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CAFFARELLI-KOHN-NIRENBERG TYPE PROBLEMS WITH BERESTYCKI-LIONS TYPE NONLINEARITIES

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Abstract. In this article we use a Palais-Smale sequence satisfying a property related to Pohozaev identity to show the existence of solution for the elliptic Caffarelli-Kohn-Nirenberg type problems

$$-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) + |x|^{-bp^*}|u|^{p-2}u = |x|^{-bp^*}h(u) \quad \text{in } \mathbb{R}^N$$

and

$$-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = |x|^{-bp^*}f(u) \quad \text{in } \mathbb{R}^N,$$

where $1 < p < N$, $0 \leq a < \frac{N-p}{p^*}$, $a < b \leq a+1$, $p^* = p^*(a,b) = \frac{pN}{N-dp}$ and $d = 1 + a - b$. and h and f are continuous functions that satisfy hypotheses considered by Berestycki and Lions in [7].

1. Introduction

Using a constrained minimization, Berestycki and Lions [7] showed the existence of positive solutions of C^2 class for the problem

$$-\Delta u = g(u) \quad \text{in } \mathbb{R}^N \tag{1.1}$$

with exponential decay and spherically symmetric, where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) = 0$. The authors assume that g is odd and satisfies the following conditions.

- (A1) $-\infty < \liminf_{s\to 0^+} g(s)/s \leq \limsup_{s\to 0^+} g(s)/s = -m \leq 0$.
- (A2) $-\infty \leq \limsup_{s\to\infty} g(s)/s^{2^*-1} \leq 0$.
- (A3) There exists $\xi > 0$ such that $G(\xi) = \int_0^\xi g(s)ds > 0$.

The constraint causes a Lagrange multiplier to appear, but it can be removed using the special homogeneity of the operator and a scale change in $\mathbb{R}^N$. The authors studied two cases: The positive mass case, $m > 0$, and the zero mass case, $m = 0$.

Alves, Montenegro and Souto [1] studied the existence of ground state solution for (1.1) with critical growth. By using the variational method, the authors in [1] gave a unified approach for the subcritical and critical cases. However, we would like

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to point out that a result due to Jeanjean and Tanaka [20] says that the Mountain-Pass value gives the least energy level, and it was the main tool used. A similar study was made for the critical case in Zhang and Zou [24].

After these pioneering papers, many researches worked on this subject, extending or improving it in several ways; see, for instance, [2, 3, 4, 5, 8, 11, 13, 14, 17] and references therein.

This article concerns the existence of nontrivial solutions for the problems

$$-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) + |x|^{-bp^*}|u|^{p-2}u = |x|^{-bp^*}h(u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

and

$$-\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = |x|^{-bp^*}f(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $1 < p < N$, $0 \le a < \frac{N-p}{p^*}$, $a < b \le a+1$, $p^* = p^*(a,b) = \frac{pN}{N-dp}$ and $d = 1 + a - b$.

Equations involving the operator $\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)$ are regarded as prototype of more general nonlinear degenerate elliptic equations from physical phenomena; see for example [15, 16, 23].

In this article we adapt some arguments found in [18] and [19]. More precisely, we find a Palais-Smale sequence satisfying a property related to Pohozaev identity. The same approach was used in [4] for a problem involving the Grushin operator.

We would like to point out that in the proof of Theorems 1.1 and 1.2, we have found some difficulties when applying variational methods. For example, for this class operator there is no a result like Jeanjean and Tanaka [20], which say that the Mountain-Pass value gives the least energy level of the Pohozaev manifold, which is crucial in order to use the arguments due to Berestycki-Lions. Furthermore, it was necessary to prove a Straus-type Lemma result for this class of problems (Lemma 3.2 and Lemma 3.3). In Chen [12] we can find a Straus-type Lemma result for this class of problems, but it does not apply to our case.

Before concluding this introduction, it is very important to say that in the literature, we find many papers where the authors study problems involving the operator $\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)$; see, Bastos, Miyagaki and Vieira [6], Catrina and Wang [10], Chen [12], Xuan [22] and references therein.

