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ABEL QUADRATIC DIFFERENTIAL SYSTEMS OF SECOND KIND

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Abstract. The Abel differential equations of second kind, named after Niels Henrik Abel, are a class of ordinary differential equations studied by many authors. Here we consider the Abel quadratic polynomial differential equations of second kind denoting this class by QS_{Ab}. Firstly we split the whole family of non-degenerate quadratic systems in four subfamilies according to the number of infinite singularities. Secondly for each one of these four subfamilies we determine necessary and sufficient affine invariant conditions for a quadratic system in this subfamily to belong to the class QS_{Ab}. Thirdly we classify all the phase portraits in the Poincaré disc of the systems in QS_{Ab} in the case when they have at infinity either one triple singularity (21 phase portraits) or an infinite number of singularities (9 phase portraits). Moreover we determine the affine invariant criteria for the realization of each one of the 30 topologically distinct phase portraits.

1. Introduction and statement of main results

We consider the class of real quadratic polynomial differential systems

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(\tilde{a},x,y), \\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(\tilde{a},x,y) \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} p_0 &= a, \quad p_1(x,y) = cx + dy, \quad p_2(x,y) = gx^2 + 2hxy + ky^2, \\ q_0 &= b, \quad q_1(x,y) = ex + fy, \quad q_2(x,y) = lx^2 + 2mxy + ny^2 \end{aligned}$$

and with $\max(\deg(p), \deg(q)) = 2$. Here the dot denotes derivative with respect to an independent variable t, usually called the time. We denote by $\tilde{a} = (a, c, d, g, h, k, b, e, f, l, m, n)$ the 12-tuple of the coefficients of systems (1.1), and by QS the class of all real quadratic polynomial differential systems, sometimes are simply called *quadratic systems*.

There are more than one thousand papers published on QS (see for instance [15]). The main difficulty of studying QS comes from the fact that they depend on twelve parameters. However modulo the affine group action and time rescaling

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this family depends on five parameters, still a large number. So people began by studying subclasses of QS depending on at most three parameters.

For a nice extensive survey on quadratic differential systems, and an extended bibliography on them, the reader is advised to consult [5, Chapter 2].

In this article we study the Abel differential equations of second kind which are of the form

$$y\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x), \tag{1.2}$$

where $A(x)$, $B(x)$ and $C(x)$ are rational functions over $\mathbb{R}$. The above equations are equivalent to the differential systems

$$\dot{x} = d(x)y, \quad \dot{y} = a(x)y^2 + b(x)y + c(x),$$

where $A(x) = a(x)/d(x)$, $B(x) = b(x)/d(x)$ and $C(x) = c(x)/d(x)$.

In this article we are interested in studying the *Abel quadratic polynomial differential systems*, i.e. differential systems of the form

$$\dot{x} = (d_0+d_1x)y \equiv \widetilde{P}(x,y), \quad \dot{y} = a_0y^2+(b_0+b_1x)y+c_0+c_1x+c_2x^2 \equiv \widetilde{Q}(x,y), \tag{1.3}$$

coming from the Abel differential equation of second kind (1.2).

Definition 1.1. We say that a non-degenerate quadratic system (1.1) is of Abel type if there exists an affine transformation which brings this system to the form (1.3). We denote the class of quadratic systems of Abel type by QS_{Ab} .

Some subclasses of QS_{Ab} have already been studied. The family of systems (1.3) with $a_0 = 0$ and with $\mathbb{Z}_2$-symmetries is considered in [13]. The family of systems (1.3) with $d_1 = 0$ and $a_0 \neq 0$ is analyzed in [10]. Finally, in [11] the family of systems (1.3) with $a_0 \neq 0$ and having a symmetry with respect to an axis or with respect to the origin is considered.

The goal of this article is firstly to determine necessary and sufficient conditions in terms of affine invariant polynomials for an arbitrary quadratic system to be of Abel type. Secondly, we classify topologically all the phase portraits in the Poincaré disc of the systems in QS_{Ab} in the case when they have at infinity either one triple singularity (when all three infinite singularities coalesce), or an infinite number of singularities. Moreover we want to determine the affine invariant criteria for the realization of each one of the 30 topologically distinct phase portraits.

The affine invariant polynomials which appear in the statement of the next theorem are defined in Section 2. Our main result is the following one.

Theorem 1.2. *A non-degenerate quadratic system* (1.1) *(i.e.* $\sum_{i=0}^{4}\mu_i^2 \neq 0$*) belongs to the class* QS_{Ab} *of Abel quadratic systems if and only if* $B_1 = 0$ *and one of the following conditions is satisfied:*

- (A) *If* $\eta > 0$ *then either*
  - (A_1) $\theta \neq 0$, *or*
  - (A_2) $\theta = 0$, $\widetilde{N} \neq 0$, $H_7 \neq 0$, *or*
  - (A_3) $\theta = 0$, $\widetilde{N} \neq 0$, $H_7 = 0$, $B_2 = 0$, *or*
  - (A_4) $\theta = 0$, $\widetilde{N} = 0$, $\theta_3 \neq 0$, *or*
  - (A_5) $\theta = 0$, $\widetilde{N} = 0$, $\theta_3 = 0$, $B_2 = 0$, $\theta_4 \neq 0$, *or*
  - (A_6) $\theta = 0$, $\widetilde{N} = 0$, $\theta_3 = 0$, $B_2 = 0$, $\theta_4 = 0$, $B_3 = 0$.
- (B) *If* $\eta < 0$ *then either*
  - (B_1) $\theta \neq 0$, $B_2 \neq 0$, *or*

(B$_2$) $\theta \neq 0$, $B_2 = 0$, $B_3 = 0$, *or*
(B$_3$) $\theta = 0$, $\widetilde{N} \neq 0$, $H_7 \neq 0$, $B_2 \neq 0$, *or*
(B$_4$) $\theta = 0$, $\widetilde{N} \neq 0$, $H_7 \neq 0$, $B_2 = 0$, $B_3 = 0$, *or*
(B$_5$) $\theta = 0$, $\widetilde{N} = 0$, $B_2 \neq 0$, *or*
(B$_6$) $\theta = 0$, $\widetilde{N} = 0$, $B_2 = 0$, $B_3 = 0$.

(C) *If* $\eta = 0$ *and* $\widetilde{M} \neq 0$ *then either*
(C$_1$) $\theta \neq 0$, *or*
(C$_2$) $\theta = 0$, $\mu_0 \neq 0$, $H_7 \neq 0$, *or*
(C$_3$) $\theta = 0$, $\mu_0 \neq 0$, $H_7 = 0$, $B_2 = 0$, *or*
(C$_4$) $\theta = 0$, $\mu_0 = 0$, $\widetilde{N} \neq 0$, $H_7 \neq 0$, *or*
(C$_5$) $\theta = 0$, $\mu_0 = 0$, $\widetilde{N} \neq 0$, $H_7 = 0$, $B_3 = 0$, *or*
(C$_6$) $\theta = 0$, $\mu_0 = 0$, $\widetilde{N} = 0$, $\widetilde{K} \neq 0$, $\theta_3 \neq 0$, *or*
(C$_7$) $\theta = 0$, $\mu_0 = 0$, $\widetilde{N} = 0$, $\widetilde{K} \neq 0$, $\theta_3 = 0$, $B_3 = 0$.

(D) *If* $\eta = 0$ *and* $\widetilde{M} = 0$ *then either*
(D$_1$) $C_2 \neq 0$, $\theta \neq 0$, *or*
(D$_2$) $C_2 \neq 0$, $\theta = 0$, $\widetilde{N} = 0$, $B_2 \neq 0$, *or*
(D$_3$) $C_2 = 0$, $H_{10} \neq 0$, *or*
(D$_4$) $C_2 = 0$, $H_{10} = 0$, $H_{12} \neq 0$.

The phase portraits of the subcase (D) will be obtained in Theorems 4.1 and 4.11 in Section 4.4. We are able to find all the phase portraits of subcase (D) because they are the less generic ones, and use less parameters. The other subcases are more generic and much more difficult to study. We have 30 phase portraits, none with limit cycles but 10 with graphics surrounding a focus or a center.

2. Main invariant polynomials associated with the class QS_{Ab}

Consider quadratic systems of the form (1.1). It is known that on the set QS acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane (cf. [17]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on QS . We can identify the set QS of systems (1.1) with a subset of $\mathbb{R}^{12}$ via the map QS $\longrightarrow \mathbb{R}^{12}$ which associates to each system (1.1) the 12–tuple $\tilde{a} = (a,c,d,g,h,k,b,e,f,l,m,n)$ of its coefficients. We associate to this group action polynomials in x, y and parameters which behave well with respect to this action, the GL–comitants (GL–invariants), the T–comitants (affine invariants) and the CT–comitants. For their definitions as well as their detailed constructions we refer the reader to the paper [17] (see also [5]).

Next we define the following 41 invariant polynomials associated with the class QS_{Ab}:

$$\Big\{\mu_0,\dots,\mu_4,\, \mathbf{D},\, \mathbf{P},\, \mathbf{R},\, \mathbf{S},\, \mathbf{T},\, \mathbf{U},\, \mathcal{T}_1,\dots,\mathcal{T}_4,\, \mathcal{F},\, \mathcal{F}_1,\dots,\mathcal{F}_4,\, \mathcal{H},\, \mathcal{B},\, \mathcal{B}_1,\, \mathcal{B}_2,\, \sigma,\\ \eta,\, \widetilde{M},\, C_2,\, \theta,\, \theta_3,\, \theta_4,\, \widetilde{K},\, \widetilde{N},\, H_7,\, H_9,\, H_{10},\, H_{11},\, H_{12},\, E_1,\, U_1,\, U_2\Big\}. \tag{2.1}$$

According to [5] (see also [7]) we apply the differential operator $\mathcal{L} = x\cdot\mathbf{L}_2 - y\cdot\mathbf{L}_1$ acting on $\mathbb{R}[\tilde{a},x,y]$ with

$$\begin{aligned}\mathbf{L}_1 =& 2a\frac{\partial}{\partial c} + c\frac{\partial}{\partial g} + \frac{1}{2}d\frac{\partial}{\partial h} + 2b\frac{\partial}{\partial e} + e\frac{\partial}{\partial l} + \frac{1}{2}f\frac{\partial}{\partial m},\\ \mathbf{L}_2 =& 2a\frac{\partial}{\partial d} + d\frac{\partial}{\partial k} + \frac{1}{2}c\frac{\partial}{\partial h} + 2b\frac{\partial}{\partial f} + f\frac{\partial}{\partial n} + \frac{1}{2}e\frac{\partial}{\partial m},\end{aligned}$$

to construct several invariant polynomials from the set. More precisely using this operator and the affine invariant $\mu_0 = \operatorname{Res}_x\big(p_2(\tilde{a},x,y), q_2(\tilde{a},x,y)\big)/y^4$ we construct the following polynomials

$$\mu_i(\tilde{a},x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0), \quad i=1,\dots,4, \text{ where } \mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0)).$$

Using these invariant polynomials we define some new ones, which according to [5] are responsible for the number and multiplicities of the finite singular points of (1.1):

$$\begin{aligned}\mathbf{D} &= \big[3\big((\mu_3,\mu_3)^{(2)},\, \mu_2\big)^{(2)} - \big(6\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\ \mu_4\big)^{(4)}\big]/48,\\ \mathbf{P} &= 12\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\\ \mathbf{R} &= 3\mu_1^2 - 8\mu_0\mu_2,\\ \mathbf{S} &= \mathbf{R}^2 - 16\mu_0^2\mathbf{P},\\ \mathbf{T} &= 18\mu_0^2(3\mu_3^2 - 8\mu_2\mu_4) + 2\mu_0(2\mu_2^3 - 9\mu_1\mu_2\mu_3 + 27\mu_1^2\mu_4) - \mathbf{PR},\\ \mathbf{U} &= \mu_3^2 - 4\mu_2\mu_4.\end{aligned}$$

In what follows we also need the so-called *transvectant of order k* (see [12, 14] of two polynomials $f,\, g \in \mathbb{R}[\tilde{a},x,y]$

$$(f,g)^{(k)} = \sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h\partial y^{k-h}}.$$

Following [22] we denote by $\sigma(\tilde{a},x,y) = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} = \sigma_0(\tilde{a}) + \sigma_1(\tilde{a},x,y)$ and we observe that the polynomial $\sigma(\tilde{a},x,y) \in \mathbb{R}[x,y]$ is an affine comitant of systems (1.1).

Next we construct the elements $\mathcal{T}_1,\dots,\mathcal{T}_4$ of the set (2.1) which are responsible for the number of the vanishing traces corresponding to the finite singularities of systems (1.1). For this we define a polynomial (which we call *trace polynomial*) as follows.

Definition 2.1 ([22])**.** We call *trace polynomial* $\mathfrak{T}(w)$ over the ring $\mathbb{R}[\tilde{a}]$ the polynomial defined as follows

$$\mathfrak{T}(w) = \sum_{i=0}^{4}\frac{1}{(i!)^2}\Big(\sigma_1^i,\ \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0)\Big)^{(i)} w^{4-i} = \sum_{i=0}^{4}\mathcal{G}_i(\tilde{a})\, w^{4-i},$$

where the coefficients $\mathcal{G}_i(\tilde{a}) = \dfrac{1}{(i!)^2}(\sigma_1^i,\ \mu_i)^{(i)} \in \mathbb{R}[\tilde{a}]$, $i=0,1,2,3,4$ $\big(\mathcal{G}_0(\tilde{a}) \equiv \mu_0(\tilde{a})\big)$ are GL-invariants.

Using the polynomial $\mathfrak{T}(w)$ we could construct the above mentioned four affine invariants $\mathcal{T}_4$, $\mathcal{T}_3, \mathcal{T}_2$ and $\mathcal{T}_1$:

$$\mathcal{T}_{4-i}(\tilde{a}) = \frac{1}{i!}\,\frac{d^i\mathfrak{T}}{dw^i}\Big|_{w=\sigma_0} \in \mathbb{R}[\tilde{a}], \quad i = 0,1,2,3 \quad \big(\mathcal{T}_4 \equiv \mathfrak{T}(\sigma_0)\big).$$

To construct the remaining invariant polynomials contained in the set (2.1) we first need to define some elementary bricks which help to construct these elements of the set.

We remark that the following polynomials in $\mathbb{R}[\tilde{a},x,y]$ are the simplest invariant polynomials of degree one with respect to the coefficients of the differential systems (1.1) and which are GL-comitants:

$$C_i(x,y) = yp_i(x,y) - xq_i(x,y),\ i=0,1,2; \quad D_i(x,y) = \frac{\partial}{\partial x}p_i(x,y) + \frac{\partial}{\partial y}q_i(x,y),$$

for $i=1,2$. Apart from these simple invariant polynomials we shall also make use of the following nine GL-invariant polynomials:

$$\begin{gathered} T_1 = (C_0,C_1)^{(1)}, \quad T_2 = (C_0,C_2)^{(1)}, \quad T_3 = (C_0,D_2)^{(1)}, \\ T_4 = (C_1,C_1)^{(2)}, \quad T_5 = (C_1,C_2)^{(1)}, \quad T_6 = (C_1,C_2)^{(2)}, \\ T_7 = (C_1,D_2)^{(1)}, \quad T_8 = (C_2,C_2)^{(2)}, \quad T_9 = (C_2,D_2)^{(1)}. \end{gathered}$$

These are of degree two with respect to the coefficients of systems (1.1).

