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CONVEXITY OF SOLUTIONS TO ELLIPTIC PDE'S

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Abstract. This article concerns the convexity or concavity of solutions to special second order elliptic partial differential equations in convex domains. We concentrate our investigation to boundary blow up solutions as well as to solutions of particular singular equations. Following a method due to Korevaar and Kennington, we find a new sufficient condition for proving convexity or concavity. This sufficient condition is useful when the semilinear component of the equation is the sum of two or more terms.

1. Introduction

The notion of convexity of a function is useful in many branches of Mathematics. In this paper we address the following question: when a solution to a second order elliptic equation in a convex domain is convex? The answer is trivial in dimension $n = 1$ only. For a discussion of this problem for a general n we refer to the monograph of Kawohl [14] and references therein. It happens that the answer depends on the boundary conditions as well as on the structure of the equation.

Brascamp and Lieb [5] proved that, if $u(x)$ is a solution to the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

then $v(x) = \log u(x)$ is concave provided Ω is convex. Here λ denotes the first eigenvalue. In [16], it is proved that if $u(x)$ is a solution to the problem

$$\Delta u + u^q = 0, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad 0 \le q < 1,$$

then $v(x) = u^{\frac{1-q}{2}}(x)$ is concave provided Ω is convex. This result can be extended (with the same proof) to the case $-1 < q < 1$. For the general case $\Delta u + f(u) = 0$ one may find a special function $\phi(t)$ (depending on f) so that $v = \phi(u(x))$ is concave [14]. Concavity results have been found for p-Laplace equations [4, 13, 21] as well as for some fully nonlinear equations [1].

A nice method to study convexity has been established by Korevaar [17] and Kennington [16]. Another powerful method uses the so called constant rank Theorem [2, 6, 18]. We also recall a method which uses the concave envelop of a function [8]. For more convexity results we refer to [3, 7, 10, 11, 13] and references therein. In [12] the authors show that if the solution u of $\Delta u + 1 = 0$ in Ω with $u = 0$ on $\partial\Omega$ is such that $\sqrt{\max u - u(x)}$ is convex then Ω must be an ellipsoid.

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In Section 1 we consider the problem

$$\Delta u = f(u) \quad \text{in } \Omega, \quad u \to +\infty \quad \text{as } x \to \partial\Omega.$$

This problem has been investigated in the seminal papers [15, 19]. The solution $u(x)$ (if it exists) is named boundary blow-up solution or large solution. It is known that, if the domain Ω is convex and if f satisfies suitable conditions, then $u(x)$ is convex. We are able to prove that, under some additional condition on f, also $\log u$ is convex. Next, for proving convexity, we describe a new method which uses a different condition on f. This new condition is very useful when f is the sum of two or more terms.

In Section 2 we consider the problem

$$\Delta u + f(u) = 0, \quad u(x) > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad u_\nu = -\infty \quad \text{on } \partial\Omega.$$

This kind of problems, named singular problems, are discussed in [9, 20]. If the domain Ω is convex and if f satisfies suitable conditions, the solution $u(x)$ exists and it is concave. Also for this problem, we describe a new method for studying concavity that uses a different condition on f.

2. Blow-up problems

Let $f(t)$ be smooth, positive and increasing for $t > 0$, with $f(0) = 0$. If

$$\int_1^\infty (F(t))^{-1/2} dt < \infty, \quad F(t) = \int_0^t f(\tau) d\tau,$$

then, the boundary blow-up problem

$$\Delta u(x) = f(u(x)) \quad \text{in } \Omega, \quad u = +\infty \quad \text{on } \partial\Omega, \tag{2.1}$$

has a positive solution [15, 19].

In this article, Ω is assumed to be bounded and convex. To investigate the convexity of $u(x)$, define the concavity function

$$c(x, y) = 2u(z) - u(x) - u(y), \quad x, y \in \Omega, \quad z = \frac{x + y}{2}.$$

Clearly, $u(x)$ is convex if and only if $c(x, y) \le 0$ in $\overline{\Omega \times \Omega}$. By the condition $u(x) = +\infty$ on $\partial\Omega$, we have $c(x, y) < 0$ on $\partial(\Omega \times \Omega)$. To show that $c(x, y) \le 0$ in $\Omega \times \Omega$, it is enough to prove that $c(x, y)$ cannot have any positive maximum in $\Omega \times \Omega$. The following result is well known (see [16, 17]).

Theorem 2.1 (Korevaar-Kennington)**.** *Let $f(t)$ be smooth and positive for $t > 0$, with $f(0) = 0$. We also assume $f(t)$ to be strictly increasing and harmonic concave. If $u \in C^2(\Omega)$ is a solution to Problem* (2.1)*, then the corresponding concavity function $c(x, y)$ cannot have any positive maximum in $\Omega \times \Omega$.*

If f satisfies some additional conditions, we can improve Theorem 2.1 as follows.

