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# EXISTENCE OF NON-GLOBAL SOLUTIONS TO BI-HARMONIC INHOMOGENEOUS NONLINEAR SCHRÖDINGER EQUATIONS WITHOUT GAUGE INVARIANCE

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Abstract. This work shows the existence of non-global mass and energy solutions to inhomogeneous nonlinear bi-harmonic Schrödinger problems without gauge invariance.

## 1. Introduction

This note studies the initial valued problem for the inhomogeneous non-linear fourth-order Schrödinger equation

$$\begin{gathered} \mathrm{i}u_t + \Delta^2 u = \lambda |x|^{-\tau}|u|^p, \\ u(0,\cdot) = \nu u_0, \end{gathered} \tag{1.1}$$

where $u : (t,x) \in \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$ for some integer $N \geq 1$ is the wave function. The source term satisfies $0 \neq \lambda \in \mathbb{C}$, $p > 1$, and $\tau > 0$. The datum satisfies $\nu \in \mathbb{R}$.

The fourth-order Schrödinger problem takes into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with a Kerr non-linearity [13, 14]. It was also considered in [12, 22, 1] to study the stability of solitons in magnetic materials once the effective quasi particle mass becomes infinite.

The inhomogeneous nonlinear Schrödinger equation (1.1) is called non gauge invariant because the source term $\mathcal{N}(u) := |x|^{-\tau}|u|^p$ satisfies

$$\mathcal{N}(e^{i\theta}u) \neq e^{i\theta}\mathcal{N}(u), \quad \theta \in \mathbb{R} - 2\pi\mathbb{Z}. \tag{1.2}$$

Well-posedness issues of the inhomogeneous non-linear Schrödinger equation with a gauge invariant source term

$$\mathrm{i}u_t + \Delta^2 u = \pm|x|^{-\tau}|u|^{p-1}u, \tag{1.3}$$

were investigated by many authors in the previous few years. Indeed, a local theory was developed in [10] using contraction mapping argument via Strichartz inequalities and revisited in [15]. A small data global existence result was proved in [11]. A sharp dichotomy of global versus non-global existence of solutions by using the existence of ground states is given by the author in [19]. The energy scattering in

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the inter-critical regime was first proved by the author [17, 18] for radial data in space dimensions $N \geq 5$ and revisited in [4, 7] for non-radial data and low space dimensions. The existence of non-global solutions was investigated in [5, 6] for the mass-critical regime with negative energy and by the author in [3] for the inter-critical regime. In the energy critical regime, a local theory was developed recently by the author [21, 20], see also [2]. To the author knowledge, there is no work dealing with the inhomogeneous bi-harmonic Shrödinger equation with non-gauge invariance (1.3).

This note aims to present some differences between the gauge invariant problem (1.1) and the non-gauge invariant one (1.3). Indeed, we prove the existence of non global mass-solutions for $1 < p < 1 + \frac{4-\tau}{N}$ and the existence non global energy-solutions for $1 < p < 1 + \frac{2(4-\tau)}{N-4}$ with small data. These results are known to be false for the gauge invariant problem (1.3). To the author knowledge, this work is the first one dealing with the inhomogeneous nonlinear bi-harmonic Schrödinger equation with non-gauge invariant source term (1.1).

The plan of this note is as follows. Section 2 contains the main results and some useful inequalities. Sections 3-4 presents the proof of the main results.

We denote the standard Lebesgue and Sobolev spaces and norms by

$$L^r := L^r(\mathbb{R}^N), \quad H^2 := \{f \in L^2,\, \Delta f \in L^2\},$$

$$\|\cdot\|_r := \|\cdot\|_{L^r}, \quad \|\cdot\| := \|\cdot\|_2, \quad \|\cdot\|_{H^2} := \Big(\|\cdot\|^2 + \|\Delta\cdot\|^2\Big)^{1/2}.$$

We define the ball of $\mathbb{R}^N$ with center at the origin and radius $R > 0$ by $B(R) := \{x \in \mathbb{R}^N,\, |x| < R\}$, and its complement by $B^c(R) := \{x \in \mathbb{R}^N,\, x \notin B(R)\}$. The annulus of $\mathbb{R}^N$ with radii $0 < R' < R$ is $C(R', R) := \{x \in \mathbb{R}^N,\, R' < |x| < R\}$. Also $x^-$ is a real number close to $x$ such that $x > x^-$ and $r' := r/(r-1)$ is the Hölder conjugate of $r > 1$.

## 2. Background and main result

This section contains the main contribution of this note and some useful standard estimates.

