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LOCAL BIFURCATION STRUCTURE AND STABILITY OF THE MEAN CURVATURE EQUATION IN THE STATIC SPACETIME

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Abstract. We consider the curvature equation in the static spacetime,
$$\operatorname{div}\Big(\frac{f(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\Big)+\frac{\nabla u\nabla f(x)}{\sqrt{1-f^2(x)|\nabla u|^2}}=\lambda NH \quad \text{in } \Omega,$$
where Ω is a bounded domain in $\mathbb{R}^N$, $N\geq 1$; the function H gives the mean curvature. We investigate the local bifurcation structure and stability of the solutions to this equation.

1. Introduction and main results

We consider a domain $\Omega\subseteq\mathbb{R}^N$, where N is greater than or equal to 1. Let f be a smooth positive function on $\overline{\Omega}$. Consider the $N+1$-dimensional product manifold $\mathcal{M}=I\times\Omega$ equipped with the Lorentzian metric
$$g=-f^2(x)\,\mathrm{d}t^2+\mathrm{d}x^2.$$
In [22, Lemma 12.37], it was established that $\mathcal{M}$ is static with respect to ∂_t/f. For each $u\in C^2(\Omega)$, let $M=\{(x,u): x\in\Omega, u\in C^2(\Omega)\}$. A spacetime M is termed static in relation to an observer field $\mathscr{Q}$ if $\mathscr{Q}$ is irrotational and if there exists a smooth positive function such that $f\mathscr{Q}$ is a Killing vector field. Then, $(M,g)=\mathscr{U}$ represents an N-dimensional hypersurface in $\mathcal{M}$ at time t, which can be depicted by the graph of $t=u$. $\mathscr{U}$ is referred to as spacelike if $|\nabla u|<1/f$ in Ω (see [20]). We define $\mathscr{U}$ as being weakly spacelike if $|\nabla u|\leq 1/f$, i.e., if it is in Ω. Given the mean curvature H for a spacelike graph $\mathscr{U}$, Problem (1.1) has implications for classical relativity [4] and cosmology research [5, 19, 21].

For the case in which f is constantly equal to 1, Calabi [9] explored the properties of maximal surfaces and demonstrated that when $N\leq 4$, equation (1.2) allows only linear solutions. Cheng and Yau [10] further investigated maximal surfaces, extending Calabi's findings to all dimensions, and proposed the Bernstein theorem. For cases in which f is constantly equal to 1, Treibergs [24] provided significant results for entire surfaces with a constant mean curvature. For cases in which f equals 1, Bartnik and Simon [4] considered the Dirichlet problem for equation (1.2) with surfaces of bounded mean curvature.

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The authors of [6, 11] used critical point theory and topological degree arguments to explore the nonexistence, existence, and multiplicity of positive solutions for $f \equiv 1$ in bounded domains. In [13], the authors investigated the nonexistence, existence, and multiplicity of positive radial solutions of equation (1.2) with $NH = -\lambda f(x, s)$ on the unit ball via the bifurcation method. This work was later extended to general domains in [14, 16]. The author in [18] studied the existence and uniqueness of classical solutions, the multiplicity of strong solutions, and the symmetry of positive solutions. The global structure of the positive solutions for this problem was also delineated. For more research results on the mean curvature equation, see references [7, 8, 17, 15] and their cited literature.

In [1], the stability of hypersurfaces with a constant mean curvature was studied through the calculus of variations. The relationship between stability and constant mean curvature was presented under the condition that the hypersurface is compact. In [2], the stability of hypersurfaces with constant mean curvature in Riemannian manifolds was studied. Barros, Brasil, and Caminha [3] investigated stability issues concerning the generalized Robertson-Walker spacetime. In this work, we investigate the local bifurcation structure and stability of the mean curvature equation in static the spacetime.

We consider the following 0-Dirichlet problem involving the mean curvature operator in Minkowski space:

$$\begin{gathered} -\operatorname{div}\Big(\frac{f^2(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\Big) = -\lambda N f(x) H(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

Here, λ is a nonnegative parameter representing the strength of the mean curvature function, the real-valued function H gives the mean curvature, Ω is a $C^{2,\alpha}$ bounded domain in $\mathbb{R}^N$ with $N \geq 1$ for some $\alpha > 0$, and $f \in C^{0,\alpha}(\overline{\Omega} \times [-d, d])$, where d is the diameter of Ω.