We next define a list of T-comitants:

$$\hat{A}(\tilde{a}) = (C_1, T_8 - 2T_9 + D_2^2)^{(2)}/144,$$

$$\begin{aligned} &\widehat{B}(\tilde{a},x,y) \\ &= \Big\{16D_1(D_2,T_8)^{(1)}(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0(D_2,T_9)^{(1)}(3D_1D_2 \\ &\quad - 5T_6 + 9T_7) + 2(D_2,T_9)^{(1)}\big(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32(C_0,T_5)^{(1)}\big) \\ &\quad + 6(D_2,T_7)^{(1)}\big[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5) \\ &\quad + C_2(9T_4 + 96T_3)\big] + 6(D_2,T_6)^{(1)}\big[32C_0T_9 - C_1(12T_7 + 52D_1D_2) \\ &\quad - 32C_2D_1^2\big] + 48D_2(D_2,T_1)^{(1)}(2D_2^2 - T_8) + 6D_1D_2T_4(T_8 - 7D_2^2 - 42T_9) \\ &\quad - 32D_1T_8(D_2,T_2)^{(1)} + 9D_2^2T_4(T_6 - 2T_7) - 16D_1(C_2,T_8)^{(1)}(D_1^2 + 4T_3) \\ &\quad + 12D_1(C_1,T_8)^{(2)}(C_1D_2 - 2C_2D_1) + 12D_1(C_1,T_8)^{(1)}(T_7 + 2D_1D_2) \\ &\quad + 96D_2^2\big[D_1(C_1,T_6t)^{(1)} + D_2(C_0,T_6)^{(1)}\big] - 4D_1^3D_2(D_2^2 + 3T_8 + 6T_9) \\ &\quad - 16D_1D_2T_3(2D_2^2 + 3T_8) + 6D_1^2D_2^2(7T_6 + 2T_7) - 252D_1D_2T_4T_9\Big\}/(2^83^3), \end{aligned}$$

$$\begin{aligned} \widehat{D}(\tilde{a},x,y) = \big[&2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6) - (C_1,T_5)^{(1)} - 9D_1^2C_2 \\ &+ 6D_1(C_1D_2 - T_5)\big]/36, \end{aligned}$$

$$\widehat{E}(\tilde{a},x,y) = \big[D_1(2T_9 - T_8) - 3(C_1,T_9)^{(1)} - D_2(3T_7 + D_1D_2)\big]/72,$$

$$\begin{aligned} \widehat{F}(\tilde{a},x,y) = \big[&6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0(D_2,T_9)^{(1)} - 9D_2^2T_4 \\ &+ 288D_1\widehat{E} - 24(C_2,\widehat{D})^{(2)} + 120(D_2,\widehat{D})^{(1)} - 36C_1(D_2,T_7)^{(1)} \\ &+ 8D_1(D_2,T_5)^{(1)}\big]/144, \end{aligned}$$

$$\widehat{K}(\tilde{a},x,y) = (T_8 + 4T_9 + 4D_2^2)/72,$$
$$\widehat{H}(\tilde{a},x,y) = (-T_8 + 8T_9 + 2D_2^2)/72,$$

as well as the needed bricks:

$$A_1(\tilde{a}) = \hat{A}, \quad A_2(\tilde{a}) = (C_2, \widehat{D})^{(3)}/12,$$
$$A_3(\tilde{a}) = [[C_2, D_2)^{(1)}, D_2\big)^{(1)}, \quad D_2\big)^{(1)}/48,$$
$$A_4(\tilde{a}) = (\widehat{H}, \widehat{H})^{(2)}, \quad A_5(\tilde{a}) = (\widehat{H}, \widehat{K})^{(2)}/2,$$
$$A_6(\tilde{a}) = (\widehat{E}, \widehat{H})^{(2)}/2, \quad A_7(\tilde{a}) = [[C_2, \widehat{E})^{(2)}, D_2\big)^{(1)}/8,$$
$$A_8(\tilde{a}) = [[\widehat{D}, \widehat{H})^{(2)}, D_2\big)^{(1)}/8, \quad A_9(\tilde{a}) = [[\widehat{D}, D_2)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/48,$$
$$A_{10}(\tilde{a}) = [[\widehat{D}, \widehat{K})^{(2)}, D_2\big)^{(1)}/8, \quad A_{11}(\tilde{a}) = (\widehat{F}, \widehat{K})^{(2)}/4,$$
$$A_{12}(\tilde{a}) = (\widehat{F}, \widehat{H})^{(2)}/4, \quad A_{14}(\tilde{a}) = (\widehat{B}, C_2)^{(3)}/36,$$
$$A_{15}(\tilde{a}) = (\widehat{E}, \widehat{F})^{(2)}/4, \quad A_{25}(\tilde{a}) = [[\widehat{D}, \widehat{D})^{(2)}, \widehat{E}\big)^{(2)}/16,$$

plus

$$A_{33}(\tilde{a}) = [[\widehat{D}, D_2)^{(1)}, \widehat{F}\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/128,$$
$$A_{34}(\tilde{a}) = [[\widehat{D}, \widehat{D})^{(2)}, D_2\big)^{(1)}, \widehat{K}\big)^{(1)}, D_2\big)^{(1)}/64.$$

In the above list the bracket "[[" means a succession of two or up to four parentheses "(" depending on the row where they appear.

Now we can define the remaining invariant polynomials of the set (2.1):

$$\mathcal{F}(\tilde{a}) = A_7, \quad \mathcal{F}_1(\tilde{a}) =, A_2,$$
$$\begin{aligned}\mathcal{F}_2(\tilde{a}) = {} & -2A_1^2A_3 + 2A_5(5A_8 + 3A_9) + A_3(A_8 - 3A_{10} + 3A_{11} + A_{12}) \\ & - A_4(10A_8 - 3A_9 + 5A_{10} + 5A_{11} + 5A_{12}),\end{aligned}$$
$$\begin{aligned}\mathcal{F}_3(\tilde{a}) = {} & -10A_1^2A_3 + 2A_5(A_8 - A_9) - A_4(2A_8 + A_9 + A_{10} + A_{11} + A_{12}) \\ & + A_3(5A_8 + A_{10} - A_{11} + 5A_{12}),\end{aligned}$$
$$\begin{aligned}\mathcal{F}_4(\tilde{a}) = {} & 20A_1^2A_2 - A_2(7A_8 - 4A_9 + A_{10} + A_{11} + 7A_{12}) \\ & + A_1(6A_{14} - 22A_{15}) - 4A_{33} + 4A_{34},\end{aligned}$$
$$\mathcal{H}(\tilde{a}) = -(A_4 + 2A_5),$$
$$\mathcal{B}(\tilde{a}) = -(3A_8 + 2A_9 + A_{10} + A_{11} + A_{12}),$$
$$\begin{aligned}\mathcal{B}_1(\tilde{a},x,y) = \Big\{ & \big(T_7, D_2\big)^{(1)}\big[12D_1T_3 + 2D_1^3 + 9D_1T_4 + 36\big(T_1, D_2\big)^{(1)}\big] \\ & - 2D_1\big(T_6, D_2\big)^{(1)}(D_1^2 + 12T_3) + D_1^2\big[D_1\big(T_8, C_1\big)^{(2)} \\ & + 6\big(\big(T_6, C_1\big)^{(1)}, D_2\big)^{(1)}\big]\Big\}/144,\end{aligned}$$
$$\begin{aligned}\mathcal{B}_2(\tilde{a},x,y) = \Big\{ & \big(T_7, D_2\big)^{(1)}\big[8T_3\big(T_6, D_2\big)^{(1)} - D_1^2\big(T_8, C_1\big)^{(2)} \\ & - 4D_1\big(\big(T_6, C_1\big)^{(1)}, D_2\big)^{(1)}\big] + \Big[\big(T_7, D_2\big)^{(1)}\Big]^2(8T_3 - 3T_4 + 2D_1^2)\Big\}/384,\end{aligned}$$
$$\widetilde{K}(\tilde{a},x,y) = 4\widehat{K} \equiv \operatorname{Jacob}\big(p_2(\tilde{a},x,y), q_2(\tilde{a},x,y)\big),$$
$$\widetilde{M}(\tilde{a},x,y) = (C_2, C_2)^{(2)} \equiv 2\operatorname{Hess}\big(C_2(\tilde{a},x,y)\big),$$

$$\widetilde{N}(\tilde{a},x,y) = \widetilde{K} - 4\widehat{H},$$
$$\eta(\tilde{a}) = (\widetilde{M},\widetilde{M})^{(2)}/384 \equiv \text{Discrim}\big(C_2(\tilde{a},x,y)\big),$$
$$\theta(\tilde{a}) = -(\widetilde{N},\widetilde{N})^{(2)}/2 \equiv \text{Discrim}\big(\widetilde{N}(\tilde{a},x,y)\big);$$
$$\theta_3(\tilde{a}) = A_8 + A_{11}, \quad \theta_4(\tilde{a}) = A_7,$$
$$B_1(\tilde{a}) = \text{Res}_x\left(C_2,\widetilde{D}\right)/y^9 = -2^{-9}3^{-8}\left(B_2,B_3\right)^{(4)},$$
$$B_2(\tilde{a},x,y) = \left(B_3,B_3\right)^{(2)} - 6B_3(C_2,\widetilde{D})^{(3)},$$
$$B_3(\tilde{a},x,y) = (C_2,\widetilde{D})^{(1)} \equiv \text{Jacob}\left(C_2,\widetilde{D}\right),$$
$$E_1(\tilde{a}) = A_{25},$$
$$\widetilde{U}_1(\tilde{a}) = A_9 - 54A_1^2, \quad \widetilde{U}_2(\tilde{a}) = 3A_8 - A_9,$$
$$H_7(\tilde{a}) = (\widetilde{N},C_1)^{(2)}, \quad H_9(\tilde{a}) = -\big[[\widetilde{D},\widetilde{D})^{(2)},\widetilde{D},\big)^{(1)},\widetilde{D}\big)^{(3)},$$
$$H_{10}(\tilde{a}) = \big[[\widetilde{N},\widetilde{D})^{(2)},\ D_2\big)^{(1)},$$
$$H_{11}(\tilde{a},x,y) = 8\widehat{H}\big[(C_2,\widetilde{D})^{(2)} + 8(\widetilde{D},D_2)^{(1)}\big] + 3\big[(C_1,\,2\widehat{H}-\widetilde{N})^{(1)} - 2D_1\widetilde{N}\big]^2,$$
$$H_{12}(\tilde{a},x,y) = (\widetilde{D},\widetilde{D})^{(2)} \equiv \text{Hessian}(\widetilde{D}).$$

We remark that the above invariant polynomials (except $\widetilde{U}_1$ and $\widetilde{U}_2$) were constructed and used in [22], [18] and [3], and only the invariant polynomials $\widetilde{U}_1$ and $\widetilde{U}_2$ are defined here.

3. Preliminary results involving the use of polynomial invariants

We remark that the invariant polynomials $\mu_i(\tilde{a},x,y)$ $(i = 0,1,\dots,4)$ defined in the previous subsection are responsible for the total multiplicity of the finite singularities of quadratic systems (1.1). Moreover they detect whether a quadratic system is degenerate or not. More exactly we have the following lemma.

Lemma 3.1 ([5, Lemma 5.2])**.** *Consider a system (S) from the family (1.1) with coefficients $\tilde{\mathbf{a}} \in \mathbb{R}^{12}$. Then:*

- (i) *The total multiplicity of the finite singularities of this system is $4-k$ if and only if for every i such that $0 \le i \le k-1$ we have $\mu_i(\tilde{\mathbf{a}},x,y) = 0$ in $\mathbb{R}[x,y]$ and $\mu_k(\tilde{\mathbf{a}},x,y) \neq 0$.*
- (i) *The system (S) is degenerate (i.e. $\gcd(P,Q) \neq \text{const}$) if and only if $\mu_i(\tilde{\mathbf{a}},x,y) = 0$ in $\mathbb{R}[x,y]$ for every $i = 0,1,2,3,4$.*

On the other hand the invariant polynomials η, $\widetilde{M}$ and C_2 govern the number of real and complex infinite singularities. More precisely, according to [19] (see also [17]) we have the next result.

Lemma 3.2. *The number of infinite singularities (real and complex) of a quadratic system in* QS *is determined by the following conditions:*

- (i) *3 real if $\eta > 0$;*
- (ii) *1 real and 2 imaginary if $\eta < 0$;*
- (iii) *2 real if $\eta = 0$ and $\widetilde{M} \neq 0$;*
- (iv) *1 real if $\eta = \widetilde{M} = 0$ and $C_2 \neq 0$;*
- (v) *∞ if $\eta = \widetilde{M} = C_2 = 0$.*

Moreover, the quadratic systems (1.1)*, for each one of these cases, can be brought via a linear transformation to the corresponding case of the following canonical systems* $(\mathbf{S}_I) - (\mathbf{S}_V)$*:*

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + (h-1)xy, \\ \dot{y} &= b + ex + fy + (g-1)xy + hy^2; \end{cases} \tag{$\mathbf{S}_I$}$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + (h+1)xy, \\ \dot{y} &= b + ex + fy - x^2 + gxy + hy^2; \end{cases} \tag{$\mathbf{S}_{II}$}$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + hxy, \\ \dot{y} &= b + ex + fy + (g-1)xy + hy^2; \end{cases} \tag{$\mathbf{S}_{III}$}$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + hxy, \\ \dot{y} &= b + ex + fy - x^2 + gxy + hy^2; \end{cases} \tag{$\mathbf{S}_{IV}$}$$

$$\begin{cases} \dot{x} &= a + cx + dy + x^2, \\ \dot{y} &= b + ex + fy + xy. \end{cases} \tag{$\mathbf{S}_V$}$$

According to [4] (see also [5]) the next proposition holds.

Proposition 3.3. *Consider a non-degenerate quadratic differential system. Then:*

(i) this system has one center if and only if one of the following sets of conditions holds

$$\begin{aligned}
&(\mathfrak{C}_1)\ \mathcal{T}_4 = 0,\ \mathcal{T}_3\mathcal{F} < 0,\ \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3\mathcal{F}_4 = 0;\\
&(\mathfrak{C}_2)\ \mathcal{T}_4 = \mathcal{T}_3 = 0,\ \mathcal{T}_2 > 0,\ \mathcal{B} < 0,\ \mathcal{F} = \mathcal{F}_1 = 0;\\
&(\mathfrak{C}_3)\ \mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = \mathcal{T}_1 = 0,\ \sigma \neq 0,\ \mathcal{F}_1 = 0, \mathcal{H} < 0,\ \mathcal{B} < 0,\ \mathcal{F} = 0;\\
&(\mathfrak{C}_4)\ \mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = \mathcal{T}_1 = 0,\ \sigma \neq 0,\ \mathcal{F}_1 = 0, \mathcal{H} = \mathcal{B}_1 = 0,\ \mathcal{B}_2 < 0;\\
&(\mathfrak{C}_5)\ \sigma = 0,\ \mu_0 < 0,\ \mathbf{D} < 0,\ \mathbf{R} > 0,\ \mathbf{S} > 0;\\
&(\mathfrak{C}_6)\ \sigma = 0,\ \mu_0 = 0,\ \mathbf{D} < 0,\ \mathbf{R} \neq 0;\\
&(\mathfrak{C}_7)\ \sigma = 0,\ \mu_0 > 0,\ \mathbf{D} > 0;\\
&(\mathfrak{C}_8)\ \sigma = 0,\ \mu_0 > 0,\ \mathbf{D} = 0,\ \mathbf{T} < 0;\\
&(\mathfrak{C}_9)\ \sigma = 0,\ \mu_0 = \mu_1 = 0,\ \mu_2 \neq 0, \mathbf{U} > 0,\ \widetilde{K} = 0;\\
&(\mathfrak{C}_{10})\ \sigma = 0,\ \mu_0 > 0, \mathbf{D} = \mathbf{T} = \mathbf{P} = 0,\ \mathbf{R} \neq 0;
\end{aligned} \tag{3.1}$$

(ii) and it has two centers if and only if one of the following sets of conditions holds

$$\begin{aligned}
&(\widehat{\mathfrak{C}}_1)\ \mathcal{T}_4 = \mathcal{T}_3 = 0,\ \mathcal{T}_2 < 0,\ \mathcal{B} < 0,\ \mathcal{H} < 0,\ \mathcal{F} = \mathcal{F}_1 = 0;\\
&(\widehat{\mathfrak{C}}_2)\ \sigma = 0,\ \mu_0 > 0,\ \mathbf{D} < 0,\ \mathbf{R} > 0,\ \mathbf{S} > 0.
\end{aligned} \tag{3.2}$$

In what follows we also need the next lemma.

Lemma 3.4 ([16])**.** *For the existence of invariant straight lines of a system* (1.1) *in one (respectively 2; 3 distinct) directions in the affine plane it is necessary that* $B_1 = 0$ *(respectively* $B_2 = 0$*;* $B_3 = 0$*).*

4. Proof of Theorem 1.2

We shall consider step by step each one of the subfamilies of quadratic systems defined by the conditions (A) - (D) which are provided by Theorem 1.2.

4.1. **Subfamily defined by (A):** $\eta > 0$**.** According to Lemma 3.2 we consider systems $(\mathbf{S}_I)$ for which calculations yield:

$$\eta = 1, \quad \theta = -8(g-1)(h-1)(g+h).$$

We examine two cases: $\theta \neq 0$ and $\theta = 0$.

4.1.1. *Case* $\theta \neq 0$. Then $(g-1)(h-1)(g+h) \neq 0$ and due to a translation we may assume $d = e = 0$, i.e. we obtain the systems

$$\dot{x} = a + cx + gx^2 + (h-1)xy, \quad \dot{y} = b + fy + (g-1)xy + hy^2, \tag{4.1}$$

for which we calculate

$$\begin{aligned} B_1 &= ab(g-1)^2(h-1)^2\big[(b-a)(g+h)^2 + cf(g-h) + c^2h - f^2g\big] \\ &\equiv ab(g-1)^2(h-1)^2\mathcal{H}. \end{aligned}$$

So because $\theta \neq 0$ the condition $B_1 = 0$ is equivalent to $ab\mathcal{H} = 0$ and we consider two subcases: $ab = 0$ and $ab \neq 0$.

1: $ab = 0$. We observe that systems (4.1) keep the form under the change $(x, y, a, b, c, f, g, h) \mapsto (y, x, b, a, f, c, h, g)$, and hence without loss of generality we may consider that the condition $a = 0$ is fulfilled. Then we arrive at the family of systems

$$\dot{x} = x\big[c + gx + (h-1)y\big], \quad \dot{y} = b + fy + (g-1)xy + hy^2,$$

possessing the invariant affine line $x = 0$. It is not too difficult to see, that after the affine transformation

$$x_1 = x, \quad y_1 = gx + (h-1)y + c,$$

we arrive at the systems

$$\dot{x}_1 = x_1y_1, \quad \dot{y}_1 = b' + e'x_1 + l'x_1^2 + (f' + 2m'x_1)y_1 + n'y^2, \tag{4.2}$$

where b', e', f', l', m' and n' are rational functions of the parameters b, c, f, g, h with the same denominator $h - 1 \neq 0$. We observe that these systems belong to the family of systems (1.3).

2: $ab \neq 0$. In this case we get $\mathcal{H} = 0$. Then from the equality

$$(b-a)(g+h)^2 + cf(g-h) + c^2h - f^2g = 0,$$

we obtain

$$b = a + \frac{(f-c)(fg+ch)}{(g+h)^2} \equiv b_0,$$

and this leads to the family of systems

$$\dot{x} = a + cx + gx^2 + (h-1)xy, \quad \dot{y} = b_0 + fy + (g-1)xy + hy^2.$$

Since $g + h \neq 0$ we can apply to these systems the transformation

$$x_1 = (g+h)(x-y) + c - f, \quad y_1 = (g+h)(gx+hy) + fg + ch, \quad t_1 = t/(g+h)$$

with the determinant $(g+h)^3 \neq 0$. Then we obtain the family of systems (4.2), where the parameters b', e', f', l', m' and n' are rational functions of the parameters a, c, f, g, h with the same denominator $g + h \neq 0$. So we again arrive at a subfamily of systems (1.3).