Theorem 2.2. *Let $f(t)$ be smooth and positive for $t > 0$, with $f(0) = 0$. We also assume that $\frac{f(e^t)}{e^t}$ is strictly increasing and harmonic concave. If $u \in C^2(\Omega)$ is a solution to Problem* (2.1)*, then the function $v(x) = \log u(x)$ is convex.*

Recall that $f(e^t)/e^t$ is harmonic concave if and only if $e^t/f(e^t)$ is convex, that is, if

$$f^2(t) - 3tf'(t)f(t) - t^2 f''(t)f(t) + 2t^2(f'(t))^2 \ge 0. \tag{2.2}$$

The latter condition is satisfied by $f(t) = t^p$ with $p > 1$.

Proof of theorem 2.2. Putting $v = \log u$, our problem reads as

$$\Delta v = -|Dv|^2 + \frac{f(e^v)}{e^v} \quad \text{in } \Omega, \quad v = +\infty \quad \text{on } \partial\Omega.$$

Clearly, the corresponding concavity function

$$c(x,y) = 2v(z) - v(x) - v(y), \quad z = \frac{x+y}{2},$$

is negative on $\partial(\Omega\times\Omega)$. Let us prove that $c(x,y)$ cannot have any positive maximum in $\Omega \times \Omega$. Arguing by contradiction, suppose $c(x,y)$ has a positive maximum at some point $(x,y) \in \Omega \times \Omega$. At this point, we have

$$\begin{gathered} D_x c(x,y) = Dv(z) - Dv(x) = 0, \\ D_y c(x,y) = Dv(z) - Dv(y) = 0. \end{gathered}$$

Moreover, we find that

$$\begin{gathered} D_{xx} c(x,y) = \frac{1}{2}\Delta v(z) - \Delta v(x), \\ D_{xy} c(x,y) = \frac{1}{2}\Delta v(z), \\ D_{yy} c(x,y) = \frac{1}{2}\Delta v(z) - \Delta v(y). \end{gathered} \tag{2.3}$$

Following [16], define

$$Lc(x,y) = r^2 D_{xx} c(x,y) + 2rs D_{xy} c(x,y) + s^2 D_{yy} c(x,y). \tag{2.4}$$

We suppose that $r^2 + s^2 > 0$. The corresponding coefficient $(2n \times 2n)$ matrix is

$$\begin{bmatrix} r^2 I & rsI \\ rsI & s^2 I \end{bmatrix},$$

where I is the $n\times n$ unitary matrix. The eigenvalues of this matrix are 0 (computed n times) and $r^2 + s^2$ (computed n times), hence, it is non-negative.

Inserting the values of $D_{xx}c(x,y)$, $D_{xy}c(x,y)$ and $D_{yy}c(x,y)$ given in (2.3) into (2.4), we have

$$\begin{aligned} Lc(x,y) &= r^2\Big(\frac{1}{2}\Delta v(z) - \Delta v(x)\Big) + rs\Delta v(z) + s^2\Big(\frac{1}{2}\Delta v(z) - \Delta v(y)\Big) \\ &= \frac{(r+s)^2}{2}\Delta v(z) - r^2\Delta v(x) - s^2\Delta v(y). \end{aligned}$$

On using our equation and recalling that $Dv(x) = Dv(z) = Dv(y)$, we have

$$\begin{aligned} Lc(x,y) = &-|Dv(z)|^2\Big[\frac{(r+s)^2}{2} - r^2 - s^2\Big] \\ &+ \frac{(r+s)^2}{2}\frac{f(e^{v(z)})}{e^{v(z)}} - r^2\frac{f(e^{v(x)})}{e^{v(x)}} - s^2\frac{f(e^{v(y)})}{e^{v(y)}}. \end{aligned}$$

On noting that

$$-\Big[\frac{(r+s)^2}{2} - r^2 - s^2\Big] = \frac{(r-s)^2}{2} \geq 0,$$

we find

$$Lc(x,y) \geq \frac{(r+s)^2}{2}\frac{f(e^{v(z)})}{e^{v(z)}} - r^2\frac{f(e^{v(x)})}{e^{v(x)}} - s^2\frac{f(e^{v(y)})}{e^{v(y)}}.$$

Putting $H(t) := f(e^t)/e^t$, we find that

$$Lc(x,y) \geq \frac{(r+s)^2}{2} H(v(z)) - r^2 H(v(x)) - s^2 H(v(y)).$$

Since we are assuming that

$$u(z) > \frac{u(x)+u(y)}{2}$$

and since $H(t)$ is strictly increasing, we have

$$Lc(x,y) > \frac{(r+s)^2}{2} H\Big(\frac{v(x)+v(y)}{2}\Big) - r^2 H(v(x)) - s^2 H(v(y)).$$