To present the main results of this article, it is necessary to state hypotheses about the nonlinearities h and f. The hypotheses on the function h in this case are the following:

(A4) There exists $q \in (p, p^*)$ such that

$$\lim_{|t|\to 0} \frac{h(t)}{|t|^{p-1}} = \lim_{|t|\to\infty} \frac{h(t)}{|t|^{q-1}} = 0.$$

(A5) There exists $\xi > 0$ such that $pH(\xi) - \xi^p > 0$, where $H(t) = \int_0^t h(r)dr$.

The first main result reads as follows.

Theorem 1.1. *Under the conditions* (A4) *and* (A5)*, problem* (1.2) *has a nontrivial solution.*

The first class of problems is called Positive Mass because $g(t) = h(t) - t$ satisfies (A1)–(A3) for $m > 0$.

The hypotheses on the function f in this case are as follws:

(A6)
$$\lim_{|t|\to 0}\frac{f(t)}{|t|^{p^*-1}}=\lim_{|t|\to\infty}\frac{f(t)}{|t|^{p^*-1}}=0.$$

(A7) There exists $\xi>0$ such that $F(\xi)>0$, where $F(t)=\int_0^t f(r)dr$.

The second main result reads as follows.

Theorem 1.2. *Under assumptions* (A6) *and* (A7), *problem* (1.3) *has a nontrivial solution.*

The second class of problems is called Zero Mass because f satisfies (A1)–(A3) for $m=0$.

The plan for this article is as follows: In section 2 we present the spaces that we find the solutions. In section 3 we prove Theorem 1.1. And in section 3 we prove Theorem 1.2.

2. Variational framework

For the zero-mass case we use $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ which is the completion of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with the norm
$$\|u\|_0^p=\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p\,dx,$$
where $\mathcal{C}_0^\infty(\mathbb{R}^N)$ is the space of smooth functions with compact support. For the Positive Mass case we use $E=\{u\in\mathcal{D}_a^{1,p}(\mathbb{R}^N):\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^p\,dx<\infty\}$ with the norm
$$\|u\|^p=\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p\,dx+\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^p\,dx.$$
Let $L_b^s(\mathbb{R}^N)$ be the weighted L^s space with weighted norm
$$|u|^s=\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^s\,dx.$$
We also define $E(B_R(0))=\{u\in\mathcal{D}_a^{1,p}(B_R(0)):\int_{B_R(0)}|x|^{-bp^*}|u|^p\,dx<\infty\}$. Let $L_b^s(B_R(0))$ be the weighted L^s space with weighted norm
$$|u|^s=\int_{B_R(0)}|x|^{-bp^*}|u|^s\,dx.$$
Let the weighted L^s space be defined by the weighted norm
$$|u|_{B_R(0)}^s=\int_{B_R(0)}|x|^{-bp^*}|u|^s\,dx.$$

Using an inequality established by Caffarelli, Kohn, and Nirenberg in [9],
$$\Big(\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^{p^*}\,dx\Big)^{p/p^*}\le S_{a,b}\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p\,dx,$$
we conclude that the embedding $\mathcal{D}_a^{1,p}(\mathbb{R}^N)\hookrightarrow L_b^{p^*}(\mathbb{R}^N)$ is continuous. Moreover, by interpolation, we also conclude that $E\hookrightarrow L_b^s(\mathbb{R}^N)$ is continuous, for $s\in[p,p^*]$.

3. Existence of solutions for the positive-mass case

Consider the functional $I : E \to \mathbb{R}$ associated given by

$$I(u) = \frac{1}{p}\|u\|^p - \int_{\mathbb{R}^N} |x|^{-bp^*} H(u)\, dx.$$

As a consequence of (A4) we obtain that I is well-defined and of C^1 class. Also note that

$$I'(u)\phi = \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}u\phi\, dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p-2}u\phi\, dx - \int_{\mathbb{R}^N} |x|^{-bp^*} h(u)\phi\, dx,$$

for all $\phi \in E$. Then, the critical points of I are weak solutions of (1.2).

To use critical point theory we firstly derive results related to the Palais-Smale compactness condition. We say that a sequence (u_n) is a Palais-Smale sequence for the functional I if

$$I(u_n) \to c_*, \quad \text{and} \quad \|I'(u_n)\| \to 0 \quad \text{in } (E)',$$

where

$$c_* = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} I(\eta(t)) > 0, \quad \Gamma := \{\eta \in C([0, 1], E) : \eta(0) = 0,\ I(\eta(1)) < 0\}.$$

If every Palais-Smale sequence of I has a strong convergent subsequence, then one says that I satisfies the Palais-Smale condition ((PS) for short).