4.1.2. *Case* $\theta = 0$. For systems $(\mathbf{S}_I)$ we have

$$\theta = -8(g-1)(h-1)(g+h), \quad \widetilde{N} = (g^2-1)x^2+2(g-1)(h-1)xy+(h^2-1)y^2, \tag{4.3}$$

and therefore the condition $\theta = 0$ yields $(h-1)(g-1)(g+h) = 0$. Without loss of generality we can consider $h = 1$. Indeed, if $g = 1$ (respectively, $g + h = 0$) we can apply the linear transformation that replaces the straight line $x = 0$ by $y = 0$ (respectively, $x = 0$ by $y = x$) reducing this case to $h = 1$.

So we assume $h = 1$ and in this case by (4.3) for systems $(\mathbf{S}_I)$ we have $\widetilde{N} = (g-1)(1+g)x^2$. We consider two subcases: $\widetilde{N} \neq 0$ and $\widetilde{N} = 0$.

1: $\widetilde{N} \neq 0$. Then $(g-1)(g+1) \neq 0$ and due to a translation we may assume $e = f = 0$. So we obtain the family of systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b + (g-1)xy + y^2, \tag{4.4}$$

for which we calculate

$$B_1 = bd^2g(g-1)^2\big[(b-a)(1+g)^2 + (c+d)(c-dg)\big] \equiv bd^2g(g-1)^2\Phi,$$
$$\mu_0 = g^2, \quad H_7 = 4d(g^2-1).$$

1.1: $H_7 \neq 0$. This implies $d(g-1) \neq 0$ and therefore the condition $B_1 = 0$ yields $bg\Phi = 0$. We consider two cases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

1.1.1: $\mu_0 \neq 0$. Then $g \neq 0$ and we obtain $b\Phi = 0$.

1.1.1.1: $b = 0$. Then systems (4.4) possess the invariant line $y = 0$ and using the transformation

$$x_1 = y, \quad y_1 = (g-1)x + y,$$

we arrive at the systems

$$\dot{x}_1 = x_1y_1, \quad \dot{y}_1 = a(g-1) - (c+d-dg)x_1 + cy_1 + \frac{1}{g-1}\big[gx_1^2 - (g-1)x_1y_1 + gy_1^2\big].$$

So we obtain a subfamily of the family of systems (1.3).

1.1.1.2: $\Phi = 0$. This condition gives

$$b = \frac{a(1+g)^2 - (c+d)(c-dg)}{(1+g)^2} \equiv b_0,$$

and systems (4.4) with $b = b_0$ possess the invariant line $(1+g)(x-y)+c+d = 0$. Then applying the transformation

$$x_1 = (1+g)(x-y) + c + d, \quad y_1 = gx + y \quad (\mathrm{Det} = (1+g)^2 \neq 0),$$

we obtain a subfamily of Abel quadratic systems of the form (1.3):

$$\dot{x}_1 = x_1y_1, \quad \dot{y}_1 = Q(x_1, y_1),$$

where $Q(x_1, y_1)$ is a quadratic polynomial whose coefficients are rational functions of the parameters a, c, d, g, h with the denominators being some powers of $g+1 \neq 0$.

1.1.1: $\mu_0 = 0$. Then we have $g = 0$ and considering systems (4.4) we obtain the systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = b - xy + y^2.$$

Since $d \neq 0$ we can apply the transformation
$$x_1 = x, \quad y_1 = cx + dy + a,$$
which brings the above systems to the form
$$\dot{x}_1 = y_1, \quad \dot{y}_1 = \frac{1}{d}\big[a^2+bd^2+a(2c+d)x_1+(cd-2a)y_1+c(c+d)x_1^2-(2c+d)x_1y_1+y_1^2\big].$$
It is clear that these systems are contained in the family of systems (1.3) (in the first equation we have $d_1 = 0$ and $d_0 = 1$).

1.2: $H_7 = 0$. Then $d = 0$ and we arrive at the family of systems
$$\dot{x} = a + cx + gx^2, \quad \dot{y} = b + (g-1)xy + y^2, \tag{4.5}$$
for which we calculate
$$B_1 = 0, \quad B_2 = -648b(-1+g)^2\big[(b-a)(1+g)^2 + c^2\big]x^4, \quad \mu_0 = g^2$$
and we consider two cases: $B_2 \neq 0$ and $B_2 = 0$.

1.2.1: $B_2 \neq 0$. We claim that for $B_2 \neq 0$, systems (4.5) could not be brought via an affine transformation to the form (1.3). To prove this claim we examine two subcases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

1.2.1.1: $\mu_0 \neq 0$. Then $g \neq 0$ and systems (4.5) possess two parallel invariant lines $a + cx + gx^2 = 0$ (which can be real or complex or coinciding).

On the other hand for systems (1.3) we have $\mu_0 = a_0c_2d_1^2$ and the condition $\mu_0 \neq 0$ implies $d_1 \neq 0$. This means that systems (1.3) possess invariant line $d_0 + d_1x = 0$ and there does not exist another parallel invariant line in the direction $x = 0$.

It remains to observe that according to Lemma 3.4 for the existence of invariant lines in two distinct directions for a quadratic system the condition $B_2 = 0$ is necessary. Therefore systems (4.5) for $B_2 \neq 0$ could not have an invariant affine line in other direction, which could be used for the construction of the needed affine transformation.

1.2.1.2: $\mu_0 = 0$. Then $g = 0$ and considering (4.5) we obtain the systems
$$\dot{x} = a + cx, \quad \dot{y} = b - xy + y^2, \tag{4.6}$$
for which we have
$$B_1 = \mu_0 = H_7 = 0, \quad \widetilde{N} = -x^2.$$
On the other hand for systems (1.3) we have $\mu_0 = a_0c_2d_1^2$ and the condition $\mu_0 = 0$ gives $a_0c_2d_1 = 0$.

If $d_1 = 0$ then for systems (1.3) we calculate
$$H_7 = -4(b_1^2 - 4a_0c_2)d_0, \quad \widetilde{N} = -(b_1^2 - 4a_0c_2)x^2$$
and the conditions $H_7 = 0$ and $\widetilde{N} \neq 0$ imply $d_0 = 0$ which leads to degenerate systems (1.3).

Assume now $d_1 \neq 0$. This means that systems (1.3) possess invariant line $d_0 + d_1x = 0$ in the direction $x = 0$ and therefore $(d_0 + d_1x)$ is a factor in $\widetilde{P}(x, y)$. Moreover the second factor of $\widetilde{P}(x, y)$ in (1.3) is y.

On the other hand systems (4.6) could possess in the direction $x = 0$ either one invariant affine line $a + cx = 0$ if $c \neq 0$ or zero lines if $c = 0$. Moreover the right hand side of the first equation does not contain the factor y.

It remains to observe that according to Lemma 3.4 systems (4.6) for $B_2 \neq 0$ could not have an invariant affine line in another direction, which could be used for the construction of the needed affine transformation. This completes the proof of the claim.

1.1.2: $B_2 = 0$. Then $b\big[(b-a)(1+g)^2 + c^2\big] = 0$. We observe that the second factor equals $\Phi\big|_{d=0}$ and we deduce that we can apply the same arguments as previously in the case $H_7 \neq 0$ repeating the steps ***1.1.1.1*** ($b = 0$) and ***1.1.1.2*** ($\Phi = 0$) and considering the condition $d = 0$.

Thus the condition $B_2 = 0$ guarantees the existence of an affine transformation which brings systems (4.5) to the form (1.3).

2: $\widetilde{N} = 0$. Considering (4.3) the condition $\widetilde{N} = 0$ yields $(g-1)(h-1) = g^2 - 1 = h^2 - 1 = 0$ and we obtain three possibilities: (a) $g = 1 = h$; (b) $g = 1 = -h$; (c) $g = -1 = -h$. The cases (b) and (c) can be brought by linear transformations to the case (a).

So $g = h = 1$ and systems $(\mathbf{S}_I)$ after an additional translation (to make $c = d = 0$) are of the form:

$$\dot{x} = a + dy + x^2, \quad \dot{y} = b + ex + y^2. \tag{4.7}$$

For these systems we calculate

$$B_1 = -d^2e^2(4a - 4b + d^2 - e^2), \quad \mu_0 = 1, \quad \theta_3 = -2de, \quad \theta_4 = -(d+e),$$

and we consider two possibilities: $\theta_3 \neq 0$ and $\theta_3 = 0$.

2.1: $\theta_3 \neq 0$. Then the condition $B_1 = 0$ implies $b = a + (d^2 - e^2)/4$ and we obtain the systems

$$\dot{x} = a + dy + x^2, \quad \dot{y} = a + (d^2 - e^2)/4 + ex + y^2, \tag{4.8}$$

possessing the invariant line $2x - 2y + d - e = 0$. So by means of the transformation

$$x_1 = 2x - 2y + d - e, \quad y_1 = x + y - (d+e)/2,$$

we arrive at the following subfamily of (1.3):

$$\dot{x}_1 = x_1y_1, \quad \dot{y}_1 = (4a + 2d^2 + e^2)/2 + (e-d)x_1 + (d+e)y_1 + x_1^2/8 + y_1^2/2. \tag{4.9}$$

2.2: $\theta_3 = 0$. Then $de = 0$ and we may consider $d = 0$ due to the change $(x, y, a, b, d, e) \mapsto (y, x, b, a, e, d)$ which conserves the systems. In this case we have

$$B_1 = 0, \quad B_2 = 648e^2(4a - 4b - e^2)x^4, \quad \theta_4 = -e,$$

and we consider two cases: $B_2 \neq 0$ and $B_2 = 0$.

2.2.1: $B_2 \neq 0$. We claim that for $B_2 \neq 0$ systems (4.7) with $d = 0$ could not be brought via an affine transformation to the form (1.3).

Indeed, for systems (1.3) we calculate $\mu_0 = a_0c_2d_1^2 \neq 0$ (since for (4.7) we have $\mu_0 = 1$). Hence $d_1 \neq 0$ and these systems possess a single invariant line $d_0 + d_1x = 0$ in the direction $x = 0$.

On the other hand systems (4.7) with $d = 0$ possess in the direction $x = 0$ two parallel invariant lines $x^2 + a = 0$, which could be real or complex or coinciding. Taking into account that by Lemma 3.4 in the case $B_2 \neq 0$ these systems could not have invariant lines in other directions we conclude that the claim is proved.

2.2.2: $B_2 = 0$. Then $e(4a - 4b - e^2) = 0$ and we examine two subcases: $\theta_4 \neq 0$ and $\theta_4 = 0$.

2.2.2.1: $\theta_4 \neq 0$. In this case we get $b = a - e^2/4$ and we obtain systems (4.8) with $d = 0$. So applying the transformation

$$x_1 = 2x - 2y - e, \quad y_1 = x + y - e/2,$$

we arrive at the family of systems (4.9) with $d = 0$ which is a subfamily of (1.3).

2.2.2.2: $\theta_4 = 0$. Then $e = 0$ and we obtain the systems

$$\dot{x} = a + x^2, \quad \dot{y} = b + y^2, \tag{4.10}$$

for which calculations yield

$$B_1 = B_2 = 0, \quad B_3 = -12(a - b)x^2y^2, \quad \mu_0 = 1.$$

These systems have two couples of parallel lines: $a + x^2 = 0$ (in the direction $x = 0$) and $b + y^2 = 0$ (in the direction $y = 0$) which could be real, or complex, or coinciding. We examine two possibilities: $B_3 \neq 0$ and $B_3 = 0$.

2.2.2.2.1: $B_3 \neq 0$. Then by Lemma 3.4 systems (4.10) could not have other invariant lines.

On the other hand, as it was mentioned earlier, since $\mu_0 \neq 0$, systems (1.3) have a single invariant line in the direction $x = 0$. So we conclude that systems (4.10) could not be brought to the form (1.3) by means of an affine transformation.

2.2.2.2.2: $B_3 = 0$. This implies $b = a$ and then systems (4.10) possess also the invariant line $y = x$. So applying the transformation $x_1 = x - y$, $y_1 = x + y$ we obtain the family of systems

$$\dot{x}_1 = x_1y_1, \quad \dot{y}_1 = 2a + x_1^2/2 + y_1^2/2,$$

which is a subfamily of (1.3).

As all the possibilities in the case $\eta > 0$ are examined we conclude that the statement (A) of Theorem 1.2 is proved.

4.2. **Subfamily defined by (B): $\eta < 0$.** In this case by Lemma 3.2 we have to consider the systems $(\mathbf{S}_{II})$ for which we calculate:

$$\begin{gathered}\eta = -4, \quad \theta = 8(1 + h)\big[g^2 + (h - 1)^2\big], \\ \widetilde{N} = (g^2 - 2h + 2)x^2 + 2g(h + 1)xy + (h^2 - 1)y^2.\end{gathered} \tag{4.11}$$

So we examine two cases: $\theta \neq 0$ and $\theta = 0$.

4.2.1. *Case* $\theta \neq 0$. Then $h + 1 \neq 0$ and by a translation we cany assume $c = d = 0$, i.e. we obtain the systems

$$\dot{x} = a + gx^2 + (h + 1)xy, \quad \dot{y} = b + ex + fy - x^2 + gxy + hy^2. \tag{4.12}$$

For these systems we calculate

$$\begin{aligned}B_1 &= -a(h + 1)^2(\alpha^2 + \beta^2), \\ B_2 &= -648\big[\alpha(\alpha + a(1 + h)^2) + \beta^2\big]x^4 + 648a(1 + h)^2\alpha y^2(6x^2 - y^2) \\ &\quad - 2592a(1 + h)^2\beta xy(x^2 - y^2),\end{aligned}$$

where

$$\begin{gathered}\alpha = a\big[g^2 - (h - 1)^2\big] - 2bg(h - 1) + f(-e + fg - eh), \\ \beta = 2ag(h - 1) + b\big[g^2 - (h - 1)^2\big] - f^2 - efg + e^2h.\end{gathered} \tag{4.13}$$

It is not too difficult to observe that the condition $\alpha = \beta = 0$ is equivalent to $B_1 = B_2 = 0$. So we consider two subcases: $B_2 \neq 0$ and $B_2 = 0$.

1: $B_2 \neq 0$. Then by $\theta \neq 0$ the condition $B_1 = 0$ implies $a = 0$ and applying the transformation $x_1 = x$, $y_1 = gx + (1+h)y$, we arrive at the family of systems

$$\begin{aligned}
&\dot{x}_1 = x_1 y_1,\\
&\dot{y}_1 = b(1+h) + (e - fg + eh)x_1 + fy_1 - \frac{1}{1+h}\big[(g^2 + (h-1)^2)x_1^2 - 2gx_1y_1 - hy_1^2\big].
\end{aligned}$$

So we obtain a subfamily of the family of systems (1.3).

2: $B_2 = 0$. Then we obtain $\alpha = \beta = 0$ and considering (4.13) this condition yields

$$\begin{aligned}
a =& \frac{1}{\big[g^2 + (h-1)^2\big]^2}\big[(e + fg - eh)(2egh + fh^2 - f - fg^2)\big] \equiv a^*,\\
b =& -\frac{1}{\big[g^2 + (h-1)^2\big]^2}\big[efg(h-1)(1+3h) - efg^3\\
&+ (h-1)^2(f^2 - e^2h) + g^2(f^2 + e^2h - 2f^2h)\big] \equiv b^*.
\end{aligned}$$

In this case clearly we obtain systems (4.12) with $a = a^*$ and $b = b^*$ which we denote by (4.12*). For these systems calculations yield $B_1 = B_2 = 0$ and

$$B_3 = \frac{3}{\big[g^2 + (h-1)^2\big]^2}(1+h)^2(e+fg-eh)(f+fg^2-2egh-fh^2)(x^2+y^2)^2. \tag{4.14}$$

We detect that systems (4.12*) possess two complex invariant lines:

$$\big[g \pm i(1-h)\big]x + (1 - h \mp ig)y - (f \pm ie) = 0$$

in two different directions (intersecting infinite line at complex singularities).

Since for systems (1.3) we have $\theta = 8d_1(b_1^2c_2 - 4a_0c_2^2 + 4a_0c_2d_1 - a_0d_1^2) \neq 0$, we deduce that in order to exist an affine transformation for bringing systems (4.12*) to the form (1.3) we need a real invariant affine line in the third (real) direction.

On the other hand according to Lemma 3.4 for the existence of invariant affine lines in three distinct directions, the condition $B_3 = 0$ is necessary.

So we conclude that in the case $\eta < 0$, $\theta \neq 0$, $B_1 = B_2 = 0$ and $B_3 \neq 0$ a quadratic system could not be brought to an Abel quadratic differential system.

Assume now $B_3 = 0$. Considering the condition $\theta \neq 0$ and (4.14) we obtain the condition

$$(e + fg - eh)(f + fg^2 - 2egh - fh^2) = 0.$$

2.1: $e(1-h) + fg = 0$. If $g = 0$ then from $\theta \neq 0$ (i.e. $g^2 + (h-1)^2 \neq 0$) we obtain $e = 0$ and in this case systems (4.12*) have the form

$$\dot{x} = (h+1)xy, \quad \dot{y} = -\frac{f^2}{(h-1)^2} + fy - x^2 + hy^2.$$

Thus we obtain Abel quadratic systems of the form (1.3).