With $a = v(x)$ and $b = v(y)$, we choose

$$r = H(b), \quad s = H(a).$$

After some computations, we find that

$$Lc(x,y) > (r+s)\Big[\frac{H(a)+H(b)}{2} H\Big(\frac{a+b}{2}\Big) - H(a)H(b)\Big].$$

Finally, since $H(t)$ is harmonic concavity, $Lc(x,y) > 0$. But, at (x,y) (point of maximum) we have $Lc(x,y) \leq 0$, a contradiction. The proof is complete. $\square$

By Theorem 2.2, if $u(x)$ is a blow-up solution to $\Delta u = u^p$, $p > 1$, the function $v(x) = \log u(x)$ is convex. In case $u(x) \geq 1$, we may ask if also $z = \big(\log u(x)\big)^\beta$ for $\beta \in (0,1)$ is convex. The answer is negative in general. Indeed, for $n > 2$ consider the problem

$$\Delta u = n(n-2) u^{\frac{n+2}{n-2}}, \quad u = +\infty \quad \text{on } \partial\Omega.$$

If Ω is the unit ball centered at the origin, a (radial) solution is

$$u(r) = (1-r^2)^{\frac{2-n}{2}}, \quad r = |x|.$$

The function $v = \log u = \frac{2-n}{2}\log(1-r^2)$, for r near to 0, behaves like $\frac{n-2}{2} r^2$, and $r^{2\beta}$ is not convex for $0 < \beta < \frac{1}{2}$.

Theorem 2.2 cannot be applied when $f(t) = t \log^\alpha(t+1)$, $\alpha > 2$, because this function does not satisfy condition (2.2). The following is a weaker result.

Theorem 2.3. *Let $0 < \beta < 1$. Let $f(t)$ be smooth and positive for $t > 0$, with $f(0) = 0$, and such that $t^{\frac{\beta-1}{\beta}} f(t^{\frac{1}{\beta}})$ is strictly increasing and harmonic concave. If $u \in C^2(\Omega)$ is a positive solution to Problem (2.1), then the function $v(x) = u^\beta(x)$ is convex.*

Note that $t^{\frac{\beta-1}{\beta}} f(t^{\frac{1}{\beta}})$ is harmonic concave if and only if $t^{\frac{1-\beta}{\beta}} \big(f(t^{\frac{1}{\beta}})\big)^{-1}$ is convex, that is, if

$$(1-\beta)(1-2\beta) f^2(t) - 3(1-\beta) t f'(t) f(t) - t^2 f''(t) f(t) + 2t^2 (f'(t))^2 \geq 0. \quad (2.5)$$

The latter condition is satisfied by $f(t) = t \log^\alpha(t+1)$ with $\alpha > 2$ and $\beta = 1/3$. Indeed, with $\beta = 1/3$ and f as in above we find

$$\begin{aligned} &(1-\beta)(1-2\beta) f^2(t) - 3(1-\beta) t f'(t) f(t) - t^2 f''(t) f(t) + 2t^2 (f'(t))^2 \\ &= t^2 \log^{2\alpha-2}(t+1) \Big[\frac{2}{9}\log^2(t+1) + \frac{\alpha t^2}{(t+1)^2}\log(t+1) + \frac{\alpha(\alpha+1)t^2}{(t+1)^2}\Big] \geq 0. \end{aligned}$$

Proof of theorem 2.3. The proof is similar to that of Theorem 2.2. Putting $v = u^\beta$, our problem reads as

$$\Delta v = \frac{\beta-1}{\beta}\frac{|Dv|^2}{v} + \beta v^{\frac{\beta-1}{\beta}} f(v^{\frac{1}{\beta}}) \quad \text{in } \Omega, \quad v = +\infty \quad \text{on } \partial\Omega.$$

Clearly, the corresponding concavity function

$$c(x,y) = 2v(z) - v(x) - v(y), \quad z = \frac{x+y}{2}$$

is negative on $\partial(\Omega\times\Omega)$. Let us prove that $c(x,y)$ cannot have any positive maximum in $\Omega\times\Omega$. Arguing by contradiction, suppose $c(x,y)$ has a positive maximum at some point $(x,y) \in \Omega\times\Omega$. If L is defined as in (2.4) and if we replace the expressions for $D_{xx}c(x,y)$, $D_{xy}c(x,y)$ and $D_{yy}c(x,y)$ given in (2.3), we find (as in the proof of Theorem 2.2)

$$Lc(x,y) = \frac{(r+s)^2}{2}\Delta v(z) - r^2\Delta v(x) - s^2\Delta v(y).$$

On using our equation for v and recalling that (at the point of maximum) $Dv(x) = Dv(z) = Dv(y)$, we find that

$$\begin{aligned} Lc(x,y) &= \frac{\beta-1}{\beta}|Dv(z)|^2\Big[\frac{(r+s)^2}{2v(z)} - \frac{r^2}{v(x)} - \frac{s^2}{v(y)}\Big] \\ &+\beta\Big[\frac{(r+s)^2}{2}v^{\frac{\beta-1}{\beta}}(z)f(v^{\frac{1}{\beta}}(z)) \quad - r^2 v^{\frac{\beta-1}{\beta}}(x)f(v^{\frac{1}{\beta}}(x)) - s^2 v^{\frac{\beta-1}{\beta}}(y)f(v^{\frac{1}{\beta}}(y))\Big]. \end{aligned}$$

