + \mathcal{I}_{\varepsilon_1}(u^1) + \mathcal{I}(\overline{u}) + o_n(1) \\ &= \mathcal{I}(z_n^3) + \mathcal{I}_{\varepsilon_2}(u^2) + \mathcal{I}_{\varepsilon_1}(u^1) + \mathcal{I}(\overline{u}) + o_n(1).\end{aligned} \tag{3.25}$$

We have two possibilities:

If $z_n^3 \to 0$ in E, i.e., $z_n^2 - u^2(x - y_n^2) = u_n - \overline{u} - u^1(x - y_n^1) - u^2(x - y_n^2) \to 0$, i.e.,

$$u_n \to \overline{u} + u^1(x - y_n^1) + u^2(x - y_n). \tag{3.26}$$

From (3.25), we obtain

$$\mathcal{I}(u_n) = \mathcal{I}(\overline{u}) + \mathcal{I}_{\varepsilon_1}(u^1) + \mathcal{I}_{\varepsilon_2}(u^2) + o_n(1). \tag{3.27}$$

Then the Lemma is proved for $k = 2$. It follows from (3.21), (3.22), (3.26), (3.27), and (3.24).

If $z_n^2 \not\to 0$ in E, we just repeat the argument.

Step k: By $\mathcal{I}(u_n) = \mathcal{I}(\overline{u}) + \sum_{i=1}^k \mathcal{I}_{\varepsilon_i}(u^i) + o_n(1)$, since $\mathcal{I}_{\varepsilon_i}(u^i) \geq c_{\varepsilon_i} \geq \min\{c_1, c_2\}$ and $\mathcal{I}(u_n)$ is bound, the iteration must stop at some finite index k. The proof is complete □

Proof of Theorem 1.1. In view of Lemma 3.3, we obtained that there exists a bounded $(PS)_c$ sequence $\{u_k\} \subset \mathcal{N}$ such that $\mathcal{I}(u_k) \to c$ and $\mathcal{I}'(u_k) \to 0$ as $k \to +\infty$. Since $\{u_k\}$ is bounded in E, the, going to a subsequence if necessary, still denoted by $\{u_k\}$, we can suppose that there exists $\overline{u} \in E$ such that $u_k \rightharpoonup \overline{u}$ in

E. With $c<\min\{c_1,c_2\}$, by Lemma 3.4, if $u_k \not\to u$, we can obtain that $k\geq 1$ and nontrivial solutions $u^1,u^2,\dots,u^j$ of $\mathcal{I}_{\varepsilon_j}$ with $\varepsilon_j=\{1,2\}$ satisfy

$$c=\lim_{k\to+\infty}\mathcal{I}(u_k)=\mathcal{I}(\overline{u})+\sum_{j=1}^{k}\mathcal{I}_{\varepsilon_j}(u^j)\geq k\min\{c_1,c_2\}\geq\min\{c_1,c_2\},$$

which is contradiction with $c<\min\{c_1,c_2\}$. Thus, $u_k\to\overline{u}$, and then $\mathcal{I}(\overline{u})=c$ and $\mathcal{I}'(\overline{u})=0$. Obviously, $\overline{u}\neq 0$. Therefore, $\overline{u}$ is a ground state solution of (1.6).

Considering $\overline{u}_0=|\overline{u}|$, it is easy to check that $\mathcal{I}(\overline{u}_0)=\mathcal{I}(\overline{u})=c$ and $\overline{u}_0\in\mathcal{N}$. From standard arguments, we infer that $\mathcal{I}'(\overline{u}_0)=0$. Thus, $\bar{u}_0$ is a non-negative solution of system (1.6). Furthermore, the strong maximum principle implies that $\overline{u}_0>0$ in $\mathbb{R}^3$, and thus, $\overline{u}_0$ is a positive ground state solution of system (1.6). □

Proof of Theorem 1.2. We just study the case of $c_1\leq c_2$. The case $c_2\leq c_1$ is analogous. Let $w_1\in\mathcal{N}_1\subset E$ be a positive ground state for the purely periodic problem for (1.6) with $i=1$. We can see from Lemma 3.1 that there exists $s>0$ satisfying $sw_1\in\mathcal{N}$. Then, from the assumption of Theorem 1.2 and Corollary 3.2, we have

$$\begin{aligned}
c&\leq\mathcal{I}(sw_1)\\
&=\frac{s^2}{2}\|w_1\|^2+\frac{s^4}{4}\int\phi_{w_1}|w_1|^2dx-\frac{s^p}{p}\|w_1\|_p^p\\
&=\frac{s^2}{2}\|w_1\|_{E_1}^2+\frac{s^4}{4}\int\phi_{w_1}|w_1|^2\,\mathrm{d}x-\frac{s^p}{p}\|w_1\|_p^p+\frac{s^2}{2}\int_{\mathbb{R}^3_-}(V_2-V_1)w_1^2\,\mathrm{d}x\\
&=\mathcal{I}_1(sw_1)+\frac{s^2}{2}\int_{\mathbb{R}^3_-}(V_2-V_1)w_1^2\,\mathrm{d}x\\
&<\mathcal{I}_1(w_1)=c_1,
\end{aligned}$$

which implies that $c<c_1$. Thus, $c<\min\{c_1,c_2\}$. □

4. Critical case

In this section, we give the proof of Theorems 1.3 and 1.4. According to [24, Theorem 1.2], we have the following result.

Theorem 4.1. *Assume that* (A1) *holdls and* $\min\sigma(-\Delta+V_i)>0$. *Then, for every* $\lambda>0$ *and* $p\in(4,6)$, (1.16) *has a positive ground state solution. Moreover, if* $p=4$, *then* (1.16) *possesses positive ground state solutions for* $\lambda>0$ *sufficiently large.*

Similar to the case of subcritical, we have some properties of $\mathcal{M}$.

