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CROSSED DIFFERENTIAL SYSTEMS OF EQUATIONS AND CLUNIE LEMMA

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Abstract. We study properties of transcendental meromorphic solutions of crossed complex differential systems of equations. For instance, we study the crossed Riccati differential system

$$f(z)^2 = 1 - g'(z),$$
$$g(z)^2 = 1 - f'(z),$$

and the crossed Weierstrass differential system

$$f(z)^3 = 1 - g'(z)^2,$$
$$g(z)^3 = 1 - f'(z)^2.$$

In addition, we establish a crossed version of Clunie lemma.

1. Introduction

Let $f(z)$ and $g(z)$ be non-constant meromorphic functions in the complex plane. Define a differential monomial

$$M_j(z,f) = f(z)^{\lambda_{0j}}(f'(z))^{\lambda_{1j}} \cdots (f^{(n)}(z))^{\lambda_{nj}},$$

where $\gamma_{M_j} := \lambda_{0j} + \lambda_{1j} + \cdots + \lambda_{nj}$ is the degree of $M_j(z,f)$. We define a differential polynomial

$$L(z,f) = \sum_{j=1}^{k} \alpha_j(z) M_j(z,f),$$

where $\alpha_j(z)$ are small functions with respect to $f(z)$ in the sense of Nevanlinna theory. The degree γ_L of $L(z,f)$ is defined by $\gamma_L = \max_{1\le j\le k}\{\gamma_{M_j}\}$. In particular, $L(z,f) = \alpha_0(z)f(z) + \alpha_1(z)f'(z) + \cdots + \alpha_k(z)f^{(k)}(z)$ is called a linear differential polynomial of $f(z)$.

Definition 1.1. Let $L(z,f)$ be a non-constant differential polynomial. If

$$L(z,f) = L(z,g)$$

implies that $f = g$, where $f(z)$ and $g(z)$ are two meromorphic functions, then $L(z,f)$ is called a unique differential polynomial of meromorphic functions (UDPM).

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Remark 1.2. (1) The definition of UDPM is a generalization of UPM. A non-constant polynomial $P(z)$ is called a unique polynomial of meromorphic functions (UPM for abbreviation), whenever $P(f) = P(g)$ implies that $f = g$. Li and Yang [18, Theorem 7] proved that any non-linear polynomial with degree two or three is not a UPM. Li and Yang [18, Theorem 8] obtained that

$$P(z) = z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

is not a UPM except that $\frac{a_3^3}{8} - \frac{a_2 a_3}{2} + a_1 \neq 0$. If $P(z) = z^m + az^n + b$ and $m \geq 5$, $1 \leq n \leq m - 2$ and $(m, n) = 1$, then $P(z)$ is a UPM, see [37]. The singularity theory, the concept of genus for algebraic curves and Nevanlinna theory always be applied to consider the UPM, see [10] and references therein.

(2) Yang and Hua [32] obtained that $f = tg$ if $f^n f' = g^n g'$ and $n \geq 3$, where $t^{n+1} = 1$. Fang and Fang [4] obtained that $f = g$ if $f^n (f-1)^2 f' = g^n (g-1)^2 g'$ and $n \geq 8$. Thus, $f^n(f-1)^2 f' (n \geq 8)$ is a UDPM, however, $f^n f'$ is not a UDPM.

As far as we know describing UDPM completely is more difficult than UPM for the reason of the complexity of differential polynomials. Nevanlinna theory plays an important part in considering the uniqueness of meromorphic functions. The standard notations of Nevanlinna theory such as the counting function $N(r, f)$, the proximity function $m(r, f)$ and the characteristic function $T(r, f)$ can be found in [13, 36] for details or [16, 21] for a short introduction. Heittokangas, Korhonen and Laine [15, Theorem 4.2] described the uniqueness of complex differential polynomials of certain types, their results can be stated as follows.