Since we are assuming $c(x,y) > 0$, we have

$$\frac{1}{2v(z)} < \frac{1}{v(x)+v(y)}$$

and

$$\begin{aligned} &\frac{\beta-1}{\beta}|Dv(z)|^2\Big[\frac{(r+s)^2}{2v(z)} - \frac{r^2}{v(x)} - \frac{s^2}{v(y)}\Big] \\ &\geq \frac{\beta-1}{\beta}|Dv(z)|^2\Big[\frac{(r+s)^2}{v(x)+v(y)} - \frac{r^2}{v(x)} - \frac{s^2}{v(y)}\Big] \\ &= \frac{1-\beta}{\beta}|Dv(z)|^2\frac{(rv(y)-sv(x))^2}{(v(x)+v(y))v(x)v(y)} \geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &Lc(x,y) \\ &\geq \beta\Big[\frac{(r+s)^2}{2}v^{\frac{\beta-1}{\beta}}(z)f(v^{\frac{1}{\beta}}(z)) - r^2 v^{\frac{\beta-1}{\beta}}(x)f(v^{\frac{1}{\beta}}(x)) - s^2 v^{\frac{\beta-1}{\beta}}(y)f(v^{\frac{1}{\beta}}(y))\Big]. \end{aligned}$$

With $K(t) := t^{\frac{\beta-1}{\beta}}f(t^{\frac{1}{\beta}})$, we have

$$Lc(x,y) \geq \beta\Big[\frac{(r+s)^2}{2}K(v(z)) - r^2K(v(x)) - s^2K(v(y))\Big].$$

Since $K(t)$ is strictly increasing and $v(z) > \frac{v(x)+v(y)}{2}$, we have

$$K(v(z)) > K\Big(\frac{v(x)+v(y)}{2}\Big).$$

Therefore,

$$Lc(x,y) > \beta\Big[\frac{(r+s)^2}{2}K\Big(\frac{u(x)+u(y)}{2}\Big) - r^2K(u(x)) - s^2K(u(y))\Big].$$

With $a = v(x)$ and $b = v(y)$, we choose $r = K(b)$ and $s = K(a)$. Then

$$Lc(x,y) > \beta(r+s)\Big[\frac{K(a)+K(b)}{2}K\Big(\frac{a+b}{2}\Big) - K(a)K(b)\Big].$$

Finally, since $K(t)$ is harmonic concavity, we find $Lc(x,y) > 0$. But, at (x,y) (point of maximum) we have $Lc(x,y) \le 0$, a contradiction. The proof is complete. □

Let us recall that Theorem 2.1 has been extended to the problem

$$\Delta u(x) = f(u(x))\big(1 + k|Du|^2\big) \quad \text{in } \Omega, \quad u = +\infty \quad \text{on } \partial\Omega,$$

where k is a positive constant. We shall prove the following version.

Theorem 2.4. *Let $\varphi(t)$ be a positive function such that $\varphi^{1/2}(t)$ is convex. Let $f(t)$ and $g(t)$ be positive, increasing smooth functions with $f(t)$ strictly increasing and with $f(0) = g(0) = 0$. We suppose that $\varphi(t)f(t)$ and $\varphi(t)g(t)$ are concave. If $u \in C^2(\Omega)$ is a solution to*

$$\Delta u(x) + k_1|Du|^2 = f(u(x)) + k_2|Du(x)|^2 g(u(x)), \quad k_1,\ k_2 \ge 0, \tag{2.6}$$

in a convex domain Ω then the corresponding concavity function $c(x,y)$ cannot have a positive maximum in $\Omega \times \Omega$.