### 2.1. **Preliminaries.**

Let us denote the free bi-harmonic Schrödinger kernel by

$$e^{\mathrm{i}t\Delta^2}u := \mathcal{F}^{-1}\big(e^{\mathrm{i}t|\cdot|^4}\mathcal{F}u\big). \tag{2.1}$$

where $\mathcal{F}$ is the Fourrier transform. Thanks to the Duhamel formula, solutions to (1.1) are fix points of the integral operator

$$f(u(t)) := e^{\mathrm{i}t\Delta^2}u_0 - \mathrm{i}\lambda\int_0^t e^{\mathrm{i}(t-s)\Delta^2}\big(|x|^{-\tau}|u|^p\big)\,ds. \tag{2.2}$$

If $u$ resolves (1.1), then so does the family $u_\kappa := \kappa^{\frac{4-\tau}{p-1}}u(\kappa^4\cdot, \kappa\cdot)$, $\kappa > 0$. Moreover, there is only one invariant Sobolev norm under the above dilatation, precisely

$$\|u_\kappa(t)\|_{\dot H^{s_c}} = \|u(\kappa^4 t)\|_{\dot H^{s_c}}, \quad s_c := \frac{N}{2} - \frac{4-\tau}{p-1}.$$

In contrast to (1.3), the mass and energy are not conserved quantities. The above problem (1.1) is said to be

$$\text{mass-sub-critical if } s_c < 0 \Leftrightarrow p < p_c := 1 + \frac{2(4-\tau)}{N}; \tag{2.3}$$

$$\text{energy-sub-critical if } s_c < 2 \Leftrightarrow p < p^c := 1 + \frac{2(4-\tau)}{N-4}. \tag{2.4}$$

In (2.4), we take $p^c = \infty$ if $1 \le N \le 4$. For future convenience, we recall the so-called Strichartz inequalities.

**Definition 2.1.** A couple of real numbers $(q, r)$ is said to be admissible if

$$2 \le r < \frac{2N}{N-4}, \quad 2 \le q, r \le \infty \quad \text{and} \quad N\big(\frac{1}{2} - \frac{1}{r}\big) = \frac{4}{q}.$$

Denote the set of admissible pairs by $\Lambda$. If $I$ is a time slab, one denotes the Strichartz spaces as

$$\Omega(I) := \cap_{(q,r)\in\Lambda} L^q(I, L^r).$$

Now we recall some Strichartz inequalities [16, 9].

**Proposition 2.2.** *Let $N \ge 1$ and $T > 0$. Then,*

1. $\sup_{(q,r)\in\Lambda} \|u\|_{L^q_T(L^r)} \lesssim \|u_0\| + \inf_{(\tilde q,\tilde r)\in\Lambda} \|iu_t + \Delta^2 u\|_{L^{\tilde q'}_T(L^{\tilde r'})}$;
2. $\sup_{(q,r)\in\Lambda} \|\Delta u\|_{L^q_T(L^r)} \lesssim \|\Delta u_0\| + \|iu_t + \Delta^2 u\|_{L^2_T(\dot W^{1,\frac{2N}{2+N}})}$ *for all $N \ge 3$;*
3. $\sup_{(q,r)\in\Lambda} \|u\|_{L^q_T(L^r)} \lesssim \|u_0\|_{\dot H^s} + \inf_{(\tilde q,\tilde r)\in\Lambda} \|iu_t + \Delta^2 u\|_{L^{\tilde q'}_T(L^{\tilde r'})}$.

The existence of $L^2$ and energy solutions to (1.1) follows as in [10].

**Proposition 2.3.** *Let $N \ge 1$.*

1. *If $0 < \tau < \min\{N, 4\}$, $1 < p < p_c$ and $u_0 \in L^2$, then there exist $T := T_{N,\tau,p,\|u_0\|} > 0$ and a unique local solution of (1.1), in the space*
$$C([0,T], L^2) \cap \Omega(0,T).$$
2. *If $N \ge 3$, $0 < \tau < \min\{4, \frac{N}{2}\}$, $\max\{1, \frac{2(1-\tau)}{N}\} < p < p^c$ and $u_0 \in H^2$, there exist $T := T_{N,\tau,p,\|u_0\|_{H^2}} > 0$ and a unique local solution of (1.1), in the space*
$$C([0,T], H^2) \cap_{(q,r)\in\Lambda} L^q_T(W^{2,r}).$$

We recall the so-called weak solution to the Schrödinger problem (1.1).

**Definition 2.4.** For $T > 0$, a function $u \in L^1_{loc}([0,T] \times \mathbb{R}^N)$ is said a weak solution of (1.1) on $[0,T)$ if for each $v \in C_0^\infty([0,T) \times \mathbb{R}^N)$,

$$\int_0^T \int_{\mathbb{R}^N} u\big(-\mathrm{i}\partial_t v + \Delta^2 v\big)\,dx\,dt = \mathrm{i}\nu \int_{\mathbb{R}^N} u_0 v(0,\cdot)\,dx + \lambda \int_0^T \int_{\mathbb{R}^N} |x|^{-\tau}|u|^p v\,dx\,dt. \tag{2.5}$$

From now on, we hide the time variable $t$ for simplicity, showing it only when necessary.