Lemma 4.2. *Suppose that* (A1), (A2) *are satisfied, then we have*

- (i) *For any* $u\in E\setminus\{0\}$, *there exists a unique* $\hat{t}_u>0$ *such that* $\hat{t}_u u\in\mathcal{M}$. *Moreover,* $\mathcal{J}(\hat{t}_u u)=\max_{t>0}\mathcal{I}(tu)$.
- (ii) $\mathcal{M}$ *is a natural constraint for the functional* $\mathcal{J}$, *i.e., critical points of* $\mathcal{J}$ *on* $\mathcal{M}$ *are critical points of* $\mathcal{J}$ *on* E.
- (iii) *The functional* $\mathcal{J}$ *is bounded away from zero on* $\mathcal{M}$, *i.e.,* $m=\inf_{u\in\mathcal{M}}\mathcal{J}(u)>0$.

Proof. (i) For $t>0, tu\in\mathcal{M}$ is equivalent to

$$t^2\|u\|^2+t^4\int\phi_u u^2\,\mathrm{d}x=t^6\|u\|_6^6+\lambda t^p\|u\|_p^p.$$

Set $a_1 = \|u\|^2 > 0$, $a_2 = \int \phi_u u^2 \,\mathrm{d}x > 0$, $a_3 = \|u\|_6^6 > 0$, $a_4 = \|u\|_p^p > 0$. Then we obtain $a_1 t^2 + a_2 t^4 = a_3 t^6 + \lambda a_4 t^p$. Let

$$\hat{g}(t) = a_1 t^2 + a_2 t^4 - a_3 t^6 - \lambda a_4 t^p.$$

Since $p > 4$, then $\hat{g}(t) \to -\infty$ as $t \to \infty$ and $\hat{g}(t) > 0$ as $t \to 0$. So there exists a solution $\hat{t} = \hat{t}_u > 0$ such that $g(\hat{t}) = 0$, i.e., $\hat{t}_u u \in \mathcal{M}$. Furthermore, since $\mathcal{J}'(tu) = \hat{g}(t)$, we deduce that $\mathcal{J}(\hat{t}_u u) = \max_{t>0} \mathcal{J}(tu)$. It remains to show the uniqueness of $\hat{t}_u$. In fact, suppose by contradiction that there exists $0 < t_1 < t_2$ such that $\hat{g}(t_1) = \hat{g}(t_2) = 0$. Then

$$\begin{gathered}
\frac{\|u\|^2}{t_1^2} + \int \phi_u u^2 \,\mathrm{d}x = t_1^2 \|u\|_6^6 + \lambda t_1^{p-4} \|u\|_p^p, \\
\frac{\|u\|^2}{t_2^2} + \int \phi_u u^2 \,\mathrm{d}x = t_2^2 \|u\|_6^6 + \lambda t_2^{p-4} \|u\|_p^p.
\end{gathered}$$

As a consequence,

$$\Big(\frac{1}{t_2^2} - \frac{1}{t_1^2}\Big)\|u\|^2 = (t_2^2 - t_1^2)\|u\|_6^6 + \lambda(t_2^{p-4} - t_1^{p-4})\|u\|_p^p,$$

which is impossible by $0 < t_1 < t_2$.

(ii) For any $u \in \mathcal{M}$, we have that

$$\begin{aligned}
0 &= \|u\|^2 + \int \phi_u u^2 \,\mathrm{d}x - \|u\|_6^6 - \lambda \|u\|_p^p \\
&\geq \|u\|^2 - \|u\|_6^6 - \lambda \|u\|_p^p \\
&\geq \|u\|^2 - S_6^6 \|u\|^6 - \lambda S_p^p \|u\|^p.
\end{aligned}$$

Since $p \in [4, 6)$, there exists a constant $\Lambda_0 > 0$ such that

$$\|u\| \geq \Lambda_0 > 0. \tag{4.1}$$

We define $\mathcal{B}(u) := \langle \mathcal{J}'(u), u \rangle$, by a direct computation,

$$\begin{aligned}
\langle \mathcal{B}'(u), u \rangle &= 2\|u\|^2 + 4 \int \phi_u u^2 \,\mathrm{d}x - 6\|u\|_6^6 - \lambda p \|u\|_p^p \\
&= (2-p)\|u\|^2 + (4-p) \int \phi_u u^2 \,\mathrm{d}x + (p-6)\|u\|_6^6 \\
&\leq (2-p)\|u\|^2 < 0.
\end{aligned} \tag{4.2}$$

Then there exists $\hat{\mu} \in \mathbb{R}$ such that $\mathcal{J}'(u) = \hat{\mu} \mathcal{B}'(u)$. Therefore

$$0 = \langle \mathcal{J}'(u), u \rangle = \hat{\mu} \langle \mathcal{B}'(u), u \rangle,$$

which implies $\hat{\mu} = 0$ by (4.2) and then $\mathcal{J}'(u) = \hat{\mu} \mathcal{B}'(u) = 0$.

(iii) For any $u \in \mathcal{M}$, we can deduce from $p \in [4, 6)$ and (4.1) that

$$\begin{aligned}
\mathcal{J}(u) &= (\frac{1}{2} - \frac{1}{p})\|u\|^2 + (\frac{1}{4} - \frac{1}{p}) \int \phi_u u^2 \,\mathrm{d}x + (\frac{1}{p} - \frac{1}{6})\|u\|_6^6 \\
&\geq (\frac{1}{2} - \frac{1}{p})\|u\|^2 \\
&\geq (\frac{1}{2} - \frac{1}{p})\Lambda_0^2 > 0.
\end{aligned}$$

This completes the proof. □

Corollary 4.3. *For any $u \in E_i \setminus \{0\}$, there exists a unique $\hat{t}_{ui} > 0$ such that $\hat{t}_{ui}u \in \mathcal{M}_i$. Moreover, $\mathcal{J}_i(\hat{t}_{ui}u) = \max_{t>0} \mathcal{J}_i(tu)$, where $i = 1, 2$.*

Lemma 4.4. *Suppose that* (A1), (A2) *are satisfied, then there exists a $(PS)_c$ sequence $\{u_n\} \subset \mathcal{M}$ such that $\mathcal{J}(u_n) \to c$ and $\mathcal{J}'(u_n) \to 0$ as $n \to +\infty$. Moreover, the sequence $\{u_n\}$ is bounded and bounded away from zero on $\mathcal{N}$.*