Proof. By contradiction, suppose $c(x,y)$ has a positive maximum at some point $(x,y) \in \Omega \times \Omega$. If L is defined as in (2.4), by using (2.3) with u in place of v we have

$$Lc(x,y) = \frac{(r+s)^2}{2}\Delta u(z) - r^2\Delta u(x) - s^2\Delta u(y), \quad z = \frac{x+y}{2}.$$

On using our equation (2.6) and recalling that $Du(x) = Du(z) = Du(y)$ we have

$$\begin{aligned} Lc(x,y) &= k_1\frac{(r-s)^2}{2}|Du(z)|^2 + \frac{(r+s)^2}{2}f(u(z)) - r^2f(u(x)) - s^2f(u(y)) \\ &\quad + k_2|Du(z)|^2\Big[\frac{(r+s)^2}{2}g(u(z)) - r^2g(u(x)) - s^2g(u(y))\Big]. \end{aligned}$$

Since we are assuming that

$$u(z) > \frac{u(x)+u(y)}{2},$$

and since $f(t)$ is strictly increasing and $g(t)$ is increasing, we find that

$$\begin{gathered} f(u(z)) > f\Big(\frac{u(x)+u(y)}{2}\Big), \\ g(u(z)) \ge g\Big(\frac{u(x)+u(y)}{2}\Big). \end{gathered}$$

Therefore,

$$\begin{aligned} Lc(x,y) &> \frac{(r+s)^2}{2}f\Big(\frac{u(x)+u(y)}{2}\Big) - r^2f(u(x)) - s^2f(u(y)) \\ &\quad + k_2|Du(z)|^2\Big[\frac{(r+s)^2}{2}g\Big(\frac{u(x)+u(y)}{2}\Big) - r^2g(u(x)) - s^2g(u(y))\Big]. \end{aligned}$$

Put $u(x)=a$, $u(y)=b$, and choosing $r=\varphi^{1/2}(a)$ and $s=\varphi^{1/2}(b)$, we find that

$$
\begin{aligned}
Lc(x,y) > \frac{1}{2}\Big(\varphi^{1/2}(a)+\varphi^{1/2}(b)\Big)^2 f\Big(\frac{a+b}{2}\Big) - \varphi(a)f(a) - \varphi(b)f(b) \\
+ k_2|Du(z)|^2\Big[\frac{1}{2}\Big(\varphi^{1/2}(a)+\varphi^{1/2}(b)\Big)^2 g\Big(\frac{a+b}{2}\Big) - \varphi(a)g(a) - \varphi(b)g(b)\Big].
\end{aligned}
$$

Since $\varphi^{1/2}(t)$ is convex, we have

$$
\frac{\varphi^{1/2}(a)+\varphi^{1/2}(b)}{2} \ge \varphi^{1/2}\Big(\frac{a+b}{2}\Big),
$$

which can be written as

$$
\frac{1}{2}\Big(\varphi^{1/2}(a)+\varphi^{1/2}(b)\Big)^2 \ge 2\varphi\Big(\frac{a+b}{2}\Big).
$$

On using this inequality, we find

$$
\begin{aligned}
&\frac{1}{2}\Big(\varphi^{1/2}(a)+\varphi^{1/2}(b)\Big)^2 f\Big(\frac{a+b}{2}\Big) - \varphi(a)f(a) - \varphi(b)f(b) \\
&\ge 2\varphi\Big(\frac{a+b}{2}\Big) f\Big(\frac{a+b}{2}\Big) - \varphi(a)f(a) - \varphi(b)f(b) \ge 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\Big(\varphi^{1/2}(a)+\varphi^{1/2}(b)\Big)^2 g\Big(\frac{a+b}{2}\Big) - \varphi(a)g(a) - \varphi(b)g(b) \\
&\ge 2\varphi\Big(\frac{a+b}{2}\Big) g\Big(\frac{a+b}{2}\Big) - \varphi(a)g(a) - \varphi(b)g(b) \ge 0,
\end{aligned}
$$

where the concavity of $\varphi(t)f(t)$ and $\varphi(t)g(t)$ have been used. Hence, $Lc(x,y)>0$. But, at (x,y) (point of maximum) we have $Lc(x,y)\le 0$, a contradiction. The proof is complete. □

Examples. If $f(t)=t^p$ and $g(t)=t^{p-\gamma}$ with $p>1$ and $0<\gamma\le 1$, we can choose $\varphi(t)=t^{1-p}$. If $f(t)=te^t$ and $g(t)=t^\gamma e^t$ with $0<\gamma\le 1$, we can choose $\varphi(t)=e^{-t}$.

3. Singular problems

First we recall the following result.

Proposition 3.1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with a smooth boundary. Let $f(t)$ be positive for $t>0$, smooth, decreasing, and such that*

$$
\lim_{t\to 0^+}\int_t^1 f(\tau)d\tau = \infty.
$$

Then the boundary value problem

$$
\Delta u(x) + f(u(x)) = 0 \quad \text{in } \Omega, \quad u=0 \quad \text{on } \partial\Omega,
$$

has a positive solution. Moreover we have $u_\nu=-\infty$ on $\partial\Omega$, where u_ν denotes external normal derivative on $\partial\Omega$.

