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GLOBAL GRADIENT ESTIMATES FOR SHEAR THINNING-TYPE STOKES SYSTEM ON NON-SMOOTH DOMAINS

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Abstract. This article presents global L^q estimates for the weak solution of the steady p-Stokes equations, which describe the motion of shear-thinning flow under the nonslip boundary condition. We focus on non-smooth domains whose boundaries extend beyond the Lipschitz category, with coefficients belonging to the BMO (Bounded Mean Oscillation) space having a sufficiently small BMO semi-norm.

1. Introduction

The Navier-Stokes equations with shear-dependent viscosity are extensively studied in the literature. The stationary generalized Navier-Stokes equations with shear-dependent viscosity are expressed as

$$\begin{aligned} \operatorname{div} \mathcal{T}(u,\pi) - (u\cdot\nabla)u &= f, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1.1}$$

where

$$\mathcal{T}(u,\pi) = -\pi I + \nu_T(u)Du \quad \text{and} \quad Du := \frac{1}{2}(\nabla u + \nabla u^T). \tag{1.2}$$

Here, $\nabla u \in \mathbb{R}^{n\times n}$ denotes a matrix-valued function where $(\nabla u)_{ij} = \partial_i u^j$, and $M^T \in \mathbb{R}^{m\times n}$ represents the transpose of the matrix $M \in \mathbb{R}^{n\times m}$. If $\nu_T(u) = (|Du| + \mu)^{p-2}$, where $\mu \ge 0$ and $p \ne 2$, then equations (1.1) and (1.2) are referred to as the p-Navier-Stokes equations. Ladyzhenskaya's fundamental works initiate the systematic study of the p-Navier-Stokes equations [25, 26]. For further details on the p-Navier-Stokes equations, readers are referred to [27, 17, 31] and the references therein.

To simplify the problem, we shall consider the p-Stokes equations, which are the equations without the convection term in (1.1). A systematic study of the p-Stokes equations is meaningful because, in many cases, the regularity of a solution to the p-Stokes equations is closely related to the regularity of a solution to the p-Navier-Stokes equations.

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The interior regularity results for p-Stokes equations are well-established in many cases. However, when we want to extend the regularity results up to the boundary, the problem becomes more difficult. The difficulty arises from the existence of a pressure term and the fact that the equation depends only on the symmetric part of the gradient Du and not on the full gradient, ∇u. This may lead to the loss of regularity in reconstructing the normal derivatives.

To simplify the boundary value problem, a cubical domain is considered for shear thickening and shear thinning cases in [2, 3], respectively. These papers study the reconstruction of the normal direction derivative. These results are extended to $C^{2,1}$ domains in [4] for the shear thickening fluid and in [6] for the shear thinning fluid. In [5, 3], the shear thinning case $p < 2$ is considered, but the range of p is restricted to be greater than $3/2$ because of the technical issue. Later, this restriction is removed, and the full range $p \in (1, 2)$ is considered in [6]

This article studies gradient L^q estimates of the shear-thinning fluid with the nonslip boundary condition.

In addition, we assume that the domain Ω belongs to the δ-Reifenberg flat domain and that the BMO semi-norm of the coefficients is sufficiently small. The global gradient L^q estimates on non-smooth domains are initiated in [9] for linear elliptic equations. Other types of equations on non-smooth domains have been studied subsequently. For instance, the linear Stokes equations are explored in [8, 18], p-Laplacian type equations are studied in [7], and shear-thickening p-Stokes equations are studied in [12].

The problem under consideration is the stationary Stokes equations

$$\begin{aligned} \operatorname{div} \mathcal{A}(x, Du) - \nabla \pi &= \operatorname{div} (\varphi''(|F|)F) \quad \text{i n}\Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where $\varphi \in C^2(\mathbb{R}^+)$ is a function defined as

$$\varphi(t) := \int_0^t (\mu + s)^{p-2} s \, ds + \frac{\kappa_2}{2} t^2. \tag{1.4}$$

for some constant $\kappa_2 > 0$, $p \in (1, 2)$ and $\mu \in (0, 1)$. In particular, if $n = 2$, we restrict the range of p to $p \in (4/3, 2)$. It is readily checked that the following properties hold:

$$\begin{gathered} p - 1 \le \frac{t\varphi''(t)}{\varphi'(t)} \le 1, \\ \varphi''(0) = \mu^{p-2} + \kappa_2 > 0, \\ \varphi''(t) \text{ is a decreasing function.} \end{gathered} \tag{1.5}$$

We shall denote by $\mathbb{R}^{n\times n}_{\rm sym}$ the class of symmetric matrices and $\mathbf{0}$ as the zero matrix. For $A, B \in \mathbb{R}^{n\times n}$, we denote $A : B = \sum_{i,j=1}^n A_{ij} B_{ij}$. The nonlinearity $\mathcal{A}(x, P) : \Omega \times \mathbb{R}^{n\times n}_{\rm sym} \to \mathbb{R}^{n\times n}_{\rm sym}$ is a given matrix-valued function such that $\mathcal{A}(x, \cdot) \in C^0\big(\mathbb{R}^{n\times n}_{\rm sym}\big) \cap C^1\big(\mathbb{R}^{n\times n}_{\rm sym} \setminus \{0\}\big)$ for each $x \in \Omega$ and satisfies the following basic structural conditions

$$\begin{gathered}\sum_{i,j,k,l=1}^{n} \frac{\partial \mathcal{A}(x,P)}{\partial P_{kl}} Q_{ij} Q_{kl} \geq \nu \varphi''(|P|)|Q|^2, \\ |\partial_{kl} \mathcal{A}_{ij}(x,P)| \leq L\varphi''(|P|), \\ \mathcal{A}(x,\mathbf{0}) = \mathbf{0},\end{gathered} \tag{1.6}$$

for all $P, Q \in \mathbb{R}^{n\times n}_{\rm sym}$ and for some $0 < \nu \leq L$.

Definition 1.1. For $F \in L^{\varphi}(\Omega)$, we say that $(u,\pi) \in W^{1,\varphi}_0(\Omega) \times L^{\varphi^*}(\Omega)$ is the weak solution pair to (1.3) if

$$\int_\Omega \mathcal{A}(x,Du) : D\xi \, dx - \int_\Omega \pi \operatorname{div} \xi \, dx = \int_\Omega \varphi''(|F|) F : D\xi \, dx,$$

for all $\xi \in W^{1,\varphi}_0(\Omega)$. Here, φ^* is the conjugate function of φ, and the details of the Orlicz functions and related function spaces shall be specified in Section 2.1.

For each open set $U \subset \Omega$, let us denote the integral average by

$$(f)_U = \fint_U f(x)\,dx = \frac{1}{|U|}\int_U f(x)\,dx.$$

For notational convenience, we write

$$\beta(\mathcal{A}, B_r(y)) := \sup_{P \in \mathbb{R}^{n\times n}_{\rm sym}\setminus\{0\}} \frac{|\mathcal{A}(x,P) - \big(\mathcal{A}(\cdot,P)\big)_{B_r(y)}|}{\varphi'(|P|)},$$

where $B_r(y)$ is the ball of radius r centered at y. By (1.6), we obtain

$$\big|\beta(\mathcal{A}, B_r(y))\big| \leq c := c(L,p). \tag{1.7}$$

We assume the domain Ω, is a bounded δ-Reifenberg flat domain for a small $\delta \in (0, 1/16)$. We further assume that $\mathcal{A}$ has a small BMO (Bounded Mean Oscillation) semi-norm. The precise assumptions are stated below.

Assumption 1.2. (i) There exist $R_1 > 0$ and $\delta \in (0, 1/16)$ such that for all $x \in \partial\Omega$ and for all $r \in (0, R_1]$, there exists a coordinate system $(z_1, z_2, \dots, z_n)$ which depends on r and x so that in this coordinate system, x is the origin and

$$B_r(0) \cap \{z_n > 0\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{z_n > -\delta r\}.$$

(ii) There exists $R_2 \leq R_1$ such that for all $x \in \partial\Omega$ and for all $r \in (0, R_2]$, there exists a coordinate system $(z_1, z_2, \dots, z_n)$ which depends on r and x so that in this coordinate system, x is the origin and

$$B_r(0) \cap \{z_n > 0\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{z_n > -\delta r\},$$

and

$$\sup_{0<r\leq R_2} \sup_{y \in \mathbb{R}^n} \fint_{B_r(y)} \beta\big(\mathcal{A}, B_r(y)\big)\,dx \leq \delta, \tag{1.8}$$

where δ is the same as the one chosen in (i).

In this work, $\delta > 0$ only shows the flatness of Ω and the smallness of the BMO semi-norm of $\mathcal{A}$ in this paper. The detailed reason for imposing two separate assumptions on the domain and coefficients shall be specified later in Remark 4.9.

For simplifying the notation, let us introduce the maximal exponent $\bar{q}$, depending on n, κ_2 and p, as

$$\bar{q} = \begin{cases} \text{any number in } (1,\infty) & \text{if } n = 2 \text{ and } 3/4 \le p < 2, \\ \frac{n}{n-2} & \text{if } n \ge 3. \end{cases} \tag{1.9}$$

We are ready to state our main results.

Theorem 1.3. *We assume that $p \in (1,2)$ if $n \ge 3$ and $p \in [\frac{4}{3}, 2)$. Let $\mathcal{A}$ satisfy (1.6), Ω be a bounded open set, $\kappa_2 > 0$ and $q \in [1, \bar{q})$ with $\bar{q}$ as in (1.9). Suppose that $\varphi(|F|) \in L^q(\Omega)$. There exists a $\delta > 0$ depending only on $|\Omega|, R_1, q, n, \nu, L, p$ and κ_2 such that if $\mathcal{A}$ and Ω satisfy Assumption 1.2, then the weak solution pair to (1.3), $(u,\pi) \in W_0^{1,\varphi}(\Omega) \times L^{\varphi^*}(\Omega)$, satisfy*

$$\int_\Omega \varphi(|Du|)^q \, dx + \int_\Omega \varphi^*(|\pi|)^q \, dx \le c \int_\Omega \varphi(|F|)^q + 1 \, dx$$

where the constant c only depends on $\mu, \nu, L, p, \kappa_2, |\Omega|, R_1$ and q.

This article is organized as follows. Section 2 provides the required preliminary materials. Section 3 presents the regularity results of the p-Stokes equations on the local flat boundary. Section 4 presents comparison estimates. Finally, Section 5 has the proof of our main theorem.

2. Preliminaries

This section consists of preliminary materials and notation. Throughout this paper, the universal constant c may vary from line to line, but it only depends on the data,

$$\text{data} := \{n, \nu, L, p, R_1/\operatorname{diam}(\Omega)\}.$$

If there exists a universal constant $c := c(\text{data}) > 1$ such that $\frac{1}{c} f \le g \le cf$, then we shall denote $f \sim g$. When the constant depends on other quantities, we shall specify them. In this paper, $\varepsilon > 0$ is a small number, and instead of writing $c\varepsilon$, we shall denote it as ε. When it is clear from the context, we shall not distinguish between scalar, vector-valued, or tensor-valued functions and their function spaces. If $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, then $u \otimes v \in \mathbb{R}^{m\times n}$ is defined as $(u \otimes v)_{ij} := u_i v_j$.

2.1. **Function spaces.** Let us introduce the definition of the Orlicz spaces and Sobolev-Orlicz spaces. A convex function $G : \mathbb{R}^+ \to \mathbb{R}^+$ is an N-function if $G'(t)$ is a non-decreasing function, $\lim_{t\to\infty} G(t)/t = \infty$ and $\lim_{t\to 0} G(t)/t = 0$. For a given N-we function, we define a conjugate of G by

$$G^*(t) := \sup_{s>0}\{st - G(s)\}.$$

For all $t > 0$, we uniformly

$$G^*(G'(t)) \sim G(t). \tag{2.1}$$

Let us define $\Delta_2(G)$ by the smallest constant $M > 0$ satisfying

$$G(2t) \le MG(t) \quad \text{for all } t \ge 0. \tag{2.2}$$

If such a $M > 0$ does not exist, we denote $\Delta_2(G) = \infty$. If $\Delta_2(G), \Delta_2(G^*) < \infty$, we define the Orlicz space by

$$L^G(\Omega) := \{f \in L^1(\Omega) : \int_\Omega G(|f|)\,dx < \infty\},$$

equipped with the Luxemburg norm

$$\|f\|_{L^G(\Omega)} := \inf_{\lambda>0}\{\lambda > 0 : \int_\Omega G\big(\frac{|f|}{\lambda}\big)\,dx \le 1\}.$$

If $\Delta_2(G), \Delta_2(G^*) < \infty$, we have

$$tG'(t) \sim G(t) \quad \text{and} \quad tG''(t) \sim G'(t), \tag{2.3}$$

and for each $\varepsilon > 0$, we have

$$st \le c_\varepsilon G(s) + \varepsilon G^*(t) \quad \text{and} \quad st \le c_\varepsilon G^*(s) + \varepsilon G(t) \tag{2.4}$$

for some constant $c_\varepsilon > 0$ depending only on $\Delta_2(G), \Delta_2(G^*)$ and $\varepsilon > 0$. By the convexity of G and (2.2), there exists a constant $c > 0$ depending only on $\Delta_2(G)$ and $\Delta_2(G^*)$ satisfying

$$G(t_1 + t_2) \le cG(t_1) + cG(t_2) \quad \text{for all } t_1, t_2 \ge 0, \tag{2.5}$$

A shifted N-function is of G is defined by

$$G'_a(t) := G'(a+t)\frac{t}{a+t}, \quad \text{for } a > 0. \tag{2.6}$$

For the shifted N function, we have

$$G_{|P|}(t) \le c_\varepsilon G_{|Q|}(t) + \varepsilon G_{|P|}(|P - Q|), \tag{2.7}$$

for the details, we refer to [28]. The Sobolev-Orlicz space $W^{1,G}(\Omega)$ is defined as

$$W^{1,G}(\Omega) := \{f \in W^{1,1}(\Omega) : |f|, |\nabla f| \in L^G(\Omega)\}$$

equipped with the norm

$$\|f\|_{W^{1,G}(\Omega)} := \|f\|_{L^G(\Omega)} + \|\nabla f\|_{L^G(\Omega)}.$$

Let us present the Poincaré inequality in the setting of the Orlicz space.