Proof. By the Ekeland Variational Principle, there exists a sequence $\{u_n\} \subset \mathcal{M}$ such that

$$\mathcal{J}(u_n) \to c \quad \text{and} \quad \mathcal{J}'|_{\mathcal{M}}(u_n) \to 0 \quad \text{as} \quad n \to +\infty.$$

From the definition of $\mathcal{B}$ in the proof of Lemma 4.2, we have, as $n \to +\infty$,

$$o_n(1) = \nabla|_{\mathcal{M}}\mathcal{J}(u_n) = \mathcal{J}'(u_n) + \hat{\mu}_n\mathcal{B}'(u_n), \tag{4.3}$$

for some $\hat{\mu}_n \in \mathbb{R}$. Taking the scalar product with u_k, we obtain that

$$o_n(1) = \langle \mathcal{J}'(u_n), u_n\rangle + \hat{\mu}_n\langle \mathcal{B}'(u_n), u_n\rangle = \hat{\mu}_n\langle \mathcal{B}'(u_n), u_n\rangle. \tag{4.4}$$

Since $u_n \in \mathcal{M}$ and $p \in [4, 6)$, we obtain

$$\begin{aligned}
\langle \mathcal{B}'(u_n), u_n\rangle &= 2\|u_n\|^2 + 4\int \phi_{u_n}u_n^2\,\mathrm{d}x - 6\int |u_n|^6\,\mathrm{d}x - \lambda p\int |u_n|^p\,\mathrm{d}x \\
&= (2-p)\|u_n\|^2 + (4-p)\int \phi_{u_n}u_n^2\,\mathrm{d}x + (p-6)\int |u_n|^6\,\mathrm{d}x \\
&\le (2-p)\|u_n\|^2 < 0.
\end{aligned}$$

It follows from (4.3) and (4.4) that $\hat{\mu}_n \to 0$ and $\mathcal{J}'(u_n) \to 0$ in E^{-1} as $n \to +\infty$. It remains to show that $\{u_n\}$ is bounded in E. Since $u_n \in \mathcal{M}$, we obtain

$$0 = \langle \mathcal{J}'(u_n), u_n\rangle = \|u_n\|^2 + \int \phi_{u_n}u_n^2\,\mathrm{d}x - \int |u_n|^6\,\mathrm{d}x - \lambda\int |u_n|^p\,\mathrm{d}x.$$

By a direct calculation, for $p \in [4, 6)$,

$$\begin{aligned}
\mathcal{J}(u_n) &= \mathcal{J}(u_n) - \frac{1}{4}\langle \mathcal{J}'(u_n), u_n\rangle \\
&= \frac{1}{4}\|u_n\|^2 + \frac{1}{12}\int |u_n|^6\,\mathrm{d}x + \lambda(\frac{1}{4} - \frac{1}{p})\int |u_n|^p\,\mathrm{d}x \\
&\ge \frac{1}{4}\|u_n\|^2.
\end{aligned}$$

Then, it follows from $\mathcal{J}(u_n) \to c$ as $n \to +\infty$ that $\{u_n\}$ is bounded. Moreover, by (4.1), $\{u_n\}$ is bounded and bounded away from zero. □

Since $\{u_n\}$ is bounded in E, passing to a subsequence, there exists $\hat{u} \in E$ such that as $n \to \infty$,

$$\begin{gathered}
u_n \rightharpoonup \hat{u} \quad \text{in } E, \\
u_n \to \hat{u} \quad \text{in } L^r_{\rm loc}(\mathbb{R}^3), r \in [1, 6), \\
u_n(x) \to \hat{u}(x) \quad \text{a.e. on } \mathbb{R}^3.
\end{gathered} \tag{4.5}$$

For $\varepsilon > 0$, let

$$\varphi_\varepsilon(x) := \frac{3^{1/4}\psi(x)\varepsilon^{1/2}}{(\varepsilon^2 + |x|^2)^{1/2}},$$

where $\psi \in C_0^\infty(\mathbb{R}^3,[0,1])$ is such that $\psi(x)=1$ for $|x|\le R$ and $\psi(x)=0$ for $|x|\ge 2R$. We need the following asymptotic estimates as $\varepsilon\to 0^+$ (see [9])

$$\|\nabla\varphi_\varepsilon\|_2^2 = S^{3/2}+O(\varepsilon), \quad \|\varphi_\varepsilon\|_6^6 = S^{3/2}+O(\varepsilon^3)$$

$$\|\varphi_\varepsilon\|_s^s = \begin{cases} O(\varepsilon^{s/2}), & \text{if } s\in[2,3) \\ O(\varepsilon^{s/2}|\ln\varepsilon|), & \text{if } s=3, \\ O(\varepsilon^{(6-s)/2}), & \text{if } s\in(3,6) \end{cases} \tag{4.6}$$

Lemma 4.5. *Suppose that* (A1) *and* (A2) *are satisfied. Then*

$$0<m_i<\frac{1}{3}S^{3/2}, \quad i=1,2, \tag{4.7}$$

where m_i defined in (1.17)*, if one of the following conditions is satisfied:*

(i) $4<p<6$ *and* $\lambda>0$*;*
(ii) $p=4$ *and* $\lambda>0$ *large enough.*

Proof. From the definition of m_i and Lemma 4.2, there exists $t_\varepsilon>0$ such that

$$0<m_i\le \mathcal{J}_i(t_\varepsilon\varphi_\varepsilon)=\max_{t\ge0}\mathcal{J}_i(t\varphi_\varepsilon). \tag{4.8}$$

On the one hand, since 0 is a local minimum of $\mathcal{J}_i$, there exists a constant $C>0$, independent of ε, such that $\mathcal{J}_i(t_\varepsilon\varphi_\varepsilon)\ge C>0$. Then from the continuity of $\mathcal{J}_i$, we may assume that $t_\varepsilon\ge t_1>0$, where t_1 is a positive constant.