Proof. For a proof of existence we refer to [9]. We claim that $u_\nu=-\infty$. Following the proof of [9, Theorem 2.2], we consider the ordinary differential equation

$$
p''(s) + f(p(s)) = 0, \quad p(0)=0, \quad p(s)>0.
$$

Integrating this equation we find, with some $a > 0$,

$$\frac{(p'(a))^2 - (p'(s))^2}{2} + \int_{p(s)}^{p(a)} f(\tau) d\tau = 0.$$

On using our assumption on f we find

$$\lim_{s \to 0^+} (p'(s))^2 = +\infty.$$

By [9, Theorem 2.2] we have that for some $\lambda > 0$,

$$\lambda p(d(x)) \le u(x),$$

where $d(x)$ denotes the distance of x from $\partial\Omega$. Recalling that $p(0) = 0$ and that $u = 0$ on $\partial\Omega$, it follows that $u_\nu = -\infty$ on $\partial\Omega$. The claim is proved.

Our aim is to prove that $u(x)$ is concave. To show this, we shall prove that the corresponding concavity function $c(x, y)$ satisfies $c(x, y) \ge 0$ in $\Omega \times \Omega$. Let us show first that $c(x, y)$ cannot have a minimum at a point (x, y) with $y \in \partial\Omega$. Indeed, suppose a minimum occurs at a point (x, y) with $y \in \partial\Omega$. If we take the derivative of $c(x, y)$ with respect to ν (external normal) and compute it for $x \in \Omega$ and $y \in \partial\Omega$, we find

$$c_\nu(x, y) = 2u_\nu(z) - u_\nu(x) + \infty = +\infty, \quad z = \frac{x + y}{2}.$$

But, if $c(x, y)$ has a minimum then $c_\nu(x, y) \le 0$, a contradiction. □

Now, to prove that $c(x, y) \ge 0$ it is sufficient to show that $c(x, y)$ cannot have a negative minimum in $\Omega \times \Omega$. The following result is well-known [16].

Theorem 3.2 (Korevaar-Kennington)**.** *Let $f(t)$ be a function smooth, strictly decreasing and harmonic concave. If $u \in C^2(\Omega)$ is a solution to*

$$\Delta u(x) + f(u(x))\big(1 + k|Du(x)|^2\big) = 0, \quad k \ge 0,$$

then, the corresponding concavity function $c(x, y)$ cannot have any negative minimum in $\Omega \times \Omega$.

Let us prove a different version of Theorem 3.2.

Theorem 3.3. *Let $\varphi(t)$ be a positive function such that $\varphi^{1/2}(t)$ is convex. Let $f(t)$ and $g(t)$ be smooth, decreasing functions for $t > 0$, with $f(t)$ strictly decreasing. Suppose that $\varphi(t)f(t)$ and $\varphi(t)g(t)$ are concave. If $u \in C^2(\Omega)$ is a solution to*

$$\Delta u(x) + f(u(x)) + k|Du(x)|^2 g(u(x)) = 0, \quad k \ge 0, \tag{3.1}$$

in a convex domain Ω then, the corresponding concavity function $c(x, y)$ cannot have any negative minimum in $\Omega \times \Omega$.

Proof. The proof is similar to that of Theorem 2.4. By contradiction, suppose $c(x, y)$ has a negative minimum at $(x, y) \in \Omega \times \Omega$. If L is defined as in (2.4), by using (2.3) with u in place of v, we find

$$Lc(x, y) = \frac{(r + s)^2}{2} \Delta u(z) - r^2 \Delta u(x) - s^2 \Delta u(y).$$

Since (x, y) is a point of minimum for $c(x, y)$, we have $Du(x) = Du(z) = Du(y)$. Hence, on using (3.1), we find that

$$Lc(x, y) = -\frac{(r + s)^2}{2} f(u(z)) + r^2 f(u(x)) + s^2 f(u(y))$$

$$+ k|Du(z)|^2\Big[-\frac{(r+s)^2}{2}g(u(z)) + r^2 g(u(x)) + s^2 g(u(y))\Big].$$

Since $f(t)$ is strictly decreasing, $g(t)$ is decreasing and $v(z) < \frac{v(x)+v(y)}{2}$, we have

$$-f(u(z)) < -f\Big(\frac{u(x)+u(y)}{2}\Big)$$

and

$$-g(u(z)) \le -g\Big(\frac{u(x)+u(y)}{2}\Big).$$

Therefore,

$$\begin{aligned} Lc(x,y) &< -\frac{(r+s)^2}{2} f\Big(\frac{u(x)+u(y)}{2}\Big) + r^2 f(u(x)) + s^2 f(u(y)) \\ &\quad + k|Du(z)|^2\Big[-\frac{(r+s)^2}{2} g\Big(\frac{u(x)+u(y)}{2}\Big) + r^2 g(u(x)) + s^2 g(u(y))\Big]. \end{aligned}$$