2.2. **Main results.** The first contribution of this note is the non-global well-posedness of (1.1) in $L^2$.

**Theorem 2.5.** *Let $N \ge 1$, $0 < \tau < \min\{N, 4\}$ and $1 < p \le 1 + \frac{4-\tau}{N}$. Then, there is $u_0 \in L^2$ such that the unique maximal solution $u \in C([0,T^+), L^2)$ of (1.1) with $\nu = 1$ is non-global.*

In view of the results stated in the above theorem, some comments are in order.

- The existence of the local solution is given by Proposition 2.3.

- In the case of a gauge invariant inhomogeneous source term, namely (1.3), any mass-sub-critical solution is global. So, the above result makes an essential difference between (1.1) and (1.3).
- The choice $\nu = 1$ is not necessary, indeed, in the proof we can pick $\nu = -1$ with some change in the choice of the datum.
- With standard method, Theorem 2.5 implies that $\lim_{T^+} \|u(t)\| = \infty$.
- For the heat equation, the Fujita exponent $p = 1 + \frac{4-\tau}{N}$ gives the threshold between the small data global existence and blow-up of solutions [8].

The second contribution of this note is the non-global well-posedness of (1.1) in $H^2$.

**Theorem 2.6.** *Let $N \geq 3$ and $\tau < \min\{4, \frac{N}{2}\}$. Take $\lambda = 1$ and $\max\{1, \frac{2(1-\tau)}{N}\} < p < p^c$. There exist $\nu > 0$ and $u_0 \in H^2$ such that the unique maximal solution $u \in C([0, T^+), H^2)$ of (1.1) is non global.*

In view of the above theorem, some comments are in order.

- The existence of the local solution is given by Proposition 2.3.
- in the proof, we see that Theorem 2.6 holds for any $\nu > \nu_0 > 0$. This gives a blow-up result for small datum. This makes a difference with (1.3), where the global existence is known for small data.
- With standard method, Theorem 2.6 implies that $\lim_{T^+} \|\Delta u(t)\| = \infty$.

2.3. **Sketch of the proofs.** The proof of Theorem 2.5 is based on the two next results.

**Proposition 2.7.** *The solution given by Proposition 2.3 is a weak solution to* (1.1).

**Proposition 2.8.** *Letting $1 < p < 1 + \frac{4-\tau}{N}$, there is a certain $0 \neq u_0 \in L^2$ such that if $u$ is a global weak solution to* (1.1)*, and then $u = 0$.*

Indeed, letting $u \in C([0, T^+), L^2)$ be a maximal solution to (1.1). If $T^+ = \infty$, then by Proposition 2.7, $u$ is a global weak solution to (1.1). So, by Proposition 2.8, $u = 0$, which contradicts $u_0 \neq 0$. This completes the proof of Theorem 2.5. The proof of Theorem 2.6 is essentially based on Proposition 2.7 and Lemma 4.1.

## 3. No global mass solutions

In this section, we fix $\nu = 1$ and we prove Theorem 2.5. It is sufficient to establish Propositions 2.7 and 2.8.

3.1. **Weak solutions.** In this sub-section, we prove Proposition 2.7. Let $v \in C_0^\infty([0, T) \times \mathbb{R}^N)$, for some $T > 0$ and $u \in C([0, T], L^2(\mathbb{R}^N)) \cap \Omega(0, T)$ be a solution to (2.2). So, $u \in L^1_{loc}([0, T] \times \mathbb{R}^N)$. By a density argument, we have

$$\int_0^T \int_{\mathbb{R}^N} e^{\mathrm{i}t\Delta^2} u_0(-\mathrm{i}\partial_t v + \Delta^2 v)\, dx\, dt = \mathrm{i} \int_{\mathbb{R}^N} u_0(x) v(0, x)\, dx. \tag{3.1}$$

So, by (2.2), (2.5) and (3.1), it is sufficient to prove that

$$\int_0^T \int_{\mathbb{R}^N} w(-\mathrm{i}\partial_t v + \Delta^2 v)\, dx\, dt = \lambda \int_0^T \int_{\mathbb{R}^N} |x|^{-\tau} |u|^p v\, dx\, dt; \tag{3.2}$$

$$w := -\mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\Delta^2} \big(|x|^{-\tau} |u|^p\big)\, ds. \tag{3.3}$$

With a density argument, we take

$$u_n \in C_0^\infty([0,T)\times\mathbb{R}^N), \quad \lim_n \|u_n - u\|_{\Omega(0,T)} = 0. \tag{3.4}$$