On the other hand, from the definition of φ_ε and (4.6), for any $\varepsilon>0$ small enough, we have

$$\mathcal{J}_i\big(t_\varepsilon\varphi_\varepsilon\big)\le (S^{3/2}+C_1)t^2+C_2t^4-\frac{S^{3/2}}{12}t^6,$$

where C_1,C_2 are positive constants, independent of ε. Thus there exists $t_2>0$ such that $t_1\le t_\varepsilon\le t_2$ for each $\varepsilon>0$.

We set

$$h(t)=\frac{t^2}{2}\int|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x-\frac{t^6}{6}\int|\varphi_\varepsilon|^6\,\mathrm{d}x.$$

By a direct calculation, we can show that g attains its maximum at

$$t_0=\Big(\frac{\int|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x}{\int|\varphi_\varepsilon|^6\,\mathrm{d}x}\Big)^{\frac14}.$$

Moreover, by (4.6), using the inequality $(a+b)^p\le a^p+p(a+b)^{p-1}b$, which holds for any $p\ge1$ and $a,b\ge0$, we deduce that

$$\begin{aligned}
\max_{t\ge0}h(t)&=h(t_0)\\
&=\frac12\Big(\frac{\int|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x}{\int|\varphi_\varepsilon|^6\,\mathrm{d}x}\Big)^{2/4}\int|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x-\frac16\Big(\frac{\int|\nabla\varphi_\varepsilon t|^2\,\mathrm{d}x}{\int|\varphi_\varepsilon|^6\,\mathrm{d}x}\Big)^{6/4}\int_{\mathbb{R}^3}|\varphi_\varepsilon|^6\,\mathrm{d}x\\
&=\frac13\frac{\|\nabla\varphi_\varepsilon\|_2^3}{\|\varphi_\varepsilon\|_6^3}\\
&\le\frac13\frac{[S^{3/2}+O(\varepsilon)]^{3/2}}{[S^{3/2}+O(\varepsilon^3)]^{1/2}}\\
&\le\frac13S^{3/2}+O(\varepsilon).
\end{aligned}$$

Then we obtain

$$\begin{aligned}\mathcal{J}_i(t_\varepsilon\varphi_\varepsilon) &= \frac{t_\varepsilon^2}{2}\int |\nabla\varphi_\varepsilon|^2\,\mathrm{d}x + \frac{t_\varepsilon^2}{2}\int V(x)\varphi_\varepsilon^2\,\mathrm{d}x + \frac{t_\varepsilon^4}{4}\int \phi_{\varphi_\varepsilon}\varphi_\varepsilon^2\,\mathrm{d}x \\ &\quad - \frac{t_\varepsilon^6}{6}\int \varphi_\varepsilon^6\,\mathrm{d}x - \frac{\lambda t_\varepsilon^4}{p}\int |\varphi_\varepsilon|^p\,\mathrm{d}x \\ &\le \frac{1}{3}S^{3/2} + O(\varepsilon) + C_1\|\varphi_\varepsilon\|_2^2 + C_2\|\varphi_\varepsilon\|_{12/5}^4 - C_3\lambda\|\varphi_\varepsilon\|_p^p.\end{aligned}$$

To complete the proof, it remains to show that

$$\lim_{\varepsilon\to 0^+}\frac{C_1\|\varphi_\varepsilon\|_2^2 + C_2\|\varphi_\varepsilon\|_{12/5}^4 - C_3\lambda\|\varphi_\varepsilon\|_p^p}{\varepsilon} = -\infty. \tag{4.9}$$

In fact, by (4.6) the following estimate holds as $\varepsilon \to 0$:

$$C_1\|\varphi_\varepsilon\|_2^2 + C_2\|\varphi_\varepsilon\|_{12/5}^4 - C_3\lambda\|\varphi_\varepsilon\|_p^p \le C_4\varepsilon + C_5\varepsilon^2 - C_6\lambda\varepsilon^{(6-q)/2}. \tag{4.10}$$

If $4 < q < 6$, it follows immediately from (4.9) for any $\lambda > 0$. If $q = 4$, one can chose $\lambda = \varepsilon^{-\mu}, \mu > 0$ in the above inequality to obtain (4.9). □

Now, we give the splitting lemma of a $(PS)_c$ sequence of $\mathcal{J}$, which plays a crucial role for subsequent discussion.

Lemma 4.6 (Splitting Lemma). *Let $\{u_n\} \subset E$ be a bounded $(PS)_c$ sequence of $\mathcal{J}$ at level $m \in (0, \frac{1}{3}S^{\frac{2}{3}})$ and assume that $u_n \rightharpoonup \hat{u}$ in E. Then, passing to a subsequence, either u_n strongly converges to $\hat{u}$, or setting $\hat{k} \in \mathbb{N}\cup\{0\}$, there are sequences $\{u^i\}_{i=1}^{\hat{k}} \subset E$ and $y_n^i \in T_1\mathbb{Z}\times T_2\mathbb{Z}\times T_3\mathbb{Z} \subset \mathbb{R}^3$ with $1 \le j \le \hat{k}$ such that*

- (i) *$(-1)^{\varepsilon_i-1}(y_n^i)_1 \to +\infty$ and $|(y_n^i)_1 - (y_n^j)_1| \to +\infty$ for $1 \le i \ne j \le \hat{k}$, as $n \to +\infty$, where $(y_n^i)_\gamma$ denotes the γ-th component of y_n^i, $1 \le \gamma \le 3$.*
- (ii) *$u_n \to \hat{u} + \sum_{i=1}^{\hat{k}} u^i(\cdot - y_n^i)$ in E.*
- (iii) *$\mathcal{J}(u_n) = \mathcal{I}(\hat{u}) + \sum_{i=1}^{\hat{k}} \mathcal{J}_{\varepsilon_i}(u^i) + o_n(1)$.*
- (iv) *$\mathcal{J}'_{\varepsilon_i}(u^i) = 0$ and $u^i \ne 0$ with $1 \le i \le \hat{k}$.*

where $\varepsilon_i = \{1, 2\}$.

Proof. Let us divide the proof in various steps.

Step 1: Let $z_n^1 = u_n - \hat{u}$. We have two possibilities:

If $z_n^1 \to 0$ in E, then the first alternative follows and the proof is concluded.