Theorem 1.3. *Let n, k be positive integers with $n \geq 4$, let $f(z)$ be a meromorphic function with $N(r, f) = S(r, f)$ and let $L(z, f)$ be a non-zero linear differential polynomial of $f(z)$ with small coefficients with respect to $f(z)$, and let $h(z)$ be a meromorphic function. For the differential equation*

$$f^n + L(z, f) = h, \tag{1.1}$$

then one of the following two situations holds:

- (a) *equation (1.1) has $f(z)$ as its unique transcendental meromorphic solution such that $N(r, f) = S(r, f)$;*
- (b) *equation (1.1) has exactly n transcendental meromorphic solutions f_j $(j = 1, 2, \cdots, n)$ such that $N(r, f_j) = S(r, f_j)$. In this case, $L(z, f_j) \equiv 0$ and $h = f_j^n$.*

Remark 1.4. (1) Theorem 1.3 is not true for $n = 3$ by observing that $f_1(z) = \sin z$, $f_2(z) = -\frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$, $f_3(z) = \frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$ are the entire solutions of the non-linear differential equation $4f^3 + 3f'' = -\sin 3z$, see [15], [34, Theorem 4].

(2) Some results on transcendental meromorphic solutions to non-linear differential equations of certain types can be found in [22, 23] and references therein, which is an active research topic recently. Yang and Laine [33, Theorem 2.6] obtained the uniqueness of complex delay-differential equations of certain type.

(3) Theorem 1.3 also shows that $f^n + L(z, f)$ $(n \geq 4)$ is UDPM in the sense of $N(r, f) = S(r, f)$ and $L(z, f)$ is a non-zero linear differential polynomial of $f(z)$ with small coefficients. Remark that the linearity of $L(z, f)$ in Theorem 1.3, it seems that there are no further results when $L(z, f)$ is a non-linear differential polynomial as far as we know.

Making a small modification of UDPM will allow us to consider the main systems in the following. For example, the existence of meromorphic solutions $f(z)$ and $g(z)$ of the functional equation $f^n + L(z,f) = g^n + L(z,g)$ is considered in Theorem 1.3, the above equation is equivalent to $f^n - L(z,g) = g^n - L(z,f)$. According to Theorem 1.3, we obtain the following corollary.

Corollary 1.5. *Let $L(z,h)$ be a non-zero linear differential polynomial. If $n \geq 4$, then the differential system*

$$\begin{aligned} f(z)^n - L(z,g) &= h(z), \\ g(z)^n - L(z,f) &= h(z), \end{aligned}$$

has at most a pair admissible transcendental entire solution (f,g) and $L(z,f) \not\equiv 0$ and $L(z,g) \not\equiv 0$. In fact, we have $f \equiv g$ provided that the pair admissible transcendental entire solution exists.

Definition 1.6. The following system of equations will be called the crossed system of equations

$$\begin{aligned} P(z,f) &= Q(z,g) + \alpha(z), \\ P(z,g) &= Q(z,f) + \alpha(z), \end{aligned} \tag{1.2}$$

in other words, two equations in the system will be consistent if we interchange the functions f and g for the equations, where $P(z,h)$ and $Q(z,h)$ are differential or difference or delay-differential polynomials of $h(z)$ with the small coefficients with respect to h and $\alpha(z)$ is a meromorphic function.

An application of Nevanlinna theory to the systems of complex differential, difference and delay-differential equations is an interesting research. The research on the complex differential systems of equations can be found in Gao [5, 6], Tu and Xiao [27], Liu, Laine and Yang [21, Chapter 11]. Recently, Wang and Liu [28], Guo and Liu [8], Li and Liu [20] have also considered the systems of complex differential and difference equations. In fact, the systems of equations considered in Li and Liu [20, Theorems 1.1, 1.3, 1.5] are the crossed systems of equations, such as Fermat type crossed differential system

$$\begin{aligned} f(z)^2 + g'(z)^2 &= 1, \\ f'(z)^2 + g(z)^2 &= 1. \end{aligned} \tag{1.3}$$

Additionally, some systems of partial difference or differential equations in several variables can be found in [29, 30, 31]. Barsegian, Laine and Yang [1] raised seven problems, related to the complex differential systems of equations, one of them is stated below.