Using the equation

$$\operatorname{div}\Big(\frac{f(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\Big) + \frac{\nabla u \nabla f(x)}{\sqrt{1-f^2(x)|\nabla u|^2}} = NH, \tag{1.2}$$

we can derive that

$$\begin{aligned} -\operatorname{div}\Big(\frac{f^2(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\Big) &= -\operatorname{div}\Big(f(x)\cdot\frac{f(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\Big) \\ &= -f(x)\operatorname{div}\Big(\frac{f(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\Big) - f(x)\frac{\nabla u \nabla f(x)}{\sqrt{1-f^2(x)|\nabla u|^2}} \\ &= -Nf(x)H. \end{aligned}$$

From [18], we have

$$\operatorname{div}\Big(\frac{f^2(x)\nabla u}{\sqrt{1-f^2(x)|\nabla u|^2}}\Big) = Nf(x)H.$$

This equation is equivalent to (1.2). Next, we present the main theorem.

Theorem 1.1. *Suppose that H is C^3 with respect to its second argument and that $H_0 \in (0, +\infty)$. Then, all the solutions of problem (1.1) near $(\lambda_1/H_0, 0)$ can be* *expressed as* $(\lambda(s), s\varphi_1 + sz(s))$ *for* s *in an open interval* $(-\delta,\delta)$*, where* $\delta > 0$*, such that* $\lambda(0) = \lambda_1/H_0$ *and*

$$\lambda'(0) = -\frac{-\frac{\lambda_1}{H_0} N \int_\Omega f(x) H_{uu}(x,0)\varphi_1^3\,dx}{2H_0 \int_\Omega f(x)\varphi_1^2\,dx}.$$

Here, $z : (-\delta,\delta) \to Z$ *is a* C^2 *function that satisfies* $z(0) = 0$*. Additionally, if* $H_{uu}(x,0)$ *is exactly zero on* Ω*, then we obtain*

$$\lambda''(0) = -\frac{-\frac{\lambda_1}{H_0} N \int_\Omega f(x) H_{uuu}(x,0)\varphi_1^4 dx}{3H_0 \int_\Omega f(x)\varphi_1^2 dx},$$

where $H_{uuu}(x,0)$ *denotes the third derivative of* H *with respect to its second variable at* 0*.*

By Theorem 1.1, we can deduce the following stability result.

Theorem 1.2. *Let* λ_1/H_0 *be a bifurcation point for the equation* $F(\lambda,u) = 0$ *in a Banach space* $\mathcal{X}$*, and assume that* 0 *is a simple eigenvalue of the linearized operator* $F_u(\lambda_1/H_0, 0)$*. Suppose further that* $H_{uu}(x,0) \equiv 0$ *in* Ω *and that* $H_{uuu}(x,0) \neq 0$ *in* Ω*. Then, the stability of the solutions* $u(s)$ *near the bifurcation point* $(\lambda_1/H_0, 0)$ *is determined as follows:*

(1) *If* $H_{uuu}(x,0) > 0$ *in* Ω*, then the solutions are asymptotically linearly stable, i.e.,* $\lambda''(0) > 0$*.*
(2) *If* $H_{uuu}(x,0) < 0$ *in* Ω*, then the solutions are asymptotically linearly unstable, i.e.,* $\lambda''(0) < 0$*.*

This article is organized as follows. Section 2 discusses the local bifurcation structure of the solution set of equation (1.1). Section 3 presents the stability results near the bifurcation point.

2. Local bifurcation structure

In this section, we provide the proof of Theorem 1.1. Consider the set X defined as

$$X = \{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$$

with the norm $\|u\| := \|f(x)\nabla u\|_\infty$. Let φ_1 be a positive eigenfunction corresponding to λ_1 with $\|\varphi_1\| = 1$. Let X_0 be a closed subspace of X such that

$$X = X_1 \oplus X_0,$$

where $X_1 = \operatorname{span}\{\varphi_1\}$. By applying the Hahn–Banach theorem, we can find a linear continuous functional $l \in X^*$ satisfying

$$l(\varphi_1) = 1 \quad \text{and} \quad X_0 = \{u \in X : l(u) = 0\}.$$

Proof of Theorem 1.1. Define $\mathcal{X} = \{u \in C^2(\overline{\Omega}) : u = 0, \text{ on } \partial\Omega\}$, $\mathcal{Y} = C(\overline{\Omega})$. Consider the function defined by

$$F(\lambda,u) = \operatorname{div}\Big(\frac{f^2(x)\nabla u}{\sqrt{1 - f^2(x)|\nabla u|^2}}\Big) - \lambda N f(x) H(x,u).$$