Lemma 2.1. *Let G be an N-function with $\Delta_2(G), \Delta_2(G^*) < \infty$. Then for all $v \in W^{1,G}(B_r)$, there exists $0 < \theta < 1$ depending only on $\Delta_2(G)$ and $\Delta_2(G^*)$, such that*

$$⨍_{B_r} G\Big(\frac{|v - (v)_{B_r}|}{r}\Big)\,dx \le c\Big(⨍_{B_r} G(|\nabla v|)^\theta\,dx\Big)^{1/\theta}. \tag{2.8}$$

Furthermore, if $Z = \{x \in B_r : v(x) = 0\}$ has a positive measure satisfying

$$|Z| \ge c_0|B_r| \tag{2.9}$$

for some constant $\alpha_0 \in (0,1)$, then we have

$$⨍_{B_r} G\big(\frac{|v|}{r}\big)\,dx \le c(c_0)\Big(⨍_{B_r} G(|\nabla v|)^\theta\,dx\Big)^{1/\theta}. \tag{2.10}$$

Proof. The proof of (2.8) can be found in [14, Theorem 7]. For second term, (2.10), we proceed as in [22, Theorem 2.5]. Since $v = 0$ on Z, we have

$$|Z|G\Big(\frac{|(v)_{B_r}|}{r}\Big) = \int_Z G\Big(\frac{|v - (v)_{B_r}|}{r}\Big)\,dx \le \int_{B_r} G\Big(\frac{|v - (v)_{B_r}|}{r}\Big)\,dx, \tag{2.11}$$

which implies

$$G\Big(\frac{|(v)_{B_r}|}{r}\Big) \le \frac{1}{|Z|}\int_{B_r} G\Big(\frac{|v-(v)_{B_r}|}{r}\Big)\,dx. \tag{2.12}$$

Using the convexity of G and (2.12), we have

$$\begin{aligned}\int_{B_r} G(\frac{|v|}{r})\,dx &\le c\int_{B_r} G\Big(\frac{|v-(v)_{B_r}|}{r}\Big)\,dx + c\int_{B_r} G\Big(\frac{|v_{B_r}|}{r}\Big)\,dx\\ &\le c\Big(1+\frac{|B_r|}{|Z|}\Big)\int_{B_r} G\Big(\frac{|v-(v)_{B_r}|}{r}\Big)\,dx.\end{aligned}$$

Inequality (2.10) holds by (2.11), (2.9) and (2.8). □

From assumptions (1.4) and (1.5), the following inequalities are well-known:

$$\min(\lambda^2,\lambda^p)\varphi(t) \le \varphi(\lambda t) \le \max(\lambda^2,\lambda^p)\varphi(t), \tag{2.13}$$

$$\min(\lambda^2,\lambda^{\frac{p}{p-1}})\varphi^*(t) \le \varphi^*(\lambda t) \le \max(\lambda^2,\lambda^{\frac{p}{p-1}})\varphi^*(t). \tag{2.14}$$

From inequalities (2.13), (2.14) and (2.3), it is straightforward to check that

$$t^p \le \frac{\varphi(t)}{\varphi(1)} + 1 \quad\text{and}\quad t^2 \le \frac{\varphi^*(t)}{\varphi^*(1)} + 1. \tag{2.15}$$

From $(2.15)_1$, it follows that

$$\varphi^{-1}(t) \le \Big(\frac{t}{\varphi(1)}\Big)^{1/p} + 1. \tag{2.16}$$

Now, let us denote

$$V(P) = \sqrt{\frac{\varphi'(|P|)}{|P|}}P,$$

then the following estimates are known:

$$\begin{aligned}\big(\mathcal{A}(x,P)-\mathcal{A}(x,Q)\big) : \big(P-Q\big) &\sim \varphi''(|P|+|P-Q|)|P-Q|^2\\ &\sim |V(P)-V(Q)|^2,\end{aligned} \tag{2.17}$$

and

$$|\mathcal{A}(x,P)-\mathcal{A}(x,Q)| \le \varphi'_{|P|}(|P-Q|) \sim \varphi''(|P|+|P-Q|)|P-Q|. \tag{2.18}$$

We refer to [23] for a proof of the relations above. Furthermore, we have the following result, since $\varphi''(t)$ is a decreasing function.

Lemma 2.2. *For φ defined in* (1.4)*, we have*

$$\varphi(|P-Q|) \le c_\varepsilon |V(P)-V(Q)|^2 + \varepsilon\varphi(|Q|). \tag{2.19}$$

Proof. We use (2.3), the increasing property of $\varphi'(t)$, Young's inequality, (2.17) and decreasing property of $\varphi''(t)$ to have

$$\begin{aligned}\varphi(|P-Q|) &\le c\varphi'(|P-Q|)|P-Q|\\ &\le c\varphi'(|P-Q|+|Q|)|P-Q|\\ &\le c\varphi''(|P-Q|+|Q|)(|P-Q|^2+|Q||P-Q|)\\ &\le c\varphi''(|P-Q|+|Q|)(c_\varepsilon|P-Q|^2+\varepsilon|Q|^2)\\ &\le c_\varepsilon|V(P)-V(Q)|^2+\varepsilon\varphi''(|Q|)|Q|^2\\ &\le c_\varepsilon|V(P)-V(Q)|^2+\varepsilon\varphi(|Q|).\end{aligned}$$

□

The subscript "div" implies an additional divergence-free condition. For example, we have:

$$C^\infty_{0,\mathrm{div}}(\Omega) = \{u \in C^\infty_0(\Omega) : \operatorname{div} u = 0 \quad \text{in } \Omega\},$$
$$W^{1,G}_{0,\mathrm{div}}(\Omega) = \{u \in W^{1,G}_0(\Omega) : \operatorname{div} u = 0 \quad \text{in } \Omega\}.$$

The subscript "0" with Orlicz spaces and Lebesgue spaces implies that additional integral zero condition holds, that is,

$$L^q_0(\Omega) := \{f \in L^q(\Omega) : \int_\Omega f\,dx = 0\},$$
$$L^G_0(\Omega) := \{f \in L^G(\Omega) : \int_\Omega f\,dx = 0\}.$$

2.2. **Existence and uniqueness of the solution.** The existence and uniqueness of the solution pairs of (1.3) shall be studied in this subsection.

Lemma 2.3 ([29, Theorem 2.2]). *Let V be a separable reflexive Banach space and $\mathcal{F} \in V^*$, the dual space of V. Assume that $\Gamma : V \to V^*$ is monotone and demicontinuous. We further assume that there exists $\rho > 0$ such that $\Gamma(v)(v) > \mathcal{F}(v)$ for all $v \in V$ with $||v|| > \rho$. Then there exists $u \in V$ satisfying $\Gamma(u) = \mathcal{F}$.*

Remark 2.4. By setting $V = W^{1,\varphi}_{0,\mathrm{div}}(\Omega)$ and using Lemma 2.3, it is a standard process to show the existence and uniqueness of $u \in W^{1,\varphi}_{0,\mathrm{div}}(\Omega)$ satisfying

$$\langle \mathcal{A}(x, Du), D\xi\rangle = \langle \varphi''(|F|)F, \nabla\xi\rangle,$$

for all $\xi \in W^{1,\varphi}_{0,\mathrm{div}}(\Omega)$ if $F \in L^\varphi(\Omega)$. Next, by testing (1.3) with the function $u \in W^{1,\varphi}_{0,\mathrm{div}}(\Omega)$, we find

$$\langle \mathcal{A}(x, Du), Du\rangle = \langle \varphi''(|F|)F, \nabla u\rangle.$$

Using (2.4), (2.17), (2.1) and Korn's inequality (2.22), we discover that

$$\begin{aligned}\int_\Omega \varphi(|Du|)\,dx &\le \frac{1}{2c}\int_\Omega \varphi(|\nabla u|)\,dx + c\int_\Omega \varphi^*(\varphi''(|F|)|F|)\,dx\\ &\le \frac{1}{2c}\int_\Omega \varphi(|\nabla u|)\,dx + c\int_\Omega \varphi(|F|)\,dx\\ &\le \frac{1}{2}\int_\Omega \varphi(|Du|)\,dx + c\int_\Omega \varphi(|F|)\,dx,\end{aligned}$$

which implies

$$\int_\Omega \varphi(|Du|)\,dx \le c\int_\Omega \varphi(|F|)\,dx. \tag{2.20}$$

Let us define a linear functional $\mathcal{F} : W^{1,\varphi}_0(\Omega) \to \mathbb{R}$ by

$$\mathcal{F}(\xi) := \int_\Omega \mathcal{A}(x, Du) : D\xi\,dx - \int_\Omega \varphi''(|F|)F : \nabla\xi\,dx \quad \forall \xi \in W^{1,\varphi}_0(\Omega).$$

By [20, Theorem III.5.3] and Lemma 2.7, there exists a unique function $\pi \in L^{\varphi^*}_0(\Omega)$ satisfying

$$\mathcal{F}(\xi) = \int_\Omega \pi \operatorname{div}\xi\,dx \quad \forall \xi \in W^{1,\varphi}_0(\Omega).$$

2.3. **Known results in a John domain.** A John domain is a very generalized domain. In particular, it is well-known that $(1/600, R_1)$-Reifenberg flat domain is an α John domain for some $\alpha = \alpha(R_1/\operatorname{diam}(\Omega))$; see [1, 24] for details. Any two points in a John domain can be connected by a curve that is not too close to the boundary, see Definition 2.5. This subsection introduces the definition and auxiliary results of an α John domain.

Definition 2.5 ([15]). A domain $\Omega \subset \mathbb{R}^n$ is called an α John domain, for some $\alpha > 0$, if for all $x, y \in \Omega$, there exists a rectifiable path $\vec{\gamma}$ such that

$$\operatorname{dist}(\vec{\gamma}(t), \partial\Omega) \geq \frac{1}{\alpha} \min\{t, |\vec{\gamma}(t)| - t\} \quad \forall t \in [0, |\vec{\gamma}|].$$

Here, we assume that $\vec{\gamma}$ is parameterized by its arclength. We also emphasize that α is a scaling invariant.

For each open set Ω, there exist an universal constants $k_1 = k_1(n)$ and $k_2 = k_2(n)$ satisfying $1 < k_1 < k_2$, a positive number $N = N(n) > 0$ and a family of cubes $\{Q_j\}_{j\in\mathbb{N}}$ such that

$$\begin{aligned} &(C1) \quad \Omega = \cup_{j\in\mathbb{N}} k_1 Q_j = gcup_{j\in\mathbb{N}} 2k_1 Q_j, \\ &(C2) \quad \frac{1}{2} k_1 \operatorname{diam}(Q_j) \leq \operatorname{dist}(Q_j, \partial\Omega) \leq k_2 \operatorname{diam}(Q_j), \\ &(C3) \quad \sum_{j\in\mathbb{N}} \chi_{2k_1 Q_j} \leq N\chi_\Omega. \end{aligned} \tag{2.21}$$

We refer to this decomposition as a Whitney covering of Ω. We present a decomposition theorem in an α John domain.

Lemma 2.6 (Decomposition Theorem). *Let $\Omega \subset \mathbb{R}^n$ be an α John domain and $1 < q < \infty$. Let us denote $\mathfrak{W} := \{W_i := \frac{3}{2} k_1 Q_i\}$ where Q_i is a Whitney covering of Ω. Then there exists a constant $k = k(\alpha) > 0$ and a family of continuous linear operators $T_i : L^q_0(\Omega) \to L^q_0(W_i)$ satisfying the following:*

1. *For each $i \in \mathbb{N}$, we have $|T_i f| \leq ck_0 \chi_{W_i} Mf$ almost everywhere, where Mf is a maximal function of f.*
2. *For all $f \in L^q_0(\Omega)$, we have*
$$f = \sum_{i\in\mathbb{N}} T_i f \quad \text{in } L^q_0(\Omega).$$
This series converges for every permutation of the sequence.
3. *There exists $c = c(k_0)$ such that*
$$\frac{1}{c} \|f\|_{L^q(\Omega)} \leq \Big(\sum_{i\in\mathbb{N}} \|T_i f\|^q_{L^q(W_i)} \Big)^{1/q} \leq c\|f\|_{L^q(\Omega)}.$$
4. *If $f \equiv 0$ in W_i, then $T_i f \equiv 0$.*

Proof. Statements (1)-(3) follow from [15, Theorem 4.2]. Statement (4) follows from the construction of $T_i f$ in [15, (4.9)-(4.14)], i.e., we have

$$|T_i f| \leq c\Big(\int_{W_i} |f| dx \Big) \frac{\chi_{W_i}}{|W_i|} = 0.$$

□

Lemma 2.7 ([16, Theorem 4.2]). *Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be a bounded α John domain and G be an N-function with $\Delta_2(G), \Delta_2(G^*) < \infty$. For each $f \in L_0^G(\Omega)$, there exists at least one $v \in W_0^{1,G}(\Omega)$ satisfying*

$$\operatorname{div} v = f \quad \text{and} \quad \int_\Omega G(|\nabla v|)\,dx \leq c \int_\Omega G(|f|)\,dx,$$

for some $c = c(\alpha, \Delta_2(G), \Delta_2(G^*)) > 0$.

Next, we present Korn's inequality in the Orlicz spaces whose proof can be found in [15, Theorem 6.10, Theorem 6.13].

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded α John domain and G be an N-function with $\Delta_2(G), \Delta_2(G^*) < \infty$. Then for all $u \in W_0^{1,G}(\Omega)^n$, we have*

$$\int_\Omega G(|\nabla u|)\,dx \leq c \int_\Omega G(|Du|)\,dx, \tag{2.22}$$

$$\int_\Omega G(|\nabla u - (\nabla u)_\Omega|)\,dx \leq c \int_\Omega G(|Du - (Du)_\Omega|)\,dx \tag{2.23}$$

for some $c = c(\alpha, \Delta_2(G), \Delta_2(G^*)) > 0$.

The proof of the next lemma can be found in [16, Lemma 4.3].

Lemma 2.9. *Let Ω be an α John domain and let G be an N-function satisfying $\Delta_2(G), \Delta_2(G^*) < \infty$. Then for all $\pi \in L_0^{G^*}(\Omega)$ there exists a constant $c_* := c_*(\alpha, \Delta_2(G), \Delta_2(G^*) > 0$ such that*

$$\|\pi\|_{L^{G^*}(\Omega)} \leq c_* \sup_{\|\xi\|_{W_0^{1,G}(\Omega)} \leq 1} \langle \pi, \operatorname{div} \xi \rangle$$

and

$$\int_\Omega G^*(|\pi|)\,dx \leq \sup_{\xi \in W_0^{1,G}(\Omega)} \Big(\int_\Omega \pi \, \operatorname{div} \xi \,dx - \frac{1}{c_*} \int_\Omega G(|\nabla \xi|)\,dx \Big). \tag{2.24}$$

Remark 2.10. For the solution pair $(u, \pi) \in W^{1,\varphi}(\Omega) \times L_0^{\varphi^*}(\Omega)$, we use (2.17), (2.4), (2.5) and (2.1) to have

$$\begin{aligned}
\int_\Omega \pi \operatorname{div} \xi \,dx &= \int_\Omega \mathcal{A}(x, Du) : D\xi \,dx + \int_\Omega \varphi''(|F|)F : D\xi \,dx \\
&\leq c \int_\Omega \varphi'(|Du|)|D\xi|\,dx + c \int_\Omega \varphi'(|F|)|D\xi|\,dx \\
&\leq c(c_*) \int_\Omega \varphi^* \left(\varphi'(|Du|) + \varphi'(|F|)\right)\,dx + \frac{1}{c_*} \int_\Omega \varphi(|\nabla \xi|)\,dx \\
&\leq c(c_*) \int_\Omega \varphi(|Du|) + \varphi(|F|)\,dx + \frac{1}{c_*} \int_\Omega \varphi(|\nabla \xi|)\,dx
\end{aligned} \tag{2.25}$$

for all $\xi \in W^{1\,\varphi}(\Omega)$. Using (2.25) and (2.20), we have

$$\begin{aligned}
\int_\Omega \pi \operatorname{div} \xi \,dx - \frac{1}{c_*} \int_\Omega \varphi(|\nabla \xi|)\,dx &\leq c \int_\Omega \varphi(|Du|)\,dx + c \int_\Omega \varphi(|F|)\,dx \\
&\leq c \int_\Omega \varphi(|F|)\,dx, \quad \forall \xi \in W_0^{1,\varphi}(\Omega).
\end{aligned} \tag{2.26}$$

Finally, by (2.24) and (2.26), we obtain

$$\begin{aligned}\int_\Omega \varphi^*(|\pi|)\,dx &\le \sup_{\xi\in W_0^{1,\varphi}(\Omega)}\Big(\int_\Omega \pi \operatorname{div}\xi\,dx - \frac{1}{c_*}\int_\Omega \varphi(|\nabla\xi|)\,dx\Big)\\ &\le c\int_\Omega \varphi(|F|)\,dx.\end{aligned}$$

2.4. **Technical lemmas.** In this subsection, we present auxiliary lemmas. Let us begin with the following scaling property in the Reifenberg flat domain.