Assume now $g \neq 0$. Then we obtain $f = e(h-1)/g$ and systems (4.12*) become

$$\dot{x} = gx^2 + (1+h)xy, \quad \dot{y} = -e^2/g^2 + e(h-1)y/g - x^2 + gxy + hy^2.$$

Applying the transformation $x_1 = x$, $y_1 = gx + (1+h)y$ we arrive at the following subfamily of the family of systems (1.3):

$$\begin{aligned}\dot{x}_1 =&x_1y_1,\\ \dot{y}_1 =&-\frac{e^2(1+h)}{g^2}+2ex_1+\frac{e(h-1)}{g}y_1+\frac{1}{1+h}\big[(g^2+(h+1)^2)x_1^2+2gx_1y_1+hy_1^2\big].\end{aligned}$$

2.2: $f(1+g^2-h^2)-2egh=0$. If $g=0$ then $h^2-1\neq 0$ and we again get $f=0$ and we arrive at the case considered above.

If $h=0$ then the condition $f(1+g^2)=0$ gives $f=0$ and this leads to the degenerate systems

$$\dot{x}=x(gx+y),\quad \dot{y}=x(e-x+gy).$$

Assume now $gh\neq 0$. Then we calculate $e=f(1+g^2-h^2)/(2gh)$, and after the same transformation applied to systems (4.12*) we obtain the systems

$$\begin{aligned}\dot{x}_1 =&x_1y_1,\\ \dot{y}_1 =&\frac{f^2(1+h)\big[g^2+(h+1)^2\big]}{4g^2h}+\frac{-f(h-1)(g^2+(h+1)^2)}{2gh}x_1+fy_1\\ &-\frac{(g^2+(h+1)^2)}{1+h}x_1^2+\frac{2g}{1+h}x_1y_1+\frac{h}{1+h}y_1^2.\end{aligned}$$

Thus we obtain the Abel quadratic systems of the form (1.3).

4.2.2. *Case $\theta=0$.* According to (4.11) we have $(h+1)[(h-1)^2+g^2]=0$, and we consider two subcases: $\widetilde{N}\neq 0$ and $\widetilde{N}=0$.

1: $\widetilde{N}\neq 0$. Then by (4.11) the condition $\theta=0$ yields $h=-1$, and in addition we may assume $f=0$ due to the translation $(x,y)\to(x,y+f/2)$. Hence we obtain the family of systems

$$\dot{x}=a+cx+dy+gx^2,\quad \dot{y}=b+ex-x^2+gxy-y^2, \tag{4.15}$$

for which calculations yield:

$$\begin{gathered}B_1=-d^2g(\hat{\alpha}^2+\hat{\beta}^2),\quad H_7=4d(4+g^2),\\ \begin{aligned}B_2=&-648\big[\hat{\alpha}(\hat{\alpha}+d^2g)+\hat{\beta}^2\big]x^4+648a(1+h)^2\hat{\alpha}y^2(6x^2-y^2)\\ &-2592a(1+h)^2\hat{\beta}xy(x^2-y^2),\end{aligned}\end{gathered} \tag{4.16}$$

where

$$\begin{aligned}\hat{\alpha}=&a(g^2-4)+4bg-2ce-d(d+e)g,\\ \hat{\beta}=&-4ag+b(g^2-4)+c^2+d^2-e^2+cdg.\end{aligned} \tag{4.17}$$

We observe that the condition $\hat{\alpha}=\hat{\beta}=0$ is equivalent to $B_1=B_2=0$, and so we examine two possibilities: $B_2\neq 0$ and $B_2=0$.

1.1: $B_2\neq 0$. Then the condition $B_1=0$ implies $dg=0$, and we discuss two cases: $H_7\neq 0$ and $H_7=0$.

1.1.1: $H_7\neq 0$. Considering (4.16) we have $d\neq 0$, and this implies $g=0$. So we obtain the family of systems

$$\dot{x}=a+cx+dy,\quad \dot{y}=b+ex-x^2-y^2,$$

and applying the transformation $x_1 = x$, $y_1 = cx + dy + a$ and $t_1 = t/d$ we arrive at the family of systems

$$\dot{x}_1 = dy_1, \quad \dot{y}_1 = bd^2 - a^2 + (d^2e - 2ac)x_1 + (2a + cd)y_1 - (c^2 + d^2)x_1^2 + 2cx_1y_1 - y_1^2.$$

So we obtain a subfamily of the family of systems (1.3).

1.1.2: $H_7 = 0$. Then $d = 0$ and we arrive at the systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = b + ex - x^2 + gxy - y^2, \tag{4.18}$$

which could have real straight lines only in the direction $x = 0$. However the right hand side of the first equation does not have as a factor y. Comparing with systems (1.3) we deduce that there could not exist an affine transformation which brought the above systems to the form (1.3).

1.2: $B_2 = 0$. Then we obtain $\hat{\alpha} = \hat{\beta} = 0$ and considering (4.17) this condition yields

$$a = \frac{1}{(4+g^2)^2}(2c + dg + eg)(-4e + 2cg + dg^2) \equiv a_*,$$
$$b = \frac{1}{(4+g^2)^2}\left[cdg^3 + (c^2 - 3d^2 - 4de - e^2)g^2 - 4c(d + 2e)g - 4(c^2 + d^2 - e^2)\right] \equiv b_*.$$

In this case clearly we obtain systems (4.15) with $a = a_*$ and $b = b_*$ which we denote by (4.15_*). For these systems calculations yield $B_1 = B_2 = 0$ and

$$B_3 = -3d^2g(x^2 + y^2)^2, \quad H_7 = 4d(4 + g^2). \tag{4.19}$$

We detect that systems $(4.15)_*$ possess two complex invariant lines:

$$(g \pm 2i)x + (2 \mp ig)y + c \mp i(d + e) = 0.$$

We consider two cases: $B_3 \neq 0$ and $B_3 = 0$.

1.2.1: $B_3 \neq 0$. In this case by the same arguments as earlier we deduce that in order to exist an affine transformation for bringing systems (4.15_*) to the form (1.3) we need a real invariant affine line in the third (real) direction. However according to Lemma 3.4 for the existence of invariant affine lines in three distinct directions the condition $B_3 = 0$ is necessary.

So we conclude that in the case considered, a quadratic system $(4.15)_*$ could not be brought to an Abel quadratic system of the form (1.3).

1.2.2: $B_3 = 0$. Considering (4.19) the condition $dg = 0$ holds, and we examine two subcases: $H_7 \neq 0$ and $H_7 = 0$.

1.2.2.1: $H_7 \neq 0$. Then $d \neq 0$ and the condition $B_3 = 0$ implies $g = 0$. Then systems (4.15_*) become

$$\dot{x} = -ce/2 + cx + dy, \quad \dot{y} = (c^2 + d^2 - e^2)/4 + ex - x^2 - y^2,$$

and applying the transformation $x_1 = x$, $y_1 = cx + dy - ce/2$ and $t_1 = t/d$ we arrive at the following subfamily of systems (1.3):

$$\dot{x}_1 = dy_1,$$
$$\dot{y}_1 = (c^2 + d^2)(d^2 - e^2)/4 + (c^2 + d^2)ex_1 + c(d - e)y_1 - (c^2 + d^2)x_1^2 + 2cx_1y_1 - y_1^2.$$

1.2.2.2: $H_7 = 0$. Then $d = 0$ which implies $B_3 = 0$. In this case considering systems (4.15_*) we arrive at the systems

$$\dot{x} = a_*\big|_{d=0} + cx + gx^2, \quad \dot{y} = b_*\big|_{d=0} + ex - x^2 + gxy - y^2.$$

These systems could possess invariant lines in the unique real direction $x = 0$. However by the same arguments as we present earlier for systems (4.18) we conclude that there could not exist an affine transformation which brings the above systems to the form (1.3).

2: $\widetilde{N} = 0$. Then from (4.11) we have $g = h - 1 = 0$ and without loss of generality we may assume $c = d = 0$ via the translation $(x, y) \to (x - d/2, y - c/2)$. Hence we obtain the systems

$$\dot{x} = a + 2xy, \quad \dot{y} = b + ex + fy - x^2 + y^2, \tag{4.20}$$

for which calculations yield:

$$\begin{aligned} B_1 =& -4a(e^2 + f^2)^2, \\ B_2 =& -648\big[(e^4 - 8aef + 2e^2f^2 + f^4)x^4 + 16a(e^2 - f^2)x^3y \\ &+ 48aefx^2y^2 + 16a(f^2 - e^2)xy^3 - 8aefy^4\big]. \end{aligned}$$

We observe that the condition $e = f = 0$ is equivalent to $B_1 = B_2 = 0$, and so we consider two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

2.1: $B_2 \neq 0$. In this case the condition $B_1 = 0$ implies $a = 0$ and evidently systems (4.20) are of the form (1.3).

2.2: $B_2 = 0$. Then considering the condition $B_1 = 0$ we obtain $e = f = 0$, and we obtain the family of systems

$$\dot{x} = a + 2xy, \quad \dot{y} = b - x^2 + y^2,$$

for which we have $B_3 = -12a(x^2 + y^2)^2$. We detect that these systems possess the following two couples of complex invariant lines:

$$b + ia - (x - iy)^2 = 0, \quad b - ia - (x + iy)^2 = 0.$$

According to Lemma 3.4 if $B_3 \neq 0$ then in the real direction $x = 0$ the above systems do not have any invariant line and this means that we could not bring them to the form (1.3) via an affine transformation.

It remains to observe that for $B_3 = 0$ (i.e. $a = 0$) the above systems are of the form (1.3).

Thus all the possibilities in the case $\eta < 0$ are examined and we conclude that the statement (B) of Theorem 1.2 is proved.

4.3. **The subfamily defined by (C):** $\eta = 0$, $\widetilde{M} \neq 0$. In this case by Lemma 3.2 we have to consider the systems ($\mathbf{S}_{III}$) for which calculations yield:

$$\theta = 8h^2(1 - g), \quad \mu_0 = gh^2, \quad \widetilde{N} = (g^2 - 1)x^2 + 2h(g - 1)xy + h^2y^2. \tag{4.21}$$

We consider two cases: $\theta \neq 0$ and $\theta = 0$.

4.3.1. *Case* $\theta \neq 0$. Then $(g-1)h \neq 0$ and by a translation we can assume $d = e = 0$, i.e. we obtain the systems

$$\dot{x} = a + cx + gx^2 + hxy, \quad \dot{y} = b + fy + (g-1)xy + hy^2, \tag{4.22}$$

for which we calculate

$$B_1 = -a^2 b(g-1)^2 h^4.$$

Therefore due to $\theta \neq 0$ the condition $B_1 = 0$ implies $ab = 0$ and we consider two subcases: $a = 0$ or $b = 0$.

1: $a = 0$. In this case applying the transformation $x_1 = x$, $y_1 = gx + hy + c$ we arrive at the family of systems

$$\dot{x}_1 = x_1 y_1, \quad \dot{y}_1 = c^2 - cf + bh + (c + cg - fg)x_1 + (f - 2c)y_1 + gx_1^2 - x_1 y_1 + y_1^2,$$

which is a subfamily of (1.3).

2: $b = 0$. Then systems (4.22) possess the invariant line $y = 0$ and using the transformation $x_1 = y, \quad y_1 = (g-1)x + hy + f$ we arrive at the systems

$$\dot{x}_1 = x_1 y_1, \quad \dot{y}_1 = b' + e' x_1 + l' x_1^2 + (f' + 2m' x_1) y_1 + n' y_1^2,$$

where b', e', f', l', m' and n' are rational functions of the parameters a, c, f, g, h with the same denominator $g-1 \neq 0$. So we obtained the systems belonging to the family of systems (1.3).

4.3.2. *Case* $\theta = 0$. By (4.21) we obtain $h(g-1) = 0$, and since $\mu_0 = gh^2$ we consider two subcases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

1: $\mu_0 \neq 0$. Considering (4.21) we obtain $h \neq 0$, $g = 1$, and then we may assume $h = 1$ due to the rescaling $y \to y/h$. Moreover we may assume $c = d = 0$ via the translation $(x, y) \to (x - d, y + 2d - c)$. So, we obtain the systems

$$\dot{x} = a + x^2 + xy, \quad \dot{y} = b + ex + fy + y^2,$$

for which calculation yields

$$B_1 = -a^2 e^2, \quad B_2 = 648\big[(4a - b)e^2 x^4 + 4ae^2 x^3 y - a^2 y^4\big], \quad H_7 = -4e.$$

The condition $B_1 = 0$ implies $ae = 0$, and we consider two possibilities: $H_7 \neq 0$ and $H_7 = 0$.

1.1: $H_7 \neq 0$. In this case $e \neq 0$ and we obtain $a = 0$. Then applying the transformation $x_1 = x$, $y_1 = x + y$ we arrive at the family of systems

$$\dot{x}_1 = x_1 y_1, \quad \dot{y}_1 = b + (e - f)x_1 + fy_1 + x_1^2 - x_1 y_1 + y_1^2, \tag{4.23}$$

which is a subfamily of (1.3).

1.2: $H_7 = 0$. Then $e = 0$ and this leads to the systems

$$\dot{x} = a + x^2 + xy, \quad \dot{y} = b + fy + y^2,$$

for which $B_1 = 0, \quad B_2 = -648a^2 y^4$. We observe that these systems possess only two (parallel) invariant lines $b + fy + y^2 = 0$ in the direction $y = 0$ which could be real or complex or could coincide. Moreover by Lemma 3.4 in the case $B_2 \neq 0$ we do not have any other invariant line in the second direction $x = 0$. Therefore by the same arguments as we presented earlier for systems (4.18), we conclude that for $B_2 \neq 0$ there cannot exist an affine transformation which brings the above systems to the form (1.3).

Assuming $B_2 = 0$ we obtain $a = 0$ and using the transformation $x_1 = x$, $y_1 = x + y$ we arrive at the systems (4.23) with $e = 0$, i.e. we obtain systems of the form (1.3).

2: $\mu_0 = 0$. Since $\theta = 0$ this implies $h = 0$ and for the systems $(\mathbf{S}_{III})$ we have $\widetilde{N} = (g^2 - 1)x^2$ and we examine two possibilities: $\widetilde{N} \neq 0$ and $\widetilde{N} = 0$.

2.1: $\widetilde{N} \neq 0$. In this case $g - 1 \neq 0$, and we may assume $e = f = 0$ via the translation $(x, y) \to \big(x + f/(1 - g), y + e/(1 - g)\big)$. This leads to the systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b + (g - 1)xy, \tag{4.24}$$

for which we have

$$B_1 = -bd^4(g - 1)^2 g^2, \quad \widetilde{N} = (g^2 - 1)x^2, \quad H_7 = 4d(g^2 - 1).$$

Because $\widetilde{N} \neq 0$ the condition $B_1 = 0$ gives $bdg = 0$ and we consider two cases: $H_7 \neq 0$ and $H_7 = 0$.

2.1.1: $H_7 \neq 0$. Then $d \neq 0$ and we obtain $bg = 0$.

If $b = 0$ then it is evident that after the interchange $x \leftrightarrow y$ systems (4.24) become of the form (1.3).

Assume now $g = 0$. Since $d \neq 0$ we can apply the transformation $x_1 = x$, $y_1 = cx + dy + a$ and this leads to the following subfamily of (1.3):

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = bd + ax_1 + cy_1 + cx_1^2 - x_1y_1.$$

2.1.2: $H_7 = 0$. Then $d = 0$ and we obtain the systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = b + (g - 1)xy, \tag{4.25}$$

which possess the invariant lines $a + cx + gx^2 = 0$ in the real double direction $x = 0$ because $C_2 = x^2y$.

We calculate

$$B_1 = B_2 = 0, \quad B_3 = -3b(g - 1)^2x^4,$$

and we conclude that an invariant line exists in the direction $y = 0$ if and only if $B_3 = 0$. So by the same arguments as we presented earlier for systems (4.18), we conclude that for $B_3 \neq 0$ there cannot exist an affine transformation which brings the above systems to the form (1.3).

Assuming $B_3 = 0$ we obtain $b = 0$ (due to $\widetilde{N} \neq 0$) and in the same manner as above by the interchange $x \leftrightarrow y$ systems (4.24) become of the form (1.3).

2.2: $\widetilde{N} = 0$. In this case $g^2 - 1 = 0$, i.e. $g = 1$ or $g = -1$.

On the other hand for systems $(\mathbf{S}_{III})$ with $h = 0$ we have $\widetilde{K} = g(g - 1)x^2$ and we consider two cases: $\widetilde{K} \neq 0$ and $\widetilde{K} = 0$.

2.2.1: $\widetilde{K} \neq 0$. Then $g - 1 \neq 0$ and this implies $g = -1$. In this case we may assume $e = f = 0$ via the translation $(x, y) \to \big(x + f/2, y + e/2\big)$ and we arrive at the family of systems

$$\dot{x} = a + cx + dy - x^2, \quad \dot{y} = b - 2xy, \tag{4.26}$$

for which calculations yield

$$B_1 = -4bd^4, \quad \theta_3 = 2d^2.$$

2.2.1.1: $\theta_3 \neq 0$. Then the condition $B_1 = 0$ gives $b = 0$ and after the interchange $x \leftrightarrow y$ the above systems become of the form (1.3).

2.2.1.1: $\theta_3 = 0$. This implies $d = 0$ and we obtain the systems (4.25) with $g = -1$. So we repeat the same steps as before in this particular case and we conclude that the systems (4.26) could be brought via an affine transformation to the form (1.3) if and only if either $\theta_3 \neq 0$, or $\theta_3 = 0$ and $B_3 = 0$.

2.2.2: $\widetilde{K} = 0$. Then $g = 1$ and we may assume $c = 0$ due to the translation $(x, y) \to (x - c/2, y)$. Then we obtain the systems

$$\dot{x} = a + dy + x^2, \quad \dot{y} = b + ex + fy.$$

It is clear that in order to have invariant lines in the direction $x = 0$ (respectively $y = 0$) the condition $d = 0$ (respectively $e = 0$) has to be satisfied.