Lemma 3.1. *The functional I satisfies the following conditions:*

(i) *There exist ρ_1, $\rho_2 > 0$ such that $I(u) \geq \rho_2$ with $\|u\| = \rho_1$.*
(ii) *There exists $e \in B^c_{\rho_2}(0)$ with $I(e) < 0$ and $\|e\| > \rho_2$.*

Proof. (i) First of all, from (A4), for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$h(t) \leq \varepsilon|t|^{p-1} + C_\varepsilon|t|^{q-1}, \quad \forall t \in \mathbb{R}. \tag{3.1}$$

Using the inequality above and taking $\epsilon > 0$ sufficiently small such, we obtain

$$I(u) \geq \big(\frac{1}{p} - \frac{\epsilon}{p}\big)\|u\|^p - \frac{C_1 C_\varepsilon}{q}\|u\|^q$$

and the result follows because $q > p$.

(ii) From (A5), there exists $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |x|^{-bp^*}\big(H(\phi) - \frac{|\phi|^p}{p}\big)\, dx > 0.$$

For $t > 0$, setting

$$\omega_t(x) = \phi\big(\frac{x}{t}\big),$$

by simple calculations, we obtain

$$I(\omega_t) = t^{N-p}\int_{\mathbb{R}^N} |y|^{-ap}|\nabla\phi(y)|^p dy - t^N \int_{\mathbb{R}^N} |y|^{-bp^*}\big(H(\phi(y)) - \frac{|\phi(y)|^p}{p}\big)dy \to -\infty,$$

as $t \to \infty$. Then, there exists $\bar{t} > 0$ large such that $e = \omega_{\bar{t}}$ satisfies $I(e) < 0$ and $\|e\| > \rho_2$. Also note that $c_* \geq \rho_2$. □

Next, we prove a compactness result, which is crucial in our approach. We denote by $\mathcal{C}^{\infty}_{0,rad}(\mathbb{R}^N)$ the collection of smooth radially symmetric functions with compact, i.e.,

$$\mathcal{C}^{\infty}_{0,rad}(\mathbb{R}^N) = \{u \in \mathcal{C}^{\infty}_0(\mathbb{R}^N) : u(x) = u(|x|), \quad x \in \mathbb{R}^N\}.$$

Let $\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$ be the completion of $\mathcal{C}^{\infty}_{0,rad}(\mathbb{R}^N)$ under the norm $\|\cdot\|_0$ and define

$$E_{\mathrm{rad}} = \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N) \cap L^r_{b,rad}(\mathbb{R}^N)$$

under the norm $\|\cdot\|$.

Lemma 3.2 (Radial Lemma in E_{rad})**.** *Let $u \in E_{\mathrm{rad}}$, then for almost every $x \in \mathbb{R}^N\backslash\{0\}$, then there exists $\overline{C} = \overline{C}(a, b, p) > 0$ such that*

$$|u(x)| \le \overline{C}\frac{1}{|x|^{\frac{(N-p)-ap^*}{p}}}\|u\|.$$

Proof. Up to a standard density argument, we only consider $u \in \mathcal{C}^{\infty}_{0,rad}(\mathbb{R}^N)$. Denote by ω_N the volume of the unit sphere in $\mathbb{R}^N$. We have

$$-u(\Upsilon) = u(\infty) - u(\Upsilon) = \int_{\Upsilon}^{\infty} u'(s)ds.$$

Thus,

$$|u(\Upsilon)| \le \int_{\Upsilon}^{\infty} |u'(s)|ds = \int_{\Upsilon}^{\infty} s^{\frac{-ap^*}{p}}|u'(s)|s^{\frac{N-1}{p}}s^{\frac{ap^*}{p}}s^{\frac{1-N}{p}}ds.$$

From Holder's inequality, we obtain

$$|u(\Upsilon)| \le \Big(\int_{\Upsilon}^{\infty} s^{-ap^*}|u'(s)|s^{N-1}ds\Big)^{1/p}\Big(\int_{\Upsilon}^{\infty} s^{\frac{ap^*}{p-1}}s^{\frac{1-N}{p-1}}ds\Big)^{(p-1)/p}.$$

Hence

$$|u(\Upsilon)| \le \omega_N^{\frac{-1}{p}}\Big(\frac{p-1}{N-p-ap^*}\Big)^{\frac{p-1}{p}}\frac{1}{|x|^{\frac{(N-p)-ap^*}{p}}}\Big(\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p\,dx\Big)^{1/p}. \qquad \square$$

Now we present a compactness result.