Lemma 2.11 ([13, Lemma 3.1])**.** *Let (u,π) be the solution pair of* (1.3) *satisfying* (1.4) *and* (1.6)*. Suppose that Ω satisfies Assumption 1.2-(i) for some δ and R_1 and $\mathcal{A}$ satisfies Assumption 1.2-(ii) for some δ and R_2. For $\lambda>0$, $M>0$, $r>0$, $P\in\mathbb{R}^{n\times n}$ and $x,y\in\Omega$, denote*

$$\begin{gathered}\widetilde{u}(x)=\frac{u(y+rx)}{\lambda r},\quad \widetilde{F}(x)=\frac{F(y+rx)}{\lambda},\\ \tilde{\pi}(x)=\frac{\lambda\pi(y+rx)}{Mr},\quad \widetilde{\Omega}=\Big\{\frac{x-y}{r}:y\in\Omega\Big\},\\ \widetilde{\mathcal{A}}(x,P)=\frac{\lambda\mathcal{A}(y+rx,\lambda P)}{M},\quad \tilde{\varphi}(t)=\frac{\varphi(\lambda t)}{M}.\end{gathered}$$

Then the following statements hold:

1. *We have $\Delta_2(\varphi)=\Delta_2(\tilde{\varphi})$ and $\Delta_2(\varphi^*)=\Delta_2(\tilde{\varphi}^*)$.*
2. *$\widetilde{\mathcal{A}}$ satisfies* (1.6) *with φ replaced by $\tilde{\varphi}$.*
3. *$\widetilde{\Omega}$ satisfies Assumption 1.2-(i) with δ and $\frac{R_1}{r}$.*
4. *$\widetilde{\mathcal{A}}$ satisfies Assumption 1.2-(ii) with δ and $\frac{R_2}{r}$,*
5. *$(\tilde{u},\tilde{\pi})$ is the solution of*

$$\begin{gathered}\operatorname{div}\widetilde{\mathcal{A}}(x,D\tilde{u})-\nabla\tilde{\pi}=\operatorname{div}\Big(\varphi''(|\tilde{F}|)\tilde{F}\Big)\quad \text{in } \widetilde{\Omega},\\ \operatorname{div}\tilde{u}=0\quad \text{in } \widetilde{\Omega},\\ \tilde{u}=0\quad \text{on } \partial\widetilde{\Omega}.\end{gathered}$$

Lemma 2.12 ([9])**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be a measurable sets with $\mathcal{C}\subset\mathcal{D}\subset\Omega$. Assume that Ω is $(\delta,1)$-Reifenberg flat for some small $\delta>0$. Furthermore, assume that following two conditions are satisfied.*
First, there exists $\epsilon>0$ such that

$$|\mathcal{C}\cap B_1(x)|<\epsilon|B_1|. \tag{2.27}$$

Second, for all $x\in\Omega$ and $r\in(0,1)$,

$$|\mathcal{C}\cap B_r(x)|>\epsilon|B_r(x)|\quad \text{implies}\quad B_r(x)\cap\Omega\subset\mathcal{D}. \tag{2.28}$$

Then, there exists $C_1:=C(n)$ satisfying

$$|\mathcal{C}|\le\Big(\frac{20}{1-\delta}\Big)\epsilon|\mathcal{D}|:=C_1\epsilon|\mathcal{D}|.$$

We would like to mention that (δ,R)-Reifenberg flat domain has a measure density condition, see [9]:

$$\sup_{0<r\le R}\sup_{y\in\Omega}\frac{|B_r(y)|}{|\Omega\cap B_r(y)|}\le\Big(\frac{20}{1-\delta}\Big)^n. \tag{2.29}$$

3. Regularity of the Limiting system up to the boundary

In this section, we focus on the following limiting equations:

$$\begin{gathered}\operatorname{div} \bar{\mathcal{A}}(Dw) - \nabla \pi_w = 0 \quad \text{in } B_8^+, \\ \operatorname{div} w = 0 \quad \text{in } B_8^+, \\ w = 0 \quad \text{on } \{x_n = 0\} \cap B_8,\end{gathered} \tag{3.1}$$

where

$$\bar{\mathcal{A}}(\cdot) = \fint_{B_8^+} \mathcal{A}(x, \cdot)\, dx.$$

In this section, we revisit the method outlined in [6], providing additional details for the sake of completeness. Throughout this section, we shall assume that $\eta \in C^\infty(B_8^+)$ is a cut-off function satisfying

$$\chi_{B_4}(x) \le \eta(x) \le \chi_{B_6}(x) \quad \text{and} \quad |\eta| + |\nabla \eta| + |\nabla^2 \eta| \le c, \tag{3.2}$$

for some $c = c(n) > 0$. We denote $\partial_\tau w$ for the derivative in the tangential direction. For $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $h \in \mathbb{R}^{n-1}$ with $0 < |h| < 1/16$, we denote the tangential translation by

$$f_\tau(x) := f(x' + h, x_n) - f(x', x_n) \quad \text{and} \quad f_{-\tau}(x) := \quad f(x' - h, x_n) - f(x', x_n),$$

and the difference operator by

$$\Delta^+ f := f_\tau - f \quad \text{and} \quad \Delta^- f := f_{-\tau} - f$$

and the difference quotient by

$$d^+ f := \frac{\Delta^+ f}{|h|} \quad \text{and} \quad d^- f := \frac{\Delta^- f}{|h|}.$$

For all $g \in W^{1,1}(B_8^+)$, we have $d^+ g \to \partial_\tau g$ as $h \to 0$ almost everywhere in B_8^+. Moreover, using the difference quotient method presented in [19, Chapter 5.8.2], we have

$$\|d^+ g\|_{L^q(B_6^+)} \le c \|\nabla g\|_{L^q(B_8^+)}. \tag{3.3}$$

Conversely, if $\operatorname{supp}(g) \subset B_6^+$ and $\|d^+ g\|_{L^q(B_8^+)} \le c$ for all $h \in \mathbb{R}^{n-1}$ satisfying $|h| \in (0, 1/2)$, then

$$\|\partial_\tau g\|_{L^q(B_8^+)} \le c.$$

For any real-valued functions f, g satisfying $\operatorname{supp}(f) \cup \operatorname{supp}(g) \subset B_6^+$ with $0 < |h| < 1/2$, we have

$$\int_{B_8^+} d^+ f g\, dx = \int_{B_8^+} f d^- g\, dx. \tag{3.4}$$

During the process of the proof, we occasionally use specific numbers such as $\varphi(1), \varphi''(0)$ or $\varphi^*(1)$. By the definition of φ in (1.4) and (1.5), these constants depend only on κ_2, μ, p and n. Now, let us begin with the following lemmas.

Lemma 3.1. *Let $w \in W^{1,\varphi}(B_8^+)$ be a weak solution of (3.1). Then,*

$$\int_{B_8^+} \varphi(\eta |d^+ Dw|)\, dx \le c \int_{B_8^+} \eta^2 |d^+ V(Dw)|^2\, dx + c \int_{B_8^+} \varphi(|\nabla w|)\, dx, \tag{3.5}$$

$$\int_{B_8^+} |\pi_w|^2\, dx \le c \int_{B_8^+} \frac{\varphi(|\nabla w|)}{\varphi^*(1)} + 1\, dx. \tag{3.6}$$

Proof. It is directly checked that the following inequalities hold:

$$\begin{aligned}
\varphi(\eta|d^+Dw|) &\overset{(2.7)}{\le} c\varphi_{|Dw|+|\Delta^+Dw|}(\eta|d^+Dw|) + c\varphi(|Dw|+|\Delta^+Dw|)\\
&\overset{(2.3)}{\le} c\varphi''_{|Dw|+|\Delta^+Dw|}(\eta|d^+Dw|)\eta^2|d^+Dw|^2 + c\varphi(|Dw|+|\Delta^+Dw|)\\
&\overset{(2.6)}{\le} c\varphi''(|Dw|+|\Delta^+Dw|+\eta|d^+Dw|)\eta^2|d^+Dw|^2\\
&\quad + c\varphi(|Dw|+|\Delta^+Dw|)\\
&\overset{(1.5)_3}{\le} c\varphi''(|Dw|+|\Delta^+Dw|)\eta^2|d^+Dw|^2 + c\varphi(|Dw|+|\Delta^+Dw|)\\
&\overset{(2.17)}{\le} c\eta^2|d^+V(Dw)|^2 + c\varphi(|Dw|+|\Delta^+Dw|).
\end{aligned}$$

Integrating on both sides over B_8^+, we have (3.5).

Next, we proceed as in Remark 2.4 and Remark 2.10 to find

$$\int_{B_8^+} \varphi^*(|\pi_w|)\,dx \le c\int_{B_8^+} \varphi(|\nabla w|)\,dx. \tag{3.7}$$

Using $(2.15)_2$ and (3.7), we find

$$\int_{B_8^+} |\pi_w|^2\,dx \le c\int_{B_8^+} \frac{\varphi^*(|\pi_w|)}{\varphi^*(1)} + 1\,dx \le c\int_{B_8^+} \frac{\varphi(|\nabla w|)}{\varphi^*(1)} + 1\,dx,$$

which implies (3.6). □

Next, we present the anisotropic embedding theorem whose proof can be found in [4, Theorem 2.1] and [30].

Lemma 3.2. *Suppose that $F \in W^{1,1}(B_8^+)$ and $\operatorname{supp}(F) \subset \operatorname{supp}(\eta) \cap B_8^+$. Let $\partial_\tau F \in L^{q_1}(B_8^+)$ and $\partial_n F \in L^{q_2}(B_8^+)$ for some $q_1, q_2 > 1$. Then $F \in L^{q_3}(B_8^+)$ with*

$$1 + \frac{n}{q_3} = \frac{n-1}{q_1} + \frac{1}{q_2}$$

and the inequality

$$\|F\|_{L^{q_3}} \le c\left(\|\partial_\tau F\|_{L^{q_1}} + \|F\|_{L^{q_1}}\right)^{\frac{n-1}{n}} \left(\|\partial_n F\|_{L^{q_2}} + \|F\|_{L^{q_2}}\right)^{1/n},$$

holds.

3.1. **Regularity in tangential direction.** This subsection presents the regularity results in the tangential direction near the boundary.

Proposition 3.3. *Suppose that $w \in W^{1,\varphi}(B_8^+)$ is a solution of (3.1) and the inequality (3.29) holds. Then*

$$\int_{B_8^+} \varphi(\eta|\partial_\tau Dw|)\,dx + \int_{B_8^+} \eta^2|\partial_\tau V(Dw)|^2\,dx \le c\int_{B_8^+} \varphi(|\nabla w|)\,dx \tag{3.8}$$

$$\int_{B_8^+} \eta^2|\partial_\tau \pi_w|^2\,dx \le c\left(\varphi''(0) + \frac{1}{\varphi^*(1)} + 1\right)\int_{B_8^+} \varphi(|\nabla w|) + 1\,dx. \tag{3.9}$$

Proof. By testing $d^-(\eta\psi)$ to the (3.1) and using (3.4), we find

$$\int_{B_8^+} d^+\bar{\mathcal{A}}(Dw) : D(\eta\psi)\,dx = \int_{B_8^+} \pi_w \operatorname{div}\left(d^-(\eta\psi)\right)\,dx. \tag{3.10}$$

Now, taking $\psi = \eta d^+ w$ and using (3.4), we have

$$\begin{aligned}
&\int_{B_8^+} \eta^2 |d^+ V(Dw)|^2 \, dx \\
&\le \int_{B_8^+} \eta^2 d^+ \bar{\mathcal{A}}(Dw) : d^+ Dw \, dx \\
&= \int_{B_8^+} \bar{\mathcal{A}}(Dw) : d^-(2\eta \nabla\eta \overset{\text{sym}}{\otimes} d^+ w) \, dx + \int_{B_8^+} \pi_w d^-(2\eta\nabla\eta \cdot d^+ w) \, dx \\
&=: I_1 + I_2.
\end{aligned} \tag{3.11}$$

We estimate the first term as

$$\begin{aligned}
I_1 &\le c \int_{B_8^+} |\bar{\mathcal{A}}(Dw)| |d^-(\eta\nabla\eta \overset{\text{sym}}{\otimes} d^+ w)| \, dx \\
&\overset{(1.6)}{\le} c \int_{B_8^+} \varphi'(|Dw|) |d^-(\eta\nabla\eta \overset{\text{sym}}{\otimes} d^+ w)| \, dx \\
&\overset{(2.4)}{\le} c_\varepsilon \int_{B_8^+} \varphi^*(\varphi'(|\nabla w|)) + \varepsilon \int_{B_8^+} \varphi(|d^-(\eta\nabla\eta \overset{\text{sym}}{\otimes} d^+ w)|) \, dx \\
&\overset{(3.3)}{\le} c_\varepsilon \int_{B_8^+} \varphi^*(\varphi'(|\nabla w|)) + \varepsilon \int_{B_8^+} \varphi(|\nabla(\eta\nabla\eta \overset{\text{sym}}{\otimes} d^+ w)|) \, dx \\
&\overset{(2.22)}{\le} c_\varepsilon \int_{B_8^+} \varphi^*(\varphi'(|\nabla w|)) + \varepsilon \int_{B_8^+} \varphi(|D(\eta\nabla\eta \overset{\text{sym}}{\otimes} d^+ w)|) \, dx \\
&\overset{(2.1)(3.3)}{\le} c_\varepsilon \int_{B_8^+} \varphi(|\nabla w|) + \varepsilon \int_{B_8^+} \varphi(\eta |D d^+ w|) \, dx.
\end{aligned} \tag{3.12}$$

For the second term I_2, we use (2.4), (2.22) and (3.3) to find that for any $\varepsilon > 0$,

$$\begin{aligned}
I_2 &\le \int_{B_8^+} |\pi_w| |d^-(2\eta\nabla\eta \cdot d^+ w)| \, dx \\
&\le c_\varepsilon \int_{B_8^+} \varphi^*(|\pi_w|) \, dx + \varepsilon \int_{B_8^+} \varphi(|\nabla(2\eta\nabla\eta \cdot d^+ w)|) \, dx \\
&\le c_\varepsilon \int_{B_8^+} \varphi^*(|\pi_w|) \, dx + \varepsilon \int_{B_8^+} \varphi(|D(2\eta\nabla\eta \cdot d^+ w)|) \, dx \\
&\le c_\varepsilon \int_{B_8^+} \varphi^*(|\pi_w|) \, dx + \varepsilon \int_{B_8^+} \varphi(|d^+ w|) \, dx + \varepsilon \int_{B_8^+} \varphi(\eta |d^+ Dw|) \, dx \\
&\le c_\varepsilon \int_{B_8^+} \varphi^*(|\pi_w|) \, dx + \varepsilon \int_{B_8^+} \varphi(|\nabla w|) \, dx + \varepsilon \int_{B_8^+} \varphi(\eta |d^+ Dw|) \, dx.
\end{aligned} \tag{3.13}$$

Using (3.5) (3.11), (3.12) and (3.13), we have

$$\begin{aligned}
&\int_{B_8^+} \varphi(\eta |d^+ \nabla w|) + \eta^2 |d^+ V(Dw)|^2 \, dx \\
&\le c \int_{B_8^+} \eta^2 |d^+ V(Dw)|^2 \, dx + c \int_{B_8^+} \varphi(|\nabla w|) \, dx \\
&\le c \int_{B_8^+} \varphi(|\nabla w|) \, dx + \varepsilon \int_{B_8^+} \varphi(\eta |d^+ \nabla w|) \, dx + c_\varepsilon \int_{B_8^+} \varphi^*(|\pi_w|) \, dx.
\end{aligned}$$

Now, we take $\varepsilon = 1/2$ and then use (3.7) to find

$$\begin{aligned} &\int_{B_8^+} \varphi(\eta|d^+Dw|) + \eta^2|d^+V(Dw)|^2\,dx \\ &\le c\int_{B_8^+} \varphi(|\nabla w|)\,dx + c\int_{B_8^+} \varphi^*(|\pi_w|)\,dx \\ &\le c\int_{B_8^+} \varphi(|\nabla w|)\,dx. \end{aligned} \tag{3.14}$$

Finally, (3.8) holds by (3.14) and different quotient argument.