Let also define the sequence

$$w_n := -\mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\Delta^2}\big(|x|^{-\tau}|u_n|^p\big)\,ds. \tag{3.5}$$

Now, by Strichartz and Hölder inequalities and taking account of [10, Lemma 3.1], for some $\alpha_1,\alpha_2>0$, we write

$$\begin{aligned}
&\|w_n - w\|_{L^\infty([0,T),L^2)}\\
&\lesssim \||x|^{-\tau}\big(|u_n|^p - |u|^p\big)\|_{\Omega'(0,T)}\\
&\lesssim \||x|^{-\tau}\big(|u_n|^{p-1} + |u|^{p-1}\big)(u_n - u)\|_{\Omega'(0,T)}\\
&\lesssim \big(T^{\alpha_1}+T^{\alpha_2}\big)\big(\|u_n\|_{\Omega(0,T)}^{p-1} + \|u\|_{\Omega(0,T)}^{p-1}\big)\|u_n - u\|_{\Omega(0,T)}\\
&\to 0, \quad \text{as} \quad n\to\infty.
\end{aligned} \tag{3.6}$$

Since $w_n(0,\cdot)=0$, the first term of the left-hand side of (3.2) reads

$$\begin{aligned}
-\mathrm{i}\int_0^T\int_{\mathbb{R}^N} w\partial_t v\,dx\,dt &= -\mathrm{i}\lim_n \int_0^T\int_{\mathbb{R}^N} w_n\partial_t v\,dx\,dt\\
&= \mathrm{i}\lim_n \int_0^T\int_{\mathbb{R}^N} \partial_t w_n v\,dx\,dt.
\end{aligned} \tag{3.7}$$

Moreover, by Strichartz and Hölder inequalities and taking account of [10, Lemma 3.2], we write for some $\beta_1,\beta_2>0$,

$$\begin{aligned}
\|\Delta w_n\|_{L^\infty([0,T),L^2)} &\lesssim \||x|^{-\tau}|u_n|^{p-1}|\nabla u_n| + |x|^{-\tau-1}|u_n|^p\|_{L^1_T(L^{\frac{2N}{2+N}})}\\
&\lesssim \big(T^{\beta_1}+T^{\beta_2}\big)\big(\|\Delta u_n\|_{\Omega(0,T)} + \|u_n\|_{\Omega(0,T)}\big)^p.
\end{aligned} \tag{3.8}$$

So, $w_n \in C([0,T),H^2)$ and we have the equality in $C([0,T),H^{-2})$,

$$\mathrm{i}\partial_t w_n = -\Delta^2 w_n + \lambda|x|^{-\tau}|u_n|^p. \tag{3.9}$$

Moreover, since $0<\tau<N$, by Hölder inequality for $\operatorname{supp}(u_n(0,\cdot))\subset B(R_n)$,

$$\begin{aligned}
\|\partial_t w_n(t)\| &= \big\|\partial_t \int_0^t e^{\mathrm{i}s\Delta^2}\big(|x|^{-\tau}|u_n(t-s)|^p\big)\,ds\big\|\\
&\le \|e^{\mathrm{i}t\Delta^2}\big(|x|^{-\tau}|u_n(0)|^p\big)\| + \|\int_0^t e^{\mathrm{i}s\Delta^2}\big(|x|^{-\tau}\partial_t|u_n|^p(t-s)\big)\,ds\|\\
&\lesssim \||x|^{-\tau}u_n^p(0)\| + \||x|^{-\tau}\partial_t|u_n|^p\|_{\Omega'(0,t)}\\
&\lesssim \||x|^{-\tau}\|_{L^{(\frac{N}{\tau})^-}(B(R_n))}\|u_n(0)\|_\infty^p R_n^{\frac{N}{r'}}\\
&\quad + \big(T^{\alpha_1}+T^{\alpha_2}\big)\|u_n\|_{\Omega(0,T)}^{p-1}\|\partial_t u_n\|_{\Omega(0,t)}\\
&\lesssim C_n + \big(T^{\alpha_1}+T^{\alpha_2}\big)\|u_n\|_{\Omega(0,T)}^{p-1}\|\partial_t u_n\|_{\Omega(0,t)}.
\end{aligned} \tag{3.10}$$

Thus, $u_n \in C_0^\infty([0,T)\times\mathbb{R}^N)$ implies that $w_n \in C^1([0,T),L^2)$ and (3.9) gives $w_n \in C([0,T),H^2)$. So, (3.7) and (3.9) imply that

$$\begin{aligned} -\mathrm{i}\int_0^T\int_{\mathbb{R}^N} w\partial_t v\,dx\,dt &= \lim_n \int_0^T\int_{\mathbb{R}^N}\big(-\Delta^2 w_n + \lambda|x|^{-\tau}|u_n|^p\big)v\,dx\,dt \\ &= -\lim_n \int_0^T\int_{\mathbb{R}^N} w_n\Delta^2 v\,dx\,dt \\ &\quad + \lambda\lim_n\int_0^T\int_{\mathbb{R}^N}|x|^{-\tau}|u_n|^p v\,dx\,dt. \end{aligned} \tag{3.11}$$