Lemma 3.3. *If $a < \frac{N-p}{p^*}$, then the embedding $E_{\mathrm{rad}} \hookrightarrow L^s_b(\mathbb{R}^N)$ is compact for all $s \in (p, p^*)$.*

Proof. Let $(u_n) \subset E_{\mathrm{rad}}(\mathbb{R}^N)$ be a bounded sequence and let $C > 0$ be such that

$$\|u_n\| \le C, \quad \forall n \in \mathbb{N}.$$

By Lemma 3.2 it follows that, for all $n \in \mathbb{N}$,

$$|u_n(x)| \le C\overline{C}\frac{1}{|x|^{\frac{(N-p)-ap^*}{p}}}, \quad \text{a.e. in } \mathbb{R}^N\backslash\{0\}.$$

Since $s > 1$, given $\epsilon > 0$, there exists $R > 0$ such that, for all $n \in \mathbb{N}$,

$$|u_n(x)|^s \le \frac{\epsilon}{2C\overline{C}}|u_n(x)| \quad \forall x \in B_R(0)^c.$$

This implies that

$$\int_{B_R(0)^c}|x|^{-bp^*}|u_n|^s\,dx \le \frac{\epsilon}{2C\overline{C}R^{bp^*}}\int_{B_R(0)^c}|u_n|dx \le \frac{\epsilon}{2R^{\frac{(N-p)-ap^*+bpp^*}{p}}} \le \frac{\epsilon}{2}, \tag{3.2}$$

for all $n \in \mathbb{N}$. Moreover, since $E(B_R(0))$ is compactly embedded into $L_b^s(B_R(0))$, there exists $u \in L_b^s(B_R(0))$ such that, up to a subsequence $u_n \to u$ in $L_b^s(B_R(0))$, as $n \to \infty$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\int_{B_R(0)} |x|^{-bp^*}|u_n - u|^s dx < \frac{\epsilon}{2}, \quad \forall n \geq n_0. \tag{3.3}$$

Let us define $\overline{u} : \mathbb{R}^N \to \mathbb{R}$ as to be equal to u in $B_R(0)$ and equal to 0 in $B_R(0)^c$. Then, by (3.2) and (3.3), it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n - \overline{u}|^s \, dx &= \int_{B_R(0)} |x|^{-bp^*}|u_n - \overline{u}|^s dx + \int_{B_R(0)^c} |x|^{-bp^*}|u_n|^s dx \\ &< \epsilon.\end{aligned}$$

Then it is clear that $u_n \to \overline{u}$ in $L_b^s(\mathbb{R}^N)$, as $n \to \infty$. □

Following [18] and [19], we consider an auxiliary functional $\widetilde{I} \in C^1(\mathbb{R} \times E_{\rm rad}), \mathbb{R})$ given by

$$\begin{aligned}\widetilde{I}(\theta, u) = \frac{\exp(N-p)\theta}{p} \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p \, dx + \frac{\exp{(N\theta)}}{p} \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^p \, dx \\ - \exp{(N\theta)} \int_{\mathbb{R}^N} |x|^{-bp^*} H(u) \, dx.\end{aligned}$$

The following properties hold, for all $(\theta, u) \in \mathbb{R} \times E_{\rm rad}$,

$$\begin{gathered}\widetilde{I}(0, u) = I(u), \\ \widetilde{I}(\theta, u) = I(u(x/\exp(\theta))).\end{gathered}$$

We equip the standard product norm

$$\|(\theta, u)\|^p_{\mathbb{R}\times E_{\rm rad}} = |\theta|^p + \|u\|^p$$

to $\mathbb{R} \times E$. Now we prove that $\widetilde{I}$ satisfies the Mountain Pass geometry.