Assume first $d = 0$. Then we obtain two parallel invariant lines and clearly we could not use them for the construction of the transformation which brings these systems to the form (1.3). So we need a real invariant line in the direction $y = 0$, i.e. the condition $e = 0$ must hold. In this case we obtain the invariant line $fy + b = 0$ if $f \neq 0$. However applying the transformation $x_1 = fy + b$, $y_1 = \gamma x + \delta y + \nu$ with free parameters γ, δ and ν, we arrive at the systems

$$\dot{x}_1 = fx_1, \quad \dot{y}_1 = \tilde{Q}(x_1, y_1).$$

As it can be observed these systems do not have the form (1.3).

So we deduce that in the case $\widetilde{N} = 0 = \widetilde{K}$ there cannot exist an affine transformation which brings a system $(\mathbf{S}_{III})$ to an Abel quadratic system of the form (1.3).

Since all the possibilities in the case $\eta = 0$ and $\widetilde{M} \neq 0$ are examined we have that the statement (C) of Theorem 1.2 is proved.

4.4. **Subfamily defined by (D):** $\eta = \widetilde{M} = 0$. According to the conditions provided by Theorem 1.2 we consider two cases: $C_2 \neq 0$ and $C_2 = 0$.

4.4.1. *Case* $C_2 \neq 0$. Then by Lemma 3.2 we examine the systems $(\mathbf{S}_{IV})$ for which calculations yield:

$$\eta = \widetilde{M} = 0, \quad C_2 = x^3, \quad \theta = 8h^3.$$

We consider two subcases: $\theta \neq 0$ and $\theta = 0$.

1: $\theta \neq 0$. Then $h \neq 0$ and by a translation we can assume $c = d = 0$, i.e. we obtain the systems

$$\dot{x} = a + gx^2 + hxy, \quad \dot{y} = b + ex + fy - x^2 + gxy + hy^2, \tag{4.27}$$

for which we calculate $B_1 = -a^3h^6$. So the condition $B_1 = 0$ gives $a = 0$ and then the above systems after the transformation $x_1 = x$, $y_1 = gx + hy$ become

$$\dot{x}_1 = x_1y_1, \quad \dot{y}_1 = bh + (eh - fg)x_1 + fy_1 - hx_1^2 + y_1^2, \tag{4.28}$$

i.e. we obtain a subfamily of (1.3).

2: $\theta = 0$. Then $h = 0$ and we calculate

$$B_1 = -d^6g^3, \quad \widetilde{N} = g^2x^2$$

and we consider two possibilities: $\widetilde{N} \neq 0$ and $\widetilde{N} = 0$.

2.1: $\widetilde{N} \neq 0$. We have $g \neq 0$ and the condition $B_1 = 0$ gives $d = 0$. In this case due to a translation we may assume $e = f = 0$, and this leads to the systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = b - x^2 + gxy. \tag{4.29}$$

Since for these systems we have $C_2 = x^3$ (i.e. we could have real invariant affine lines only in this direction) we conclude, that besides the parallel invariant lines $a + cx + gx^2 = 0$ the above systems cannot have other invariant lines.

Thus applying the same arguments as we present earlier for systems (4.18), we deduce that for $\widetilde{N} \neq 0$ there cannot exist an affine transformation which brings systems (4.29) to the form (1.3).

2.2: $\widetilde{N} = 0$. Then $g = 0$ (this implies $B_1 = 0$) and we arrive at the systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = b + ex + fy - x^2,$$

for which $B_2 = -648d^4x^4$.

2.2.1: $B_2 \neq 0$. We obtain $d \neq 0$ and applying the transformation $x_1 = x$, $y_1 = cx + dy + a$ we obtain the following subfamily of (1.3):

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = bd - af + (de - cf)x_1 + (c + f)y_1 - dx_1^2. \tag{4.30}$$

2.2.2: $B_2 = 0$. Then we obtain the systems

$$\dot{x} = a + cx, \quad \dot{y} = b + ex + fy - x^2,$$

and since the right hand side of the first equation does not have as a factor y we deduce that there could not exist an affine transformation which brings the above systems to the form (1.3).

4.4.2. *Case* $C_2 = 0$. Then by Lemma 3.2 we examine the systems $(\mathbf{S}_V)$ which have the infinite line filled up with singularities. This family of systems is considered in [18], where a total of 9 canonical forms are given for this family: $C_2.1$ – $C_2.9$ (see Table 1, page 741).

We observe that the canonical systems $C_2.1$ – $C_2.4$ for $H_{10} \neq 0$ as well as $C_2.5$ – $C_2.7$ for $H_{10} = 0$ and $H_{12} \neq 0$ after the additional interchange $x \leftrightarrow y$ have the form

$$\dot{x} = xy, \quad \dot{y} = Q_i(x, y), \quad (i = 1, \dots, 7)$$

where $Q_i(x, y)$ is the corresponding to $C_2.i$ quadratic polynomial depending of at least one parameter. It is evident that these canonical systems belong to the family (1.3).

It remains to consider two canonical systems given in Table 1 of [18]:

$$(C_2.8): \begin{cases} \dot{x} = x + x^2, \\ \dot{y} = 1 + xy; \end{cases} \quad (C_2.9): \begin{cases} \dot{x} = x^2, \\ \dot{y} = 1 + xy, \end{cases}$$

and we claim that there does not exist an affine transformation bringing any of these two systems to the form (1.3). Indeed, for both systems $(C_2.8)$ and $(C_2.9)$ we have: $C_2 = 0$ and $H_{10} = 0 = H_{12}$.

On the other hand for systems (1.3) we calculate

$$C_2 = -c_2x^3 - b_1x^2y + (d_1 - a_0)xy^2,$$

and hence the condition $C_2 = 0$ implies $c_2 = b_1 = 0$ and $d_1 = a_0$. Then we obtain the systems

$$\dot{x} = (d_0 + a_0x)y, \quad \dot{y} = c_0 + c_1x + b_0y + a_0x^2,$$

for which calculations yield

$$H_{10} = 36a_0^4c_1^2 = 0, \quad H_{12}\big|_{\{a_0c_1=0\}} = -8a_0^4c_0^2y^2 = 0.$$

If $a_0 \neq 0$ then we obtain $c_1 = 0 = c_0$ and this leads to the degenerate systems

$$\dot{x} = (d_0 + a_0x)y, \quad \dot{y} = y(b_0 + a_0y).$$

On the other hand assuming $a_0 = 0$ we obtain the linear systems

$$\dot{x} = d_0y, \quad \dot{y} = c_0 + c_1x + b_0y.$$

This completes the proof of our claim.

Thus all the cases are examined and the proof of Theorem 1.2 is complete. □

4.5. **Phase portraits of the quadratic systems from the family (D) defined by Theorem 1.2.** According to Lemma 3.2 the systems from the family (D) defined by the condition $\eta = \widetilde{M} = 0$ could be brought via an affine transformation either to the systems ($\mathbf{S}_{IV}$) (if $C_2 \neq 0$), or to the systems ($\mathbf{S}_V$) (if $C_2 = 0$). So we examine these two subfamilies separately. We give examples for the realization of each one of the phase portraits of systems (1.1) constructed, and which belong to one of the above mentioned two classes in the form (a, c, d, g, h, k), (b, e, f, l, m, n).

4.5.1. *Systems* ($\mathbf{S}_{IV}$): $\eta = \widetilde{M} = 0$, $C_2 \neq 0$.

Theorem 4.1. *Assume that for a quadratic system the conditions* $\eta = \widetilde{M} = 0$, *and* $C_2 \neq 0$ *hold. In agreement to Theorem 1.2 this system belongs to the class* QS_{Ab} *if and only if either* $\theta \neq 0$, *or* $\theta = \widetilde{N} = 0$ *and* $B_2 \neq 0$. *In this case its phase portrait is topologically equivalent to one of the pictures given in Figure 1 if and only if the following corresponding conditions are verified:*

Picture $S_{IV}.1 \Leftrightarrow \theta \neq 0,\ \mathbf{D} < 0,\ \mathbf{R} > 0,\ \mathbf{S} > 0,\ \mu_0 < 0,\ \neg(\mathfrak{C}_2),\ \widetilde{U}_1\widetilde{U}_2 < 0$; (3)

Picture $S_{IV}.2 \Leftrightarrow \theta \neq 0,\ \mathbf{D} < 0,\ \mathbf{R} > 0,\ \mathbf{S} > 0,\ \mu_0 < 0,\ \neg(\mathfrak{C}_2),\ \widetilde{U}_1\widetilde{U}_2 > 0$; (3)

Picture $S_{IV}.3 \Leftrightarrow \theta \neq 0,\ \mathbf{D} < 0,\ \mathbf{R} > 0,\ \mathbf{S} > 0,\ \mu_0 < 0,\ (\mathfrak{C}_2)$; (4)

Picture $S_{IV}.4 \Leftrightarrow \theta \neq 0,\ \mathbf{D} < 0,\ \mathbf{R} > 0,\ \mathbf{S} > 0,\ \mu_0 > 0$; (8)

Picture $S_{IV}.5 \Leftrightarrow \begin{cases} \theta \neq 0, \mathbf{D} < 0, (\mathbf{R} \leq 0) \vee (\mathbf{S} \leq 0) \ \text{or} \\ \theta = \widetilde{N} = 0, B_2 \neq 0, \mathbf{U} < 0; \end{cases}$ (12)

Picture $S_{IV}.6 \Leftrightarrow \begin{cases} \theta \neq 0, \mathbf{D} > 0, \mu_0 < 0, \neg(\widehat{\mathfrak{C}}_1), \widetilde{U}_1 < 0, \ \text{or} \\ \theta \neq 0, \mathbf{D} = \mathbf{T} = \mathbf{P} = 0, \mathbf{R} \neq 0, \mathcal{T}_4 \neq 0, \mu_0 < 0; \end{cases}$ (16)

Picture $S_{IV}.7 \Leftrightarrow \theta \neq 0,\ \mathbf{D} > 0,\ \mu_0 < 0,\ \neg(\widehat{\mathfrak{C}}_1), \widetilde{U}_1 > 0$; (16)

Picture $S_{IV}.8 \Leftrightarrow \theta \neq 0,\ \mathbf{D} > 0,\ \mu_0 < 0,\ (\widehat{\mathfrak{C}}_1)$; (18)

Picture $S_{IV}.9 \Leftrightarrow \begin{cases} \theta \neq 0, \mathbf{D} > 0, \mu_0 > 0, \neg(\mathfrak{C}_2), \ \text{or} \\ \theta \neq 0, \mathbf{D} = \mathbf{T} = \mathbf{P} = 0, \mathbf{R} \neq 0, \mathcal{T}_4 \neq 0, \mu_0 > 0, \ \text{or} \\ \theta = \widetilde{N} = 0, B_2 \neq 0, \mathbf{U} > 0, \neg(\mathfrak{C}_9); \end{cases}$ (23)

Picture $S_{IV}.10 \Leftrightarrow \begin{cases} \theta \neq 0, \mathbf{D} > 0, \mu_0 > 0, (\mathfrak{C}_2), \ \text{or} \\ \theta = \widetilde{N} = 0, B_2 \neq 0, \mathbf{U} > 0, (\mathfrak{C}_9); \end{cases}$ (24)

Picture $S_{IV}.11 \Leftrightarrow \theta \neq 0,\ \mathbf{D} = 0,\ \mathbf{T} < 0\ \mu_0 < 0,\ B_2\widetilde{U}_1 \neq 0,\ E_1 \neq 0$; (30)

Picture $S_{IV}.12 \Leftrightarrow \theta \neq 0,\ \mathbf{D} = 0,\ \mathbf{T} < 0\ \mu_0 < 0,\ B_2\widetilde{U}_1 \neq 0,\ E_1 = 0$; (34)

Picture $S_{IV}.13 \Leftrightarrow \theta \neq 0,\ \mathbf{D} = 0,\ \mathbf{T} < 0\ \mu_0 < 0,\ B_2 = 0$; (30)

Picture $S_{IV}.14 \Leftrightarrow \theta \neq 0,\ \mathbf{D} = 0,\ \mathbf{T} < 0\ \mu_0 < 0,\ \widetilde{U}_1 = 0$; (30)

Picture $S_{IV}.15 \Leftrightarrow \theta \neq 0,\ \mathbf{D}=0,\ \mathbf{T}<0\ \mu_0>0$; (37)

Picture $S_{IV}.16 \Leftrightarrow \begin{cases} \theta \neq 0, \mathbf{D}=0, \mathbf{T}>0, E_1 \neq 0 \text{ or} \\ \theta \neq 0, \mathbf{D}=\mathbf{T}=\mathbf{P}=\mathbf{R}=0, \mu_0>0; \end{cases}$ (44)

Picture $S_{IV}.17 \Leftrightarrow \begin{cases} \theta \neq 0, \mathbf{D}=0, \mathbf{T}>0, E_1 = 0 \text{ or} \\ \theta = \widetilde{N}=0, B_2 \neq 0, \mathbf{U}=0; \end{cases}$ (47)

Picture $S_{IV}.18 \Leftrightarrow \theta \neq 0,\ \mathbf{D}=0,\ \mathbf{T}=0\ \mathbf{P}\neq 0$; (50)

Picture $S_{IV}.19 \Leftrightarrow \theta \neq 0,\ \mathbf{D}=0,\ \mathbf{T}=0\ \mathbf{P}=0,\ \mathbf{R}\neq 0,\ \mathcal{T}_4=0,\ \mu_0<0$; (60)

Picture $S_{IV}.20 \Leftrightarrow \theta \neq 0,\ \mathbf{D}=0,\ \mathbf{T}=0\ \mathbf{P}=0,\ \mathbf{R}\neq 0,\ \mathcal{T}_4=0,\ \mu_0>0$; (64)

Picture $S_{IV}.21 \Leftrightarrow \theta \neq 0,\ \mathbf{D}=0,\ \mathbf{T}=0\ \mathbf{P}=0,\ \mathbf{R}=0,\ \mu_0<0$. (67)

The right-most entry in the table above corresponds to the global topological configurations of the singularities (finite and infinite), according to the notation in the set of diagrams provided by [4, Main Theorem].

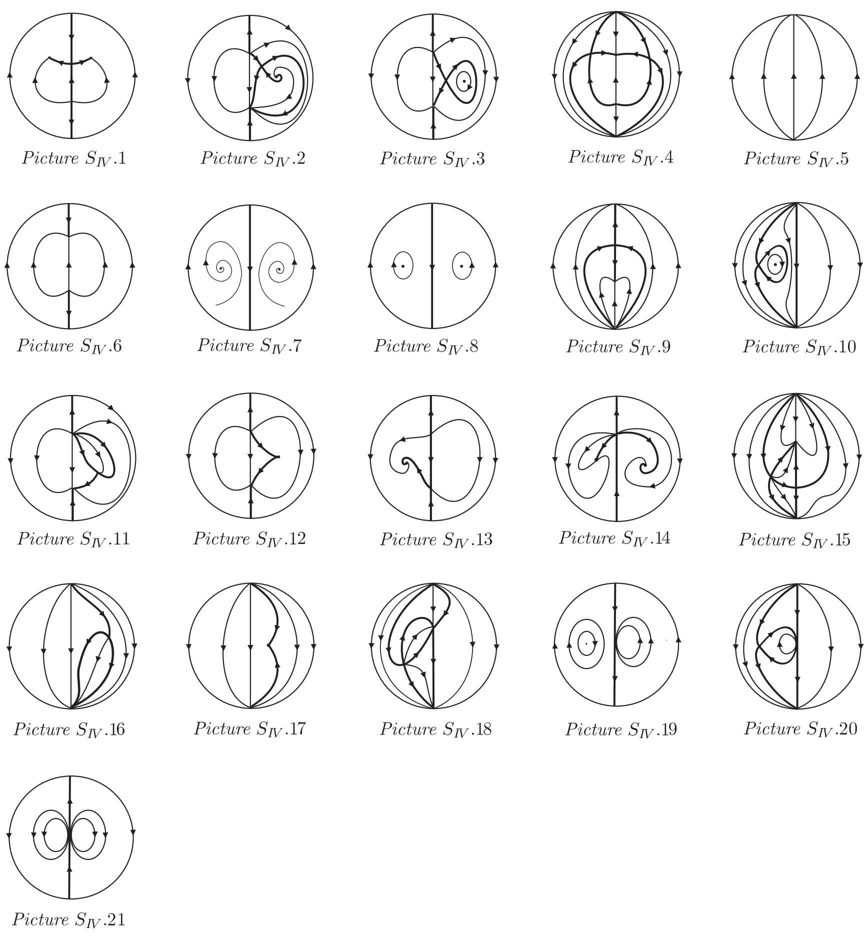


FIGURE 1. Global phase portraits of quadratic systems with $\eta = \widetilde{M} = 0$, $C_2 \neq 0$.

Proof. We prove this theorem following the conditions provided by the statement (D) of Theorem 1.2 in the case $C_2 \neq 0$. So we discuss two cases: $\theta \neq 0$ and $\theta = \widetilde{N} = 0$, $B_2 \neq 0$.

1: *Case* $\theta \neq 0$. Considering the systems (4.27) and the corresponding transformed systems (4.28) we examine the family:

$$\dot{x} = xy \equiv P(x,y), \quad \dot{y} = b + ex + fy - hx^2 + y^2 \equiv Q(x,y). \tag{4.31}$$

We shall consider step by step the conditions provided by the Diagrams 1-6 from [4], taking into account that the conditions $\eta = \widetilde{M} = 0$, and $C_2 \neq 0$ are satisfied.