Let us present the estimate of $\partial_\tau \pi_w$. Rewriting (3.10), we have

$$\begin{aligned} &\int_{B_8^+} \eta d^+\pi_w \operatorname{div}\psi\,dx \\ &= \int_{B_8^+} \eta d^+\bar{A}(Dw) : D\psi\,dx + \int_{B_8^+} d^+\bar{A}(Dw) : \nabla\eta \overset{\text{sym}}{\otimes} \psi\,dx \\ &\quad - \int_{B_8^+} \pi_w d^-(\nabla\eta\cdot\psi)\,dx, \end{aligned} \tag{3.15}$$

for all $\psi \in W_0^{1,\varphi}(B_8^+)$. By Lemma 2.7, there exists $\psi \in W_0^{1,\varphi}(B_8^+)$ satisfying $\operatorname{div}\psi = \eta d^+\pi_w - (\eta d^+\pi_w)_{B_8^+}$ and

$$\|\nabla\psi\|_{L^2(B_8^+)} \le c\|\eta d^+\pi_w - \big(\eta d^+\pi_w\big)_{B_8^+}\|_{L^2(B_8^+)}. \tag{3.16}$$

Testing ψ to (3.15) yields

$$\begin{aligned} &\int_{B_8^+} |\eta d^+\pi_w - \big(\eta d^+\pi_w\big)_{B_8^+}|^2\,dx \\ &= \int_{B_8^+} \eta d^+\bar{A}(Dw) : D\psi\,dx + \int_{B_8^+} d^+\bar{A}(Dw) : \nabla\eta \overset{\text{sym}}{\otimes} \psi\,dx \\ &\quad - \int_{B_8^+} \pi_w d^-(\nabla\eta\cdot\psi)\,dx \\ &:= J_1 + J_2 + J_3. \end{aligned} \tag{3.17}$$

By (2.18), Young's inequality, (1.5) and (3.16), we have

$$\begin{aligned} |J_1| &\le c\int_{B_8^+} \eta\varphi''(|Dw| + |\Delta^+Dw|)|d^+Dw||D\psi|\,dx \\ &\le \frac{c}{\varepsilon}\int_{B_8^+} \eta^2\varphi''(|Dw| + |\Delta^+Dw|)^2|d^+Dw|^2\,dx + \varepsilon\int_{B_8^+} |\nabla\psi|^2\,dx \\ &\le \frac{c\varphi''(0)}{\varepsilon}\int_{B_8^+} \eta^2|d^+V(Dw)|^2\,dx + \varepsilon\int_{B_8^+} |\nabla\psi|^2\,dx \\ &\le \frac{c\varphi''(0)}{\varepsilon}\int_{B_8^+} \eta^2|d^+V(Dw)|^2\,dx \\ &\quad + \varepsilon\int_{B_8^+} |\eta d^+\pi_w - \big(\eta d^+\pi_w\big)_{B_8^+}|^2\,dx. \end{aligned} \tag{3.18}$$

For the second term, J_2, we use (3.4), $(1.6)_2$, Young's inequality, Poincaré inequality, (3.3), (3.2), (2.3), (1.5) and (3.16) to find

$$\begin{aligned}
|J_2| &\le c\int_{B_8^+}|\bar{\mathcal{A}}(Dw)||d^-(\nabla\eta\otimes\psi)|\,dx\\
&\le c\int_{B_8^+}\varphi'(|Dw|)|d^-(\nabla\eta\otimes\psi)|\,dx\\
&\le \frac{c}{\varepsilon}\int_{B_8^+}\varphi'(|Dw|)^2\,dx+\varepsilon\int_{B_8^+}|d^-(\nabla\eta\otimes\psi)|^2\,dx\\
&\le \frac{c}{\varepsilon}\int_{B_8^+}\varphi'(|Dw|)^2\,dx+\varepsilon\int_{B_8^+}|\psi|^2+|\nabla\psi|^2\,dx\\
&= \frac{c}{\varepsilon}\int_{B_8^+}\varphi'(|Dw|)|Dw|\frac{\varphi'(|Dw|)}{|Dw|}\,dx+\varepsilon\int_{B_8^+}|\nabla\psi|^2\,dx\\
&\le \frac{c}{\varepsilon}\int_{B_8^+}\varphi(|Dw|)\varphi''(|Dw|)\,dx+\varepsilon\int_{B_8^+}|\nabla\psi|^2\,dx\\
&\le \frac{c\varphi''(0)}{\varepsilon}\int_{B_8^+}\varphi(|\nabla w|)\,dx+\varepsilon\int_{B_8^+}|\nabla\psi|^2\,dx\\
&\le \frac{c\varphi''(0)}{\varepsilon}\int_{B_8^+}\varphi(|\nabla w|)\,dx+\varepsilon\int_{B_8^+}|\eta d^+\pi_w-\big(\eta d^+\pi_w\big)_{B_8^+}|^2\,dx.
\end{aligned}\tag{3.19}$$

Similarly, we use Young's inequality, (3.3) and Poincaré inequality to find

$$\begin{aligned}
|J_3| &\le \varepsilon\int_{B_8^+}|d^-(\nabla\eta\cdot\psi)|^2\,dx+\frac{c}{\varepsilon}\int_{B_8^+}|\pi_w|^2\,dx\\
&\le \varepsilon\int_{B_8^+}|\nabla\psi|^2\,dx+\frac{c}{\varepsilon}\int_{B_8^+}|\pi_w|^2\,dx\\
&\le \varepsilon\int_{B_8^+}|\eta d^+\pi_w-\big(\eta d^+\pi_w\big)_{B_8^+}|^2\,dx+\frac{c}{\varepsilon}\int_{B_8^+}|\pi_w|^2\,dx.
\end{aligned}\tag{3.20}$$

Combining (3.17) $\sim$ (3.20), we have

$$\begin{aligned}
&\int_{B_8^+}|\eta d^+\pi_w-\big(\eta d^+\pi_w\big)_{B_8^+}|^2\,dx\\
&\le \frac{c\varphi''(0)}{\varepsilon}\int_{B_8^+}\eta^2|d^+V(Dw)|^2\,dx+\frac{c\varphi''(0)}{\varepsilon}\int_{B_8^+}\varphi(|\nabla w|)\,dx\\
&\quad+\frac{c}{\varepsilon}\int_{B_8^+}|\pi_w|^2\,dx+\varepsilon\int_{B_8^+}|\eta d^+\pi_w-\big(\eta d^+\pi_w\big)_{B_8^+}|^2\,dx.
\end{aligned}\tag{3.21}$$

Taking $\varepsilon=1/2$, the last term in the right hand side of (3.21) can be absorbed to the left hand side. Consequently,

$$\begin{aligned}
&\int_{B_8^+}|\eta d^+\pi_w-\big(\eta d^+\pi_w\big)_{B_8^+}|^2\,dx\\
&\le c\varphi''(0)\int_{B_8^+}\eta^2|d^+V(Dw)|^2\,dx\\
&\quad+c\varphi''(0)\int_{B_8^+}\varphi(|\nabla w|)\,dx+c\int_{B_8^+}|\pi_w|^2\,dx.
\end{aligned}$$

By (3.8), we have

$$\begin{aligned} &\int_{B_8^+} |\eta d^+\pi_w - \big(\eta d^+\pi_w\big)_{B_8^+}|^2\,dx \\ &\le c\varphi''(0)\int_{B_8^+}\varphi(|\nabla w|)\,dx + c\int_{B_8^+}|\pi_w|^2\,dx. \end{aligned} \tag{3.22}$$

By a direct calculation, it is clear that

$$\begin{aligned} \int_{B_8^+}\eta^2|d^+\pi_w|^2\,dx \le 2\int_{B_8^+}|\eta d^+\pi_w - \big(\eta d^+\pi_w\big)_{B_8^+}|^2\,dx \\ + c\Big|\int_{B_8^+}\eta d^+\pi_w\,dx\Big|^2. \end{aligned} \tag{3.23}$$

Using (3.4), Hölder's inequality (3.3) and (3.2), we have

$$\Big|\int_{B_8^+}\eta d^+\pi_w\,dx\Big|^2 = \Big|\int_{B_8^+} d^-\eta\pi_w\,dx\Big|^2 \le c\int_{B_8^+}|\pi_w|^2\,dx. \tag{3.24}$$

Combining (3.22), (3.23), (3.24) and (3.6), one finds that

$$\int_{B_8^+}\eta^2|d^+\pi_w|^2\,dx \le c\Big(\varphi''(0)+\frac{1}{\varphi^*(1)}+1\Big)\Big(\int_{B_8^+}\varphi(|\nabla w|)\,dx+1\Big)$$

which implies (3.9). $\square$

Remark 3.4. Proceeding as in the Proposition 3.3, the interior regularity holds

$$\int_{B_8^+}\varphi(\eta|\nabla Dw|)\,dx + \int_{B_8^+}\eta^2|\nabla V(Dw)|^2\,dx \le c(\eta)\int_{B_8^+}\varphi(|\nabla w|)\,dx,$$

where $\eta\in C_0^\infty(B_8^+)$ with $\operatorname{supp}(\eta)\subset\subset B_8^+$. Therefore, $\nabla V(Dw)$ belongs to $L^2_{loc}(B_8^+)$.

Remark 3.5. *As a result of Proposition 3.3, we have the following inequalities:*

$$\begin{aligned} \int_{B_8^+}\varphi(\eta|\partial_\tau\nabla w|)\,dx &\le \int_{B_8^+}\varphi(|\nabla(\eta\partial_\tau w)|+|\nabla\eta||\partial_\tau w|)\,dx \\ &\overset{(3.2)}{\underset{(2.5)}{\le}} c\int_{B_8^+}\varphi(|\nabla(\eta\partial_\tau w)|)\,dx + c\int_{B_8^+}\varphi(|\nabla w|)\,dx \\ &\overset{(3.2)}{\le} c\int_{B_8^+}\varphi(|\nabla(\eta\partial_\tau w)|)\,dx + c\int_{B_8^+}\varphi(|\nabla w|)\,dx \\ &\overset{(2.22)}{\le} c\int_{B_8^+}\varphi(|D(\eta\partial_\tau w)|)\,dx + c\int_{B_8^+}\varphi(|\nabla w|)\,dx \\ &\overset{(3.2)}{\underset{(2.5)}{\le}} c\int_{B_8^+}\varphi(\eta|D\partial_\tau w|)\,dx + c\int_{B_8^+}\varphi(|\nabla w|)\,dx \\ &\overset{(3.8)}{\le} c\int_{B_8^+}\varphi(|\nabla w|)\,dx. \end{aligned} \tag{3.25}$$

3.2. **Regularity in the normal direction.** For simplicity of notation, we use D instead of Dw. Rewriting (3.1) for each $\alpha = 1,\dots,n-1$, we have

$$-\sum_{\beta=1}^{n} \partial_\beta \bar{\mathcal{A}}_{\alpha\beta}(D) = \partial_\alpha \pi_w \quad \text{a.e. in } B_8^+.$$

From a direct calculation, we have

$$-\sum_{i,j,\beta=1}^{n} \partial_{ij} \bar{\mathcal{A}}_{\alpha\beta}(D)\partial_\beta D_{ij} = \partial_\alpha \pi_w$$

For brevity of notation, let us use

$$A_{j\alpha} := \partial_{nj}\bar{\mathcal{A}}_{\alpha n}, \quad b_j := \partial_n D_{nj}$$

$$\begin{aligned} f_\alpha := \partial_\alpha \pi_w + \partial_{nn} A_{\alpha n}(D)\partial_n D_{nn} + \sum_{i,j=1}^{n-1} \partial_{ij}\bar{\mathcal{A}}_{\alpha n}(D)\partial_n D_{ij} \\ + 2\sum_{i,\beta=1}^{n-1} \partial_{in}\bar{\mathcal{A}}_{\alpha\beta}(D)\partial_\beta D_{in} + \sum_{i,j,\beta=1}^{n-1} \partial_{ij}\bar{\mathcal{A}}_{\alpha\beta}(D)\partial_\beta D_{ij}. \end{aligned}$$

Since both $\bar{\mathcal{A}}$ and D are symmetric, we find that

$$2\sum_{j=1}^{n-1} A_{j\alpha} b_j = f_\alpha \quad \text{a.e. in } B_8^+$$

for $\alpha = 1,\dots,n-1$. Multiplying by b_α on both side and then taking a summation from $\alpha = 1$ to $n-1$, we obtain

$$\lambda\varphi''(|D|)|b|^2 \le 2\sum_{\alpha,j=1}^{n-1} A_{j\alpha} b_j b_\alpha \le |f||b| \quad \text{a.e. in } B_8^+. \tag{3.26}$$

Using the divergence-free condition on w, we have

$$\partial_n D_{nn} = \partial_n \partial_n w_n = \sum_{i=1}^{n-1} \partial_n \partial_i w_i.$$

From the structural condition (1.6), we find that

$$|f| \le c|\partial_\tau \pi_w| + c\varphi''(|D|)|\partial_\tau \nabla w|. \tag{3.27}$$

Using the divergence-free condition on w again, we have

$$b_\alpha = \frac{1}{2}\partial_n\partial_n w_\alpha - \frac{1}{2}\sum_{\beta=1}^{n-1} \partial_\alpha \partial_\beta w_\beta.$$

Denoting $\tilde{b}_\alpha = \partial_n \partial_n w_\alpha$, it is readily checked that

$$2|b| \ge |\tilde{b}| - |\nabla\partial_\tau w| \quad \text{and} \quad |\nabla^2 w| \le c(|\tilde{b}| + |\nabla\partial_\tau w|). \tag{3.28}$$

Let us now assume that

$$\int_{B_8^+} \varphi(|\nabla w|)\,dx \le K \tag{3.29}$$

for some constant $K > 1$.

From a direct calculation, we have

$$
\begin{aligned}
\nu\varphi''(|D|)|\nabla^2 w|^2 &\overset{(3.28)_2}{\leq} \nu\varphi''(|D|)(|b|^2 + |\nabla\partial_\tau w|^2)\\
&\overset{(3.26)}{\leq} |f||b| + c\varphi''(|D|)|\nabla\partial_\tau w|^2\\
&\overset{(3.27)}{\leq} c|\partial_\tau\pi_w||b| + c\varphi''(|D|)|\partial_\tau\nabla w||b| + c\varphi''(|D|)|\nabla\partial_\tau w|^2\\
&\overset{Young's}{\leq} \frac{c}{\varepsilon_1}|\partial_\tau\pi_w|^2 + \varepsilon_1|b|^2\\
&\quad + \frac{c}{\varepsilon_2}\varphi''(|D|)|\partial_\tau\nabla w|^2 + \varepsilon_2\varphi''(|D|)|b|^2\\
&\quad + c\varphi''(|D|)|\nabla\partial_\tau w|^2\\
&\leq \frac{c}{\varepsilon_1}|\partial_\tau\pi_w|^2 + \varepsilon_1|\nabla^2 w|^2\\
&\quad + \frac{c}{\varepsilon_2}\varphi''(|D|)|\partial_\tau\nabla w|^2 + \varepsilon_2\varphi''(|D|)|\nabla^2 w|^2\\
&\quad + c\varphi''(|D|)|\nabla\partial_\tau w|^2.
\end{aligned}
$$

Now, we take $\varepsilon_2 = \frac{\nu}{2}$ to find that

$$
\frac{\nu}{2}\varphi''(|D|)|\nabla^2 w|^2 \leq \frac{c}{\varepsilon_1}|\partial_\tau\pi_w|^2 + \varepsilon_1|\nabla^2 w|^2 + c\varphi''(|D|)|\partial_\tau\nabla w|^2.
$$

Then we use (1.4) and take $\varepsilon_1 = \frac{\nu\kappa_2}{4}$ to obtain

$$
\frac{\nu\kappa_2}{4}|\nabla^2 w|^2 \leq \frac{c}{\kappa_2}|\partial_\tau\pi_w|^2 + c\varphi''(|D|)|\partial_\tau\nabla w|^2 \quad \text{a.e. in } B_8^+. \tag{3.30}
$$