Since $H_0 \in (0,+\infty)$ and $H(x,0) = 0$ holds for any $x \in \Omega$, we have

$$F(\lambda,0) = \operatorname{div}\Big(\frac{f^2(x)\nabla 0}{\sqrt{1 - f^2(x)|\nabla 0|^2}}\Big) - \lambda N f(x) H(x,0) = 0$$

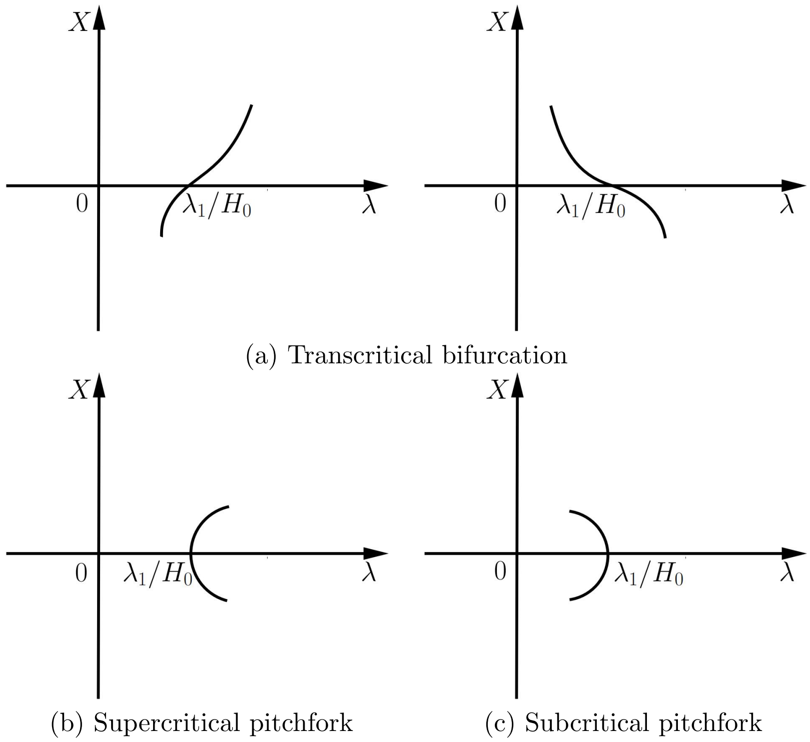

FIGURE 1. Bifurcation diagrams of Theorem 1.1

for any λ. The partial derivative of $F(\lambda, u)$ with respect to λ is

$$F_\lambda(\lambda, u) = -Nf(x)H(x, u).$$

According to [18], we have

$$\lim_{t\to 0^+} \frac{NH(x,t)}{t} = -H_0.$$

This holds because H is a function with third-order continuous derivatives with respect to its second variable, and F is C^3 with respect to u in some small neighborhood $V \subset \mathcal{X}$ of 0.

By calculation, we obtain

$$F_u(\lambda, 0)[\varphi_1] = \operatorname{div}(f^2(x)\nabla\varphi_1) + \lambda f(x)H_0\varphi_1,$$
$$F_u(\frac{\lambda_1}{H_0}, 0)[\varphi] = \operatorname{div}(f^2(x)\nabla\varphi) + \frac{\lambda_1}{H_0} f(x)H_0\varphi,$$

where $\varphi \in \mathcal{X}$. The function φ_1 is a positive eigenfunction corresponding to the principal eigenvalue λ_1 of the linearized problem associated with equation (1.1). Specifically, φ_1 is a solution of $F_u(\lambda_1/H_0, 0)[\varphi] = 0$. Then, we have

$$\operatorname{div}(f^2(x)\nabla\varphi_1) + \frac{\lambda_1}{H_0} f(x)H_0\varphi_1 = 0.$$

Since φ_1 is a nontrivial solution, it follows that $\varphi_1 \neq 0$ and $\varphi_1^2 \geq 0$. Hence, the integral $\int_\Omega \varphi_1^2 \,\mathrm{d}x > 0$. Thus, the kernel space is

$$\mathscr{N}\Big(F_u(\frac{\lambda_1}{H_0}, 0)\Big) = \operatorname{span}\{\varphi_1\}.$$