For these systems calculations yield:

$$\begin{gathered} C_2 = hx^3, \quad \mu_0 = -h, \quad \mathbf{D} = -48b^2(f^2-4b)(e^2+4bh), \quad B_2 = -648b^2h^2x^4, \\ \mathcal{T}_4 = -f^2h(9b-2f^2), \quad \mathcal{T}_3 = -fh(18b-5f^2), \quad \mathcal{T}_2 = -3(3b-f^2)h, \\ \mathcal{F} = 9fh/8, \quad \mathcal{F}_1 = 0, \quad \mathcal{F}_2 = -9f^2h^2/2 = -\mathcal{F}_3, \\ \mathcal{B} = -9(9e^2+36bh-8f^2h)/8, \quad \mathcal{H} = -9h/2, \quad \sigma = f+3y. \end{gathered} \tag{4.32}$$

We observe that condition $C_2 \neq 0$ implies $\mu_0 \neq 0$, and according with [5, Table 6.2] the above systems possess finite singularities of total multiplicity four. More exactly we have the singularities $M_{1,2}(0,y_{1,2})$ and $M_{3,4}(x_{3,4},0)$, where

$$y_{1,2} = \big(-f \pm \sqrt{f^2-4b}\big)/2, \quad x_{3,4} = \big(e \pm \sqrt{e^2+4bh}\big)/(2h). \tag{4.33}$$

It is clear that the singularities $M_{1,2}(0,y_{1,2})$ are located on the real invariant line $x = 0$, and these singularities are real if $f^2 - 4b > 0$ and they are complex if $f^2 - 4b < 0$.

First we prove the following lemma:

Lemma 4.2. *For a system* (4.31) *the conditions* $(\mathfrak{C}_1)$ *as well as the conditions* $(\mathfrak{C}_5)$ - $(\mathfrak{C}_{10})$ *and* $(\widehat{\mathfrak{C}}_2)$ *could not be satisfied.*

Proof. First, from (4.32) we obtain that for systems (4.31) the condition $\sigma = f + 3y \neq 0$ holds. Therefore considering (3.1) and (3.2) we deduce that the conditions $(\mathfrak{C}_5)$ - $(\mathfrak{C}_{10})$ and $(\widehat{\mathfrak{C}}_2)$ could not be satisfied for these systems.

It remains to examine the conditions $(\mathfrak{C}_1)$. According to (3.1) these conditions imply $\mathcal{T}_3 \neq 0$ and $\mathcal{F}_2 = 0$. However considering (4.32) it is clear that the condition $\mathcal{T}_3 \neq 0$ (i.e. $fh \neq 0$) implies $\mathcal{F}_2 \neq 0$. This completes the proof of the lemma. □

According to [5, Table 6.2] all the finite singularities of systems (4.31) are distinct if $\mathbf{D} \neq 0$ and we have multiple singular points if $\mathbf{D} = 0$. So we examine three subcases: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

1.1: $\mathbf{D} < 0$. According to [5, Table 6.2] systems (4.31) possess either four real distinct finite singularities in the case $\mathbf{R} > 0$ and $\mathbf{S} > 0$, or four complex finite singularities if $(\mathbf{R} \leq 0) \vee (\mathbf{S} \leq 0)$.

1.1.1: $\mathbf{R} > 0$ and $\mathbf{S} > 0$. So systems (4.31) possess four real distinct finite singularities and following [4, Diagram 1, page 3] we consider two cases: $\mu_0 < 0$ and $\mu_0 > 0$.

1.1.1.1: $\mu_0 < 0$. According to this diagram we could either have the topological configuration (3) $s, a, a, a; S$ if $\neg(\mathfrak{C}_2)$, or (4) $s, a, a, c; S$ if $(\mathfrak{C}_2)$.

Consider first the configuration (3). It is clear that if the saddle is located on the invariant line $x = 0$ then we have the separatrix connection between the finite saddle and the infinite one. So we need a condition to distinguish whether the saddle is located on the invariant line or not.

On the other hand denoting by Δ_i $(i = 1, 2, 3, 4)$ the determinant of the linear matrix corresponding to the singular point M_i we calculate

$$\Delta_{1,2} = -2b + (f^2 \pm f\sqrt{f^2 - 4b}\,)/2 \ \Rightarrow\ \Delta_1\Delta_2 = b(4b - f^2).$$

We remark that when two finite singularities coalesce (M_1 with M_2 if $4b-f^2 = 0$, or M_3 with M_4 if $e^2 + 4bh = 0$, or M_1 with M_4 if $b = 0$) it is important to distinguish if both singularities are located on the invariant line. For systems (4.31) we have:

$$\widetilde{U}_1 = -27(f^2 - 4b)h/8, \quad \widetilde{U}_2 = 9bh/2. \tag{4.34}$$

Therefore $\widetilde{U}_1\widetilde{U}_2 = 243b(4b - f^2)h^2/16 = 243\Delta_1\Delta_2 h^2/16$ and we conclude that the following remark is valid:

Remark 4.3. Assume that the singularities of a system (4.31) located on the invariant line $x = 0$ are real and in addition the condition $\widetilde{U}_1\widetilde{U}_2 \neq 0$ holds. Then $\operatorname{sign}(\Delta_1\Delta_2) = \operatorname{sign}(\widetilde{U}_1\widetilde{U}_2)$, i.e. on the invariant line of this system lies exactly one saddle if and only if $\widetilde{U}_1\widetilde{U}_2 < 0$.

So, considering the above remark and the fact that we have a single saddle, in the case of the topological configuration (3) $s, a, a, a; S$ (see [4]) we obtain Picture $S_{IV}.1$ if $\widetilde{U}_1\widetilde{U}_2 < 0$ and Picture $S_{IV}.2$ if $\widetilde{U}_1\widetilde{U}_2 > 0$. The corresponding examples are:
Picture $S_{IV}.1$ if $\neg(\mathfrak{C}_2)$ and $\widetilde{U}_1\widetilde{U}_2 < 0$ [Ex: $(0, 0, 0, 0, 1/2, 0), (1/8, 0, 1, -1, 0, 1)$];
Picture $S_{IV}.2$ if $\neg(\mathfrak{C}_2)$ and $\widetilde{U}_1\widetilde{U}_2 > 0$ [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(-3/4, 2, 1, -1, 0, 1)$];

Consider now the configuration (4): $s, a, a, c; S$. Since we have a center (i.e. the conditions $(\mathfrak{C}_2)$ hold), considering [20] (see also [21]) we obtain the unique phase portrait given by Picture $S_{IV}.3$ [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(-1, \sqrt{5}, 0, -1, 0, 1)$].

1.1.1.2: $\mu_0 > 0$. In this case from [4, Diagram 1, page 3] we could either have the topological configuration (8) $s, s, a, a; N$ if $\neg((\widehat{\mathfrak{C}}_1) \vee (\widehat{\mathfrak{C}}_2))$, or configuration (9) $s, s, c, c; N$ if $(\widehat{\mathfrak{C}}_1) \vee (\widehat{\mathfrak{C}}_2)$.

We claim that configuration (9) with two centers is not realizable for systems (4.31). According to Lemma 4.2 the conditions $(\widehat{\mathfrak{C}}_2)$ are ruled out for these systems.

Consider now the conditions $(\widehat{\mathfrak{C}}_1)$. According to (4.32) and (3.2) the condition $\mathcal{H} = -9h/2 < 0$ is necessary, but this implies $h > 0$ which contradicts $\mu_0 = -h > 0$. This completes the proof of the claim.

It remains to examine the configuration (8) $s, s, a, a; N$. It is not too difficult to convince ourself that both saddles could not be located on the invariant line $x = 0$ (since $\Delta_1 + \Delta_2 = f^2 - 4b > 0$ by (4.33)). If both singularities on $x = 0$ are nodes, then we obtain Picture $S_{IV}.4$ [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(-3/4, 1, 1, 1, 0, 1)$].

Assume now that on the invariant line $x = 0$ we have a saddle and a node.

Lemma 4.4. *Systems* (4.31) *with configuration* $s, s, a, a; N$ *and both a saddle and a node on* $x = 0$ *possess only one phase portrait which is topologically equivalent to Picture* $S_{IV}.4$.

Proof. First of all we simplify the canonical systems (4.31) using the corresponding conditions for this case. Since we assume that systems (4.31) with $\mu_0 > 0$ have four real singularities and in addition on the invariant line $x = 0$ we have a node and a saddle then, as it was mentioned earlier, for these systems the conditions

$$h < 0, \quad f^2 - 4b > 0, \quad e^2 + 4bh > 0, \quad \Delta_1\Delta_2 = b(4b - f^2) < 0,$$

hold. These conditions imply $b > 0$ and $ef \neq 0$, and therefore we may assume $h = -1 = e$ due to the rescaling $(x, y, t) \mapsto (ex/h, -ey/\sqrt{-h}, -\sqrt{-h}\, t/e)$. So we arrive at the 2-parameter family of systems

$$\dot{x} = xy, \quad \dot{y} = b - x + fy + x^2 + y^2 \tag{4.35}$$

for which in addition we may assume $f < 0$ due to the rescaling $(x, y, t) \mapsto (x, -y, -t)$. Therefore considering the above conditions depending on four parameters, for systems (4.35) the conditions $0 < b < 1/4$ and $f < -2\sqrt{b}$ are satisfied.

We recall that the above systems possess four real singularities $M_{1,2}(0, y_{1,2})$ and $M_{3,4}(x_{3,4}, 0)$ with the coordinates

$$y_{1,2} = \big(-f \pm \sqrt{f^2 - 4b}\,\big)/2, \quad x_{3,4} = \big(1 \mp \sqrt{1 - 4b}\,\big)/2).$$

Since $0 < b < 1/4$, $f < -2\sqrt{b}$, $\Delta_1\Delta_2 < 0$ and $\Delta_3\Delta_4 < 0$, it is not too difficult to show that the following conditions hold:

$$0 < y_2 < y_1, \quad \Delta_2 < 0 < \Delta_1, \quad 0 < x_3 < x_4, \quad \Delta_4 < 0 < \Delta_3.$$

In other words the two saddles are located at M_2 and M_4 which is in accordance with Berlinskiĭ theorem [8].

We observe that the isocline on which $\dot{y} = 0$ is a real ellipse, because the determinant of the conic is $(-1 + 4b - f^2)/4 < 0$ due to the condition $f^2 - 4b > 0$. We plot this isocline in dotted line as well as the isocline $\dot{x} = xy = 0$ and the line conecting the two saddles M_2 and M_4 (see Figure 2).

We claim that the non vertical eigenvector of the saddle M_2 has a negative slope Sl_{M_2} which is smaller that the slope Sl_L of the line joining both saddles, and it is bigger than the slope Sl_{TE} of the tangent to the ellipse at the point M_2. Indeed, it is not difficult to determine that

$$Sl_{M_2} = \frac{\sqrt{1-4b}+1}{f - \sqrt{f^2-4b}}, \quad Sl_L = \frac{2}{f - \sqrt{f^2-4b}}, \quad Sl_{TE} = -\frac{1}{\sqrt{f^2-4b}},$$

and then we obtain

$$Sl_{M_2} - Sl_L = \frac{\sqrt{1-4b}+1}{f - \sqrt{f^2-4b}} < 0, \; Sl_{M_2} - Sl_{TE} = -\frac{\sqrt{f^2-4b}+f}{\sqrt{f^2-4b}\left(\sqrt{f^2-4b}-f\right)} > 0,$$

because of the conditions $0 < b < 1/4$ and $f < -2\sqrt{b}$. This completes the proof of the claim.

On the other hand we observe that the line connecting the two saddles, i.e. the line

$$y = Sl_L x + m, \quad m = \frac{2b}{\sqrt{f^2-4b}-f},$$

is an isocline for systems (4.35). Indeed, calculations yield

$$\frac{dy}{dx}\Big|_{y=Sl_L x+m} = \frac{b - x + x^2 + fy + y^2}{xy}\Big|_{y=Sl_L x+m} = \frac{f - \sqrt{1-4b}\sqrt{f^2-4b}}{2b}.$$

Moreover we detect that the slope of the line is greater than the slope of the flow on the line, because we have

$$Sl_L - \frac{f - \sqrt{1-4b}\sqrt{f^2-4b}}{2b} = \frac{(\sqrt{1-4b}+1)(\sqrt{f^2-4b}-f)}{4b} > 0.$$

We remark that all this information is presented in Figure 2.

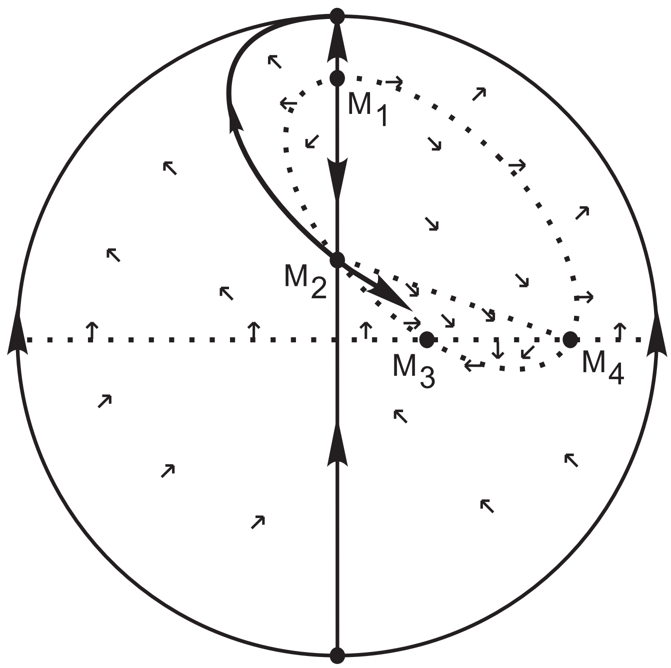


FIGURE 2. A scheme of the flow on the phase plane of systems (4.35).

Thus the separatrix of the saddle M_2 located in the first quadrant cannot go to the infinite singularity $[0:1:0]$, and hence it can only either go to the anti-saddle M_3, or connect with a separatrix of M_4. In the first case we arrive in the unique way to the phase portrait equivalent to Picture $S_{IV}.4$.

We claim that the connection of M_2 with M_4 is not possible. Indeed, we observe that the ellipse and the isocline $y = 0$ intersect at the saddle M_4 producing four regions around M_4. It is easy to determine that due to the horizontal flow on the ellipse and the vertical flow on the isocline $y = 0$, in each one of these regions there must be one separatrix of this saddle in each of the four regions. But the separatrix which is inside the ellipse and over the x-axis (which is the separatrix supposed to produce the connection) cannot be below the line joining the saddles because of the flow along all these lines. This proves the claim and completes the proof of the lemma. □

1.1.2: $(\mathbf{R} \le 0) \vee (\mathbf{S} \le 0)$. It was mentioned earlier that in this case we have four complex singularities. According to [4, Diagram 1, page 3] we could have a single topological configuration (12) N, which leads to Picture $S_{IV}.5$: [Ex: $(0,0,0,0,1/2,0)$, $(1,0,0,1,0,1)$].

1.2: $\mathbf{D} > 0$. According to [5, Table 6.2] systems (4.31) possess two real and two complex finite singularities. Considering the coordinates (4.33) of the singularities $M_{1,2}(0, y_{1,2})$ we observe that they are real if $f^2 - 4b > 0$, and they are complex if $f^2 - 4b < 0$.

On the other hand for systems (4.31) we have $\widetilde{U}_1 = -27(f^2-4b)h/8$ and $\mu_0 = -h$ and hence, $\operatorname{sign}(f^2 - 4b) = \operatorname{sign}(\mu_0 \widetilde{U}_1)$. So we arrive at the next remark.

Remark 4.5. Assume that for a quadratic system (4.31) the condition $\mathbf{D}\mu_0 \neq 0$ holds. Then the finite singularities located on the invariant line $x = 0$ of this system are real if $\mu_0 \widetilde{U}_1 > 0$, and they are complex if $\mu_0 \widetilde{U}_1 < 0$.

1.2.1: $\mu_0 < 0$. Since $\eta = \widetilde{M} = 0$, by [4, Diagram 1, page 4] systems (4.31) could either have the topological configuration (16) a, a; if $\neg((\mathfrak{C}_1) \vee (\widehat{\mathfrak{C}}_1))$, or (17) $a, c; S$ if $(\mathfrak{C}_1)$, or (18) $c, c; S$ if $(\widehat{\mathfrak{C}}_1)$.

We observe that by Lemma 4.2 the conditions $(\mathfrak{C}_1)$ from (3.1) are incompatible with systems (4.31). This means that the topological configuration (17) could not be realized for these systems.

Since $\mu_0 < 0$, considering Remark 4.5 it is not difficult to show that in the case of the configuration (16) $a, a; S$ we obtain Picture $S_{IV}.6$ if $\widetilde{U}_1 < 0$ and Picture $S_{IV}.7$ if $\widetilde{U}_1 > 0$.

On the other hand the configuration (18) $c, c; S$ leads to the Picture $S_{IV}.8$. We exhibit three examples of realization of the pictures:
Picture $S_{IV}.6$: [Ex: $(0,0,0,0,1/2,0)$, $(-1,2,1,-2,0,1)$];
Picture $S_{IV}.7$: [Ex: $(0,0,0,0,1/2,0)$, $(2,0,-1,-1,0,1)$];
Picture $S_{IV}.8$: [Ex: $(0,0,0,0,1/2,0)$, $(1,0,0,-1,0,1)$].

1.2.2: $\mu_0 > 0$. Considering the condition $\eta = \widetilde{M} = 0$, by [4, Diagram 1, page 4] systems (4.31) could either have the topological configuration (23) $s, a; N$ if $\neg((\mathfrak{C}_2) \vee (\mathfrak{C}_7))$, or (24) $s, c; N$ $(\mathfrak{C}_2) \vee (\mathfrak{C}_7)$. However by Lemma 4.2 the conditions $(\mathfrak{C}_7)$ could not be satisfied.