If $z_n^1 \not\to 0$ in E. Let $\varepsilon_i = \{1, 2\}$ and

$$\mathbb{R}^3_{(-1)^{\varepsilon_i}} = \begin{cases}\mathbb{R}^3_-, & \varepsilon_i = 1,\\ \mathbb{R}^3_+, & \varepsilon_i = 2.\end{cases} \tag{4.11}$$

Since $z_n^1 \rightharpoonup 0$ in E, it follows from Lemmas 2.2 and 2.3 that, as $n \to \infty$,

$$\mathcal{J}(u_n) = \mathcal{J}(z_n^1) + \mathcal{J}(\hat{u}) + o_n(1),$$

and for any $\varphi \in C_0^\infty(E, \mathbb{R})$, $\varepsilon_1 = \{1, 2\}$

$$\begin{aligned}\langle \mathcal{J}'(u_n), \varphi\rangle &= \langle \mathcal{J}'(\hat{u}), \varphi\rangle + \langle \mathcal{J}'(z_n^1), \varphi\rangle + o_n(1) \\ &= \langle \mathcal{J}'(\hat{u}), \varphi\rangle + \langle \mathcal{J}'_{\varepsilon_1}(z_n^1), \varphi\rangle + \int_{\mathbb{R}^3_{(-1)^{\varepsilon_1}}} (-1)^{\varepsilon_1}(V_1 - V_2) z_n^1\varphi\,\mathrm{d}x + o_n(1) \\ &= \langle \mathcal{J}'(\hat{u}), \varphi\rangle + \langle \mathcal{J}'_{\varepsilon_1}(z_n^1), \varphi\rangle + o_n(1),\end{aligned}$$

which together with $\mathcal{J}'(u_n) \to 0$ and $\mathcal{J}'(\hat{u}) = 0$ imply that

$$\langle \mathcal{J}'_{\varepsilon_1}(z_n^1), \varphi \rangle \to 0, \quad \text{as} \quad n \to \infty. \tag{4.12}$$

In addition, since $\langle \mathcal{J}'(u_n), \varphi\rangle = \langle \mathcal{J}'(\hat{u}), \varphi\rangle + \langle \mathcal{J}'(z_n^1), \varphi\rangle + o_n(1)$, which together with $\mathcal{J}'(u_n) \to 0$ and $\mathcal{J}'(\hat{u}) = 0$ implies that

$$\langle \mathcal{J}'(z_n^1), \varphi\rangle \to 0, \quad \text{as } n \to \infty.$$

Setting $\varphi = z_n^1$, we obtain $\langle \mathcal{J}'(z_n^1), z_n^1\rangle \to 0$, i.e.,

$$\|z_n^1\|^2 + \int \phi_{z_n^1}(z_n^1)^2 \,\mathrm{d}x - \int |z_n^1|^6 \,\mathrm{d}x - \lambda \int |z_n^1|^p \,\mathrm{d}x \to 0. \tag{4.13}$$

Let

$$\delta_1 := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |z_n^1|^2 \,\mathrm{d}x.$$

We claim that $\delta_1 \neq 0$. If not, by Lion's lemma, $z_n^1 \to 0$ in $L^r(\mathbb{R}^3)$ for $r \in (2,6)$. By Hölder's inequality, we obtain that

$$\int \phi_{z_n^1}(z_n^1)^2 \,\mathrm{d}x \le \Big(\int |\phi_{z_n^1}|^6 \,\mathrm{d}x\Big)^{1/6} \Big(\int |z_n^1|^{12/5} \,\mathrm{d}x\Big)^{5/6} \le CS\|z_n^1\|_{12/5}^2 \to 0.$$

It follows from (4.13) that $\|z_n^1\|^2 = \int |z_n^1|^6 \,\mathrm{d}x + o_n(1)$. Asume that $\|z_n^1\|^2 \to \eta_1$, so $\int |z_n^1|^6 \,\mathrm{d}x \to \eta_1$, as $n \to \infty$. By Sobolev embdding, we obtain $\int |z_n^1|^6 \,\mathrm{d}x \le S^{-1}\|z_n^1\|^6$, which implies that $\eta_1 \le S^{-3}\eta_1^3$. Hence, $\eta_1 = 0$ or $\eta_1 \ge S^{3/2}$. If $\eta_1 = 0$, then $z_n^1 \to 0$ in E. If $\eta_1 \le S^{-3}\eta_1^3$, then

$$\begin{aligned}
m &= \lim_{n\to\infty} \mathcal{J}(u_n) \\
&= \lim_{n\to\infty} \mathcal{J}(z_n^1) + \mathcal{J}(\hat{u}) \\
&\ge \lim_{n\to\infty} \mathcal{J}(z_n^1) \\
&= \frac{1}{2}\|z_n^1\|^2 + \frac{1}{4}\int \phi_{z_n^1}(z_n^1)^2 \,\mathrm{d}x - \frac{1}{6}\int |z_n^1|^6 \,\mathrm{d}x - \frac{\lambda}{p}\int |z_n^1|^p \,\mathrm{d}x \\
&= \frac{1}{2}\eta_1 - \frac{1}{6}\eta_1 = \frac{1}{3}\eta_1 \ge \frac{1}{3}S^{3/2},
\end{aligned}$$

which contradicts $m \in (0, \frac{1}{3}S^{3/2})$. Then $\delta_1 \neq 0$. Hence, there exists $y_n^1 \in T_1\mathbb{Z} \times T_2\mathbb{Z} \times T_3\mathbb{Z} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n^1)} |z_n^1|^2 dx \ge \frac{\delta_1}{2}. \tag{4.14}$$