The codimension of the image space is

$$\mathscr{R}\Big(F_u(\frac{\lambda_1}{H_0}, 0)\Big) = \big\{v \in \mathcal{Y} : \int_\Omega v\varphi_1 \,\mathrm{d}x = 0\big\}.$$

Therefore,

$$\dim \mathscr{N}\Big(F_u(\frac{\lambda_1}{H_0}, 0)\Big) = \operatorname{codim} \mathscr{R}\Big(F_u(\frac{\lambda_1}{H_0}, 0)\Big) = 1. \tag{2.1}$$

Clearly, F is C^1 with respect to λ, and $F_{\lambda u}$ exists and remains continuous in a small neighborhood of $(\lambda_1/H_0, 0)$. From the calculations, we obtain

$$F_{\lambda u}(\frac{\lambda_1}{H_0}, 0)[\varphi_1] = fH_0\varphi_1, \tag{2.2}$$

$$F_{uu}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]^2 = -\frac{\lambda_1}{H_0} NfH_{uu}(x, 0)\varphi_1^2$$

and

$$F_{uuu}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]^3 = -\frac{\lambda_1}{H_0} NfH_{uuu}(x, 0)\varphi_1^3.$$

Combining this with $H_0 > 0$, we obtain

$$H_0 \int_\Omega \varphi_1^2 \,\mathrm{d}x \neq 0.$$

Thus,

$$\int_\Omega fH_0\varphi_1^2 \,\mathrm{d}x \neq 0.$$

This leads to the conclusion that

$$F_{\lambda u}(\lambda_1/H_0, 0)[\varphi_1] \notin \mathscr{R}(F_u(\lambda_1/H_0, 0)). \tag{2.3}$$

By applying [16], we deduce that all the solutions near $(\lambda_1/H_0, 0)$ for problem (1.1) can be expressed as $(\lambda(s), s\varphi_1 + sz(s))$, where s belongs to the interval $(-\delta, \delta)$ for some positive value of δ, and that they satisfy the conditions $\lambda(0) = \lambda_1/H_0$ and $z(0) = 0$.

We rescale φ_1 so that

$$\int_\Omega \varphi_1^2 \,\mathrm{d}x = 1. \tag{2.4}$$

Then, we define the linear functional

$$l(u) = \int_\Omega u\varphi_1 \,\mathrm{d}x \tag{2.5}$$

and

$$\mathscr{N}(l) = \mathscr{R}(F_u(\frac{\lambda_1}{H_0}, 0)).$$

Furthermore, by employing formula (4.5) derived in [23], we can deduce that

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]^2\rangle}{2\langle l, F_{\lambda u}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]\rangle}.$$

Subsequently,

$$\begin{aligned}\langle l, F_{uu}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]^2\rangle &= \int_\Omega F_{uu}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]^2\varphi_1 \,\mathrm{d}x \\ &= -\frac{\lambda_1}{H_0} N \int_\Omega f(x) H_{uu}(x,0)\varphi_1^3 \,\mathrm{d}x\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}2\langle l, F_{\lambda u}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]\rangle &= 2\int_\Omega F_{\lambda u}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]\varphi_1 \,\mathrm{d}x \\ &= 2H_0 \int_\Omega f(x)\varphi_1^2 \,\mathrm{d}x.\end{aligned} \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$\begin{aligned}\lambda'(0) &= -\frac{\langle l, F_{uu}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]^2\rangle}{2\langle l, F_{\lambda u}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]\rangle} \\ &= -\frac{-\frac{\lambda_1}{H_0} N \int_\Omega f(x) H_{uu}(x,0)\varphi_1^3 \,\mathrm{d}x}{2H_0 \int_\Omega f(x)\varphi_1^2 \,\mathrm{d}x}.\end{aligned}$$

If $H_{uu}(x,0) \equiv 0$ in Ω, using [23, (4.6)], we deduce that

$$\lambda''(0) = -\frac{\langle l, F_{uuu}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]^3\rangle}{3\langle l, F_{\lambda u}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]\rangle},$$

where

$$\begin{aligned}\langle l, F_{uuu}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]^3\rangle &= \int_\Omega F_{uuu}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]^3\varphi_1 \,\mathrm{d}x \\ &= -\frac{\lambda_1}{H_0} N \int_\Omega f(x) H_{uuu}(x,0)\varphi_1^4 \,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}3\langle l, F_{\lambda u}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]\rangle &= 3\int_\Omega F_{\lambda u}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]\varphi_1 \,\mathrm{d}x \\ &= 3H_0 \int_\Omega f(x)\varphi_1^2 \,\mathrm{d}x.\end{aligned}$$