Problem. What conditions can assure that all meromorphic solutions (f,g) of complex differential system

$$\begin{aligned} P_1\big(z,f,g,f^{(n)},g^{(m)}\big) &= 0, \\ P_2\big(z,f,g,f^{(n)},g^{(m)}\big) &= 0, \end{aligned} \tag{1.4}$$

are of finite order, namely $f(z)$ and $g(z)$ are finite order meromorphic functions, where $n,m \geq 2$ and P_1, P_2 are polynomials in all variables.

Remark 1.7. (1) If $m = n = 1$, then the entire solutions of (1.4) may be of infinite order. For example

$$\begin{aligned} f' - fg &= 0, \\ f' - fg' &= 0, \end{aligned} \tag{1.5}$$

can be solved by $f(z) = e^{e^z}$ and $g(z) = e^z$.

(2) We remark that (1.5) is not the crossed differential systems of equations from the point of view of this paper. Therefore, we may consider the system

$$\begin{aligned} f' - f^2 g &= 0, \\ g' - g^2 f &= 0. \end{aligned}$$

Then, we conclude that $f = ag$ and $ag' = a^2 g^3$, which implies that $f(z)$ is a constant, and so $g(z)$ is. We may also consider the system

$$\begin{aligned} f' - fg' &= 0, \\ g' - gf' &= 0. \end{aligned}$$

Then, we obtain $f = 1/g$, by the basic computation, we obtain that $g(z)$ must be a constant.

This article is organized as follows. In Section 2, we discuss the meromorphic solutions of the crossed Riccati differential systems of equations. In Section 3, we consider some other crossed differential systems of equations related to some classical complex differential equations. In Section 4, we give a crossed version of Clunie lemma.

2. Crossed Riccati differential systems of equations

In this section, we first consider the crossed Riccati differential system of equations. Gao [5] considered a Riccati differential system of equations, but it is not the crossed Riccati differential system of equations. The crossed complex Riccati differential system of equations with small coefficients can be expressed as

$$\begin{aligned} f'(z) &= a_0(z) + a_1(z)g(z) + a_2(z)g(z)^2, \\ g'(z) &= a_0(z) + a_1(z)f(z) + a_2(z)f(z)^2. \end{aligned} \tag{2.1}$$

If $a_2(z)$ is a constant, then (2.1) can be changed to the special case

$$\begin{aligned} u'(z) &= A(z) + v(z)^2, \\ v'(z) &= A(z) + u(z)^2, \end{aligned} \tag{2.2}$$

by replacing $f(z) = \frac{u(z)}{a_2} - \frac{a_1(z)}{2a_2}$, $g(z) = \frac{v(z)}{a_2} - \frac{a_1(z)}{2a_2}$, where $A(z) = a_0(z)a_2 - \frac{a_1(z)^2}{4} + \frac{a_1'(z)}{2}$. We will deal with the case of $a_0(z) \equiv 1$, $a_1(z) \equiv 0$ and $a_2(z) \equiv -1$ in the following theorem. A meromorphic function $f(z)$ in the complex plane is called properly meromorphic if $f(z)$ has at least one pole.

Theorem 2.1. *Let $f(z)$ and $g(z)$ be meromorphic solutions of the complex differential system of equations*

$$\begin{aligned} f(z)^2 &= 1 - g'(z), \\ g(z)^2 &= 1 - f'(z). \end{aligned} \tag{2.3}$$

Then, we obtain

(1) *$f(z)$ and $g(z)$ must be properly meromorphic functions and all poles are simple poles;*
(2) *$m(r,f)=S(r,f)$, $m(r,g)=S(r,g)$ and $T(r,f)=T(r,g)+S(r,f)$.*