Let $\xi_n^1 := (0, (y_n^1)_2, (y_n^1)_3), \sigma_n^1 := ((y_n^1)_1, 0, 0)$, and $w_n^1 = z_n^1(\cdot + \xi_n^1)$. Clearly, $\|w_n^1\| = \|z_n^1\|$ and $w_n^1 \rightharpoonup 0$ in E but not strongly. Therefore, by (4.14), we obtain

$$\int_{B_1(\sigma_n^1)} |w_n^1|^2 dx \ge \frac{\delta_1}{2}. \tag{4.15}$$

It is easy to check that $|(y_n^1)_1| = |\sigma_n^1| \to +\infty$, that is,

$$(-1)^{\varepsilon_1 - 1}(y_n^1)_1 \to +\infty. \tag{4.16}$$

Considering the sequence $\{w_n^1(\cdot + \sigma_n^1)\}$, which is bounded in E_{ε_1}, then there exists $u^1 \in E_{\varepsilon_1}$ satisfying

$$\begin{gathered}
w_n^1(\cdot + \sigma_n^1) \rightharpoonup u^1 \quad \text{in } E_{\varepsilon_1}, \\
w_n^1(\cdot + \sigma_n^1) \to u^1 \quad \text{in } L^r_{\mathrm{loc}}(\mathbb{R}^3),
\end{gathered}$$

$$w_n^1(x+\sigma_n^1) \to u^1(x) \text{ a.e. on } \mathbb{R}^3.$$

From (4.15), we obtain $u^1 \neq 0$. From (4.11), for any $\varphi \in C_0^\infty(E,\mathbb{R})$, we obtain

$$\langle \mathcal{J}'_{\varepsilon_1}(w_n^1(\cdot+\sigma_n^1)),\varphi\rangle = \langle \mathcal{J}'_{\varepsilon_1}(z_n^1(\cdot+y_n^1)),\varphi\rangle = \langle \mathcal{J}'_{\varepsilon_1}(z_n^1),\varphi(\cdot-y_n^1)\rangle \to 0.$$

Hence, we have $\mathcal{J}'_{\varepsilon_1}(w_n^1(\cdot+\sigma_n^1)) \to 0$ since $C_0^\infty(E,\mathbb{R})$ is dense in E, and then

$$\mathcal{J}'_{\varepsilon_1}(u^1) = 0. \tag{4.17}$$

Step 2: Let $z_n^2 = z_n^1 - u^1(x-y_n^1)$. Then, $z_n^2 \rightharpoonup 0$ in E due to the norm of E is equivalent to the norm of E_{ε_1}. In addition, from Lemmas 2.2 and 2.3, by the simple calculation, as $n\to\infty$,

$$\begin{aligned}
\mathcal{J}(u_n) &= \frac{1}{2}\|u_n\|^2 + \frac{1}{4}\int \phi_{u_n}u_n^2\,\mathrm{d}x - \frac{1}{6}\|u_n\|_6^6 - \frac{\lambda}{p}\|u_n\|_p^p\\
&= \mathcal{J}(z_n^1) + \mathcal{J}(\hat{u}) + o_n(1)\\
&= \mathcal{J}(z_n^2) + \mathcal{J}(u^1) + \mathcal{J}(\hat{u}) + o_n(1)\\
&= \mathcal{J}(z_n^2) + \mathcal{J}_{\varepsilon_1}(u^1) + \int_{\mathbb{R}^3_{(-1)^{\varepsilon_1}}}(-1)^{\varepsilon_1}(V_1-V_2)u^1\,\mathrm{d}x + \mathcal{J}(\bar{u}) + o_n(1)\\
&= \mathcal{J}(z_n^2) + \mathcal{J}_{\varepsilon_1}(u^1) + \mathcal{J}(\bar{u}) + o_n(1).
\end{aligned} \tag{4.18}$$

We have two possibilities:

If $z_n^2 \to 0$ in E, i.e., $z_n^1 - u^1(x-y_n^1) = u_n - \hat{u} - u^1(x-y_n^1) \to 0$, i.e.,

$$u_n \to \hat{u} + u^1(x-y_n^1). \tag{4.19}$$

From (4.18), we obtain

$$\mathcal{J}(u_n) = \mathcal{J}(\hat{u}) + \mathcal{J}_{\varepsilon_1}(u^1) + o_n(1). \tag{4.20}$$

Then the Lemma is proved for $k=1$. It follows from (4.16), (4.17), (4.19), and (4.20).

If $z_n^2 \not\to 0$ in E. From Lemmas 2.2 and 2.3, we obtain for any $\varphi \in C_0^\infty(E,\mathbb{R})$, as $n\to\infty$,

$$\begin{aligned}
\langle \mathcal{J}'(z_n^1),\varphi\rangle &= \langle \mathcal{J}'(z_n^2),\varphi\rangle + \langle \mathcal{J}'(u^1),\varphi(x+y_n^1)\rangle + o_n(1)\\
&= \langle \mathcal{J}'(z_n^2),\varphi\rangle + \langle \mathcal{J}'_{\varepsilon_1}(u^1),\varphi(x+y_n^1)\rangle\\
&\quad + \int_{\mathbb{R}^3_{(-1)^{\varepsilon_1}}}(-1)^{\varepsilon_1}(V_1-V_2)u^1\varphi(x+y_n^1)dx + o_n(1)\\
&= \langle \mathcal{J}'(z_n^2),\varphi\rangle + \langle \mathcal{J}'_{\varepsilon_1}(u^1),\varphi(x+y_n^1)\rangle + o_n(1).
\end{aligned} \tag{4.21}$$

It follows from $\langle \mathcal{J}'(z_n^1),\varphi\rangle \to 0$ and $\mathcal{J}'_{\varepsilon_1}(u^1)=0$ that we obtain $\langle \mathcal{J}'(z_n^2),\varphi\rangle \to 0$. Setting $\varphi = z_n^2$, we obtain $\langle \mathcal{J}'(z_n^2),z_n^2\rangle \to 0$, i.e.,

$$\|z_n^2\|^2 + \int \phi_{z_n^2}(z_n^2)^2\,\mathrm{d}x - \int |z_n^1|^6\,\mathrm{d}x - \lambda\int |z_n^2|^p\,\mathrm{d}x \to 0. \tag{4.22}$$

Let

$$\delta_2 := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)}|z_n^2|^2dx.$$

Simlar to $\delta_1 \neq 0$, we obtain $\delta_2 \neq 0$. Then, there exists $y_n^2 \in T_1\mathbb{Z}\times T_2\mathbb{Z}\times T_3\mathbb{Z} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n^2)}|z_n^2|^2dx \geq \frac{\delta_2}{2}.$$

