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SUFFICIENT CONDITIONS FOR THE EXISTENCE OF INTERIOR POINTS FOR POSITIVE CONES

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Abstract. Using partial ordering methods we give a sufficient condition for a positive cone to have nonempty interior.

1. Introduction

Let $(E, \|\cdot\|_E)$ be a real Banach space and P be a nonempty closed convex set in E. P is called a cone if it satisfies the following two conditions:

(i) $x \in P$ and $\lambda \geq 0$ imply $\lambda x \in P$,
(ii) $x \in P$ and $-x \in P$ implies $x = \theta$, where θ denotes the zero element in E.

A cone P is said to be generating (or reproducing) if $E = P - P$, i.e., every element $x \in E$ can be represented in the form $x = u - v$ where $u, v \in P$.

A cone P is called solid if there exists an element u_0 which belongs to the interior of the cone P, that is, there exists positive constant r such that

$$B(u_0, r) = \{x \in E : \|u_0 - x\| \leq r\} \subset P.$$

A cone P defines a linear ordering in E by

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

A cone P is said to be normal if there exists a constant $N > 0$ such that

$$\theta \leq x \leq y \implies \|x\| \leq N\|y\|, \quad x, y \in P.$$

We denote by u_0 some fixed non-zero element of P. Our main result reads as follows.

Theorem 1.1. *If u_0 be a non-zero element of P such that for any $x \in E$ there exists positive constant $\alpha_x > 0$ such that $x \leq \alpha_x u_0$, then u_0 belongs to the interior of the cone P. That is, there exists positive constant r such that*

$$B(u_0, r) = \{x \in E : \|u_0 - x\| \leq r\} \subset P.$$

In Section 3 we introduce the u_0-norm and the space E_{u_0}, where u_0 is a given nonzero element of P. It is well-known that if P is a solid cone and $u_0 \in \mathring{P}$, then $E = E_{u_0}$. In this paper we shall study the converse statement and give an improvement and generalization of [2, Theorem 1.5.1].

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2. Proof of Theorem 1.1

To prove Theorem 1.1 we establish the following two lemmas. The first one is based on [2, Lemma 1.4.2].

Lemma 2.1. *Let u_0 be a non-zero element of P such that for any $x \in E$ there exists positive constant $\alpha_x > 0$ satisfying $x \leq \alpha_x u_0$. Then a constant $\tau > 0$ can be found such that for any $x \in E$ there exists positive constant $\beta(x) > 0$ such that $x \leq \beta(x) u_0$ and $\|\beta(x)u_0\| \leq \tau \|x\|$.*

Proof. It is clear that $E = \cup_{n=1}^{\infty} E_n$, where

$$E_n = \{x \in E : \text{there is } \beta(x) > 0 \text{ such that } x \leq \beta(x)u_0 \text{ and } \|\beta(x)u_0\| \leq n\|x\|\},$$

for $n = 1, 2, 3, \dots$. By the Baire-Hausdorff's Theorem (that is, a nonempty complete metric space is a second Baire set), there exist positive integer n_1, $x_0 \in E$ and $R > r > 0$ satisfying

$$B_0 = \{x \in E : r < \|x - x_0\| < R\} \subset \overline{E_{n_1}}.$$

Let $\beta_0 > 0$ and n_2 be a positive integer such that $-x_0 \leq \beta_0 u_0$, and $\|\beta_0 u_0\| \leq n_2 \|x_0\|$. Let $B = \{x \in E : r < \|x\| < R\}$, and choose an integer n_3 satisfying

$$n_3 > n_1 + \frac{1}{r}(n_1 + n_2)\|x_0\|.$$

In what follows, we prove that $B \subset \overline{E_{n_3}}$. Indeed, for any $x \in B$, we have $y = x_0 + x \in B_0$, then there exists a sequence $\{x_i\} \subset E_{n_1}$ such that $x_i \to y$ as $i \to \infty$. Clearly, we can assume that $x_i \in B_0$ for $i = 1, 2, 3, \dots$. Take constants $\beta_i > 0$ such that $x_i \leq \beta_i u_0$ and $\|\beta_i u_0\| \leq n_1 \|x_i\|$. Then we obtain $x_i - x_0 \leq (\beta_i + \beta_0)u_0$ and

$$\begin{aligned}
\|(\beta_i + \beta_0)u_0\| &\leq n_1\|x_i\| + n_2\|x_0\| \\
&\leq (n_1 + n_2)\|x_0\| + n_1\|x_i - x_0\| \\
&\leq \Big[(n_1 + n_2)\frac{\|x_0\|}{r} + n_1\Big]\|x_i - x_0\| \\
&\leq n_3\|x_i - x_0\|.
\end{aligned}$$

from which it follows that $x_i - x_0 \in E_{n_3}$ for $n = 1, 2, 3, \dots$. From the fact that $x_i - x_0 \to y - x_0$ as $i \to \infty$ we obtain $x \in \overline{E_{n_3}}$. Therefore $B \subset \overline{E_{n_3}}$.