Furthermore, by Hölder inequality via (3.6), we have

$$\Big|\int_0^T\int_{\mathbb{R}^N}(w_n-w)\Delta^2 v\,dx\,dt\Big| \le T\|w_n-w\|_{L_T^\infty(L^2)}\|\Delta^2 v\|_{L_T^\infty(L^2)} \to 0. \tag{3.12}$$

Also, by Hölder and Strichartz inequalities and arguing as previously, we have

$$\begin{aligned} &\Big|\int_0^T |x|^{-\tau}\big(|u_n|^p-|u|^p\big)v\,dx\,dt\Big| \\ &\le \||x|^{-\tau}\big(|u_n|^p-|u|^p\big)\|_{\Omega'(0,T)}\|v\|_{\Omega(0,T)} \\ &\lesssim \||x|^{-\tau}\big(|u_n|^p-|u|^p\big)\|_{\Omega'(0,T)}\|v\|_{\Omega(0,T)} \\ &\lesssim \big(T^{\alpha_1}+T^{\alpha_2}\big)\big(\|u_n\|^{p-1}_{\Omega(0,T)}+\|u\|^{p-1}_{\Omega(0,T)}\big)\|u_n-u\|_{\Omega(0,T)}\|v\|_{\Omega(0,T)} \to 0. \end{aligned} \tag{3.13}$$

So, by (3.11), (3.12) and (3.13), we obtain

$$-\mathrm{i}\int_0^T\int_{\mathbb{R}^N} w\partial_t v\,dx\,dt = -\int_0^T\int_{\mathbb{R}^N} w\Delta^2 v\,dx\,dt + \lambda\int_0^T\int_{\mathbb{R}^N}|x|^{-\tau}|u|^p v\,dx\,dt. \tag{3.14}$$

Then by (3.2) and (3.14) the proof is complete.

3.2. **Vanishing solutions.** This sub-section proves Proposition 2.8. Let us define some smooth functions:

$$\eta \in C_0^\infty([0,\infty)),\quad 0\le\eta\le 1,\quad \eta :\equiv \begin{cases} 1, & \text{on } [0,\frac12]; \\ 0, & \text{on } [1,\infty). \end{cases} \tag{3.15}$$

Also we define $\phi \in C_0^\infty([0,\infty))$ such that

$$0\le\phi\le 1,\quad \sup_{\{x\in B(1)\}}\frac{|\nabla\phi(x)|^2}{\phi(x)} \lesssim 1,\quad \phi :\equiv \begin{cases} 1, & \text{on } B(\frac12); \\ 0, & \text{on } B^c(1). \end{cases} \tag{3.16}$$

For $R>0$, we define the cut-off functions

$$\eta_R :\equiv \eta\big(\frac{\cdot}{R^4}\big),\quad \phi_R :\equiv \phi\big(\frac{\cdot}{R}\big),\quad \psi_R :\equiv \eta_R\phi_R. \tag{3.17}$$

Assume that

$$\Re(u_0)=0,\quad u_0\in L^1 \quad\text{and}\quad \Re(\lambda)\int_{\mathbb{R}^N}\Im(u_0)\,dx<0. \tag{3.18}$$

Without loss of generality, we assume that

$$\Re(\lambda)>0 \quad\text{and}\quad \int_{\mathbb{R}^N}\Im(u_0)\,dx<0. \tag{3.19}$$

For $q > \max\{1, 3 - \frac{3}{p}\}$, let us denote

$$J_R := J_R(p,q) := \Re\Big(\lambda \int_0^{R^4} \int_{B(R)} |x|^{-\tau} |u|^p \psi_R^q \, dx \, dt\Big). \tag{3.20}$$

Since $u$ is a global weak solution to (1.1), by (2.5), we have

$$\begin{aligned} &\int_0^{R^4} \int_{\mathbb{R}^N} u\big( -\mathrm{i}\partial_t(\psi_R^q) + \Delta^2(\psi_R^q)\big) \, dx \, dt \\ &= \mathrm{i} \int_{\mathbb{R}^N} u_0 \psi_R^q(0,\cdot) \, dx + \lambda \int_0^{R^4} \int_{\mathbb{R}^N} |x|^{-\tau} |u|^p \psi_R^q \, dx \, dt. \end{aligned} \tag{3.21}$$