Let $\xi_n^2 := (0, (y_n^2)_2, (y_n^2)_3), \sigma_n^2 := ((y_n^2)_1, 0, 0)$, and $w_n^2 = z_n^2(x+\xi_n^2)$. Clearly, $\|w_n^2\| = \|z_n^2\|$ and $w_n^2 \rightharpoonup 0$ in E. Therefore,

$$\int_{B_1(\sigma_n^2)} |w_n^2|^2 dx \geq \frac{\delta_2}{2}. \tag{4.23}$$

It is easy to check that as $n \to \infty$, $|(y_n^2)_1| = |\sigma_n^2| \to +\infty$, that is

$$(-1)^{\varepsilon_2 - 1}(y_n^2)_1 \to +\infty. \tag{4.24}$$

Then $w_n^2(\cdot + \sigma_n^2) \not\rightharpoonup 0$ in E. In addition, we claim that

$$|(y_n^2)_1 - (y_n^1)_1| \to +\infty. \tag{4.25}$$

To see this, first observe that

$$\begin{aligned}
& w_n^2(\cdot + \sigma_n^2 + (y_n^2)_1 - (y_n^1)_1) \\
&= z_n^2(\cdot + \sigma_n^2 + (y_n^2)_1 - (y_n^1)_1 + \xi_n^2) \\
&= z_n^1(\cdot + \sigma_n^2 + (y_n^2)_1 - (y_n^1)_1 + \xi_n^2) - u^1(\cdot + \sigma_n^2 + (y_n^2)_1 - (y_n^1)_1 + \xi_n^2 - y_n^1) \\
&= w_n^1(\cdot + \sigma_n^2 + (y_n^2)_1 - (y_n^1)_1 + \xi_n^2 - \xi_n^1) - u^1(\cdot + \sigma_n^2 + (y_n^2)_1 - (y_n^1)_1 + \xi_n^2 - y_n^1) \\
&= w_n^1(\cdot + \sigma_n^2 + y_n^2 - y_n^1) - u^1(\cdot + \sigma_n^2 + y_n^2 - y_n^1 - \sigma_n^1).
\end{aligned}$$

From Lemma 3.4 (ii), since $w_n^2(\cdot + \sigma_n^2) \not\rightharpoonup 0$ in E, if it were $|(y_n^2)_1 - (y_n^1)_1| \not\to +\infty$, we obtain

$$w_n^2(\cdot + \sigma_n^2 + (y_n^2)_1 - (y_n^1)_1) \not\rightharpoonup 0.$$

On the other hand, since $w_n^1(\cdot + \sigma_n^1) \rightharpoonup u^1$, we obtain

$$w_n^1(\cdot + \sigma_n^2 + y_n^2 - y_n^1) - u^1(\cdot + \sigma_n^2 + y_n^2 - y_n^1 - \sigma_n^1) \rightharpoonup 0,$$

which is a contradiction. So we obtain (4.25). Moreover, for any $\varphi \in C_0^\infty(E, \mathbb{R})$, we obtain as $n \to \infty$,

$$\begin{aligned}
& \langle \mathcal{J}'_{\varepsilon_2}(z_n^1), \varphi \rangle \\
&= \langle \mathcal{J}'_{\varepsilon_2}(z_n^2), \varphi \rangle + \langle \mathcal{J}'_{\varepsilon_1}(u^1), \varphi(x + y_n^1) \rangle + \int (V_{\varepsilon_2} - V_{\varepsilon_1}) u^1 \varphi(x + y_n^1) dx + o_n(1) \\
&= \langle \mathcal{J}'_{\varepsilon_2}(z_n^2), \varphi \rangle + \langle \mathcal{J}'_{\varepsilon_1}(u^1), \varphi(x + y_n^1) \rangle + o_n(1).
\end{aligned}$$

By $\langle \mathcal{J}'_{\varepsilon_2}(z_n^1), \varphi \rangle \to 0$ and $\mathcal{J}'_{\varepsilon_1}(u^1) = 0$, we obtain $\mathcal{J}'_{\varepsilon_2}(z_n^2) \to 0$ since $C_0^\infty(E, \mathbb{R})$ is dense in E. Considering the sequence $\{w_n^2(\cdot + \sigma_n^2)\}$, which is bounded in E_{ε_2}, there exists $u^2 \in E_{\varepsilon_2}$ satisfying

$$\begin{gathered}
w_n^2(\cdot + \sigma_n^2) \rightharpoonup u^2 \quad \text{in } E_{\varepsilon_2}, \\
w_n^2(\cdot + \sigma_n^2) \to u^2 \quad \text{in } L^r_{\rm loc}(\mathbb{R}^3), \\
w_n^2(x + \sigma_n^2) \to u^2(x) \quad \text{a.e. on } \mathbb{R}^3.
\end{gathered}$$

We can see that $u^2 \neq 0$. For any $\varphi \in C_0^\infty(E, \mathbb{R})$, we obtain

$$\langle \mathcal{J}'_{\varepsilon_2}(u^2), \varphi \rangle = \langle \mathcal{J}'_{\varepsilon_2}(w_n^2(\cdot + \sigma_n^2)), \varphi \rangle = \langle \mathcal{J}'_{\varepsilon_2}(z_n^2(\cdot + y_n^2)), \varphi \rangle = \langle \mathcal{J}'_{\varepsilon_2}(z_n^2), \varphi(\cdot - y_n^2) \rangle \to 0.$$

Hence, since $C_0^\infty(E, \mathbb{R})$ is dense in E, we have

$$\mathcal{I}'_{\varepsilon_2}(u^2) = 0. \tag{4.26}$$

Step 3: Let $z_n^3 = z_n^2 - u^2(x - y_n^2)$. From (4.18), we have

$$\begin{aligned} \mathcal{J}(u_n) &= \mathcal{J}(z_n^2) + \mathcal{J}_{\varepsilon_1}(u^1) + \mathcal{J}(\bar{u}) + o_n(1) \\ &= \mathcal{J}(z_n^3) + \mathcal{J}_{\varepsilon_2}(u^2) + \mathcal{I}_{\varepsilon_1}(u^1) + \mathcal{J}(\bar{u}) + o_n(1). \end{aligned} \tag{4.27}$$

We have two possibilities:

If $z_n^3 \to 0$ in E, i.e., $z_n^2 - u^2(x - y_n^2) = u_n - \bar{u} - u^1(x - y_n^1) - u^2(x - y_n^2) \to 0$, i.e.,

$$u_n \to \bar{u} + u^1(x - y_n^1) + u^2(x - y_n). \tag{4.28}$$

From (4.27), we obtain

$$\mathcal{J}(u_n) = \mathcal{J}(\bar{u}) + \mathcal{J}_{\varepsilon_1}(u^1) + \mathcal{J}_{\varepsilon_2}(u^2) + o_n(1). \tag{4.29}$$

Then the Lemma is proved for $k = 2$ follows from (4.24), (4.25), (4.28), (4.29) and (4.26).