Clearly, from $x \in \overline{E_{n_3}}$, we can easily prove that $tx \in \overline{E_{n_3}}$, for all $t \geq 0$. Consequently, $E = \overline{E_{n_3}}$.

Finally, we show that $E = E_{3n_3}$. Taking $x \in E$ such that $x \neq \theta$, then there exists $x_1 \in E_{n_3}$ satisfying

$$\|x - x_1\| < \frac{1}{2}\|x\|.$$

Since $x_1 \in E_{n_3}$, there exists $\beta_1 > 0$ such that

$$x_1 \leq \beta_1 u_0, \quad \|\beta_1 u_0\| \leq n_3\|x_1\|.$$

Similarly, there exist $x_2 \in E_{n_3}$ and $\beta_2 > 0$ such that

$$\|x - x_1 - x_2\| < \frac{1}{2^2}\|x\|, \quad x_2 \leq \beta_2 u_0, \quad \|\beta_2 u_0\| \leq n_3\|x_2\|.$$

Inductively, we find sequences $\{x_k\} \subset E_{n_3}$ and $\{\beta_k\} > 0$, $k = 1, 2, \dots$, satisfying

$$\|x - x_1 - x_2 - \dots - x_k\| < \frac{1}{2^k}\|x\|, \quad x_k \leq \beta_k u_0, \quad \text{and} \quad \|\beta_k u_0\| \leq n_3\|x_k\|,$$

for $k = 1, 2, 3, \dots$.

Clearly, $x = \sum_{k=1}^{\infty} x_k$ and

$$\|x_k\| \le \|x - \sum_{i=1}^{k-1} x_i\| + \|x - \sum_{i=1}^{k} x_i\| < \frac{3\|x\|}{2^k} \quad k = 1, 2, \dots .$$

From which it follows that

$$\sum_{k=1}^{\infty} \|\beta_k u_0\| \le n_3 \sum_{k=1}^{\infty} \|x_k\| \le 3n_3\|x\| < \infty.$$

Consequently the series $\sum_{k=1}^{\infty} \beta_k$ converges to some constant $\beta > 0$. Clearly

$$x = \sum_{k=1}^{\infty} x_k \le \sum_{k=1}^{\infty} \beta_k u_0 = \beta u_0,$$

and

$$\|\beta u_0\| \le \sum_{k=1}^{\infty} \|\beta_k u_0\| \le 3n_3\|x\|.$$

Therefore, $x \in E_{3n_3}$, which implies that $E = E_{3n_3}$. □

As a consequence of the previous lemma we have.

Lemma 2.2. *Let u_0 be a non-zero element of P such that for each $x \in E$ there exists positive constant $\alpha_x > 0$ satisfying $x \le \alpha_x u_0$. Then there is a constant $\beta > 0$, not depending on x, such that for every $x \in E$ satisfying $\|x\| \le 1$ we have $x \le \beta u_0$.*

Proof. By using Lemma 2.1, for every $x \in E$ satisfying $\|x\| \le 1$ there exists positive constant $\beta(x) > 0$ such that $x \le \beta(x)u_0$ and $\|\beta(x)u_0\| \le \tau\|x\| \le \tau$. Then for all constant $\beta > \frac{\tau}{\|u_0\|}$ we have $x \le \beta u_0$. □

Proof of Theorem 1.1. By Lemma 2.2 there is a constant $\beta > 0$ such that for every $x \in E$ satisfying $\|x\| \le 1$ we have $x \le \beta u_0$. By taking $r = \frac{1}{\beta}$ we have for every $x \in E$ satisfying $\|x\| \le 1$, $u_0 - rx \ge 0$. Taking an element $x \in E$ ($x \ne u_0$) such that $\|u_0 - x\| \le r$, we obtain

$$\begin{aligned} x &= u_0 - (u_0 - x) \\ &= u_0 - r\frac{\|u_0 - x\|}{r}\frac{u_0 - x}{\|u_0 - x\|} \ge 0. \end{aligned}$$

Consequently $x \in P$, which completes the proof. □

3. Space E_{u_0}

In what follows, we suppose that P is a cone in E and let u_0 be a non-zero element of P. We define the space E_{u_0} and u_0-norm as follows (see [7]),

$$\begin{gathered} E_{u_0} = \{x \in E : \text{there exists } \lambda > 0 \text{ such that } -\lambda u_0 \le x \le \lambda u_0\}, \\ \|x\|_{u_0} = \inf\{\lambda > 0 : -\lambda u_0 \le x \le \lambda u_0\}, \quad x \in E_{u_0}. \end{gathered}$$