We multiply both sides by η^2, defined as in (3.2) and then take an integral on the both side of (3.30) to find

$$
\begin{aligned}
\int_{B_8^+} \eta^2|\nabla^2 w|^2\,dx &\leq \frac{c}{\kappa_2^2}\int_{B_8^+}\eta^2|\partial_\tau\pi|^2\,dx\\
&\quad + \frac{c}{\kappa_2}\int_{B_8^+}\eta^2\varphi''(|Dw|)|\partial_\tau\nabla w|^2\,dx.
\end{aligned}\tag{3.31}
$$

By (3.9) and (3.29), we have

$$
\frac{c}{\kappa_2^2}\int_{B_8^+}\eta^2|\partial_\tau\pi|^2\,dx \leq c(\mu,\kappa_2)K. \tag{3.32}
$$

For the second term, we have

$$\begin{aligned}
&\frac{c}{\kappa_2}\int_{B_8^+}\eta^2\varphi''(|Dw|)|\partial_\tau\nabla w|^2\,dx\\
&\overset{(1.5)_3}{\le} \frac{c\varphi''(0)}{\kappa_2}\int_{B_8^+}\eta^2|\partial_\tau\nabla w|^2\,dx\\
&\overset{(1.5)_2}{\le} \frac{c\varphi''(0)}{\kappa_2^2}\int_{B_8^+}\eta^2\varphi''(\eta|\partial_\tau\nabla w|)|\partial_\tau\nabla w|^2\,dx\\
&\overset{(2.3)}{\le} \frac{c\varphi''(0)}{\kappa_2^2}\int_{B_8^+}\varphi(\eta|\partial_\tau\nabla w|)\,dx\\
&\overset{(3.25)}{\le} \frac{c\varphi''(0)}{\kappa_2^2}\int_{B_8^+}\varphi(|\nabla w|)\,dx\\
&\overset{(3.29)}{\le} c(\mu,\kappa_2)K.
\end{aligned} \tag{3.33}$$

From the estimates (3.31), (3.32) and (3.33), one can see to it that

$$\int_{B_8^+}\eta^2|\nabla^2 w|^2\,dx \le c(\mu,\kappa_2)K. \tag{3.34}$$

When $n\ge 3$, we use the Poincaré-Sobolev inequality, (3.34) and (3.29), to find

$$\begin{aligned}
\Big(\int_{B_8^+}(\eta|\nabla w|)^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}} &\le c\int_{B_8^+}\eta^2|\nabla w|^2\,dx + c\int_{B_8^+}\eta^2|\nabla^2 w|^2\,dx\\
&\le c(\mu,\kappa_2)K.
\end{aligned} \tag{3.35}$$

Using Hölder's inequality and the fact that $K>1$, one has

$$\begin{aligned}
\Big(\int_{B_8^+}(\eta|\nabla w|)^{\frac{pn}{n-2}}\,dx\Big)^{\frac{n-2}{n}} &\le \Big(\int_{B_8^+}(\eta|\nabla w|)^{\frac{2n}{n-2}}\,dx\Big)^{\frac{p(n-2)}{2n}}\\
&\le c(\mu,\kappa_2)K^{\frac{p}{2}}\\
&\le c(\mu,\kappa_2)K.
\end{aligned} \tag{3.36}$$

Using (2.13), (3.35) and (3.36), we have

$$\begin{aligned}
&\Big(\int_{B_4^+}\varphi(|\nabla w|)^{\frac{n}{n-2}}\,dx\Big)^{\frac{n-2}{n}}\\
&\le c\varphi(1)\Big(\int_{B_8^+}(\eta|\nabla w|)^{\frac{2n}{n-2}}\,dx + \int_{B_8^+}(\eta|\nabla w|)^{\frac{pn}{n-2}}\,dx\Big)^{\frac{n-2}{n}}\\
&\le c(\mu,\kappa_2)K.
\end{aligned} \tag{3.37}$$

If $n=2$, the above inequality holds with q for all $1<q<\infty$ instead of $n/(n-2)$.

3.3. **Conclusion.** In this subsection, we conclude the results in the previous subsections.

Theorem 3.6. *Suppose that $(w,\pi_w)\in$ is a solution pair of* (3.1) *satisfying* (3.29). *Let $\bar q>1$ be a constant as in* (1.9). *Then there exists a constant depending only on $\mu>0$, $\bar q$ and* data *satisfying*

$$\int_{B_4^+}\varphi(|\nabla w|)^{\bar q}\,dx \le c(\mu,\kappa_2,\bar q)K^{\bar q}. \tag{3.38}$$

4. Comparison estimates near the boundary

To prove the main theorem, we need comparison estimates between the solution of the localized original problem and the solutions of the associated homogeneous problems. The arguments in this section are similar to those in [12]. However, we have provided detailed explanations to make this document self-contained. Let us introduce some notation:

$$\begin{gathered}\Omega_\rho := B_\rho(x_0)\cap\Omega, \quad B_\rho^+(x_0) := \{x\in B_\rho(x_0) : x_n>0\},\\ \partial_w\Omega_\rho(x_0) := B_\rho(x_0)\cap\partial\Omega, \quad T_\rho := \{x\in B_\rho(x_0) : x_n=0\},\end{gathered}$$

where, as usual, $B_\rho(x_0)$ is the ball with center $x_0\in\mathbb{R}^n$ and radius $\rho>0$. We shall omit a point x_0 when it is clear from the context. For simplicity, assume that we are under the following geometric setting:

$$B_{16}^+\subset\Omega_{16}\subset B_{16}\cap\{x_n>-32\delta\}.$$

Without further clarification, we assume that the constants in the rest of this article also depend on R_1. We specify issues regarding dependency later in Remark 4.9.

Let us introduce the localized original equations and the corresponding homogeneous equations as follows:

$$\begin{aligned}\operatorname{div}\mathcal{A}(x,Du)-\nabla\pi &= \operatorname{div}(\varphi''(|F|)F) \quad \text{i n}\Omega_{16},\\ \operatorname{div} &= 0 \quad \text{in } \Omega_{16},\\ u &= 0 \quad \text{on } \partial_w\Omega_{16},\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}\operatorname{div}\mathcal{A}(x,Dv)-\nabla\pi_v &= 0 \quad \text{in } \Omega_{16},\\ \operatorname{div} v &= 0 \quad \text{in } \Omega_{16},\\ v &= u \quad \text{on } \partial\Omega_{16}.\end{aligned}\tag{4.2}$$

To introduce the limiting equations, we introduce auxiliary functions. Let $\psi_\delta = \psi_\delta(x_n)\in C^\infty(\mathbb{R}^+)$ be a smooth cut-off function satisfying

- (i) $0\le\psi_\delta\le 1$,
- (ii) $\psi_\delta(x_n)=1$ on $[2\delta,8]$,
- (iii) $|\nabla\psi_\delta|\le\frac{2}{\delta}$,
- (iv) $\psi_\delta(x_n)=0$ on $[0,\delta]$.

Next, we choose $\xi_0\in W_0^{1,\varphi}(B_8^+)$ that satisfies

$$\operatorname{div}\xi_0 = -\nabla\psi_\delta\cdot v + \fint_{B_8^+}\nabla\psi_\delta\cdot v\,dx.$$

The existence of ξ_0 follows from Lemma 2.7 with the estimate

$$\int_{B_8^+}\varphi(|\nabla\xi_0|)\,dx \le c\int_{B_8^+}\varphi\left(|\nabla\psi_\delta\cdot v|\right)dx \le c\int_{B_8^+\cap\{x_n\le 2\delta\}}\varphi\left(\Big|\frac{v}{\delta}\Big|\right)dx.\tag{4.3}$$

We next define $h(x)$ as $h(x)=(0,0,\dots,-\big(\nabla\psi_\delta\cdot v\big)_{B_8^+}x_n\chi_{\{x_n\ge 0\}})$. From a direct calculation, we have

$$|\nabla h(x)|\le c|\nabla\psi_\delta\cdot v|\le c\Big|\frac{v}{\delta}\Big|.\tag{4.4}$$

Now, let us introduce the limiting system by

$$\begin{gathered} \operatorname{div} \bar{\mathcal{A}}(Dw) - \nabla \pi_w = 0 \quad \text{in } B_8^+, \\ \operatorname{div} w = 0 \quad \text{in } B_8^+, \\ w = \psi_\delta v + \xi_0 + h \quad \text{on } \partial B_8^+. \end{gathered} \tag{4.5}$$

We shall start with the standard estimates for (4.1) and (4.2).

Lemma 4.1. *Let (u, π) be a solution of* (4.1) *and (v, π_v) be the solution of* (4.2)*. Then we have the estimate*

$$\fint_{\Omega_{16}} \varphi(|Dv|)\, dx \le c \fint_{\Omega_{16}} \varphi(|Du|)\, dx.$$

Proof. We test (4.2) with $u - v$ to have

$$\int_{\Omega_{16}} \mathcal{A}(x, Dv) : Dv\, dx = \int_{\Omega_{16}} \mathcal{A}(x, Dv) : Du\, dx.$$

By (2.17) and (2.4), we find that

$$\int_{\Omega_{16}} \varphi(|Dv|)\, dx \le c_\varepsilon \int_{\Omega_{16}} \varphi(|Du|)\, dx + \varepsilon \int_{\Omega_{16}} \varphi(|Dv|)\, dx.$$

Taking $\varepsilon = \frac{1}{2}$, we complete our proof. □

Next we present a higher integrability result of $\varphi(|\nabla v|)$ near the boundary. We are motivated by the arguments presented in [10, Theorem 5.5].

Lemma 4.2. *Let (v, π_v) be the solution of* (4.2)*. Then there exists $\theta > 1$ such that*

$$\Big(\fint_{\Omega_8} \varphi(|\nabla v|)^\theta dx \Big)^{1/\theta} \le c \fint_{\Omega_{16}} \varphi(|\nabla v|) dx.$$

Proof. Let us extend v to be zero on $B_{16} \setminus \Omega_{16}$ and still denote it as v. We intend to show

$$\fint_{B_\rho(x_0)} \varphi(|\nabla v|)\, dx \le c \Big(\fint_{B_{2\rho}(x_0)} \varphi(|\nabla v|)^\sigma\, dx \Big)^{1/\sigma} dx \tag{4.6}$$

holds for all $B_{2\rho}(x_0) \subset B_8$ and for some $\sigma \in (0, 1)$. Once inequality (4.6) holds, we use Gerhings Lemma, [21, Proposition V.1.1] to complete our proof.

First, we consider interior case $B_{2\rho}(x_0) \subset\subset \Omega_8$. Now choose a cut-off function $\eta \in C_0^\infty(B_{2\rho})$ where $\eta \equiv 1$ on B_ρ and $|\nabla \eta| \le c/\rho$. We define

$$\begin{aligned} f(x) &:= p\eta^{p-1} \nabla \eta \cdot (v - (v)_{B_{2\rho}(x_0)}) - \fint_{B_{2\rho}(x_0)} p\eta^{p-1} \nabla \eta \cdot (v - (v)_{B_{2\rho}(x_0)})\, dx \\ &= p\eta^{p-1} \nabla \eta \cdot (v - (v)_{B_{2\rho}(x_0)}). \end{aligned}$$

Using that $\eta = 0$ on $\partial B_{2\rho}(x_0)$, $\operatorname{div} v = 0$ and Gauss-Green theorem, we have

$$\int_{B_{2\rho}(x_0)} p\eta^{p-1} \nabla \eta \cdot (v - (v)_{B_{2\rho}(x_0)})\, dx = \int_{B_{2\rho}(x_0)} \operatorname{div}(\eta^p (v - (v)_{B_{2\rho}(x_0)}))\, dx = 0. \tag{4.7}$$

By Lemma 2.7, there exists $\psi \in W_0^{1,\varphi}(B_{2\rho}(x_0))$ satisfying $\operatorname{div} \psi = f$ in $B_{2\rho}(x_0)$ and

$$\begin{aligned} \int_{B_{2\rho}(x_0)} \varphi(|\nabla \psi|)\, dx &\le c \int_{B_{2\rho}(x_0)} \varphi(|f|)\, dx \\ &\le c \int_{B_{2\rho}(x_0)} \varphi\Big(\frac{|v - (v)_{B_{2\rho}(x_0)}|}{\rho} \Big)\, dx. \end{aligned} \tag{4.8}$$

Testing (4.2) with $\xi = \eta^p (v - (v)_{B_{2\rho}(x_0)}) - \psi$ gives

$$
\begin{aligned}
&\int_{\Omega_{16}} \eta^p \mathcal{A}(x, Dv) : Dv \, dx \\
&= - \int_{\Omega_{16}} \mathcal{A}(x, Dv) : (p\eta^{p-1} \nabla\eta \overset{\text{sys}}{\otimes} (v - (v)_{B_{2\rho}(x_0)})) - D\psi) \, dx.
\end{aligned}
$$

Now, we use (2.17), (2.18), (2.4) and (4.8) to find

$$
\begin{aligned}
&\int_{B_\rho(x_0)} \varphi(|Dv|) \, dx \\
&\le c \int_{B_{2\rho}(x_0)} \eta^p \mathcal{A}(x, Dv) : Dv \, dx \\
&= -c \int_{B_{2\rho}(x_0)} \mathcal{A}(x, Dv) : (p\eta^{p-1} \nabla\eta \overset{\text{sys}}{\otimes} (v - (v)_{B_{2\rho}(x_0)}) - D\psi) \, dx \\
&\le c \int_{B_{2\rho}(x_0)} \varphi'(|Dv|) \Big(\frac{|v - (v)_{B_{2\rho}(x_0)}|}{\rho} + |\nabla\psi| \Big) \, dx \\
&\le \varepsilon \int_{B_{2\rho}(x_0)} \varphi(|Dv|) \, dx + c_\varepsilon \int_{B_{2\rho}(x_0)} \varphi\Big(\frac{|v - (v)_{B_{2\rho}(x_0)}|}{\rho} \Big) + \varphi(|\nabla\psi|) \, dx \\
&\le \varepsilon \int_{B_{2\rho}(x_0)} \varphi(|Dv|) \, dx + c_\varepsilon \int_{B_{2\rho}(x_0)} \varphi\Big(\frac{|v - (v)_{B_{2\rho}(x_0)}|}{\rho} \Big) \, dx
\end{aligned}
$$

Taking $\varepsilon = \frac{1}{2}$ and using (2.8), we have

$$
\begin{aligned}
\fint_{B_\rho(x_0)} \varphi(|Dv|) \, dx &\le c \fint_{B_{2\rho}(x_0)} \varphi\Big(\frac{|v - (v)_{B_{2\rho}(x_0)}|}{\rho} \Big) \, dx \\
&\le c \Big(\fint_{B_{2\rho}(x_0)} \varphi(|\nabla v|)^\sigma \, dx \Big)^{1/\sigma}.
\end{aligned} \tag{4.9}
$$

Next consider the case when $B_{2\rho}(x_0) \not\subset \Omega_8$. If $B_{\frac{3}{2}\rho}(x_0) \subset\subset \Omega_8$, then we proceed as in the interior case. If $B_{\frac{3}{2}\rho}(x_0) \not\subset \Omega_8$, we may further assume that$B_\rho(x_0) \not\subset B_8 \setminus \Omega_8$. Otherwise, (4.6) holds, and there is nothing to show. Because of (2.29), we can guarantee that

$$
\frac{|B_{2\rho}((x_0)) \setminus \Omega_8|}{|B_{2\rho}(x_0)|} \ge c(n).
$$

Now, we take $\rho_0 \le \frac{1}{1600(k_2+1)}$, where $k_2 = k_2(n)$ is a universal constant in (2.21). We assume that $0 < \rho \le \rho_0$. If $B_{2\rho}(x_0) \cap W_i \ne \emptyset$ for some $W_i \in \mathfrak{W}$, then $\operatorname{dist}(x, W_i) \le 4\rho_0$. By (C2)-(2.21), $\operatorname{diam}(W_i) \le 4\rho_0 k_2$. Therefore for all $x_1 \in B_{2\rho}(x_0)$ and $x_2 \in W_i$, we have $\operatorname{dist}(x_1, x_2) \le 16(1 + k_2)\rho_0 \le \frac{1}{100}$. Thus, $W_i \subset B_1(x_0)$ and $W_i \subset B_{12}$. Since $W_i \subset \Omega$, it is clear that $W_i \subset \Omega_{12}$. Note that the choice of ρ_0 only depends on n. Now, we denote

$$
f(x) := p\eta^{p-1} \nabla\eta \cdot v - \fint_{B_{2\rho}(x_0)} p\eta^{p-1} \nabla\eta \cdot v \, dx = p\eta^{p-1} \nabla\eta \cdot v \quad \text{in } B_{2\rho}(x_0) \cap \Omega_8.
$$

We extend $f(x) = 0$ in $B_{2\rho}(x_0) \setminus \Omega_8$. Proceeding as in (4.7), we have $f \in L^\varphi_0(\Omega)$.