Puting $u(x) = a$, $u(y) = b$, and choosing $r = \varphi^{1/2}(a)$ and $s = \varphi^{1/2}(b)$, we find that

$$\begin{aligned} Lc(x,y) &< -\frac{1}{2}\Big(\varphi^{1/2}(a) + \varphi^{1/2}(b)\Big)^2 f\Big(\frac{a+b}{2}\Big) + \varphi(a)f(a) + \varphi(b)f(b) \\ &\quad + k|Du(z)|^2\Big[-\frac{1}{2}\Big(\varphi^{1/2}(a) + \varphi^{1/2}(b)\Big)^2 g\Big(\frac{a+b}{2}\Big) + \varphi(a)g(a) + \varphi(b)g(b)\Big]. \end{aligned}$$

Since $\varphi^{1/2}(t)$ is convex, we have

$$\frac{\varphi^{1/2}(a) + \varphi^{1/2}(b)}{2} \ge \varphi^{1/2}\Big(\frac{a+b}{2}\Big),$$

which can be written as

$$-\frac{1}{2}\Big(\varphi^{1/2}(a) + \varphi^{1/2}(b)\Big)^2 \le -2\varphi\Big(\frac{a+b}{2}\Big).$$

On using this inequality, we find

$$\begin{aligned} &-\frac{1}{2}\Big(\varphi^{1/2}(a) + \varphi^{1/2}(b)\Big)^2 f\Big(\frac{a+b}{2}\Big) + \varphi(a)f(a) + \varphi(b)f(b) \\ &\le -2\varphi\Big(\frac{a+b}{2}\Big) f\Big(\frac{a+b}{2}\Big) + \varphi(a)f(a) + \varphi(b)f(b) \le 0, \end{aligned}$$

and

$$\begin{aligned} &-\frac{1}{2}\Big(\varphi^{1/2}(a) + \varphi^{1/2}(b)\Big)^2 g\Big(\frac{a+b}{2}\Big) + \varphi(a)g(a) + \varphi(b)g(b) \\ &\le -2\varphi\Big(\frac{a+b}{2}\Big) g\Big(\frac{a+b}{2}\Big) + \varphi(a)g(a) + \varphi(b)g(b) \le 0, \end{aligned}$$

where the concavity of $\varphi(t)f(t)$ and $\varphi(t)g(t)$ have been used. Hence, $Lc(x,y) < 0$. But, at (x,y) (point of minimum) we have $Lc(x,y) \ge 0$, a contradiction. The proof is complete. □

Examples. If $f(t) = 1/t^\gamma$ and $g(t) = 1/t^{\gamma+1}$ with $\gamma \ge 1$, we can choose $\varphi(t) = t^{\gamma+1}$. If $f(t) = \frac{1}{te^t}$ and $g(t) = \frac{1}{t^2e^t}$, we can chose $\varphi(t) = t^2e^t$.

Recall that Ω is assumed to be convex. By Theorem 3.2, the solution u to the problem

$$\Delta u + u^{-\gamma} = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

with $\gamma > 1$ concave in Ω.

Can we say that u^α is concave for some $\alpha > 1$? Let us prove the following results. Recall that Ω is a bounded convex domain with a smooth boundary.

Proposition 3.4. *Let $\gamma > 1$ and $a \geq \frac{\gamma-1}{2}$. If u is a positive solution to*

$$\Delta u + a\frac{|Du|^2}{u} + u^{-\gamma} = 0 \quad in\ \Omega, \quad u = 0 \quad on\ \partial\Omega,$$

then the function $v(x) = u^{\frac{\gamma+1}{2}}(x)$ is concave.

Proof. The equation in terms of v reads as follows

$$\Delta v + \lambda\frac{|Dv|^2}{v} + \frac{\gamma+1}{2v} = 0, \quad \lambda = \frac{2}{\gamma+1}\Big(a - \frac{\gamma-1}{2}\Big).$$

Clearly, $u = 0$ implies $v = 0$. Moreover, the condition $a \geq \frac{\gamma-1}{2}$ implies $\lambda \geq 0$.

One proves (for example by using Theorem 3.3 with $\varphi(t) = t^2$) that the corresponding concavity function $c(x, y)$ cannot have a negative minimum in $\Omega \times \Omega$.

To conclude that $c(x, y) \geq 0$ in $\Omega \times \Omega$, it suffices to show that $v_\nu = -\infty$ on $\partial\Omega$. To prove this, we note that from

$$\Delta v + \frac{\gamma+1}{2v} < 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

and

$$\Delta z + \frac{\gamma+1}{2z} = 0 \quad \text{in } \Omega, \quad z = 0 \quad \text{on } \partial\Omega$$

it follows that $v(x) \geq z(x)$ in Ω and $|v_\nu| \geq |z_\nu|$ on $\partial\Omega$. By Proposition 3.1 we have $z_\nu = -\infty$ on $\partial\Omega$. It follows that $v_\nu = -\infty$ on $\partial\Omega$. The proof is complete. □

Sometimes it is convenient to transform a blow-up problem into a singular problem.