It is easy to see that E_{u_0} is a normed linear space with the norm $\|\cdot\|_{u_0}$. Then $\|x\|_{u_0}$ is called a u_0-norm of $x \in E_{u_0}$ (see [7] for more details). The following theorem can be found in [2, Theorem 1.5.1]

Theorem 3.1. *If P is a normal cone, then:*

(i) *The space E_{u_0} is a Banach space.*
(ii) *$P_{u_0} = P \cap E_{u_0}$ is a normal solid cone in space E_{u_0} and*

$$\begin{aligned}\mathring{P}_{u_0} &= \{x \in E_{u_0} : \textit{there exists } \tau > 0 \textit{ such that } x \geq \tau u_0\}\\ &= \{x \in E : \textit{there exists } \lambda > \tau > 0 \textit{ such that} \tau u_0 \leq x \leq \lambda u_0\}.\end{aligned}$$

Remark 3.2. If $x \in E$ and there exists positive constant $\alpha_x > 0$ such that $x \leq \alpha_x u_0$, then from the inequality $-x \leq \alpha_{-x} u_0$, for some $\alpha_{-x} > 0$ one has $-\alpha_{-x} u_0 \leq x \leq \alpha_x u_0$. Then $x \in E_{u_0}$ and thus $E = E_{u_0}$.

Theorem 3.3. *A necessary and sufficient condition for a cone P to be solid is that $E = E_{u_0}$.*

Proof. Suppose that $E = E_{u_0}$ then for any $x \in E$ there exists $\lambda > 0$ such that $x \leq \lambda u_0$ hence by Theorem 3.1, $u_0 \in \mathring{P}$ and thus P is a solid cone.

Conversely, suppose that $u_0 \in \mathring{P}$, then there exists positive constant $r > 0$ such that $B(u_0, r) = \{x \in E : \|u_0 - x\| \leq r\} \subset P$. For each $x \in E$, $(x \neq 0)$, we have $u_0 \pm \frac{r}{\|x\|} x \in P$ and then $-\frac{\|x\|}{r} u_0 \leq x \leq \frac{\|x\|}{r} u_0$. Therefore, $x \in E_{u_0}$ and $E = E_{u_0}$. □

In what follows, we assume that P is a normal cone.

Theorem 3.4. *If P is a solid cone, then $u_0 \in \mathring{P}$ if and only if the u_0-norm $\|\cdot\|_{u_0}$ is equivalent to the original norm $\|\cdot\|$.*

Proof. Suppose that $u_0 \in \mathring{P}$, then there exists positive constant $r > 0$ such that $B(u_0, r) = \{x \in E : \|u_0 - x\| \leq r\} \subset P$. For each $x \in E$, $(x \neq 0)$, we have $-\frac{\|x\|}{r} u_0 \leq x \leq \frac{\|x\|}{r} u_0$. Then

$$\|x\|_{u_0} \leq \frac{1}{r}\|x\|, \quad x \in E.$$

On the other hand, for each $x \in E_{u_0}$, we have $-\alpha u_0 \leq x \leq \alpha u_0$, where $\alpha = \|x\|_{u_0}$, and then $0 \leq x + \alpha u_0 \leq 2\alpha u_0$. Thus, by the normality of P, we obtain

$$\|x + \alpha u_0\| \leq 2\alpha N \|u_0\|,$$

where N is the normal constant of P, which implies that

$$\|x\| \leq \|x + \alpha u_0\| + \| - \alpha u_0\| \leq M\|x\|_{u_0},$$

where $M = (2N + 1)\|u_0\|$. Consequently, the u_0-norm $\|\cdot\|_{u_0}$ is equivalent to the original norm $\|\cdot\|$.

Conversely, suppose that for any $x \in E$ there exist two positive constants c and C satisfying

$$c\|x\|_{u_0} \leq \|x\| \leq C\|x\|_{u_0}$$

then it is easy to show that $E = E_{u_0}$, and then by Theorem 3.3, $u_0 \in \mathring{P}$. □

Remark 3.5. Theorem 3.3 does not assume P to be normal.

Remark 3.6. It is well-known that if P is a solid cone and $u_0 \in \mathring{P}$, then $E = E_{u_0}$ and the u_0-norm $\|\cdot\|_{u_0}$ is equivalent to the original norm $\|\cdot\|$. But here we have studied the converse statement and then our work improves and generalizes [2, Theorem 1.5.1].

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