Proof. (1) Firstly, $f(z)$ and $g(z)$ are not polynomials by the basic observations. Assume that $f(z)$ is a transcendental entire function, then so $g(z)$ is. By Valiron-Mohon'ko theorem [16, Theorem 2.2.5], we obtain

$$\begin{aligned}2T(r,f)&=T(r,g')+O(1)\le T(r,g)+S(r,g)=\frac{1}{2}T(r,g^2)+S(r,g)\\&\le \frac{1}{2}T(r,f')+S(r,g)\le \frac{1}{2}T(r,f)+S(r,f)+S(r,g).\end{aligned}$$

Similarly, we have $2T(r,g)\le \frac{1}{2}T(r,g)+S(r,f)+S(r,g)$. Hence,

$$2(T(r,f)+T(r,g))\le \frac{1}{2}(T(r,f)+T(r,g))+S(r,f)+S(r,g),$$

which is impossible. Thus, $f(z)$ and $g(z)$ must be properly meromorphic functions. Let z_0 be the pole of $f(z)$ with multiplicity p. Therefore, z_0 must be a pole of $g'(z)$ with multiplicity $2p$, then $p=1$ follows by the estimation of poles of the second equation of (2.3).

(2) To estimate the proximate functions of $f(z)$ and $g(z)$, taking the first derivative of the second equation of (2.3), we conclude that

$$2gg'=-f''.$$

Combining the first equation of (2.3) with the above equation, we have

$$2g(1-f^2)=-f''.$$

Hence, we have $4g^2(1-f^2)^2=(f'')^2$ and $4(1-f')(1-f^2)^2=(f'')^2$. Using the lemma of the logarithmic derivative, we obtain

$$\begin{aligned}m(r,1-f^2)&\le m\Big(r,\frac{(f'')^2}{(1-f')(1-f^2)}\Big)+O(1)\\&\le m\Big(r,\frac{f''}{1-f'}\Big)+m\Big(r,\frac{f''}{1-f^2}\Big)+O(1)\\&\le m\Big(r,\frac{f''}{1-f'}\Big)+m\Big(r,\frac{f''}{1-f}\Big)+m\Big(r,\frac{f''}{1+f}\Big)+O(1)\\&=S(r,f),\end{aligned}$$

then $2m(r,f)=m(r,f^2)\le m(r,1-f^2)+O(1)=S(r,f)$. Exchange f with g, we obtain

$$m(r,g)=S(r,g).$$

Since f and g are properly meromorphic functions, then

$$\begin{aligned}2T(r,f)&\le T(r,g')+O(1)\le 2T(r,g)+S(r,g)=T(r,g^2)+S(r,g)\\&\le T(r,f')+S(r,g)\le 2T(r,f)+S(r,f)+S(r,g).\end{aligned}$$

Hence, $T(r,f)=T(r,g)+S(r,f)$. □

So far, not all transcendental meromorphic solutions of (2.3) have been presented completely. We will give the partial consideration below. From (2.3), we conclude that

$$f(z)^2(1-f'(z))=g(z)^2(1-g'(z)),$$

and

$$f(z)^2 - g(z)^2 = f'(z) - g'(z),$$

hence

$$f(z)^2 f'(z) - g(z)^2 g'(z) = f'(z) - g'(z). \tag{2.4}$$

Integrating (2.4), it follows that

$$\frac{1}{3}f(z)^3 - \frac{1}{3}g(z)^3 = f(z) - g(z) + B. \tag{2.5}$$

Case 1. If $B = 0$, then

$$\frac{1}{3}(f(z)^3 - g(z)^3) = f(z) - g(z).$$

If $f(z) \neq g(z)$, then the above equation implies that

$$f(z)^2 + f(z)g(z) + g(z)^2 = 3.$$

We rewrite the above equation as

$$\Big(\frac{f(z)}{\sqrt{3}}\Big)^2 + \frac{f(z)}{\sqrt{3}}\frac{g(z)}{\sqrt{3}} + \Big(\frac{g(z)}{\sqrt{3}}\Big)^2 = 1.$$