Lemma 3.4. *The functional $\widetilde{I}$ satisfies the following conditions:*

(i) *There exist ρ_1, $\rho_2 > 0$ such that $\widetilde{I}(\theta, u) \geq \rho_2$ with $\|(\theta, u)\| = \rho_1$.*
(ii) *There exists $\widetilde{e} \in B^c_{\rho_2}(0)$ with $\widetilde{I}(\widetilde{e}) < 0$ and $\|\widetilde{e}\| > \rho_2$.*

Proof. Item (i) follows by using the same argument as in Lemma 3.1. For item (ii) it is sufficient to take $\widetilde{e} = (0, e)$. □

In what follows, we define the Mountain Pass level $\widetilde{c}_*$ for $\widetilde{I}$ by

$$\widetilde{c}_* = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} \widetilde{I}(\eta(t)) > 0$$

and

$$\widetilde{\Gamma} := \{\eta \in C([0, 1], \mathbb{R} \times E_{\rm rad}) : \eta(0) = 0,\ \widetilde{I}(\eta(1)) < 0\}.$$

Note that $\widetilde{c}_* \geq \rho_2$.

Lemma 3.5. *The Mountain Pass levels of I and $\widetilde{I}$ coincide, namely $c_* = \widetilde{c}_* > 0$.*

Proof. For our problem we adapt the approach explored in [18, Lemma 4.1]. Note that $\Gamma \cong \{0\} \times \Gamma \subset \widetilde{\Gamma}$, which implies $\widetilde{c}_* \leq c_*$. On the other hand, consider $\widetilde{\gamma} \in \widetilde{\Gamma}$ arbitrary. Then, for each $t \in [0, 1]$, we have $\widetilde{\gamma}(t) = (\theta_t, u_t)$. Define $\gamma(t) := u_t(\frac{x}{\exp(\theta_t)})$. From the definition of $\widetilde{I}$, we conclude that $\widetilde{I}(\widetilde{\gamma}_t) = \widetilde{I}(\theta_t, u_t) = I(u_t(x/\exp(\theta))) = I(\gamma(t))$ for each $t \in [0, 1]$. Hence $\gamma \in \Gamma$, where we derive $\widetilde{c}_* \geq c_*$. □

The proof of next lemma is the same as the proof of [18, Lemma 4.3].

Lemma 3.6. *Let $\epsilon > 0$. Suppose that $\widetilde{\eta} \in \widetilde{\Gamma}$ satisfies*
$$\max_{t\in[0,1]} \widetilde{I}(\widetilde{\eta}) \le c_* + \epsilon,$$
then, there exists $(\theta, u) \in \mathbb{R} \times E_{\rm rad}$ such that

- $\operatorname{dist}_{\mathbb{R}\times E_{\rm rad}}((\theta, u), \widetilde{\eta}([0,1])) \le 2\sqrt{\epsilon}$;
- $\widetilde{I}(\theta, u) \in [c_* - \epsilon, c_* + \epsilon]$;
- $\|D\widetilde{I}(\theta, u)\|_{\mathbb{R}\times E^*_{\rm rad}} \le 2\sqrt{\epsilon}$.

As in [18, Proposition 4.2], the proof of the next lemma is a consequence of Lemma 3.6.

Lemma 3.7. *There exists a sequence $((\theta_n, u_n)) \subset \mathbb{R} \times E_{\rm rad}$ such that, as $n \to \infty$, we obtain*

- $\theta_n \to 0$;
- $\widetilde{I}(\theta_n, u_n) \to c_*$;
- $\partial_\theta \widetilde{I}(\theta_n, u_n) \to 0$;
- $\partial_u \widetilde{I}(\theta_n, u_n) \to 0$ *strongly in* $E^*_{\rm rad}$.

Proof. For each $j \in \mathbb{N}$, we can find a $\gamma_j \in \Gamma$ such that
$$\max_{t\in[0,1]} I(\gamma_j(t)) \le c_* + \frac{1}{j}.$$
Since $\widetilde{c}_* = c_*$ and $\widetilde{\gamma}_j(t) = (0, \gamma_j(t)) \in \widetilde{\Gamma}$ satisfies $\max_{t\in[0,1]} \widetilde{I}(\widetilde{\gamma}_j)(t) \le \widetilde{c}_* + \frac{1}{j}$, we can find a (θ_j, u_j) such that

- $\operatorname{dist}_{\mathbb{R}\times E_{\rm rad}}((\theta, u), \widetilde{\gamma}_j([0,1])) \le 2/\sqrt{j}$;
- $\widetilde{I}(\theta, u) \in [c_* - 1/j, c_* + 1/j]$;
- $\|D\widetilde{I}(\theta, u)\|_{\mathbb{R}\times E^*_{\rm rad}} \le 2/\sqrt{j}$.