So, (3.21) via (3.17), gives

$$\begin{aligned} J_R &= \Im\big( \int_{B(R)} u_0 \psi_R^q(0,\cdot) \, dx\big) + \Im\big( \int_0^{R^4} \int_{B(R)} u \partial_t(\psi_R^q) \, dx \, dt\big) \\ &\quad + \Re\Big( \int_0^{R^4} \int_{B(R)} u \Delta^2(\psi_R^q) \, dx \, dt\Big) \\ &= \int_{B(R)} \Im u_0 \psi_R^q(0,\cdot) \, dx + \int_0^{R^4} \int_{B(R)} \Im u \partial_t(\psi_R^q) \, dx \, dt \\ &\quad + \int_0^{R^4} \int_{B(R)} \Re u \Delta^2(\psi_R^q) \, dx \, dt. \end{aligned} \tag{3.22}$$

Now, by (3.19), for $R >> 1$, we have

$$\begin{aligned} J_R &< \int_0^{R^4} \int_{B(R)} \Im u \partial_t(\psi_R^q) \, dx \, dt + \int_0^{R^4} \int_{B(R)} \Re u \Delta^2(\psi_R^q) \, dx \, dt \\ &\lesssim \int_0^{R^4} \int_{B(R)} |u| \psi_R^{q-1} |\partial_t \psi_R| \, dx \, dt + \int_0^{R^4} \int_{B(R)} |u| |\Delta^2(\psi_R^q)| \, dx \, dt \\ &:= J_R^1 + J_R^2. \end{aligned} \tag{3.23}$$

Using (3.15) and Hölder inequality, because $q \geq p'$, we write

$$\begin{aligned} J_R^1 &\lesssim R^{-4} \int_{\frac{R^4}{2}}^{R^4} \int_{B(R)} |u| \psi_R^{q-1} \, dx \, dt \\ &\lesssim R^{\frac{\tau}{p}-4} \int_{\frac{R^4}{2}}^{R^4} \int_{B(R)} |x|^{-\tau/p} |u| \psi_R^{\frac{q}{p}} \, dx \, dt \\ &\lesssim R^{\frac{\tau}{p}-4} \Big( \int_{\frac{R^4}{2}}^{R^4} \int_{B(R)} |x|^{-\tau} |u|^p \psi_R^q \, dx \, dt\Big)^{1/p} \big| [\frac{R^4}{2}, R^4] \times B(R)\big|^{1-\frac{1}{p}} \\ &\lesssim R^{(4+N)(1-\frac{1}{p}) + \frac{\tau}{p} - 4} \Big( \int_{\frac{R^4}{2}}^{R^4} \int_{B(R)} |x|^{-\tau} |u|^p \psi_R^q \, dx \, dt\Big)^{1/p}. \end{aligned} \tag{3.24}$$

Furthermore, with (3.16) and (3.17), we obtain

$$
\begin{aligned}
&J_R^2\\
&= \int_0^{R^4}\int_{B(R)} |u||\Delta^2(\psi_R^q)|\,dx\,dt\\
&\lesssim R^{-4}\int_0^{R^4}|\eta_R|^q\int_{B(R)}|u|\Big(\phi_R^{q-1}|(\Delta^2\phi)(\frac{x}{R})| + \phi_R^{q-2}|\nabla\phi(\frac{x}{R})||\nabla(\Delta\phi)(\frac{x}{R})| \\
&\quad + \phi_R^{q-2}|(\Delta\phi)(\frac{x}{R})|^2 + \phi_R^{q-3}|(\Delta\phi)(\frac{x}{R})||\nabla\phi(\frac{x}{R})|^2 + \phi_R^{q-4}|\nabla\phi(\frac{x}{R})|^4\Big)\,dx\,dt\\
&\lesssim R^{-4}\int_0^{R^4}\int_{C(\frac{R}{2},R)}|u|\psi_R^{q-3}\,dx\,dt.
\end{aligned} \tag{3.25}
$$

Since $qp' \geq 3$, by (3.25) via Hölder inequality and arguing as in (3.24), it follows that

$$
J_R^2 \lesssim R^{(4+N)(1-\frac{1}{p})+\frac{\tau}{p}-4}\Big(\int_0^{R^4}\int_{C(\frac{R}{2},R)}|x|^{-\tau}|u|^p\psi_R^q\,dx\,dt\Big)^{1/p}. \tag{3.26}
$$

Now, by (3.23), (3.24) and (3.26), we obtain

$$
J_R \lesssim R^{(4+N)(1-\frac{1}{p})+\frac{\tau}{p}-4} J_R^{1/p}. \tag{3.27}
$$