If $z_n^2 \not\to 0$ in E, we just repeat the argument.

Step $\hat{k}$: By $\mathcal{J}(u_n) = \mathcal{J}(\bar{u}) + \sum_{i=1}^{\hat{k}} \mathcal{J}_{\varepsilon_i}(u^i) + o_n(1)$, since $\mathcal{J}_{\varepsilon_i}(u^i) \geq c_{\varepsilon_i} \geq \min\{c_1, c_2\}$ and $\mathcal{J}(u_n)$ is bound, the iteration must stop at some finite index $\hat{k}$. The proof is complete. □

Proof of Theorem 1.3. In view of Lemma 4.4, there exists a bounded $(PS)_c$ sequence $\{u_k\} \subset \mathcal{M}$ such that $\mathcal{J}(u_k) \to m$ and $\mathcal{J}'(u_k) \to 0$ as $k \to +\infty$. Since $\{u_k\}$ is bounded in E, going to a subsequence if necessary, still denoted by $\{u_k\}$, we can suppose that there exists $\hat{u} \in E$ such that $u_k \rightharpoonup \hat{u}$ in E. With $m < \min\{m_1, m_2\}$, by Lemma 4.6, if $u_k \not\to u$, we can show that $k \geq 1$ and nontrivial solutions $u^1, u^2, \dots, u^j$ of $\mathcal{J}_{\varepsilon_j}$ with $\varepsilon_j = \{1, 2\}$ satisfy

$$m = \lim_{k \to +\infty} \mathcal{J}(u_k) = \mathcal{I}(\hat{u}) + \sum_{j=1}^{\hat{k}} \mathcal{J}_{\varepsilon_j}(u^j) \geq \hat{k} \min\{m_1, m_2\} \geq \min\{m_1, m_2\},$$

which contradicts $m < \min\{m_1, m_2\}$. Thus, $u_k \to \hat{u}$, and then $\mathcal{J}(\hat{u}) = m$ and $\mathcal{J}'(\hat{u}) = 0$. Obviously, $\hat{u} \neq 0$. Therefore, $\hat{u}$ is a ground state solution of (1.14).

Considering $\hat{u}_0 = |\hat{u}|$, it is easy to check that $\mathcal{J}(\hat{u}_0) = \mathcal{J}(\hat{u}) = m$ and $\hat{u}_0 \in \mathcal{M}$. From standard arguments, we infer that $\mathcal{J}'(\hat{u}_0) = 0$. Thus, $\hat{u}_0$ is a non-negative solution of system (1.14). Furthermore, the strong maximum principle implies that $\hat{u}_0 > 0$ in $\mathbb{R}^3$, and thus, $\hat{u}_0$ is a positive ground state solution of system (1.14). □

Proof of Theorem 1.4. We just study the case of $m_1 \leq m_2$ since the case $m_2 \leq m_1$ is analogous.

Let $v_1 \in \mathcal{M}_1 \subset E$ be a positive ground state for the purely periodic problem for (1.16) with $i = 1$. We can see from Lemma 2.2 that there exists $\hat{s} > 0$ satisfying $\hat{s}v_1 \in \mathcal{M}$. Then, from the assumption of Theorem 1.4 and Corollary 4.3, we have

$$\begin{aligned} m &\leq \mathcal{J}(\hat{s}v_1) \\ &= \frac{\hat{s}^2}{2}\|v_1\|^2 + \frac{\hat{s}^4}{4}\int \phi_{v_1}|v_1|^2 \,\mathrm{d}x - \frac{\hat{s}^6}{6}\|v_1\|_p^p - \frac{\lambda \hat{s}^p}{p}\|v_1\|_p^p \\ &= \frac{\hat{s}^2}{2}\|v_1\|_{E_1}^2 + \frac{\hat{s}^4}{4}\int \phi_{v_1}|v_1|^2 \,\mathrm{d}x - \frac{\hat{s}^6}{6}\|v_1\|_p^p - \frac{\lambda \hat{s}^p}{p}\|v_1\|_p^p + \frac{\hat{s}^2}{2}\int_{\mathbb{R}^3_-} (V_2 - V_1) v_1^2 \,\mathrm{d}x \\ &= \mathcal{J}_1(\hat{s}v_1) + \frac{\hat{s}^2}{2}\int_{\mathbb{R}^3_-} (V_2 - V_1) v_1^2 \,\mathrm{d}x \end{aligned}$$

$$< \mathcal{J}_1(v_1) = m_1,$$

which implies that $m < m_1$. Thus, $m < \min\{m_1, m_2\}$. □

Acknowledgements. This work is supported by the Research Fund of the National Natural Science Foundation of China (No. 12361024), the Team Building Project for Graduate thenTutors in Chongqing (No. yds223010) and the Natural Science Foundation of Chongqing, China (No. 2024NSCQ-MSX4817). We also wish to thank the referees for their careful reading of the manuscript which greatly improved the exposition.

G. Siciliano was supported by Capes, CNPq (grant no. 304244/2023-6), FAPDF Edital 04/2021 - Demanda Espontânea, Fapesp grants no. 2022/16407-1 and 2022/16097-2 (Brazil) , and PRIN PNRR P2022YFAJH "Linear and Nonlinear PDEs: New directions and applications", and by INdAM-GNAMPA (Italy).

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