Proposition 3.5. *Let $a > 0$ and $q > 0$. If u is a solution to*

$$\Delta u = a|Du|^2 + e^{qu} \quad in\ \Omega, \quad u = \infty \quad on\ \partial\Omega,$$

then $u(x)$ is convex.

Proof. Putting $v(x) = e^{-au(x)}$ our problem reads as follows

$$\Delta v + av^{1-\frac{q}{a}} = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega.$$

If $1 - \frac{q}{a} \leq -1$ (that is $q \geq 2a$) then v is concave by Theorem 3.3 with $k = 0$ and $\varphi(t) = t^{q/a}$. Clearly, also $\log v(x)$ is concave, so, $u(x)$ is convex. If $-1 < 1 - \frac{q}{a} < 1$ (that is $0 < q < 2a$) then $v^{\frac{q}{2a}}(x)$ is concave (as remarked in the Introduction to the present paper). This means that $e^{-qu(x)/2}$ is concave, which implies that $u(x)$ is convex. □

References

[1] O. Alvarez, J. M. Lasry, P. L. Lions; *Convex viscosity solutions and state constraints*, J. Math. Pures Appl., 76 (1997), 265–288.

[2] B. Bian, P. Guan; *A microscopic convexity principle for nonlinear partial differential equations*, Invent. Math., 177 (2009), 307–335.

[3] M. Bianchini, P. Salani; *Power concavity for solutions of nonlinear elliptic problems in convex domains*, in Geometric properties for parabolic and elliptic PDE's, vol. 2 of Springer INdAM Ser., Springer, Milan, 2013, 35–48.

[4] W. Borrelli, S. Mosconi, M. Squassina; *Concavity properties for solutions to p-Laplace equations with concave nonlinearities*, to appear in Advances in Calculus of Variations, (2024). Preprint arXiv; 2111.14801v3 [math.AP] 31 Jan. 2022.

[5] H. J. Brascamp, E. H. Lieb; *On extension of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Funct. Anal. 22 (1976), 366–389.
[6] L. A. Caffarelli, A. Friedman; *Convexity of solutions of semilinear elliptic equations*, Duke Math. J. 52 (1985), 431–456.
[7] L. A Caffarelli, J. Spruck; *Convexity properties of solutions to some classical varational problems*, Comm. P. D. E. 7 (1982), 1337–1379.
[8] A. Colesanti, P. Salani; *Quasi-concave envelope of a function and convexity of level sets of solutions to elliptic equations*, Math. Nachr., 258 (2003), 3–15.
[9] M. G. Crandall, P. H. Rabinowitz, L. Tartar; *On a Dirichlet problem with a singular nonlinearity*, Comm. in Partial Differential Equations, (1977), 193–222.
[10] A. Greco, B. Kawohl; *On the convexity of some free boundaries.* Interfaces and Free Boundaries, 11 (2009), 503–514.
[11] A. Greco, G. Porru; *Convexity of solutions to some elliptic partial differential equations*, SIAM J. Math. Anal., 24 (1993), 833–839.
[12] A. Henrot, C. Nitsch, P. Salani, C. Trombetti; *Optimal concavity of the torsion function*, J. Optim. Theory Appl., 178 (2018), 26–35.
[13] P. Juutinen; *Convexity of solutions to boundary blow-up problems*, Commun. Pure Appl. Anal., 12 (2013), 2267–2275.
[14] B. Kawohl; *Rearrangements and Convexity of Level Sets in PDE's*, Lectures Notes in Methamaics, 1150, Springer-Verlag, Berlin, Heidelberg, New York, Tokio. 1985.
[15] J. B. Keller; *On solutions of* $\Delta u = f(u)$, Comm. Pure Appl. Math. 10 (1957), 503–510.
[16] A. U. Kennington; *Power concavity and boundary value problems*, Indiana Univ. Math. J. 34 (1985), 687–704.
[17] N. Korevaar; *Convex solutions to nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J. 32 (1983), 603–614.
[18] N. Korevaar, J.L. Lewis; *Convex solutions of certain elliptic equations have constant rank Hessians*, Arch. Rational Mech. Anal. 97 (1987), 19–32.
[19] R. Osserman; *On the inequality* $\Delta u \geq f(u)$, Pacific J. Math. 7 (1957), 1641–1647.
[20] G. Porru, A. Vitolo; *Problems for elliptic singular equations with a quadratic gradient term*, J. Math. Anal. Appl. 334 (2007), 467–486
[21] S. Sakaguchi; *Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 14, no. 3 (1987), 403–421.

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