Thus,

$$\begin{aligned}\lambda''(0) &= -\frac{\langle l, F_{uuu}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]^3\rangle}{3\langle l, F_{\lambda u}(\frac{\lambda_1}{H_0}, 0)[\varphi_1]\rangle} \\ &= -\frac{-\frac{\lambda_1}{H_0} N \int_\Omega f(x) H_{uuu}(x,0)\varphi_1^4 \,\mathrm{d}x}{3H_0 \int_\Omega f(x)\varphi_1^2 \,\mathrm{d}x}.\end{aligned} \tag{2.8}$$

We observe that $\lambda'(0) \neq 0$ if $H_{uu}(x,0) \neq 0$ in Ω. This indicates the occurrence of a transcritical bifurcation, characterized by $\lambda'(0) \neq 0$ (see Figure 1). If $H_{uu}(x,0) \equiv 0$ in Ω but $H_{uuu}(x,0) \neq 0$ in Ω, it follows that $\lambda'(0) = 0$ and $\lambda''(0) \neq 0$, implying a pitchfork bifurcation, characterized by $\lambda''(0) \neq 0$. Specifically, if $\lambda''(0) > 0$, a supercritical pitchfork bifurcation occurs. If $\lambda''(0) < 0$, a subcritical pitchfork bifurcation occurs (see Figure 1). Hence, the desired conclusions are obtained. □

3. Stability properties

In this section, we provide the formal stability results near the bifurcation point. The stability properties are obtained via the exchange of stability theorem presented in [12], which is our fundamental tool.

Theorem 3.1 (Crandall-Rabinowitz). *Let X and Y be real Banach spaces, and let $K : X \to Y$ be two bounded linear operators. Assume that $F : \mathbb{R} \times X \to Y$ is C^2 near $(\lambda_*, 0) \in \mathbb{R} \times X$ with $F(\lambda, 0) = 0$ for a sufficiently small $|\lambda_* - \lambda|$. Let $T = F_u(\lambda_*, 0)$. If $\beta = 0$ is a $F_{\lambda u}(\lambda_*, 0)$-simple eigenvalue of operator T and a K-simple eigenvalue of T, then there locally exists a curve $(\lambda(s), u(s)) \in \mathbb{R} \times X$ such that*

$$(\lambda(0), u(0)) = (\lambda_*, 0) \quad \text{and} \quad F(\lambda(s), u(s)) = 0.$$

Moreover, if $F(\lambda, u) = 0$ with $u \neq 0$ and (λ, u) near $(\lambda_*, 0)$, then

$$(\lambda, u) = (\lambda(s), u(s)) \quad \text{for some } s \neq 0.$$

Furthermore, there are eigenvalues $\beta(s)$ and $\beta_{\rm triv}(\lambda) \in \mathbb{R}$ with eigenvectors $\varphi(s)$ and $\varphi_{\rm triv}(\lambda) \in X$ such that

$$F_u(\lambda(s), u(s))\varphi(s) = \beta(s) K\varphi(s),$$
$$F_u(\lambda, 0)\varphi_{\rm triv}(\lambda) = \beta_{\rm triv}(\lambda) K\varphi_{\rm triv}(\lambda),$$

with

$$\beta(0) = \beta_{\rm triv}(\lambda_*) = 0, \quad \varphi(0) = \varphi_{\rm triv}(\lambda_*) = \varphi^*.$$

Each curve is C^1 if F is C^2. Then,

$$\frac{d\beta_{\rm triv}(\lambda)}{d\lambda}|_{\lambda=\lambda_*} \neq 0, \ \lim_{s\to 0, \beta(s)\neq 0} \frac{s\lambda'(s)}{\beta(s)} = -\frac{1}{\beta'_{\rm triv}(\lambda_*)}.$$

By Theorem 3.1, we obtain the following formula, which is convenient to be used.