Since $\widetilde{\gamma}([0,1]) \subset \{0\} \times E_{\rm rad}$, the first inequality implies $|\theta_j| \le 2/\sqrt{j}$ and, consequently, $\theta_j \to 0$. The second item implies $\widetilde{I}(\theta_j, u_j) \to c_*$ and the last item implies the last two items of these lemma. □

3.1. **Proof of Theorem 1.1.** By Lemma 3.7, there exists a sequence $((\theta_n, u_n)) \subset \mathbb{R} \times E_{\rm rad}$ such that
$$\begin{aligned} &\frac{\exp((N-p)\theta_n)}{p}\|u_n\|_0^p + \frac{\exp(N\theta_n)}{p}\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^p\,dx \\ &- \exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*} H(u_n)\,dx = c_* + o_n(1); \end{aligned} \tag{3.4}$$
$$\begin{aligned} &(N-p)\frac{\exp((N-p)\theta_n)}{p}\|u_n\|_0^p + N\frac{\exp(N\theta_n)}{p}\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^p\,dx \\ &- N\exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*} H(u_n)\,dx = o_n(1); \end{aligned} \tag{3.5}$$
$$\begin{aligned} &\exp((N-p)\theta_n)\|u_n\|_0^p + \exp((N-p)\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^p\,dx \\ &- \exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*} h(u_n)u_n\,dx = o_n(1)\|u_n\|. \end{aligned} \tag{3.6}$$

From (3.4) and (3.5) and since $p < N$, we have

$$\exp((N-p)\theta_n)\|u_n\|_0^p = Nc_* + o_n(1). \tag{3.7}$$

Since $\theta_n \to 0$, we have that (u_n) is bounded in $\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$ and $L_b^{p^*}(\mathbb{R}^N)$.

From (A4), there exists $C > 0$ such that

$$h(t)t \le \frac{1}{2}|t|^p + C|t|^{p^*}, \quad \text{for all } t \in \mathbb{R}.$$

Using the last inequality in (3.6), we obtain

$$\frac{1}{2}\exp((N-p)\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p\,dx \le C\exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p^*}\,dx,$$

which implies that (u_n) is bounded in E_{rad}. Hence, there exists $u \in E_{\mathrm{rad}}$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E_{rad}. From Lemma 3.7, for all $v \in E_{\mathrm{rad}}$, we have $\partial_u\widetilde{I}(\theta_n,u_n)v = o_n(1)$; that is,

$$\begin{aligned}
&\exp((N-p)\theta_n)\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^{p-2}u_n v\,dx \\
&+\exp((N-p)\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p-2}u_n v\,dx \\
&-\exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}h(u_n)v\,dx = o_n(1).
\end{aligned}$$

Since $\theta_n \to 0$ in $\mathbb{R}$ and from weak convergence, for all $v \in E_{\mathrm{rad}}$, we obtain

$$\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^{p-2}uv\,dx + \int_{\mathbb{R}^N}|x|^{-bp^*}|u|^{p-2}uv\,dx - \int_{\mathbb{R}^N}|x|^{-bp^*}h(u)v\,dx = 0,$$

showing that $I'(u)v = 0$, for all $v \in E_{\mathrm{rad}}$, that is u is a critical point of I. We are going to show that u is not trivial. Suppose that $u = 0$. From (A4) there exist $\epsilon > 0$ and $C_\epsilon > 0$ such that

$$\int_{\mathbb{R}^N}|x|^{-bp^*}h(u_n)u_n\,dx \le \epsilon\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p\,dx + C_\epsilon\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^q\,dx.$$

Since (u_n) is bounded in E_{rad} and since $E_{\mathrm{rad}} \hookrightarrow L_b^q(\mathbb{R}^N)$ is compact, we conclude that

$$\int_{\mathbb{R}^N}|x|^{-bp^*}h(u_n)u_n\,dx = o_n(1).$$

This limit combined with the limit $\partial_u\widetilde{I}(\theta_n,u_n)u_n = o_n(1)$ allows to deduce that $u_n \to 0$ in E_{rad}. Hence, $\widetilde{I}(\theta_n,u_n) \to 0 = c_*$, which is absurd. Thus, u is a nontrivial critical point of I in E_{rad}. Finally, u is a nontrivial critical point of I in E using the Principle of Symmetric Criticality [21] or [25, Theorem 1.28].