Thus considering Remark 4.5 and the condition $\mu_0 > 0$ we deduce that the configuration (23) $s, a; N$ with the condition $\neg(\mathfrak{C}_2)$ leads to Picture $S_{IV}.9$ if $\widetilde{U}_1 > 0$. If $\widetilde{U}_1 < 0$ then by Remark 4.5 the real finite singularities are located outside the invariant straight line and hence the anti-saddle could be a focus. Therefore we conclude that there could be three topologically distinct phase portraits presented in Figure 3. In the case when the anti-saddle is a node only the phase portrait given by Picture $S_{IV}.9$ (which is topologically equivalent to Picture (a) in Figure 3) is possible. But when besides the saddle we have a focus, then we could have two more topologically distinct possibilities, given by Picture (b) (possessing a loop) and Picture (c) (possessing a limit cycle).

Lemma 4.6. *The phase portraits given by Pictures* (b) *and* (c) *of Figure 3 cannot be realizable for systems* (4.31).

Proof. By Dulac's theorem (see [9, Theorem 7.12]) since taking the function $B(x, y) = x^{-3}$ in the simply connected region $R = \{x > 0\}$ (or $R = \{x < 0\}$), the divergence $\partial(BP)/\partial x + \partial(BQ)/\partial y = f x^{-3}$ is strictly positive or negative in R if $f \neq 0$ with

P and Q given as in systems (4.31), these systems cannot have periodic orbits lying entirely in R. This proves that Picture (c) cannot be realizable when $f \neq 0$. Moreover since the proof of Dulac's theorem also works for a homoclinic loop and its singular point, it follows that in the case $f \neq 0$ Picture (b) is also non realizable.

Assume now $f = 0$. Then the systems $\dot{x} = BP$ and $\dot{y} = BQ$ are Hamiltonian in R, because their divergence is identically zero. So these systems cannot have a focus, and consequently systems (4.31) also cannot have a focus. This completes the proof of the lemma. □

It is clear that the configuration (24) $s, c; N$ (with the conditions $(\mathfrak{C}_2)$) leads to Picture $S_{IV}.10$. The realization of the phase portraits in the case $\mu_0 > 0$ is proved by the following examples:
Picture $S_{IV}.9$: [Ex: $(0,0,0,0,1/2,0)$, $(1/8,0,1,1,0,1)$];
Picture (a), Figure 3: [Ex: $(0,0,0,0,1/2,0)$, $(2,3,-1,1,0,1)$] $\cong$ Picture $S_{IV}.9$;
Picture $S_{IV}.10$: [Ex: $(0,0,0,0,1/2,0)$, $(1/2,2,0,1,0,1)$].

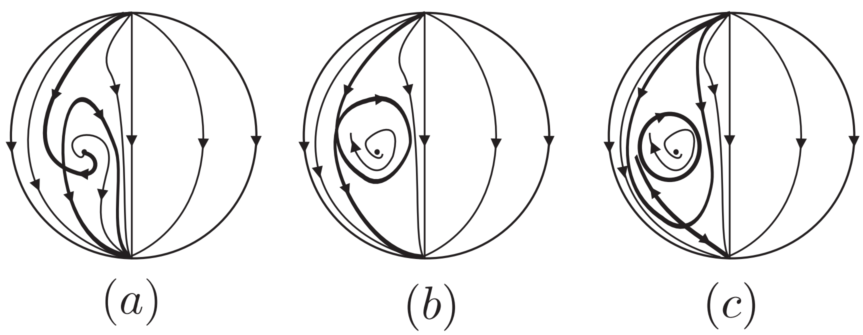


FIGURE 3. Some phase portraits of quadratic systems with $\eta = \widetilde{M} = 0$, $C_2 \neq 0$.

1.3: $\mathbf{D} = 0$. If $\mathbf{T} \neq 0$ then according to [5, Table 6.2] systems (4.31) possess one double real singular point and two distinct finite singularities. Moreover these two singular points are real if $\mathbf{T} < 0$ and complex if $\mathbf{T} > 0$. In the case $\mathbf{T} = 0$ and $\mu_0 \neq 0$ by [5, Table 6.2] these systems possess at most two finite singularities of total multiplicity four.

Considering (4.32) we detect that the condition $\mathbf{D} = 0$ gives three possibilities: (1) $b = 0$; (2) $f^2 - 4b = 0$ and (3) $e^2 + 4bh = 0$. Taking into account the values of the invariant polynomials $\mathbf{D}$, B_2 and $\widetilde{U}_1$ from (4.32) and (4.34) it is easy to determine, that due to $\mu_0 \neq 0$ the three mentioned possibilities could be distinguished by means of these invariant polynomials. More precisely, considering also the coordinates (4.33) of the finite singularities $M_{1,2}(0, y_{1,2})$ and $M_{3,4}(x_{3,4}, 0)$ we have the next remark.

Remark 4.7. (i) The following conditions are equivalent:

(1) $b = 0 \Leftrightarrow B_2 = 0$;

(2) $f^2 - 4b = 0 \Leftrightarrow \widetilde{U}_1 = 0$;

(3) $e^2 + 4bh = 0 \Leftrightarrow \mathbf{D} = 0$ and $B_2\widetilde{U}_1 \neq 0$.

(ii) In the case $B_2 = 0$ (respectively $\widetilde{U}_1 = 0$; $\mathbf{D} = 0$, $B_2\widetilde{U}_1 \neq 0$) the singular point M_4 coalesces with M_1 (respectively M_2 with M_1; M_4 with M_3).
(iii) the condition $B_2 = 0$ (i.e. $b = 0$) implies $\mathbf{T} \leq 0$ since $\mathbf{T} = -3e^2f^2x^2y^2(fhx - ey)^2$.

In what follows we consider three cases: $\mathbf{T} < 0$, $\mathbf{T} > 0$ and $\mathbf{T} = 0$.

1.3.1: $\mathbf{T} < 0$. Then all three finite singularities (one of them is double) are real and following [4, Diagram 1, page 4] we consider two cases: $\mu_0 < 0$ and $\mu_0 > 0$.

1.3.1.1: $\mu_0 < 0$. Since $\eta = \widetilde{M} = 0$, according to this diagram we could either have the topological configuration (30) $a, a, sn; S$ if $E_1 \neq 0$, or (34) $a, a, cp; S$ if $E_1 = 0$.

Considering Remark 4.7 (i) we examine three subcases: $B_2\widetilde{U}_1 \neq 0$; $B_2 = 0$ and $\widetilde{U}_1 = 0$.

1.3.1.1.1: $B_2\widetilde{U}_1 \neq 0$. Then by Remark 4.7 (i) the condition $\mathbf{D} = 0$ yields $e^2 + 4bh = 0$ and we obtain $b = -e^2/(4h)$. In this case for systems (4.31) we calculate:

$$\begin{aligned}\mathbf{T} &= -\,3e^2(e^2 + f^2h)x^2(ehx^2 + 2fhxy - ey^2)^2/(16h),\\ E_1 &= -\,e^2f(e^2 + f^2h)/(8h),\end{aligned} \tag{4.36}$$

and since $\mathbf{T} < 0$, the condition $E_1 = 0$ is equivalent to $f = 0$.

By Remark 4.7 (ii) we deduce that in this case the singularities located outside the invariant line coalesced. So in the case $E_1 \neq 0$ the configuration (30) $a, a, sn; S$ leads to the phase portrait given by Picture $S_{IV}.11$ (see Figure 1).

If $E_1 = 0$ we have the topological configuration (34) $a, a, cp; S$ which leads to the Picture $S_{IV}.12$.

Corresponding examples are the following:
Picture $S_{IV}.11$: [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(-1, 2, -1, -1, 0, 1)$];
Picture $S_{IV}.12$: [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(-1, -2, 0, -1, 0, 1)$].

1.3.1.1.2: $B_2 = 0$. Then by Remark 4.7 we have $b = 0$ and in this case we obtain:

$$\mathbf{T} = -3e^2f^2x^2y^2(fhx - ey)^2, \quad E_1 = -e^2f^3/2, \tag{4.37}$$

and evidently the condition $\mathbf{T} \neq 0$ implies $E_1 \neq 0$. So in this case we could only have the configuration (30) $a, a, sn; S$. Taking into account Remark 4.7 (ii) we arrive at Picture $S_{IV}.13$: [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(0, -1, -1, -1, 0, 1)$].

1.3.1.1.3:$\widetilde{U}_1 = 0$. By Remark 4.7 we have $b = f^2/4$ and in this case we obtain:

$$\mathbf{T} = -3f^2(e^2 + f^2h)y^2(-fhx^2 + 2exy + fy^2)^2/16, \quad E_1 = f^3(e^2 + f^2h)/16. \tag{4.38}$$

Clearly the condition $\mathbf{T} \neq 0$ implies $E_1 \neq 0$ and again we could have only the configuration (30) $a, a, sn; S$. In this case according to Remark 4.7 (ii) the singularities located on the invariant line coalesced. Therefore we arrive at Picture $S_{IV}.14$: [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(1/4, 1, -1, -1, 0, 1)$].

1.3.1.2: $\mu_0 > 0$. In this case by [4, Diagram 1, page 4] we could either have the topological configuration (37) $s, a, sn; N$ if $E_1 \neq 0$, or (40) $s, a, cp; N$ if $E_1 = 0$ and $\neg(\mathfrak{C}_8)$, or (41) $s, c, cp; N$ if $E_1 = 0$ and $(\mathfrak{C}_8)$. However by Lemma 4.2 the conditions $(\mathfrak{C}_8)$ are incompatible with systems (4.31). So it remains to examine the phase portraits given by the topological configurations (37) and (40).

1.3.1.2.1: $B_2\widetilde{U}_1 \neq 0$. Then by Remark 4.7 (i) we have $e^2 + 4bh = 0$, i.e. $b = -e^2/(4h)$ and we calculate:

$$\mathbf{T} = -3e^2(e^2 + f^2h)x^2(ehx^2 + 2fhxy - ey^2)^2/(16h), \quad E_1 = -e^2f(e^2 + f^2h)/(8h),$$

and since $\mathbf{T} < 0$, the condition $E_1 = 0$ is equivalent to $f = 0$. However for $f = 0$ we obtain $\mathbf{T} = -3e^6x^2(hx^2 - y^2)^2/(16h)$ and therefore the condition $\mathbf{T} < 0$ implies $h > 0$, and this contradicts $\mu_0 = -h > 0$.

Thus in the case $B_2\widetilde{U}_1 \neq 0$ we could only have the configuration (37) $s, a, sn; N$ and considering Remark 4.7 (ii) there are two singularities (saddle and node) on the invariant line $x = 0$ and a saddle-node outside.

In the previous case with the configuration (8) $s, s, a, a; N$ we have proved (see Lemma 4.4) that systems (4.31) could only have one phase portrait which is topologically equivalent to Picture $S_{IV}.4$. We proved Lemma 4.4 by means of a deep study of slopes of the flow on several lines and isoclines and the slopes of some eigenvectors. Perhaps we could use the same technique here, but the existence of a saddle-node would complicate essentially such kind of proof because of the different properties of its eigenvectors. So we will give here another proof of the topological possibilities of the phase portraits and their perturbations.

Lemma 4.8. *Systems* (4.31) *with configuration (37)* $s, a, sn; N$ *possess only one phase portrait which is topologically equivalent to Picture* $S_{IV}.15$.

Proof. Even though systems (4.31) with configuration (8) $s, s, a, a; N$ have a triple node at infinity, this node behaves topologically as a simple node, and therefore the configuration (8) is structurally stable for systems (4.31). We have proved in Lemma 4.4 that the only phase portrait is Picture $S_{IV}.4$ which is the same as phase portrait $\mathbb{S}^2_{3,4}$ from [1]. So we have proved that the phase portrait could not coincide with any other four structurally stable phase portraits with configuration (8) given in [1] and it could neither have separatrix connections.

In the case of configuration (37) $s, a, sn; N$ for systems (4.31), again the triple infinite node behaves topologically as a simple node. Inside systems (4.31) the configuration (37) (corresponding to the condition $\mathbf{D} = 0$) is part of the border of configuration (8) (corresponding to the condition $\mathbf{D} > 0$). So since Picture $S_{IV}.4$ is structurally stable, all the phase portraits with configuration (37) must be unstable of codimension one. According to [2] there are 9 codimension one structurally unstable phase portraits ranging from $\mathbb{U}^1_{A,2}$ to $\mathbb{U}^1_{A,10}$ with configuration (37). We do not plot them to save space. From these 9 phase portraits the only one which can be perturbed into phase portrait $\mathbb{S}^2_{3,4}$ inside the family (4.31) (having 4 finite singular points) is the phase portrait $\mathbb{U}^1_{A,7}$ from [2].

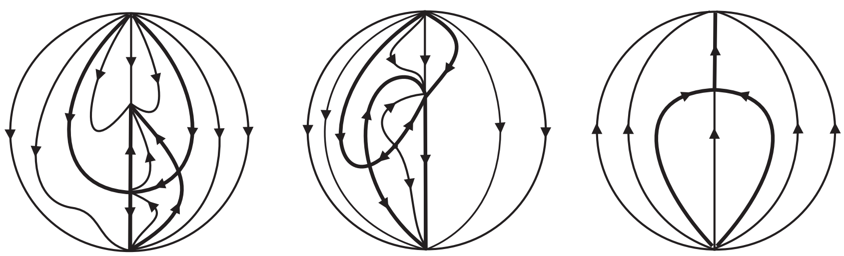
Since we have proved that only $\mathbb{S}^2_{3,4}$ is realizable with configuration (8), then the only codimension one realizable phase portrait for these systems is $\mathbb{U}^1_{A,7}$ which is topologically equivalent Picture $S_{IV}.15$.

For the same reason, as we proved that $\mathbb{S}^2_{3,4}$ was the only possibility for configuration (8) in QS_{Ab}, here we cannot have other phase portraits with separatrix connections and hence $\mathbb{U}^1_{A,7}$ is also the only possibility for configuration (37). As an example of Picture $S_{IV}.15$ we may take [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(1/8, 1, -1, 2, 0, 1)$]. □

1.3.1.2.2: $B_2 = 0$. According to (4.37) in this case the condition $\mathbf{T} \neq 0$ implies $E_1 \neq 0$, and we could only have the configuration (37) $s, a, sn; N$. Taking into account Remark 4.7 (ii) we arrive at Picture (a) of Figure 4: [Ex: $(0,0,0,0,1/2,0)$, $(0,-1,-1,1,0,1)$].

1.3.1.2.3: $\widetilde{U}_1 = 0$. Considering (4.38) we conclude that the condition $\mathbf{T} \neq 0$ implies $E_1 \neq 0$ and again systems (4.31) could only have the configuration (37) $s, a, sn; N$. In this case taking into account Remark 4.7 (ii) we obtain Picture (b) of Figure 4: [Ex: $(0,0,0,0,1/2,0)$, $(1/4,1,-1,1/2,0,1)$].

We remark that the phase portraits Picture (a) and Picture (b) from Figure 4 are topologically equivalent to Picture $S_{IV}.15$.



(a) (b) (c)

FIGURE 4. Some phase portraits of quadratic systems with $\eta = \widetilde{M} = 0$, $C_2 \neq 0$.

1.3.2: $\mathbf{T} > 0$. According to [4, Diagram 1, page 4] systems (4.31) possess one real (double) and two complex singularities. Moreover in this case we could either have the topological configuration (44) $sn; N$ if $E_1 \neq 0$, or (47) $cp; N$ if $E_1 = 0$.

According to Remark 4.7 (iii) the condition $B_2 = 0$ implies $\mathbf{T} < 0$ and therefore we examine two cases: $\widetilde{U}_1 \neq 0$ and $\widetilde{U}_1 = 0$.

1.3.2.1: $\widetilde{U}_1 \neq 0$. Then the condition $\mathbf{D} = 0$ gives $b = -e^2/(4h)$ and we obtain the values of $\mathbf{T}$ and E_1 given in (4.36). Clearly the condition $\mathbf{T} > 0$ implies $e^2h(e^2 + f^2h) < 0$, and then the condition $E_1 = 0$ is equivalent to $f = 0$.

So in the case $E_1 \neq 0$ we obtain Picture (c) of Figure 4: [Ex: $(0,0,0,0,1/2,0)$, $(1,-2,-1,1,0,1)$], which is topologically equivalent to Picture $S_{IV}.16$.

If $E_1 = 0$ we have a cusp and this leads to the phase portrait given in Picture $S_{IV}.17$: [Ex: $(0,0,0,0,1/2,0)$, $(1,-2,0,1,0,1)$].

1.3.2.2: $\widetilde{U}_1 = 0$. Then $b = f^2/4$ and in this case we obtain the values of $\mathbf{T}$ and E_1 given in (4.38). Evidently the condition $\mathbf{T} > 0$ implies $E_1 \neq 0$ and we could have only the configuration (44) $sn; N$. Then we obtain a phase portrait topologically equivalent to Picture $S_{IV}.16$: [Ex: $(0,0,0,0,1/2,0)$, $(1/4,0,-1,1,0,1)$].

1.3.3:$\mathbf{T} = 0$. Since $\mathbf{D} = 0$, according to [4, Diagram 1, page 5] we consider two cases: $\mathbf{P} \neq 0$ and $\mathbf{P} = 0$.

1.3.3.1: $\mathbf{P} \neq 0$. Then by [5, Table 6.2] systems (4.31) possess two double finite singularities, which are real if $\mathbf{PR} > 0$ and complex if $\mathbf{PR} < 0$. However we have the next lemma.

Lemma 4.9. *For a system* (4.31) *the conditions* $\mathbf{D} = 0 = \mathbf{T}$ *and* $\mathbf{P} \neq 0$ *imply* $\mathbf{PR} > 0$ *and* $\mathcal{T}_4 B_2 \neq 0$.