Using Lemma 2.6-(2) and Lemma 2.6-(3), we can decompose $f = \sum T_i f$, where $T_i f \in L^2_0(W_i)$ and

$$\frac{1}{c}\|f\|^2_{L^2(\Omega)} \le \sum_{i\in\mathbb{N}} \|T_i f\|^2_{L^2(W_i)} \le c\|f\|^2_{L^2(\Omega)}. \tag{4.10}$$

Now, for each $T_i f$, we choose $\psi_i \in W^{1,\bar p}_0(W_i)$ satisfying

$$\operatorname{div}\psi_i = T_i f \quad \text{in } W_i \quad \text{and} \quad \|\psi_i\|_{W^{1,\bar p}(W_i)} \le c\|T_i f\|_{L^{\bar p}(W_i)}. \tag{4.11}$$

By Lemma 2.6-(4), $W_i \cap B_{2\rho} = \emptyset$ implies $T_i f \equiv 0$ and again by (4.11), we have $\psi_i \equiv 0$. After prolongation by zero outside W_i, we may assume that $\psi_i \in W^{1,\bar p}_0(\Omega_{12})$ from the choice of ρ_0, $W_i \cap B_{2\rho} \ne \emptyset$ implies $W_i \subset \Omega_{12}$, if not $\psi_i \equiv 0$. Let $\psi = \sum_{i\in\mathbb{N}} \psi_i$. By (4.11) and (4.10), the summation converges and therefore $\psi \in W^{1,\bar p}_0(\Omega_{12})$. Moreover, we have

$$\operatorname{div}\psi = \operatorname{div}\sum_{i\in\mathbb{N}}\psi_i = \sum_{i\in\mathbb{N}}\operatorname{div}\psi_i = \sum_{i\in\mathbb{N}} T_i f = f,$$

$$\|\nabla\psi\|^{\bar p}_{L^{\bar p}(\Omega)} \le \sum_{i\in\mathbb{N}} \|\nabla\psi_i\|^{\bar p}_{L^{\bar p}(W_i)} \le c\sum_{i\in\mathbb{N}} \|T_i f\|^{\bar p}_{L^{\bar p}(W_i)} \le c\|f\|^{\bar p}_{L^{\bar p}(\Omega_{12})}.$$

The rigorous verification of these inequalities can be found in [15, Theorem 5.2]. So we can choose $\xi = \eta^p v - \psi$ as a test function, to find

$$\begin{aligned}
\int_{B_\rho(x_0)} \varphi(|Dv|)\,dx &\le c\int_{B_{2\rho}(x_0)} \eta^p \mathcal{A}(x,Dv) : Dv\,dx \\
&= -c\int_{B_{2\rho}(x_0)} \mathcal{A}(x,Dv) : (p\eta^{p-1}\nabla\eta \overset{\text{sys}}{\otimes} v - D\psi)\,dx \\
&\le c\int_{B_{2\rho}(x_0)} \varphi'(|Dv|)\left(\frac{|v|}{\rho} + |\nabla\psi|\right)dx \\
&\le \varepsilon\int_{B_{2\rho}(x_0)} \varphi(|Dv|)\,dx + c_\varepsilon\int_{B_{2\rho}(x_0)} \varphi\Big(\frac{|v|}{\rho}\Big) + \varphi(|\nabla\psi|)\,dx
\end{aligned}$$

Taking $\varepsilon = 1/2$ and using (4.8) and (2.10), we find that

$$⨍_{B_\rho(x_0)}\varphi(|Dv|)\,dx \le c⨍_{B_{2\rho}(x_0)}\varphi\Big(\frac{|v|}{\rho}\Big)\,dx \le c\Big(⨍_{B_r}\varphi(|\nabla v|)^\sigma\,dx\Big)^{1/\sigma}. \tag{4.12}$$

To conclude, we use (4.9) for interior case and use (4.12), to find

$$\begin{aligned}
⨍_{B_\rho(x_0)}\varphi(|\nabla v|)\,dx &\le c⨍_{B_{2\rho}(x_0)}\varphi(|\nabla v - (\nabla v)_{B_\rho(x_0)}|)\,dx + c\varphi\left((\nabla v)_{B_{2\rho}(x_0)}\right) \\
&\le c⨍_{B_{2\rho}(x_0)}\varphi(|Dv - (Dv)_{B_\rho(x_0)}|)\,dx + c\varphi\left((\nabla v)_{B_{2\rho}(x_0)}\right) \\
&\le c⨍_{B_{2\rho}(x_0)}\varphi(|Dv|)\,dx + c\varphi\left((\nabla v)_{B_{2\rho}(x_0)}\right) \\
&\le c\Big(⨍_{B_{2\rho}(x_0)}\varphi(|\nabla v|)^\sigma\,dx\Big)^{1/\sigma} + c\varphi\left((\nabla v)_{B_{2\rho}(x_0)}\right).
\end{aligned}$$

Let $\Phi(t) := \varphi(t)^\sigma$ where $\sigma \in (0,1)$ is a small number defined in Lemma 2.1 which is chosen so that that $\Phi(t)$ is also an N-function. Then by the Jensen's inequality,

$$\varphi\Big(⨍_{B_{2\rho}(x_0)}|\nabla v|\,dx\Big) \le \varphi\circ\Phi^{-1}\Big(⨍_{B_{2\rho}(x_0)}\Phi(|\nabla v|)\,dx\Big)$$

$$= \Big(\fint_{B_{2\rho}(x_0)} \varphi(|\nabla v|)^\sigma \, dx \Big)^{1/\sigma}.$$

Combining the last two inequalities, we have (4.6) and complete our proof. □

Lemma 4.3. *Let (v, π_v) be the solution of (4.2) and (w, π_w) be the solution of (4.5). We shall extend w by zero on $\Omega_8 \setminus B_8^+$ and still denote it as w. Then, we have*

$$\int_{\Omega_8} \varphi(|Dw|) \, dx \le c \int_{\Omega_{16}} \varphi(|Dv|) \, dx + c \int_{\Omega_{16} \cap \{x_n \le 2\delta\}} \varphi(|\nabla v|) \, dx. \tag{4.13}$$

Proof. We may assume that w is also defined on Ω_8 by the zero extension. Test (4.5) with $w - \psi_\delta v - \xi_0 - h$ to have the equality

$$\int_{B_8^+} \bar{A}(Dw) : Dw \, dx = \int_{B_8^+} \bar{A}(Dw) : D(\psi_\delta v + D\xi_0 + Dh) \, dx.$$

We use (2.17), properties of ψ, Young's inequality, (1.6), (4.3) and (4.4) to discover that

$$\begin{aligned}
&\int_{B_8^+} \varphi(|Dw|) \, dx \\
&\le \int_{B_8^+} \bar{A}(Dw) : (v \otimes \nabla \psi_\delta) \, dx, \\
&\quad + \int_{B_8^+} \bar{A}(Dw) : D\xi_0 \, dx + \int_{B_8^+} \bar{A}(Dw) : Dh \, dx, \\
&\le \epsilon \int_{\Omega_8} \varphi(|Dw|) \, dx + c_\epsilon \int_{\Omega_8} \varphi(|Dv|) \, dx + c_\epsilon \int_{\Omega_8} \varphi(|v \otimes \nabla \psi_\delta|) \, dx \\
&\quad + c_\epsilon \int_{\Omega_8} \varphi(|\nabla \xi_0|) \, dx + c_\epsilon \int_{\Omega_8} \varphi(|Dh|) \, dx, \\
&\le \epsilon \int_{\Omega_8} \varphi(|Dw|) \, dx + c_\epsilon \int_{\Omega_8} \varphi(|Dv|) \, dx + c_\epsilon \int_{\Omega_8 \cap \{x_n \le 2\delta\}} \varphi\big(\frac{|v|}{\delta}\big) \, dx.
\end{aligned} \tag{4.14}$$

Consequently, after taking $\epsilon = 1/2$, we have

$$\int_{\Omega_8} \varphi(|Dw|) \, dx \le c \int_{\Omega_8} \varphi(|Dv|) \, dx + c \int_{\Omega_8 \cap \{x_n \le 2\delta\}} \varphi\big(\frac{|v|}{\delta}\big) \, dx. \tag{4.15}$$

We need to estimate the last term. Using Jensen's inequality, we have the following inequalities

$$\begin{aligned}
&\int_{\Omega_8 \cap \{x_n \le 2\delta\}} \varphi\big(\frac{|v|}{\delta}\big) \, dx \\
&\le \int_{\Omega_8 \cap \{x_n \le 2\delta\}} \varphi\Big(\sum_i \int_{-2\delta}^{x_n} \Big| \frac{\partial_n v^i(x', y)}{\delta} \Big| \, dy \Big) \, dx_n dx', \\
&\le \int_{\Omega_8 \cap \{x_n \le 2\delta\}} \varphi\Big(c \fint_{-2\delta}^{2\delta} |\nabla v(x', y)| \, dy \Big) \, dx_n \, dx', \\
&\le c \int_{\Omega_8 \cap \{x_n \le 2\delta\}} \fint_{-2\delta}^{2\delta} \varphi(|\nabla v(x', y)|) \, dy dx_n dx', \\
&\le c \int_{\Omega_8 \cap \{x_n \le 2\delta\}} \varphi(|\nabla v|) \, dx.
\end{aligned} \tag{4.16}$$

By (4.15) and (4.16), the required estimate (4.13) follows. □

Lemma 4.4. *Let (u,π) be a solution of* (4.1) *and (w,π_w) be the of solution of* (4.5)*, we then discover*

$$\fint_{\Omega_8}\varphi(|\nabla w|)\,dx \le c\fint_{\Omega_{16}}\varphi(|\nabla u|)\,dx.$$

Proof. By (2.22) and similar calculation as in (4.14), we see that

$$\begin{aligned}
&\fint_{\Omega_8}\varphi(|\nabla w|)\,dx\\
&\le c\fint_{\Omega_8}\varphi(|\nabla w-\nabla(\psi_\delta v+\xi_0+h)|)\,dx + c\fint_{\Omega_8}\varphi(|\nabla(\psi_\delta v+\xi_0+h)|)\,dx,\\
&\le c\fint_{\Omega_8}\varphi(|Dw-D(\psi_\delta v+\xi_0+h)|)\,dx + c\fint_{\Omega_8}\varphi(|\nabla(\psi_\delta v+\xi_0+h)|)\,dx,\\
&\le c\fint_{\Omega_{16}}\varphi(|\nabla v|)\,dx + c\fint_{\Omega_{16}\cap\{x_n\le 2\delta\}}\varphi\big(\frac{|v|}{\delta}\big)\,dx.
\end{aligned}\tag{4.17}$$

Following the same procedure by (4.16), we have

$$\fint_{\Omega_{16}\cap\{x_n\le 2\delta\}}\varphi\big(\frac{|v|}{\delta}\big)\,dx \le c\fint_{\Omega_{16}}\varphi(|\nabla v|)\,dx. \tag{4.18}$$

By (2.22), Lemma 4.1, (4.3) and (4.4) we achieve

$$\begin{aligned}
\fint_{\Omega_8}\varphi(|\nabla v|)\,dx &\le c\fint_{\Omega_8}\varphi(|\nabla v-\nabla u|)\,dx + c\fint_{\Omega_8}\varphi(|\nabla u|)\,dx,\\
&\le c\fint_{\Omega_8}\varphi(|Dv-Du|)\,dx + c\fint_{\Omega_8}\varphi(|\nabla u|)\,dx,\\
&\le c\fint_{\Omega_{16}}\varphi(|\nabla u|)\,dx.
\end{aligned}\tag{4.19}$$

Combining (4.17), (4.18) and(4.19), the lemma is proved. □

The following lemma states the regularity of the limiting system near the boundary, which is the result of Theorem 3.6.

Lemma 4.5. *Let (u,π) be the solution of* (4.1) *with*

$$\fint_{\Omega_{16}}\varphi(|\nabla u|)\,dx \le cK.$$

Let (w,π_w) be the solution of (4.5)*. For $\bar q$, as in* (1.9)*, we have*

$$\fint_{B_4^+}\varphi(|\nabla w|)^{\bar q}\,dx \le c(\mu,\kappa_2)K^{\bar q}.$$

For the rest of the paper, we extend (w,π_w), the solution pair of (4.5), to be zero on $\Omega_8\setminus B_8^+$, and we still denote it as (w,π_w) for simplicity of the notation. Since a Reifenberg flat domain is an extension domain, we have $(w,\pi_w)\in W^{1,\varphi}(B_8)\times L^{\varphi^*}(B_8)$. Also, we shall assume that $K>1$.

Lemma 4.6. *Suppose that Ω is a (δ,R_1)-Reifenberg flat domain, (v,π_v) is the weak solution of* (4.2) *and the following inequalities hold:*

$$\fint_{\Omega_{16}}\varphi(|\nabla u|)\,dx \le K \quad\textit{and}\quad \fint_{\Omega_{16}}\beta(\mathcal{A},B_{16})\,dx \le \delta. \tag{4.20}$$

Then for any $0 < \epsilon < 1$, there exists a sufficiently small $\delta = \delta(\epsilon, \mathrm{data}, R_1) > 0$ such that if (w, π_w) is the weak solution of (4.5)*, then we have*

$$\fint_{\Omega_8} |V(Dw) - V(Dv)|^2 \, dx \le \epsilon K. \tag{4.21}$$

Proof. We first test $w - (\psi_\delta v + \xi_0 + h)$ with (4.5) and (4.2). Then, we obtain

$$0 = \fint_{\Omega_8} \big(\bar{\mathcal{A}}(Dw) - \mathcal{A}(x, Dv)\big) : D\big(w - (\psi_\delta v + \xi_0 + h)\big) \, dx.$$

From a direct calculation, we find that

$$\begin{aligned}
&\fint_{\Omega_8} \big(\bar{\mathcal{A}}(Dw) - \bar{\mathcal{A}}(Dv)\big) : (Dw - Dv) \, dx \\
&= \fint_{\Omega_8} \big(\bar{\mathcal{A}}(Dw) - \bar{\mathcal{A}}(Dv)\big) : D((\psi_\delta - 1)v) \, dx \\
&\quad + \fint_{\Omega_8} \big(\bar{\mathcal{A}}(Dw) - \bar{\mathcal{A}}(Dv)\big) : D(\xi_0 + h) \, dx \\
&\quad - \fint_{\Omega_8} \big(\bar{\mathcal{A}}(Dv) - \mathcal{A}(x, Dv)\big) : D(w - \psi_\delta v - \xi_0 - h) \, dx \\
&:= I + II + III.
\end{aligned}$$

With the help of Lemma 2.17, one has

$$\frac{1}{c} \fint_{\Omega_8} |V(Dw) - V(Dv)|^2 \, dx \le I + II + III.$$

For the first term I, we have

$$\begin{aligned}
I &= \fint_{\Omega_8} \big(\bar{\mathcal{A}}(Dw) - \bar{\mathcal{A}}(Dv)\big) : D((\psi_\delta - 1)v) \, dx, \\
&= \fint_{\Omega_8} \big(\bar{\mathcal{A}}(Dw) - \bar{\mathcal{A}}(Dv)\big) : ((\psi_\delta - 1)\nabla v + v \otimes \nabla \psi) \, dx.
\end{aligned}$$