4. Existence of solution for zero-mass case

Consider the functional $I_0 : \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I_0(u) = \frac{1}{p}\|u\|_0^p - \int_{\mathbb{R}^N}|x|^{-bp^*}F(u)\,dx.$$

Note that I_0 is well-defined and of C^1 class. Moreover, note that

$$I_0'(u)\phi = \int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla\phi\,dx - \int_{\mathbb{R}^N}|x|^{-bp^*}f(u)\phi\,dx,$$

for all $\phi \in \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$. Then, the critical points of I_0 are weak solutions of (1.3) in $\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$.

We say that a sequence (u_n) is a Palais-Smale sequence for the functional I_0 if

$$I_0(u_n) \to c_0, \quad \text{and} \quad \|I_0'(u_n)\| \to 0 \quad \text{in } (\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N))',$$

where

$$c_0 = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} I_0(\eta(t)) > 0,$$

$$\Gamma_0 := \{\eta \in C([0,1], \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)) : \eta(0) = 0,\ I_0(\eta(1)) < 0\}.$$

If every Palais-Smale sequence of I_0 has a strong convergent subsequence, then one says that I_0 satisfies the Palais-Smale condition ((PS) for short).

Lemma 4.1. *The functional I_0 satisfies the following conditions:*

(i) *There exist ρ_1, $\rho_2 > 0$ such that*

$$I_0(u) \ge \rho_2 \quad \text{with } \|u\|_0 = \rho_1.$$

(ii) *There exists $e \in B^c_{\rho_2}(0)$ with $I_0(e) < 0$ and $\|e\|_0 > \rho_2$.*

The proof of the above lemma is similar to the one in Lemma 3.1. As in the previous section, we consider an auxiliary functional $\widetilde{I_0} \in C^1(\mathbb{R} \times \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N), \mathbb{R})$ given by

$$\widetilde{I_0}(\theta, u) = \frac{\exp(N-p)\theta}{p} \|u\|_0^p - \exp(N\theta) \int_{\mathbb{R}^N} |x|^{-bp^*} F(u)\, dx.$$

The following properties hold, for all $(\theta, u) \in \mathbb{R} \times \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$:

$$\widetilde{I_0}(0, u) = I_0(u),$$

$$\widetilde{I_0}(\theta, u) = I_0(u(x/\exp(\theta)).$$

We equip the standard product norm

$$\|(\theta, u)\|^2_{\mathbb{R}\times\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)} = |\theta|^p + \|u\|_0^p$$

to $\mathbb{R} \times \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$. Now we prove that $\widetilde{I_0}$ satisfies the Mountain Pass geometry.

Lemma 4.2. *The functional $\widetilde{I_0}$ satisfies the following conditions:*

(i) *There exist ρ_1, $\rho_2 > 0$ such that*

$$\widetilde{I_0}(\theta, u) \ge \rho_2 \quad \text{with } \|(\theta, u)\|_{\mathbb{R}\times\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)} = \rho_1.$$

(ii) *There exists $\widetilde{e} \in B^c_{\rho_2}(0)$ with $\widetilde{I}(\widetilde{e}) < 0$ and $\|\widetilde{e}\|_{\mathbb{R}\times\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)} > \rho_2$.*

Proof. Item (i) follows by using the same argument as in Lemma 3.1. For item (ii) it is sufficient to take $\widetilde{e} = (0, e)$. □

In what follows, we define the Mountain Pass level $\widetilde{c}_0$ for $\widetilde{I_0}$ by

$$\widetilde{c}_0 = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} \widetilde{I_0}(\eta(t)) > 0,$$

$$\widetilde{\Gamma} := \{\eta \in C([0,1], \mathbb{R} \times \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)) : \eta(0) = 0,\ \widetilde{I_0}(\eta(1)) < 0\}.$$

Note that $\widetilde{c}_* \ge \rho_2$.

Lemma 4.3. *The Mountain Pass levels of I_0 and $\widetilde{I_0}$ coincide, namely $c_0 = \widetilde{c}_0$.*

The proof of the above lemma is the same as that of Lemma 3.5. The proof ofthe next lemma is the same proof of [18, Lemma 4.3].