Proof. Suppose first that the condition $B_2 = 0$. Then $b = 0$ and for systems (4.31) we have

$$\mathbf{D} = 0, \quad \mathbf{T} = -3e^2f^2x^2y^2(fhx - ey)^2, \quad \mathbf{P} = e^2f^2x^2y^2,$$

and clearly the condition $\mathbf{P} \neq 0$ implies $\mathbf{T} \neq 0$, i.e. we obtain a contradiction.

So $B_2 \neq 0$ and then the condition $\mathbf{D} = -48b^2(f^2 - 4b)(e^2 + 4bh) = 0$ gives $(f^2 - 4b)(e^2 + 4bh) = 0$. We claim that the condition $\mathbf{D} = \mathbf{T} = 0 \neq \mathbf{P}$ implies $f^2 - 4b = e^2 + 4bh = 0$.

Indeed, assuming $b = f^2/4$ we obtain:

$$\mathbf{D} = 0, \quad \mathbf{T} = -3(e^2 + f^2h)y^2\mathbf{P}, \quad \mathbf{P} = f^2(fhx^2 - 2exy - fy^2)^2/16,$$

and therefore the conditions $\mathbf{T} = 0$ and $\mathbf{P} \neq 0$ imply $e^2 + f^2h = 0$ and $f \neq 0$. So we have $h = -e^2/f^2$ and this implies $e^2 + 4bh = 0$. In a similar way could be shown that the condition $e^2 + 4bh = 0$ leads to the condition $f^2 - 4b = 0$ in the case $\mathbf{D} = \mathbf{T} = 0 \neq \mathbf{P}B_2$ and we deduce that the claim is proved.

On the other hand for $b = f^2/4$ and $h = -e^2/f^2$ calculations yield

$$\mathbf{D} = \mathbf{T} = 0, \quad \mathbf{P} = (ex + fy)^4/16, \quad \mathbf{R} = e^2(ex + fy)^2/f^2, \quad \mathcal{T}_4 = e^2f^2/4.$$

We observe that $\mathbf{PR} > 0$ and $\mathcal{T}_4 \neq 0$ and this completes the proof. □

Considering the conditions $\mathbf{D} = \mathbf{T} = 0$, $\mathbf{PR} > 0$ and $\mathcal{T}_4 \neq 0$, according to [4, Diagram 1, page 5] we arrive at the unique topological configuration (50) $sn, sn; N$. According to Remark 4.7 (*ii*) we have one saddle-node on the invariant line and another outside. As a result we arrive at Picture $S_{IV}.18$: [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(1/4, 0, -1, 1, 0, 1)$]. There are other topologically different phase portraits with two finite saddle-nodes and one infinite node as it is pointed out in [6] but in this case, the existence of the invariant straight line, or simply using continuity from the cases already studied $s, a, sn; N$ and $s, s, a, a; N$, give that there is only one possible phase portrait in these conditions.

1.3.3.2: $\mathbf{P} = 0$. We prove the following lemma.

Lemma 4.10. *Assume that for a system* (4.31) *the conditions* $\mu_0 \neq 0$ *and* $\mathbf{D} = \mathbf{T} = \mathbf{P} = 0$ *hold. Then the following conditions are equivalent:*

$$\mathbf{R} \neq 0, \mathcal{T}_4 \neq 0 \Leftrightarrow b = e = 0, f \neq 0;$$
$$\mathbf{R} \neq 0, \mathcal{T}_4 = 0 \Leftrightarrow b = f = 0, e \neq 0;$$
$$\mathbf{R} = 0 \Leftrightarrow b = e = f = 0.$$

Moreover in the case $\mathbf{R} \neq 0$ *this system possesses one simple and one triple real singularities, whereas for* $\mathbf{R} = 0$ *it possess one singularity of multiplicity four.*

Proof. First we note that the number and multiplicities of the singular points of systems (4.31) follows directly from [5, Diagram 6.1].

Assume now that the condition $\mathbf{D} = 0$ is fulfilled. Then we obtain $b(f^2 - 4b)(e^2 + 4bh) = 0$ and we consider all three cases given by this relation.

(i) If $b = 0$ then for systems (4.31) we have

$$\mathbf{T} = -3e^2f^2x^2y^2(fhx - ey)^2, \quad \mathbf{P} = e^2f^2x^2y^2, \quad \mathcal{T}_4 = 2f^4h.$$

It is clear that the condition $\mathbf{T} = \mathbf{P} = 0$ implies $ef = 0$ and therefore $\mathbf{R} = 3f^2h^2x^2 + 3e^2y^2$. We observe that the condition $f = 0$ is equivalent to $\mathcal{T}_4 = 0$.

Thus in the case $\mathbf{R} \neq 0$ we either have $b = e = 0$ and $f \neq 0$ if $\mathcal{T}_4 \neq 0$, or $b = f = 0$ and $e \neq 0$ if $\mathcal{T}_4 = 0$.

(ii) Assuming $b = f^2/4$ we obtain:

$$\mathbf{T} = -3(e^2 + f^2h)y^2\mathbf{P}, \quad \mathbf{P} = f^2(fhx^2 - 2exy - fy^2)^2/16, \quad \mathcal{T}_4 = -f^4h/4,$$

and clearly the condition $\mathbf{T} = \mathbf{P} = 0$ yields $f = 0$ and we have $\mathcal{T}_4 = 0$ and $\mathbf{R} = 3e^2y^2$. So if $\mathbf{R} \neq 0$ we obtain the conditions $b = f = 0$ and $e \neq 0$.

(iii) Suppose now that the condition $b = -e^2/(4h)$ holds. Then we calculate:

$$\mathbf{T} = -3h(e^2 + f^2h)x^2\mathbf{P}, \quad \mathbf{P} = e^2(ehx^2 + 2fhxy - ey^2)^2/(16h^2),$$
$$\mathcal{T}_4 = f^2(9e^2 + 8f^2h)/4,$$

and evidently the condition $\mathbf{T} = \mathbf{P} = 0$ gives $e = 0$ and in this case we obtain $\mathbf{R} = 3f^2h^2x^2$ and $\mathcal{T}_4 = 2f^4h$. Therefore the condition $\mathbf{R} \neq 0$ implies $\mathcal{T}_4 \neq 0$ and in this case we have $b = e = 0$ and $f \neq 0$.

It remains to observe that in all three cases *(i)*, *(ii)* and *(iii)*, the conditions $\mathbf{T} = \mathbf{P} = \mathbf{R} = 0$ give $b = e = f = 0$ and this completes the proof. □

In what follows we consider each one of the subcases provided by Lemma 4.10.

1.3.3.2.1: $\mathbf{R} \neq 0, \mathcal{T}_4 \neq 0$. By Lemma 4.10 we have $b = e = 0$ and considering [4, Diagram 1, page 6] we calculate:

$$E_3 = -f^2h/4, \quad \mathcal{T}_4 = 2f^4h, \quad \mu_0 = -h.$$

If $\mu_0 < 0$ then $h > 0$ and this implies $E_3 < 0$. Then by [4, Diagram 1, page 6] we arrive at the configuration (16) $a, a; S$. This leads to Picture $S_{IV}.6$: [Ex: $(0,0,0,0,1/2,0)$, $(0,0,1,-1,0,1)$].

Assuming $\mu_0 > 0$ we obtain $E_3 > 0$ and by the same Diagram 1 from [4] we obtain either the configuration (23) $s, a; N$ if $\neg(\mathfrak{C}_{10})$, or (24) $s, c; N$ if $(\mathfrak{C}_{10})$. However by Lemma 4.2 the conditions $(\mathfrak{C}_{10})$ are not compatible with systems (4.31).

Considering (4.33) we observe that if $b = e = 0$ both the triple and the simple singular points are located on the invariant line $x = 0$. So we obtain that the anti-saddle could not be a focus and hence the configuration (23) $s, a; N$ leads to a phase portrait equivalent to Picture $S_{IV}.9$: [Ex: $(0,0,0,0,1/2,0)$, $(0,0,1,1,0,1)$].

1.3.3.2.2: $\mathbf{R} \neq 0, \mathcal{T}_4 = 0$. By Lemma 4.10 we have $b = f = 0$ (this implies $\mathcal{T}_3 = 0$) and $e \neq 0$. Therefore we obtain $E_3 = -e^2/4 < 0$ and following [4, Diagram 1, page 6] we need to distinguish two cases: $\mu_0 < 0$ and $\mu_0 > 0$.

If $\mu_0 < 0$ then by [4, Diagram 1, page 5] we obtain either the configuration (59) $a, es; S$ if $\neg(\mathfrak{C}_3)$, or (60) $c, es; S$ if $(\mathfrak{C}_3)$. Considering the conditions $(\mathfrak{C}_3)$ from (3.1) in the case $b = f = 0$ we obtain:

$$\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = \mathcal{T}_1 = 0, \sigma = 3y \neq 0, \mathcal{F} = \mathcal{F}_1 = 0, \mathcal{H} = -9h/2, \mathcal{B} = -81e^2/8 < 0.$$

Since $\mu_0 < 0$ (i.e. $h > 0$) we have $\mathcal{H} < 0$ and we deduce that the conditions $(\mathfrak{C}_3)$ are satisfied in the considered case. So we could have only the configuration (60) $c, es; S$ which leads to Picture $S_{IV}.19$: [Ex: $(0,0,0,0,1/2,0)$, $(0,1,0,-1,0,1)$].

Assume now $\mu_0 > 0$. Since $E_3 < 0$ and $\mathcal{T}_4 = \mathcal{T}_3 = 0$ by [4, Diagram 1, page 5] we have the unique configuration (64) $s, es; N$ which leads to Picture $S_{IV}.20$: [Ex: $(0,0,0,0,1/2,0)$, $(0,1,0,1,0,1)$].

1.3.3.2.3: $\mathbf{R} = 0$. By Lemma 4.10 we have $b = e = f = 0$ and this leads to the homogeneous quadratic systems

$$\dot{x} = xy, \quad \dot{y} = -hx^2 + y^2.$$

In this case by [4, Diagram 1, page 6] we could either have the topological configuration (67) $ee;S$ if $\mu_0 < 0$, or (47) $hh;N$ if $\mu_0 > 0$.

In the first case we arrive at Picture $S_{IV}.21$: [Ex: $(0,0,0,0,1/2,0)$, $(0,0,0,-1,0,1)$].

The configuration (47) $hh;N$ leads to a phase portrait topologically equivalent with Picture $S_{IV}.17$: [Ex: $(0,0,0,0,1/2,0)$, $(0,0,0,1,0,1)$].

2: *Case* $\theta = \widetilde{N} = 0$, $B_2 \neq 0$. Considering the systems (4.30) after renaming the parameters we shall examine the family:

$$\dot{x} = y, \quad \dot{y} = b + ex + fy + hx^2. \tag{4.39}$$

For these systems calculations yield:

$$\begin{gathered} C_2 = -hx^3,\ \eta = \widetilde{M} = 0, \quad \mu_0 = \mu_1 = 0, \quad \mu_2 = h^2x^2, \\ \mathbf{U} = h^2(e^2 - 4bh)x^4y^2, \quad \kappa = \widetilde{K} = \tilde{L} = 0, \quad \mathcal{T}_4 = \mathcal{B}_1 = 0, \quad \sigma = f. \end{gathered} \tag{4.40}$$

Since $\mu_0 = \mu_1 = 0$ and the condition $C_2 \neq 0$ implies $\mu_2 \neq 0$, according to [5, Table 6.2] the above systems possess finite singularities of total multiplicity two. More exactly we have the singularities $M_{1,2}(x_{1,2}, 0)$, where

$$x_{1,2} = \big(-e \pm \sqrt{e^2 - 4bh}\,\big)/(2h), \quad \text{and} \quad \operatorname{sign}(e^2 - 4bh) = \operatorname{sign}(\mathbf{U}).$$

So we examine three subcases: $\mathbf{U} > 0$, $\mathbf{U} < 0$ and $\mathbf{U} = 0$.

2.1: $\mathbf{U} > 0$. Then considering (4.40) by [4, Diagram 3, page 9] we obtain either the configuration (23) $s,a;N$ if $\neg(\mathfrak{C}_9)$, or (24) $s,c;N$ if $(\mathfrak{C}_9)$.

On the other hand comparing the conditions $(\mathfrak{C}_9)$ from (3.1) with (4.40) we deduce that all the conditions are satisfied except $\sigma = 0$, because for systems (4.39) we have $\sigma = f$. So we deduce that for this case the conditions $(\mathfrak{C}_9)$ are satisfied if and only if $f = 0$.

Thus we obtain that the configuration (24) $s,c;N$ leads to a phase portrait topologically equivalent with Picture $S_{IV}.10$: [Ex: $(0,0,1,0,0,0)$, $(0,1,0,-1,0,0)$].

In the case of the configuration (23) $s,a;N$ we arrive at the same possible phase portraits as in Figure 3, for which we confirm the existence of the *Picture (a)* topologically equivalent with Picture $S_{IV}.9$: [Ex: $(0,0,1,0,0,0)$, $(0,1,-1,-1,0,0)$].

Regarding the other two phase portraits given by Picture (b) and Picture (c) (see Figure 3), they cannot exist by a similar argument as we did in Lemma 4.6. Indeed, if $f = 0$ the systems are Hamiltonian, and so, they cannot have a focus. And if $f \neq 0$, by Bendixson's theorem (see [9, Theorem 7.10]) the divergence $\partial P/\partial x + \partial Q/\partial y = f$ (for P and Q given in systems (4.39)) is strictly positive or negative in all the plane. Consequently these systems cannot have a periodic orbit. This proves that Picture (c) cannot be realizable when $f \neq 0$. Moreover since the proof of Bendixson's theorem also works for a loop formed by a singular point and a homoclinic orbit to it, it follows that in the case $f \neq 0$ Picture (b) is also non realizable.

2.2: $\mathbf{U} < 0$. By [4, Diagram 3, page 9] we obtain the topological configuration (12) N. This configuration leads to a phase portrait topologically equivalent with Picture $S_{IV}.5$: [Ex: $(0,0,1,0,0,0)$, $(-1,0,-1,-1,0,0)$].

2.3: $\mathbf{U} = 0$. In this case systems (4.39) possess a double singular point which could be a saddle-node or a cusp. But since for these systems the conditions $\kappa = \widetilde{K} = \tilde{L} = \mathcal{T}_4 = \mathcal{B}_1 = 0$ hold, according to [4, Diagram 3, page 11] we arrive at the topological configuration (47) $cp; N$. This configuration leads to a phase portrait topologically equivalent with Picture $S_{IV}.17$: [Ex: $(0,0,1,0,0,0)$, $(-1,2,0,-1,0,0)$].

Since all the cases are examined Theorem 4.1 is proved. □

4.5.2. *Systems* $(\mathbf{S}_V)$*:* $C_2 = 0$*.* These systems have the infinite line filled up with singularities and this family is considered in [18], where a total of 9 canonical forms of this family are presented: $C_2.1 - C_2.9$ (see Table 1, page 741).

Directly from [18] and Theorem 1.2 we arrive at the next result.

Theorem 4.11. *Assume that for a quadratic system the condition* $C_2 = 0$ *holds. Then this system belongs to the class* QS_{Ab} *if and only if the condition* $H_{10}^2 + H_{12}^2 \neq 0$ *is satisfied. Moreover its phase portrait is topologically equivalent to one of the pictures given in Figure 5 if and only the following corresponding conditions are verified:*
Picture $C_2.1 \Leftrightarrow H_{10} \neq 0,\ H_9 < 0$*;*
Picture $C_2.2(a) \Leftrightarrow H_{10} \neq 0,\ H_9 > 0,\ H_7 \neq 0$*;*
Picture $C_2.2(b) \Leftrightarrow H_{10} \neq 0,\ H_9 > 0,\ H_7 = 0$*;*
Picture $C_2.3 \Leftrightarrow H_{10} \neq 0,\ H_9 = 0,\ H_{12} \neq 0$*;*
Picture $C_2.4 \Leftrightarrow H_{10} \neq 0,\ H_9 = 0,\ H_{12} = 0$*;*
Picture $C_2.5(a) \Leftrightarrow H_{10} = 0,\ H_{12} \neq 0,\ H_{11} > 0,\ \mu_2 < 0$*;*
Picture $C_2.5(b) \Leftrightarrow H_{10} = 0,\ H_{12} \neq 0,\ H_{11} > 0,\ \mu_2 > 0$*;*
Picture $C_2.6 \Leftrightarrow H_{10} = 0,\ H_{12} \neq 0,\ H_{11} < 0$*;*
Picture $C_2.7 \Leftrightarrow H_{10} = 0,\ H_{12} \neq 0,\ H_{11} = 0$*.*

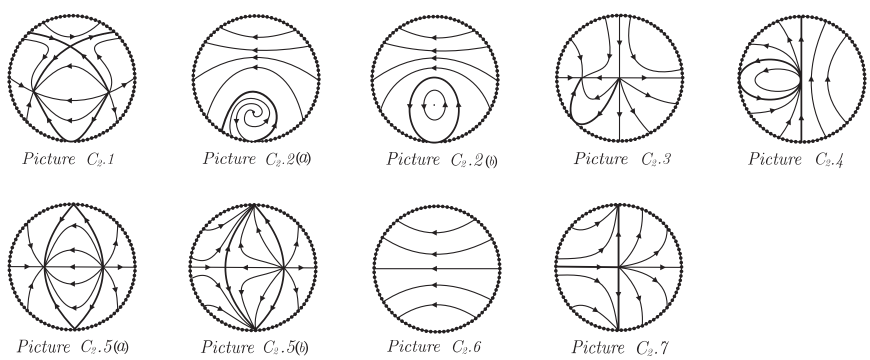


FIGURE 5. Global phase portraits of quadratic systems with $C_2 = 0$.

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