By Young's inequality, properties of ψ and (1.6) we find that

$$I \le \frac{\epsilon_1}{|\Omega_8|} \int_{\Omega_8 \cap \{x_n \le 2\delta\}} \varphi(|Dw|) \, dx + \frac{c_{\epsilon_1}}{|\Omega_8|} \int_{\Omega_8 \cap \{x_n \le 2\delta\}} \varphi(|\nabla v|) + \varphi\big(\frac{|v|}{\delta}\big) \, dx,$$

for some $\epsilon_1 > 0$. Applying (4.16), Lemma 4.1, Lemma 4.3 and (4.20) in order provides

$$I \le \epsilon_1 K + \frac{c_{\epsilon_1}}{|\Omega_8|} \int_{\Omega_8 \cap \{x_n \le 2\delta\}} \varphi(|\nabla v|) \, dx.$$

For the second term, we apply (2.4), (4.3), (4.4) and Jensen's inequality to obtain

$$II \le \epsilon_1 \fint_{\Omega_8} \varphi(|Dv|) + \varphi(|Dw|) \, dx + c_{\epsilon_1} \fint_{\Omega_8} \varphi(|v \cdot \nabla \psi_\delta|) \, dx,$$

and then we follow the same process as in estimate of I, to derive

$$II \le \epsilon_1 K + \frac{C_{\epsilon_1}}{|\Omega_8|} \int_{\Omega_8 \cap \{x_n \le 2\delta\}} \varphi(|\nabla v|) \, dx.$$

From Definition 1.2 and properties of ψ, we have

$$III \le \fint_{\Omega_8} \beta(\mathcal{A}, B_8) \varphi'(|Dv|) \big| Dw - Dv - D((\psi_\delta - 1)v + \xi_0 + h) \big| \, dx,$$

$$
\begin{aligned}
&\le \fint_{\Omega_8} \beta(\mathcal{A}, B_8)\varphi'(|\nabla v|)(|\nabla w| + |\nabla v|)\,dx \\
&\quad + \frac{c}{|\Omega_8|}\int_{\Omega_8\cap\{x_n\le 2\delta\}} \beta(\mathcal{A}, B_8)\varphi'(|\nabla v|)\Big(|\nabla v| + \big|\frac{v}{\delta}\big| + |\nabla \xi_0| + |\nabla h|\Big)\,dx \\
&:= III_1 + III_2.
\end{aligned}
$$

Using Young's inequality and (2.17), we obtain

$$
III_1 \le c_{\epsilon_1}\fint_{\Omega_8}\beta(\mathcal{A}, B_8)\varphi(|\nabla v|)\,dx + \epsilon_1\fint_{\Omega_8}\beta(\mathcal{A}, B_8)\varphi(|\nabla w|)\,dx.
$$

Using Hölder's inequality, (1.7), Lemma 4.1, (4.20), Lemma 4.2, Lemma 4.3 and Assumption 1.2, we have

$$
\begin{aligned}
III_1 &\le c_{\epsilon_1}\Big(\fint_{\Omega_8}\beta(\mathcal{A}, B_8)^{\frac{\theta}{\theta-1}}\,dx\Big)^{\frac{\theta-1}{\theta}}\Big(\fint_{\Omega_8}\varphi(|\nabla v|)^{\theta}\,dx\Big)^{1/\theta} + \epsilon_1 K, \\
&\le (c_{\epsilon_1}\delta + \epsilon_1)K.
\end{aligned}
$$

For the second term, we apply Young's inequality, (4.4), (4.3) and (1.7) to obtain

$$
III_2 \le \frac{c}{|\Omega_8|}\int_{\Omega_8\cap\{x_n\le 2\delta\}}\varphi(|\nabla v|)\,dx. \tag{4.22}
$$

Combining the previous inequalities, we derive

$$
\begin{aligned}
\fint_{\Omega_8}|V(Dw) - V(Dv)|^2\,dx &\le (\epsilon_1 + \delta)K + \frac{c}{|\Omega_8|}\int_{\Omega_8\cap\{x_n\le 2\delta\}}\varphi(|\nabla v|)\,dx, \\
&\le 2\delta K + \frac{c}{|\Omega_8|}\int_{\Omega_8\cap\{x_n\le 2\delta\}}\varphi(|\nabla v|)\,dx
\end{aligned} \tag{4.23}
$$

by taking $\epsilon_1 = \delta$. We are left to show that

$$
\int_{\Omega_8\cap\{x_n\le 2\delta\}}\varphi(|\nabla v|)\,dx
$$

can be sufficiently small if we choose $\delta > 0$ small enough.

Using Lemma 4.2, Lemma 4.4 and Hölder's inequality, we discover that

$$
\begin{aligned}
&\int_{\Omega_8\cap\{x_n\le 2\delta\}}\varphi(|\nabla v|)\,dx \\
&\le \Big(\int_{\Omega_8\cap\{x_n\le 2\delta\}}\varphi(|\nabla v|)^{\theta}\,dx\Big)^{1/\theta}\big|\Omega_8\cap\{x_n\le 2\delta\}\big|^{\frac{\theta-1}{\theta}}, \\
&\le C\delta^{\frac{\theta-1}{\theta}}|B_8|^{c}\int_{\Omega_8}\varphi(|\nabla u|)\,dx, \\
&\le C\delta^{\frac{\theta-1}{\theta}}|B_8|^{c}K,
\end{aligned} \tag{4.24}
$$

for some constant $c = c(\theta, n)$. We first combine (4.23) and (4.24), then choose $\delta > 0$ small enough depending only on $\epsilon > 0$ and data to reach (4.21). □

Lemma 4.7. *Suppose that (u, π) is a weak solution of (4.1) with the following normalization condition and the small BMO condition:*

$$
\fint_{\Omega_{16}}\varphi(|\nabla u|)\,dx \le K \quad \text{and} \quad \fint_{\Omega_{16}}\beta(\mathcal{A}, B_{16})\,dx \le \delta \tag{4.25}
$$

where Ω is a (δ, R_1)-Reifenberg flat domain. Then, for any $0 < \epsilon < 1$, there exists a sufficiently small $\delta = \delta(\epsilon, \mathrm{data}, R_1) > 0$ such that if (w, π_w) is the weak solution of (4.5)*, we have*

$$\fint_{\Omega_8} |V(Du) - V(Dw)|^2 \, dx \le \epsilon K + c_\epsilon \fint_{\Omega_{16}} \varphi(|F|) \, dx. \tag{4.26}$$

Proof. We subtract (4.2) from (4.1) to obtain the following:

$$\begin{gathered} \operatorname{div}(\mathcal{A}(x, Du) - \mathcal{A}(x, Dv)) - \nabla(\pi - \pi_v) = \operatorname{div}(\varphi''(|F|)F) \quad \text{in } \Omega_{16}, \\ \operatorname{div}\big(u - v\big) = 0 \quad \text{in } \Omega_{16}, \\ u - v = 0 \quad \text{on } \partial\Omega_{16}. \end{gathered}$$

We test $u - v$ with previous system of equation, then use (2.17), (2.4), (4.25) and Lemma 4.1, to have

$$\begin{aligned} \fint_{\Omega_{16}} |V(Du) - V(Dv)|^2 \, dx &\le c_{\epsilon_1} \fint_{\Omega_{16}} \varphi(|F|) \, dx + \epsilon_1 \fint_{\Omega_{16}} \varphi(|Du|) \, dx, \\ &\le c_{\epsilon_1} \fint_{\Omega_{16}} \varphi(|F|) \, dx + \epsilon_1 K \end{aligned} \tag{4.27}$$

for any $\epsilon_1 > 0$. By Lemma 4.6, there exists $\delta(\epsilon_2, \mathrm{data}) > 0$ and the solution of (4.5), (w, π_w) satisfying

$$\fint_{\Omega_8} |V(Dw) - V(Dv)|^2 \, dx \le \epsilon_2 K. \tag{4.28}$$

provided that Ω is a (δ, R)-Reifenberg flat for such δ. Combining (4.27) and (4.28), we have

$$\begin{aligned} &\fint_{\Omega_8} |V(Du) - V(Dw)|^2 \, dx \\ &\le C \fint_{\Omega_8} |V(Du) - V(Dv)|^2 \, dx + C \fint_{\Omega_8} |V(Dv) - V(Dw)|^2 \, dx, \\ &\le C \fint_{\Omega_{16}} |V(Du) - V(Dv)|^2 \, dx + C \fint_{\Omega_8} |V(Dv) - V(Dw)|^2 \, dx, \\ &\le C_{\epsilon_1} \fint_{\Omega_{16}} \varphi(|F|) \, dx + (\epsilon_1 + \epsilon_2) K. \end{aligned}$$

By taking $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2} > 0$, we obtain (4.26) with some small $\delta = \delta(\mathrm{data}, \epsilon) > 0$. $\square$

To establish a global Calderón-Zygmund estimate, we need a comparison estimate of full gradients of the localized original solution, (4.1), and the limiting system, (4.5), rather than just the symmetric gradient.

Lemma 4.8. *Under the assumptions in Lemma 4.7, we have*

$$\fint_{\Omega_8} \varphi(|\nabla u - \nabla w|) \, dx \le \epsilon K + c_\epsilon \fint_{\Omega_{16}} \varphi(|F|) \, dx. \tag{4.29}$$

Proof. By (2.22), we find that

$$\begin{aligned} &\fint_{\Omega_8} \varphi(|\nabla u - \nabla w|) \, dx \\ &\le c \fint_{\Omega_8} \varphi(|\nabla u - \nabla v|) \, dx + c \fint_{\Omega_8} \varphi(|\nabla v - \nabla(w - \psi_\delta v - \xi_0 - h)|) \, dx \end{aligned}$$

$$
\begin{aligned}
&\quad + c \fint_{\Omega_8} \varphi(|\nabla((\psi_\delta - 1)v + \xi_0 + h)|)\,dx \\
&\le c \fint_{\Omega_{16}} \varphi(|Du - Dv|)\,dx + c \fint_{\Omega_8} \varphi(|Dv - D(w + \psi_\delta v + \xi_0 + h)|)\,dx \\
&\quad + c \fint_{\Omega_8} \varphi(|\nabla((\psi_\delta - 1)v + \xi_0 + h)|)\,dx \\
&\le c \fint_{\Omega_{16}} \varphi(|Du - Dv|)\,dx + c \fint_{\Omega_8} \varphi(|Dv - Dw)|)\,dx \\
&\quad + c \fint_{\Omega_8} \varphi(|\nabla((\psi_\delta - 1)v + \xi_0 + h)|)\,dx \\
&=: I + II + III.
\end{aligned}
$$

Now, using (2.19), (4.25) and (4.27), we have following estimates

$$
\begin{aligned}
I &\le c_\varepsilon \fint_{\Omega_{16}} |V(Du) - V(Dv)|^2\,dx + \varepsilon \fint_{\Omega_{16}} \varphi(|Du|)\,dx \\
&\le (c_\varepsilon \varepsilon_1 + \varepsilon)K + c_{\varepsilon_1} \fint_{\Omega_{16}} \varphi(|F|)\,dx.
\end{aligned}
$$

We choose $\varepsilon_1 > 0$ small enough so that $c_\varepsilon \varepsilon_1 < \varepsilon$ we have

$$
I \le \varepsilon K + c_\varepsilon \fint_{\Omega_{16}} \varphi(|F|)\,dx.
$$

Similarly, we use (2.19), (4.21), Lemma 4.1 and (4.25) to discover

$$
\begin{aligned}
II &\le c_\varepsilon \fint_{\Omega_8} |V(Dv) - V(Dw)|^2\,dx + \varepsilon \fint_{\Omega_8} \varphi(|Dv|)\,dx \\
&\le (c_\varepsilon \varepsilon_1 + \varepsilon)K \le 2\varepsilon K
\end{aligned}
$$

after choosing $\varepsilon_1 > 0$ small enough so that $c_\varepsilon \varepsilon_1 < \varepsilon$. For the last term III, we proceed as in (4.22) to discover

$$
III \le \frac{c}{|\Omega_8|} \int_{\Omega_8 \cap \{x_n \le 2\delta\}} \varphi(|\nabla v|)\,dx \overset{(4.24)}{\le} \varepsilon K.
$$

Combining estimates for I, II and III, we have (4.29). $\square$

Remark 4.9. (Dependency of the constants and the regularity of the coefficient) We want to end this section with an important issue regarding the dependency of the constant. The constants in Lemma 2.7 and Lemma 2.8 depend on the domain Ω and δ, and again, the choice of δ depends the on constants c in Lemma 2.7 and Lemma 2.8. This implies that there is a circular reasoning in choosing δ. As a consequence, the class of the coefficient of $\mathcal{A}(\cdot, P)$ is restricted, and even a continuous coefficient may not be allowed. Thus we need an additional assumption on the coefficient.

Note that if Ω is a (δ, R)-Reifenberg flat domain, the constants in Lemma 2.7 and Lemma 2.8 are decreasing functions of δ and increasing functions on R. Now, to overcome such a dependency issue in our Assumption 1.2-(i) and Assumption 1.2-(ii) have different radii, R_1 and R_2. Then δ may depend on R_1 but does not dependent on R_2. After δ is chosen, we choose R_2 small enough to satisfy (1.8). In this way, we can include the wider class of the coefficients of $\mathcal{A}(\cdot, P)$. This argument follows from [18, Remark 2.3].

Remark 4.10. For simplicity, we only presented estimates near the boundary. With the same procedure, we have similar results.

Suppose that $B_{16} \subset\subset \Omega$ and $u \in W^{1,\varphi}_{0,\mathrm{div}}(\Omega)$ is a solution of (1.3). We then consider the following equations:

$$\begin{gathered} \operatorname{div} \mathcal{A}(x, Dv) - \nabla \pi_v = 0 \quad \text{in } B_{16}, \\ \operatorname{div} v = 0 \quad \text{in } B16, \\ v = u \quad \text{on } \partial B_{16}, \end{gathered}$$

and

$$\begin{gathered} \operatorname{div} \bar{\mathcal{A}}(Dw) - \nabla \pi_w = 0 \quad \text{in } B_8, \\ \operatorname{div} w = 0 \quad \text{in } B_8^+, \\ w = v \quad \text{on } \partial B_8. \end{gathered}$$

Then there exists $\delta = \delta(\varepsilon, \mathrm{data})$ such that if

$$⨍_{B_{16}} \varphi(|\nabla u|)\, dx \le K \quad \text{and} \quad ⨍_{B_{16}} \beta(\mathcal{A}, B_{16})\, dx \le \delta$$

holds, then

$$\int_{B_{16}} \varphi(|\nabla u - \nabla w|)\, dx \le \epsilon K + C_\epsilon ⨍_{\Omega_{16}} \varphi(|F|)\, dx ⨍_{B_4} \varphi(|\nabla w|)^{\bar{q}}\, dx \le c(\mu, \kappa_2) K^{\bar{q}}.$$

for $\bar{q}$ defined in (1.9).

5. Proof of the main theorem

The following lemmas were originally introduced in [11, Lemma 2.7] and [9]. Since we are considering a very general system with a pressure term, we do not have a Lipschitz regularity of the limiting equations, (4.5). Moreover, the equations only depend on the symmetric gradient, which makes the problem harder. Thus, we need some modification. The comparison estimate in Section 4 shall carry out the proof of the main theorem, the scaling argument, Lemma 2.11, and a Vitali-type covering lemma, Lemma 2.12. To do this, let us begin with the definition of the Hardy-Littlewood maximal function, which is defined below:

$$M(f)(x) := \sup_{x \in B} ⨍_B \chi_\Omega |f|\, dx,$$

where B represents a ball. We drop out the index Ω when it is clearly from the context.