Since $(4+N)(1-\frac{1}{p})+\frac{\tau}{p}-4 \leq 0$ because $p \leq 1+\frac{4-\tau}{N}$, then (3.27) implies that

$$
1 \gtrsim \lim_{R\to\infty} J_R = \Re\Big(\lambda\int_0^\infty\int_{\mathbb{R}^N}|x|^{-\tau}|u|^p\,dx\,dt\Big). \tag{3.28}
$$

Thus, $|x|^{-\tau/p}u \in L^p([0,\infty)\times\mathbb{R}^N)$ and so

$$
\lim_{R\to\infty} J_R^1 = \lim_{R\to\infty} J_R^2 = 0. \tag{3.29}
$$

So, (3.23) via (3.29) gives

$$
\lim_{R\to\infty} J_R = 0. \tag{3.30}
$$

Finally, (3.30) implies that $u = 0$ and the proof is complete.

## 4. No global energy solutions

This section proves Theorem 2.6. The proof is reduced to the next Lemma via Proposition 2.7.

**Lemma 4.1.** *Let $p > 1$, $\nu > 0$, $0 < \tau < \min\{4, N\}$ and an integer $n \geq 1 + 3p'$. Take $u_0 \in L^1_{loc}(\mathbb{R}^N)$ such that $\Re(u_0) = 0$ and $u$ be a weak solution to* (1.1) *on $[0, T_\nu^+)$. Then, there is $C > 0$ such that for any $0 < R < (T_\nu^+)^{1/4}$,*

$$
\mathrm{i}\nu\int_{\mathbb{R}^N} u_0\phi_R^n\,dx \leq CR^{N-\frac{4-\tau}{p-1}}. \tag{4.1}
$$

*Moreover, if for some $0 < b < \min\{N, \frac{4-\tau}{p-1}\}$, it holds*

$$
\Im(u_0) \leq -|x|^{-b}\chi_{B(1)}, \tag{4.2}
$$

*then, for any* $0 < R < (T_\nu^+)^{1/4}$,

$$\nu \le C \frac{R^{-\left(\frac{4-\tau}{p-1}-b\right)}}{\int_{B(\frac{1}{R})} |x|^{-b}\phi^n \, dx}. \tag{4.3}$$

*Furthermore, there is a* $\nu_0 > 0$ *such that*

$$T_\nu^+ \le C_{N,p,\tau,b}\nu^{-\frac{4}{\frac{4-\tau}{p-1}-b}}, \quad \forall \nu > \nu_0. \tag{4.4}$$

*Proof.* For some integer number $n \ge 1 + 3p'$, let us define

$$I_n := I_n(R) := \int_0^{R^4} \int_{B(R)} |x|^{-\tau}|u|^p \psi_R^n \, dx \, dt, \tag{4.5}$$

$$J_n := J_n(R) := \int_{B(R)} u_0 \phi_R^n \, dx. \tag{4.6}$$

Taking account of (2.5), we write

$$\begin{aligned}
&\Re\big(\mathrm{i}\nu J_n + I_n\big) \\
&= \Re\Big( - \mathrm{i} \int_0^{R^4} \int_{B(R)} u \partial_t \big(\psi_R^n\big) \, dx \, dt + \int_0^{R^4} \int_{B(R)} u \Delta^2 \big(\psi_R^n\big) \, dx \, dt\Big) \\
&= \int_0^{R^4} \int_{B(R)} \Im u \partial_t \big(\psi_R^n\big) \, dx \, dt + \int_0^{R^4} \int_{B(R)} \Re u \Delta^2 \big(\psi_R^n\big) \, dx \, dt \\
&:= (A) + (B).
\end{aligned} \tag{4.7}$$

By (3.17) and $n \ge p'$, we have

$$\begin{aligned}
(A) &\lesssim R^{-4} \int_0^{R^4} \int_{B(R)} |u|\psi_R^{n-1} \, dx \, dt \\
&\lesssim R^{-4} \int_0^{R^4} \int_{B(R)} |u|\psi_R^{n/p} \, dx \, dt \\
&\lesssim R^{\frac{\tau}{p}-4} \int_0^{R^4} \int_{B(R)} |x|^{-\tau/p}|u|\psi_R^{n/p} \, dx \, dt.
\end{aligned} \tag{4.8}$$

So, with Hölder inequality via (4.8), we write

$$(A) \lesssim R^{\frac{\tau}{p}-4} I_n^{1/p} |[0, R^4] \times B(R)|^{1-\frac{1}{p}} \lesssim R^{(4+N)(1-\frac{1}{p})+\frac{\tau}{p}-4} I_n^{1/p}. \tag{4.9}$$