Thus, we have $F(z)^2 + F(z)G(z) + G(z)^2 = 1$ by setting $F(z) = \frac{f(z)}{\sqrt{3}}$ and $G(z) = \frac{g(z)}{\sqrt{3}}$. From Saleeby's result [26, Theorem 2.1.] with $f(z)$ and $g(z)$ are properly meromorphic functions, we have

$$\begin{gathered} F(z) = \frac{1+\sqrt{3}i}{2\sqrt{3}i}h + \frac{-1+\sqrt{3}i}{2\sqrt{3}i}\frac{1}{h}, \\ G(z) = -\frac{1}{\sqrt{3}i}h + \frac{1}{\sqrt{3}ih}. \end{gathered} \tag{2.6}$$

Therefore,

$$\begin{gathered} f(z) = \frac{1+\sqrt{3}i}{2i}h + \frac{-1+\sqrt{3}i}{2i}\frac{1}{h}, \\ g(z) = ih - \frac{i}{h}, \end{gathered} \tag{2.7}$$

where h is a non-zero meromorphic function. Substituting (2.7) into (2.3), we obtain

$$\begin{gathered} \Big(\frac{1+\sqrt{3}i}{2i}h + \frac{-1+\sqrt{3}i}{2i}\frac{1}{h}\Big)^2 = 1 - \Big(ih - \frac{i}{h}\Big)', \\ \Big(ih - \frac{i}{h}\Big)^2 = 1 - \Big(\frac{1+\sqrt{3}i}{2i}h + \frac{-1+\sqrt{3}i}{2i}\frac{1}{h}\Big)'. \end{gathered} \tag{2.8}$$

Hence,

$$\begin{gathered} \frac{1-\sqrt{3}i}{2}h^2 + \frac{1+\sqrt{3}i}{2}\frac{1}{h^2} = -1 - ih' - i\frac{h'}{h^2}, \\ -h^2 - \frac{1}{h^2} = -1 - \frac{1+\sqrt{3}i}{2i}h' + \frac{-1+\sqrt{3}i}{2i}\frac{h'}{h^2}. \end{gathered} \tag{2.9}$$

Cancelling the term $\frac{h'}{h^2}$, we have

$$\Big(\frac{1-\sqrt{3}i}{2}\frac{-1+\sqrt{3}i}{2i} - i\Big)h^2 + \Big(\frac{1+\sqrt{3}i}{2}\frac{-1+\sqrt{3}i}{2i} - i\Big)\frac{1}{h^2} = \Big(\frac{1-\sqrt{3}i}{2i} - i\Big) - \sqrt{3}ih',$$

which implies that

$$\frac{\sqrt{3}-3i}{2}h^2 = \Big(\frac{1-\sqrt{3}i}{2i} - i\Big) - \sqrt{3}ih' = \frac{3-\sqrt{3}i}{2i} - \sqrt{3}ih'.$$

Thus, h satisfies

$$h' = \frac{\sqrt{3}+i}{2}h^2 - \frac{\sqrt{3}-i}{2}$$

which is a Riccati differential equation with constant coefficients and can be solved completely.

Case 2. If $B \neq 0$, [19, Corollary 1] affirms that there exist meromorphic solutions of (2.5), but it is difficult to obtain all meromorphic solutions at present. Here, we give the following details to describe the meromorphic solutions. Consider the equation

$$\frac{f^3}{3} - \frac{g^3}{3} - f + g = B.$$

Let $f = \sqrt[3]{3}\Big(G - F\Big)$ and $g = \sqrt[3]{3}\Big(G + F\Big)$. Then, the above equation changes into

$$\frac{\big(\sqrt[3]{3}(G-F)\big)^3}{3} - \frac{\big(\sqrt[3]{3}(G+F)\big)^3}{3} - \sqrt[3]{3}(G-F) + \sqrt[3]{3}(G+F) - B = 0,$$

that is

$$2F^3 + 6G^2F - 2\sqrt[3]{3}F + B = 0.$$

Let $F = 1/F_1$ and $G = G_1/F_1$. We obtain

$$\frac{1}{F_1^3}\big(BF_1^3 - 2\sqrt[3]{3}F_1^2 + 6G_1^2 + 2\big) = 0.$$

Thus, we have

$$G_1^2 = \frac{-B}{6}F_1^3 + \frac{1}{3}\sqrt[3]{3}F_1^2 - \frac{1}{3} = \frac{-B}{6}(F_1 - b_1)(F_1 - b_2)(F_1 - b_3).$$