Now let us define the sets

$$\begin{gathered} \mathcal{C} = \{y \in \Omega : M(\varphi(|\nabla u|)) > K^i\}, \\ \mathcal{D} = \{y \in \Omega : M(\varphi(|\nabla u|)) > K^{i-1}\} \cup \{x \in \Omega : M(\varphi(|F|)) > \sigma K^{i-1}\}. \end{gathered}$$

We need to verify that assumptions of Lemma 2.12 are satisfied for $\mathcal{C}$ and $\mathcal{D}$.

Lemma 5.1. *Let $(u, \pi) \in W^{1,\varphi}_{0,\mathrm{div}}(\Omega) \times L^{\varphi^*}(\Omega)$ be a solution of (1.3) and $0 < \epsilon < 1$. Then there exists $\delta = \delta(\mathrm{data}, \epsilon) > 0$ so that if Ω and $\mathcal{A}$ satisfies Assumption 1.2 with some $R_1 > R_2 = 128$, there are some large number $K = K(\mathrm{data}, \epsilon, R_1) > 1$ and some small number $\sigma = \sigma(\mathrm{data}, \epsilon) > 0$ such that the following holds:*

$$|\mathcal{C} \cap B_r(z)| > \epsilon |B_r(z)| \quad \textit{implies} \quad B_r(z) \cap \Omega \subset \mathcal{D}$$

whenever $z \in \Omega$ and $0 < r < 1$.

Proof. We prove this lemma by contradiction. If the statement is not true, we have

$$|\mathcal{C} \cap B_r(z)| > \epsilon |B_r(z)| \tag{5.1}$$

and $\big(B_r(z) \cap \Omega\big) \setminus \mathcal{D} \neq \emptyset$. Then one can find a point $z_1 \in \big(B_r(z) \cap \Omega\big) \setminus \mathcal{D}$ such that

$$M\big(\varphi(|\nabla u|)\big)(z_1) \le K^{i-1} \quad \text{and} \quad M\big(\varphi(|F|)\big)(z_1) \le \sigma K^{i-1}. \tag{5.2}$$

Let $M^*(|f|) := M(\chi_{B_{3r}(z)\cap\Omega}|f|)$. Then by (5.2), we have

$$M\big(\varphi(|\nabla u|)\big)(y) \le \max\Big\{M^*\big(\varphi(|\nabla u|)\big)(y), 128^n\Big\} \quad \text{for all } y \in B_r(z). \tag{5.3}$$

We need to consider two cases: the interior case and the boundary case.

Assume first that $B_{16r}(z) \subset \Omega$. As $z_1 \in B_{16r}(z)$, we have

$$\fint_{B_{16r}(z)} \varphi(|\nabla u|)\,dx \le K^{i-1} \quad \text{and} \quad \fint_{B_{16r}(z)} \varphi(|F|)\,dx \le \sigma K^{i-1}. \tag{5.4}$$

We then define the functions

$$\tilde{u}(x) = \frac{u(z+rx)}{r}, \quad \tilde{\pi}(z+rx) = \frac{\pi(z+rx)}{r},$$
$$\tilde{\mathcal{A}}(x,P) = \mathcal{A}(z+rx,P), \quad \tilde{F}(x) = F(y+rx), \quad \tilde{\varphi}(t) = \varphi(t).$$

Then by Lemma 2.11, $(\tilde{u},\tilde{\pi})$ is a solution of (1.3) with $\tilde{u},\tilde{\pi},\tilde{F}$, $\tilde{\mathcal{A}}$ and $\tilde{\varphi}$ replacing $u,\pi,F,\mathcal{A}$ and φ. Then, by (5.4), we have

$$\fint_{B_{16}} \tilde{\varphi}(|\nabla \tilde{u}|)\,dx \le K^{i-1}.$$

Then one can see that we are under the hypotheses of Remark 4.10 along with the scaling invariant property. We discover that there exists a $\delta = \delta(\tilde{\epsilon}, R_1, \text{data})$ such that if $\mathcal{A}$ is $(\delta,128)$-vanishing, then we can find $w \in W^{1,\varphi}_{0,\mathrm{div}}(B_{16r})$ satisfying

$$\fint_{B_{8r}(z)} \varphi(|\nabla u - \nabla w|)\,dx \le \Big(\tilde{\epsilon} K^{i-1} + c_{\tilde{\epsilon}} \fint_{B_{16r}(z)} \varphi(|F|)\,dx\Big) \le \big(\tilde{\epsilon} + c_{\tilde{\epsilon}}\sigma\big) K^{i-1}, \tag{5.5}$$

$$\fint_{B_{4r}(z)} \varphi(|\nabla w|)^{\bar{q}}\,dx \le c K^{i-1} \tag{5.6}$$

for small $\tilde{\epsilon}, \sigma > 0$, which shall be determined precisely in Remark 5.2. For $K > 128^n$, we have

$$\begin{aligned}
&|\{y \in B_r(z) : M\big(\varphi(|\nabla u|)\big) > K^i\}| \\
&\overset{(5.3)}{\le} |\{y \in B_r(z) : M^*\big(\varphi(|\nabla u|)\big) > K^i\}|, \\
&\le \Big|\Big\{y \in B_r(z) : M^*\big(\varphi(|\nabla u - \nabla w|)\big) + M^*\big(\varphi(|\nabla w|)\big) > \frac{K^i}{2}\Big\}\Big|, \\
&\le \Big|\Big\{y \in B_r(z) : M^*\big(\varphi(|\nabla u - \nabla w|)\big) > \frac{K^i}{4}\Big\}\Big| \\
&\quad + \Big|\Big\{y \in B_r(z) : M^*\big(\varphi(|\nabla w|)\big) > \frac{K^i}{4}\Big\}\Big| =: I + II.
\end{aligned}$$

By weak (1,1) estimate and (5.5), we obtain

$$I \le \frac{c}{K^i} \int_{B_{8r}(z)} \varphi(|\nabla u - \nabla w|)\,dx \le \frac{C_2}{K}(\tilde{\epsilon} + c_{\tilde{\epsilon}}\sigma)|B_r|, \tag{5.7}$$

where $C_2 = C_2(\text{data})$. We use Chebysebev inequality, strong $(\bar{q},\bar{q})$ estimates and (5.4) to discover that

$$
\begin{aligned}
II &\le \Big|\Big\{y \in B_r(z) : M^*\Big(\varphi(|\nabla w|)\Big)^{\bar{q}} > \Big(\frac{K^i}{2^{3p}}\Big)^{\bar{q}}\Big\}\Big|,\\
&\le \frac{c}{K^{\bar{q}i}} \int_{B_{4r}(z)} M^*\Big(\varphi(|\nabla w|)\Big)^{\bar{q}}\,dx,\\
&\le \frac{c}{K^{\bar{q}i}} \int_{B_{4r}(z)} \varphi(|\nabla w|)^{\bar{q}}\,dx,\\
&\overset{(5.6)}{\le} \frac{C_3(\mu,\bar{q})}{K^{\bar{q}}}|B_r|.
\end{aligned}
\tag{5.8}
$$

Note that the constant C_3 depends only on data, μ and $\bar{q}$.

Combining (5.7) and (5.8), one obtains

$$
|\mathcal{C} \cap B_r(z)| \le I + II \le \Big(\frac{C_2}{K}(\tilde{\epsilon} + C_{\tilde{\epsilon}}\sigma) + \frac{C_3}{K^{\bar{q}}}\Big)|B_r| \le \varepsilon|B_r|, \tag{5.9}
$$

by taking $K > 128^n$ large enough and $\sigma, \tilde{\varepsilon} > 0$ small enough; see below Remark 5.2. Thus we obtain a contradiction to (5.1).

Now, let us consider the boundary case when $B_{16r} \not\subset \Omega$. First, we choose $y_0 \in \partial\Omega \cap B_{16r}(z)$ and by the definition of $(\delta, 128)$-Reifenberg flatness of Ω, there exists a new coordinate, $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$, depending on y_0 and r, so that origin is $y_1 := y_0 + 128\delta r\vec{n}_0$ for some inward unit vector $\vec{n}_0$. Also, we have

$$
B^+_{128r} \subset \Omega_{128r} \subset \{\tilde{x}_n > -256r\delta\}
$$

and

$$
\fint_{\Omega_{128r}} \beta(\mathcal{A}, B_{128r})\, d\tilde{x} \le \delta.
$$

Since $0 < \delta < 1/32$, from a direct calculation, we have that $z_1 \in \Omega_{64r}$. Thus by (5.2), we have

$$
\fint_{\Omega_{64r}} \varphi(|\nabla u|)\,dx \le \frac{|B_{128r}|}{|B^+_{64r}|}\fint_{\Omega_{128r}} \varphi(|\nabla u|)\,dx \le 2^n K^{i-1}.
$$

Let us consider the functions

$$
\begin{gathered}
\tilde{u}(\tilde{x}) = \frac{u(y_1 + 4r\tilde{x})}{2^n(4r)}, \quad \tilde{\pi}(\tilde{x}) = \frac{\pi(y_1 + 4r\tilde{x})}{4r},\\
\tilde{\mathcal{A}}(\tilde{x}, P) = \mathcal{A}(y_1 + 4r\tilde{x}, 2^n P), \quad \tilde{F}(\tilde{x}) = \frac{F(y_1 + 4r\tilde{x})}{2^n}.
\end{gathered}
$$

As in the interior case, one can see that we are under the hypotheses of Lemma 4.8. Again, we discover that there exists a $\delta = \delta(\tilde{\epsilon}, R_1, \text{data})$ such that if $\mathcal{A}$ and Ω satisfies Assumption 1.2 with δ and $R_2 = 128$, then we can find $w \in W^{1,\varphi}_{0,\text{div}}(\Omega_{8r})$ satisfying for a given $\tilde{\epsilon} > 0$,

$$
\fint_{\Omega_{8r}} \varphi(|\nabla u - \nabla w|)\,d\tilde{x} \le \tilde{\epsilon}K^{i-1} + c_{\tilde{\epsilon}}\fint_{\Omega_{16r}} \varphi(|F|)\,d\tilde{x} \tag{5.10}
$$

$$
\fint_{\Omega_{8r}} \varphi(|\nabla w|)^{\bar{q}}\,d\tilde{x} \le cK^{i-1} \quad \text{and} \quad \fint_{\Omega_{8r}} \varphi(|\nabla w|)\,dx \le cK^{i-1}. \tag{5.11}
$$

Combining (5.10), (5.11) and (5.4), one can see that

$$
\fint_{\Omega_{24r}} \varphi(|\nabla u - \nabla w|)\,dx \le \Big(\tilde{\epsilon} + c_{\tilde{\epsilon}}\fint_{\Omega_{64r}} \varphi(|F|)\,dx\Big) \le \big(\tilde{\epsilon} + c_{\tilde{\epsilon}}\sigma\big)K^{i-1}. \tag{5.12}
$$

Once we obtain (5.11) and (5.12), we follow similar steps done in the interior case to reach a contradiction to (5.1). □

Remark 5.2. To prove our main result, Theorem 1.3, we need to specify the constants K, $\tilde{\epsilon}$ and σ. We want to mention that C_2 and C_3 differ in the interior case and the boundary case. We shall choose a bigger one and then choose

$$K = 2[\frac{2C_3}{\epsilon}]^{1/\bar{q}}. \tag{5.13}$$

After that, we choose ϵ small enough so that

$$K > 128^n \quad \text{and} \quad C_1 \epsilon K^q = 2^q C_1 \epsilon^{1-q/\bar{q}} (2C_3)^{q/\bar{q}} < 1. \tag{5.14}$$

It is possible since $1 \le q < \bar{q}$. Then choose $\tilde{\epsilon} > 0$ and $\sigma > 0$ small enough to obtain

$$\frac{C_2}{K}(\tilde{\epsilon} + C_{\tilde{\epsilon}}\sigma) < \frac{\epsilon}{2}. \tag{5.15}$$

Combining (5.13) and (5.15), we have

$$\frac{C_2}{K}(\tilde{\epsilon} + C_{\tilde{\epsilon}}\sigma) + \frac{C_3}{K^{\bar{q}}} \le \epsilon$$

which is (5.9). Here, we remark that C_1, C_2 and C_3 depend only on data, q, R_1, μ, κ_2 and ϵ, so are K, σ, $\tilde{\epsilon}$.

We are now in a position to prove our main theorem.

Proof of Theorem 1.3. We intend to apply Lemma 2.12. For a non-negative function g, we denote a upper level function by

$$\mathcal{U}_g(t) := |\{x \in \Omega : M(|g|) > t\}|$$

In view of (5.7), (5.8) and Remark 5.2, we obtain $|\mathcal{C} \cap B_1(z)| < \epsilon |B_1|$, which implies (2.27). On the otherhand, one can obtain (2.28) by Lemma 5.1. Therefore, according to Lemma 2.12 and the fact that $\gamma = 1$, we have

$$\mathcal{U}_{\varphi(|\nabla u|)}(K^i) \le C_1 \epsilon \Big(\mathcal{U}_{\varphi(|\nabla u|)}(K^{i-1}) + \mathcal{U}_{\varphi(|F|)}(\sigma K^{i-1}) \Big).$$

As a consequence, we have

$$\begin{aligned} \mathcal{U}_{\varphi(|\nabla u|)}(K^i) &\le C_1 \epsilon \Big(\mathcal{U}_{\varphi(|\nabla u|)}(K^{i-1}) + \mathcal{U}_{\varphi(|F|)}(\sigma K^{i-1}) \Big), \\ &\le (C_1\epsilon)^i \mathcal{U}_{\varphi(|\nabla u|)}(1) + \sum_{j=1}^{i} (C_1\epsilon)^j \Big(\mathcal{U}_{\varphi(|F|)}(\sigma K^{i-j}) \Big), \end{aligned}$$

and

$$\begin{aligned} &\sum_{i=1}^{\infty} K^{qi} \mathcal{U}_{\varphi(|\nabla u|)}(K^i) \\ &\le \sum_{i=1}^{\infty} (K^q C_1 \epsilon)^i \mathcal{U}_{\varphi(|\nabla u|)}(1) + \sum_{i=1}^{\infty} \sum_{j=1}^{i} K^{q(i-j)} (K^q C_1 \epsilon)^j \mathcal{U}_{\varphi(|F|)}(\sigma K^{i-j}), \\ &\le \sum_{i=1}^{\infty} (K^q C_1 \epsilon)^i \mathcal{U}_{\varphi(|\nabla u|)}(1) + \sum_{j=1}^{\infty} (K^q C_1 \epsilon)^j \sum_{i=j}^{\infty} K^{qi} \mathcal{U}_{\varphi(|F|)}(\sigma K^i). \end{aligned}$$

By (5.14), we have $\sum_{i=1}^{\infty}(C_1\epsilon K^q)^i < \infty$ and by the classical measure theory, we can compute as follows:

$$\begin{aligned} S &:= \sum_{i=1}^{\infty} K^{qi}\mathcal{U}_{\varphi(|\nabla u|)}(K^i),\\ &\le c|\Omega| + c\sum_{i=j}^{\infty} K^{qi}\mathcal{U}_{\varphi(|F|)}\big(\sigma K^i\big)\\ &\le c|\Omega| + c\int_\Omega |F|^q\,dx. \end{aligned}$$

Next, we have

$$\int_\Omega \varphi(|\nabla u|)^q\,dx \le c|\Omega| + S \le c\int_\Omega |F|^q + 1\,dx.$$

By (2.22), we have

$$\int_\Omega \varphi(|Du|)^q\,dx \le c\int_\Omega |F|^q + 1\,dx$$

for a constant $c = c(\kappa_2, \mu, q, \text{data})$. We use Lemma 2.9 with $G^*(t) = \varphi^*(t)^q$ and following the procedure of Remark 2.10,

$$\int_\Omega \varphi^*(|\pi|)^q\,dx \le c\int_\Omega \varphi(|\nabla u|)^q + \varphi(|F|)^q\,dx \le c\int_\Omega \varphi(|F|)^q + 1\,dx$$

where $c = c(\kappa_2, \mu, q, \text{data})$, which completes the proof. □

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