By (3.15), (3.16) and (3.17), we have

$$\begin{aligned}
&(B) \\
&\le \int_0^{R^4} \int_{B(R)} |u||\Delta^2(\psi_R^n)| \, dx \, dt \\
&\lesssim R^{-4} \int_0^{R^4} |\eta_R|^n \int_{B(R)} |u| \Big(\phi_R^{n-1}|(\Delta^2\phi)(\frac{x}{R})| + \phi_R^{n-2}|\nabla\phi(\frac{x}{R})||\nabla(\Delta\phi)(\frac{x}{R})| \\
&+ \phi_R^{n-2}|(\Delta\phi)(\frac{x}{R})|^2 + \phi_R^{n-3}|(\Delta\phi)(\frac{x}{R})||\nabla\phi(\frac{x}{R})|^2 + \phi_R^{n-4}|\nabla\phi(\frac{x}{R})|^4\Big) \, dx \, dt \\
&\lesssim R^{-4} \int_0^{R^4} \int_{C(\frac{R}{2}, R)} |u|\psi_R^{n-3} \, dx \, dt.
\end{aligned} \tag{4.10}$$

Since $n \geq 3p'$, by (4.10) via Hölder inequality and arguing as in (4.9), it follows that

$$(B) \lesssim R^{(4+N)(1-\frac{1}{p})+\frac{\tau}{p}-4}\Big(\int_0^{R^4}\int_{C(\frac{R}{2},R)}|x|^{-\tau}|u|^p\psi_R^n\,dx\,dt\Big)^{1/p}. \tag{4.11}$$

So, (4.11), (4.9) and (4.7), via Young inequality give

$$\begin{aligned} -\nu\Im J_n &= \Re\big(\mathrm{i}\nu J_n\big) \\ &\leq CR^{(4+N)(1-\frac{1}{p})+\frac{\tau}{p}-4}I_n^{1/p} - I_n \\ &\leq \frac{1}{p'}\big(CR^{(4+N)(1-\frac{1}{p})+\frac{\tau}{p}-4}p^{-\frac{1}{p}}\big)^{p'} + I_n - I_n \\ &\lesssim R^{4+N+\frac{\tau}{p-1}-4p'}. \end{aligned} \tag{4.12}$$

This proves (4.1). By (4.6) and (4.2), we obtain

$$\begin{aligned} -\Im J_n &= -\int_{\mathbb{R}^N}\Im u_0\phi_R^n\,dx \\ &= -R^N\int_{\mathbb{R}^N}\Im u_0(Rx)\phi^n\,dx \\ &\geq R^{N-b}\int_{B(\frac{1}{R})}|x|^{-b}\phi^n\,dx. \end{aligned} \tag{4.13}$$

Thus, (4.1) and (4.13) give

$$\nu R^{N-b}\int_{B(\frac{1}{R})}|x|^{-b}\phi^n\,dx \lesssim -\nu\Im J_n = \mathrm{i}\nu\int_{\mathbb{R}^N}u_0\phi_R^n\,dx \leq CR^{4+N+\frac{\tau}{p-1}-4p'}. \tag{4.14}$$

This proves (4.3). Now, there is a $\nu_0 > 0$ such that for any $\nu > \nu_0$, we have $T_\nu^+ \leq 16$. Indeed, otherwise we take $R = 2$ in (4.3) and we obtain because $b < N$, the contradiction

$$\begin{aligned} \nu &\leq C\frac{2^{4+b+\frac{\tau}{p-1}-4p'}}{\int_{B(\frac{1}{2})}|x|^{-b}\phi^n\,dx} \\ &= C\frac{2^{4+b+\frac{\tau}{p-1}-4p'}}{\int_{B(\frac{1}{2})}|x|^{-b}\,dx} \\ &\leq C2^{4+b+\frac{\tau}{p-1}-4p'} := \nu_0. \end{aligned} \tag{4.15}$$

So, taking $\nu > \nu_0$ and $0 < R < (T_\nu^+)^{1/4} \leq 2$, we write by (4.3),

$$\begin{aligned} \nu &\leq C\frac{R^{4+b+\frac{\tau}{p-1}-4p'}}{\int_{B(\frac{1}{R})}|x|^{-b}\phi^n\,dx} \\ &= C\frac{R^{4+b+\frac{\tau}{p-1}-4p'}}{\int_{B(\frac{1}{2})}|x|^{-b}\,dx} \\ &\lesssim R^{4+b+\frac{\tau}{p-1}-4p'}. \end{aligned} \tag{4.16}$$

Now, since $b < \frac{4-\tau}{p-1}$, (4.16) gives $4 + b + \frac{\tau}{p-1} - 4p' < 0$ and so

$$T_\nu^+ \lesssim \nu^{-\frac{4}{4p'-4-b-\frac{\tau}{p-1}}} = \nu^{-\frac{4}{\frac{4-\tau}{p-1}-b}}. \tag{4.17}$$

This establishes (4.4) and completes the proof. □

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