Proposition 3.2. *Under the assumption of Theorem 3.1, we have that*

$$\lim_{s\to 0, \beta(s)\neq 0} \frac{s\lambda'(s)}{\beta(s)} \frac{l(F_{\lambda u}(\frac{\lambda_1}{H_0}, 0)\varphi_1)}{l(K\varphi_1)} = -1,$$

where $l \in X^*$ satisfies $\mathscr{N}(l) = \mathscr{R}(F_u(\lambda_1/H_0, 0))$, with X^* being the dual space of X. In particular, if $K = F_{\lambda u}(\lambda_1/H_0, 0)$, then

$$\lim_{s\to 0, \beta(s)\neq 0} \frac{s\lambda'(s)}{\beta(s)} = -1,$$

and $\beta'_{\rm triv}(\lambda_1/H_0) = 1$.

Proof. By differentiating $F_u(\lambda, 0)\varphi_{\rm triv}(\lambda) = \beta_{\rm triv}(\lambda)K\varphi_{\rm triv}(\lambda)$, we have

$$F_{\lambda u}(\lambda, 0)\varphi_{\rm triv}(\lambda) + F_u(\lambda, 0)\varphi'_{\rm triv}(\lambda) = \beta_{\rm triv}(\lambda)K\varphi'_{\rm triv}(\lambda) + \beta'_{\rm triv}(\lambda)K\varphi_{\rm triv}(\lambda).$$

Taking $\lambda = \lambda_1/H_0$, we can obtain

$$F_{\lambda u}\Big(\frac{\lambda_1}{H_0}, 0\Big)\varphi_1 + F_u\Big(\frac{\lambda_1}{H_0}, 0\Big)\varphi'_{\rm triv}\Big(\frac{\lambda_1}{H_0}\Big) = \beta'_{\rm triv}\Big(\frac{\lambda_1}{H_0}\Big)K\varphi_1. \tag{3.1}$$

Since $\beta = 0$ is an $F_{\lambda u}(\lambda_1/H_0, 0)$-simple eigenvalue of operator $F_u(\lambda_1/H_0, 0)$, we have

$$F_{\lambda u}\Big(\frac{\lambda_1}{H_0}, 0\Big)\varphi_1 \notin \mathscr{R}\Big(F_u\big(\frac{\lambda_1}{H_0}, 0\big)\Big).$$

By taking l on both sides of equation (3.1) and using the fact that

$$\mathscr{N}(l) = \mathscr{R}(F_u(\frac{\lambda_1}{H_0}, 0)),$$

we obtain that

$$l\Big(F_{\lambda u}(\frac{\lambda_1}{H_0}, 0)\varphi_1\Big) = \beta'_{\rm triv}(\frac{\lambda_1}{H_0}) l(K\varphi_1).$$

It can be deduced that $l(K\varphi_1) \neq 0$; consequently,

$$\beta'_{\rm triv}\Big(\frac{\lambda_1}{H_0}\Big) = \frac{l\big(F_{\lambda u}(\frac{\lambda_1}{H_0}, 0)\varphi_1\big)}{l(K\varphi_1)},$$

which yields the desired formula. □

Before providing the stability result (in the linearized sense) for (1.1) near the bifurcation point, we review the concept of stability. The operator equation $F(\lambda, x) = 0$ represents the equilibrium form of the evolution equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(\lambda, x). \tag{3.2}$$

Suppose that $F(\lambda_0, x_0) = 0$. If all the eigenvalues of $F_x(\lambda_0, x_0)$ are negative, then x_0 is called an asymptotically linearly stable solution of (3.2). On the other hand, if a positive eigenvalue of $F_x(\lambda_0, x_0)$ exists, then x_0 is called an unstable solution of (3.2).

Proof of Theorem 1.2. Let $\mathcal{X}$ be a Banach space. From equations (2.1) and (2.3), it can be inferred that 0 is a simple eigenvalue of $F_u(\lambda_1/H_0, 0) := T$. Let φ_1 represent the eigenfunction corresponding to the eigenvalue 0 with $\|\varphi_1\| = 1$. We denote $T = F_u(\lambda_1/H_0, 0)$ and $K = F_{\lambda u}(\lambda_1/H_0, 0)$.