Lemma 4.4. *Let $\epsilon > 0$. Suppose that $\widetilde{\eta} \in \widetilde{\Gamma_0}$ satisfies*

$$\max_{t\in[0,1]} \widetilde{I}_0(\widetilde{\eta}) \le c_0 + \epsilon,$$

then, there exists $(\theta, u) \in \mathbb{R} \times \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$ such that

- $\operatorname{dist}_{\mathbb{R}\times\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)}, \widetilde{\eta}([0,1])) \le 2\sqrt{\epsilon}$;
- $\widetilde{I}_0(\theta, u) \in [c_0 - \epsilon, c_0 + \epsilon]$;
- $\|D\widetilde{I}_0(\theta, u)\|_{\mathbb{R}\times(\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)^*)} \le 2\sqrt{\epsilon}$.

The proof of the next lemma is the same proof of Lemma 3.7.

Lemma 4.5. *There exists a sequence $((\theta_n, u_n)) \subset \mathbb{R} \times \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$ such that, as $n \to \infty$, we obtain*

- $\theta_n \to 0$;
- $\widetilde{I}_0(\theta_n, u_n) \to c_0$;
- $\partial_\theta \widetilde{I}_0(\theta_n, u_n) \to 0$;
- $\partial_u \widetilde{I}_0(\theta_n, u_n) \to 0$ *strongly in* $(\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N))^*$.

Proof of Theorem 1.2. By Lemma 4.5, there exists a sequence $((\theta_n, u_n)) \subset \mathbb{R} \times \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$ such that

$$\frac{\exp(N-p)\theta_n}{p}\|u_n\|_0^p - \exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*} F(u_n)\,dx = c_0 + o_n(1); \tag{4.1}$$

$$(N-p)\frac{\exp(N-p)\theta_n}{p}\|u_n\|_0^p - N\exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*} F(u_n)\,dx = o_n(1); \tag{4.2}$$

$$\exp((N-p)\theta_n)\|u_n\|_0^p - \exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*} f(u_n)u_n\,dx = o_n(1)\|u_n\|_0. \tag{4.3}$$

From (4.1) and (4.2) and since $N > p$, we have

$$\exp(N-p)\theta_n\|u_n\|_0^p = Nc_* + o_n(1). \tag{4.4}$$

Since $\theta_n \to 0$, we have that (u_n) is bounded in $\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$ and $L_b^{p^*}(\mathbb{R}^N)$. Hence, there exists $u \in \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \rightharpoonup u$ in $\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$. From Lemma 4.5, for all $v \in \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$, we have $\partial_u\widetilde{I}_0(\theta_n, u_n)v = o_n(1)$; that is,

$$\exp((N-p)\theta_n)\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^{p-2}u_n v\,dx - \exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*} f(u_n)v\,dx = o_n(1).$$

Since $\theta_n \to 0$ in $\mathbb{R}$ and from weak convergence, for all $v \in \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}uv\,dx - \int_{\mathbb{R}^N} |x|^{-bp^*} f(u)v\,dx = 0,$$

showing that $I_0'(u)v = 0$, for all $v \in \mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$, that is u is a critical point of I_0. We are going to show that u is not trivial. Suppose that $u = 0$. From f_1) there exist $\epsilon > 0$ and $C_\epsilon > 0$ such that

$$\int_{\mathbb{R}^N} |x|^{-bp^*} f(u_n)u_n\,dx \le \epsilon\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^{p^*}\,dx + C_\epsilon\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^q\,dx.$$

Since (u_n) is bounded in $\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$ and since $\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^q_b(\mathbb{R}^N)$ is compact, we conclude that

$$\int_{\mathbb{R}^N} |x|^{-bp^*} f(u_n)u_n \, dx = o_n(1).$$

This limit combined together with the limit $\partial_u \widetilde{I}_0(\theta_n, u_n)u_n = o_n(1)$ allows to deduce that $u_n \to 0$ in $\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$. Hence, $\widetilde{I}_0(\theta_n, u_n) \to 0 = c_0$, which is absurd. Thus, u is a nontrivial critical point of I_0 in $\mathcal{D}^{1,p}_{a,\mathrm{rad}}(\mathbb{R}^N)$. Finally, u is a nontrivial critical point of I_0 in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ using the Principle of Symmetric Criticality [21] or [25, Theorem 1.28]. $\square$

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