According to Theorem 3.1, we have

$$K[\varphi_1] = F_\lambda(\lambda, u)\big|_{(\lambda,u)=(\lambda_1/H_0,0)}[\varphi_1].$$

Thus, we obtain

$$K[\varphi_1] = -Nf(x)H(x, 0)\varphi_1.$$

Next, we need to verify whether 0 is a simple eigenvalue of K. Consider the eigenvalue problem

$$K[\varphi_1] = -Nf(x)H(x, 0)\varphi_1 = 0.$$

This implies that φ_1 satisfies $f(x)H(x, 0)\varphi_1 = 0$. For nonzero φ_1, this can only hold if $f(x)H(x, 0) = 0$. However, since $f(x) \neq 0$ and $H(x, 0) \neq 0$, φ_1 must be zero, which contradicts our assumption. This indicates that $K[\varphi_1] \notin \mathscr{R}(T)$. Therefore, 0 is a K-simple eigenvalue of $F_u(\lambda_1/H_0, 0)$.

According to Theorem 3.1, there exist eigenvalues $\beta(s)$ and $\beta_{\rm triv}(\lambda) \in \mathbb{R}$ and eigenvectors $\phi_1(s)$ and $\phi_{\rm triv}(\lambda)$ in the vector space $\mathcal{X}$ such that

$$F_u(\lambda(s), u(s))\phi_1(s) = \beta(s)F_{\lambda u}(\lambda_1/H_0, 0)\phi_1(s),$$
$$F_u(\lambda, 0)\phi_{\rm triv}(\lambda) = \beta_{\rm triv}(\lambda)F_{\lambda u}(\lambda_1/H_0, 0)\phi_{\rm triv}(\lambda)$$

with

$$\beta(0) = \beta_{\rm triv}(\lambda_1/H_0) = 0,\ \phi_1(0) = \phi_{\rm triv}(\lambda_1/H_0) = \varphi_1.$$

Each curve is C^1 if F belongs to C^2; then,

$$\frac{d\beta_{\rm triv}(\lambda)}{d\lambda}\Big|_{\lambda=\lambda_1/H_0} \neq 0, \quad \lim_{s\to 0, \beta(s)\neq 0} \frac{s\lambda'(s)}{\beta(s)} = -\frac{1}{\beta'_{\rm triv}(\lambda_1/H_0)}.$$

By (2.2) and (2.5), we obtain

$$l\Big(F_{\lambda u}\big(\frac{\lambda_1}{H_0},0\big)\varphi_1\Big)=\int_\Omega F_{\lambda u}\big(\frac{\lambda_1}{H_0},0\big)\varphi_1\cdot\varphi_1\,\mathrm{d}x.$$

From (2.4), we have

$$l\Big(F_{\lambda u}\big(\frac{\lambda_1}{H_0},0\big)\varphi_1\Big)=fH_0\int_\Omega\varphi_1^2\,\mathrm{d}x=fH_0\cdot 1=fH_0>0.$$

This suggests that if $\beta(s)>0$ ($\beta(s)<0$), $u(s)$ is (formally) unstable (stable).

By Proposition 3.2, we can deduce that

$$\lim_{s\to 0}\frac{s\lambda'(s)}{\beta(s)}=-1.$$

If $H_{uu}(x,0)\equiv 0$ in Ω but $H_{uuu}(x,0)\neq 0$ in Ω, it has been demonstrated that $\lambda'(0)=0$. We can write

$$\lambda'(s)=s\lambda''(0)+O(s^2).$$

Thus, we see that

$$\lim_{s\to 0}\frac{s^2\lambda''(0)+O(s^3)}{\beta(s)}=-1.$$

Furthermore, it can be observed that

$$\lim_{s\to 0}\frac{\lambda''(0)}{\frac{\beta(s)}{s^2}}=-1.$$

We therefore have

$$\lim_{s\to 0}\frac{\beta(s)}{s^2}=-\lambda''(0). \tag{3.3}$$

If $\lambda''(0)>0$, we can conclude from (3.3) that for small values of $|s|$, $\beta(s)$ is negative. On the other hand, if $\lambda''(0)<0$, we can conclude from (3.3) that for small values of $|s|$, $\beta(s)$ is positive.

Therefore, when $H_{uuu}(x,0)>0$ in Ω, we have $\lambda''(0)>0$ along the nontrivial bifurcating curve passing through $(\lambda_1/H_0,0)$, indicating asymptotic linear stability of the nontrivial solutions. Similarly, when $H_{uuu}(x,0)<0$ in Ω, we have $\lambda''(0)<0$ along the nontrivial bifurcating curve passing through $(\lambda_1/H_0,0)$, indicating the asymptotic linear instability of the nontrivial solutions. □

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