If two of b_1, b_2, b_3 are equal, say $b_1 = b_2$, the above equation is

$$\Big(\frac{G_1}{F_1 - b_1}\Big)^2 = \frac{B}{6}(F_1 - b_3),$$

which admits obviously the non-constant meromorphic solutions F_1 and G_1, for example $F_1 = b_3 + H(z)^2$. If b_1, b_2, b_3 are different, then we can assume that $G_1 = F_1'$, thus there exists Weierstrass elliptic function F_1 satisfying

$$F_1'^2 = \frac{-B}{6}(F_1 - b_1)(F_1 - b_2)(F_1 - b_3).$$

Remark 2.2. (1) Obviously, $f = g = \tanh z$ satisfies (2.3). If $f = g$, then (2.3) reduces to Riccati differential equation $f'(z) = 1 - f(z)^2$ and all meromorphic solutions of the above equation can be stated as $f(z) = \frac{e^{2z+c}+1}{e^{2z+c}-1}$, where c is any constant.

(2) [16, Proposition 9.1.11] shows that all meromorphic solutions of (4.1) with rational coefficients are of finite order on growth and pseudo-prime with polynomials coefficients in [24]. It is an interesting question for the growth and pseudo-primeness of transcendental meromorphic solutions of (2.3), where the pseudo-primeness of $F(z)$ means that every factorization of $F = f \circ g$ implies that f is rational or g is a polynomial.

3. Crossed differential systems of equations in certain types

Recall the Weierstrass $\wp$-function which satisfies the Weierstrass differential equation $\wp'(z)^2 = 4\wp(z)^3 - 1$. The following theorem is to present the properties on the crossed Weierstrass differential system of equations. Unfortunately, we will only describe the proximate functions of the transcendental meromorphic solutions and can not describe the precise expressions of the transcendental meromorphic solutions.

Theorem 3.1. *Let $f(z)$ and $g(z)$ be transcendental meromorphic solutions of the crossed Weierstrass differential system*

$$\begin{aligned} f(z)^3 &= 1 - g'(z)^2, \\ g(z)^3 &= 1 - f'(z)^2. \end{aligned} \tag{3.1}$$

Then,

1. *$f(z)$ and $g(z)$ are properly transcendental meromorphic functions and all poles are double poles;*
2. *$m(r, f) = S(r, f)$ and $m(r, g) = S(r, g)$.*

Proof. (1) By the basic observations from (3.1), then $f(z)$ and $g(z)$ are not polynomials. Let $f(z)$ be a transcendental entire function. So $g(z)$ is entire. By Valiron-Mohon'ko theorem [16, Theorem 2.2.5] again, we obtain

$$\begin{aligned} 3T(r, f) &= 2T(r, g') + O(1) \le 2T(r, g) + S(r, g) = \frac{2}{3}T(r, g^3) + S(r, g) \\ &\le \frac{4}{3}T(r, f') + S(r, g) \le \frac{4}{3}T(r, f) + S(r, f) + S(r, g). \end{aligned}$$

Similarly, we have

$$3T(r, g) \le \frac{4}{3}T(r, g) + S(r, f) + S(r, g).$$

Combining the above two inequalities implies that

$$3(T(r, f) + T(r, g)) \le \frac{4}{3}(T(r, f) + T(r, g)) + S(r, f) + S(r, g),$$

which is impossible. Hence, $f(z)$ and $g(z)$ must be properly transcendental meromorphic functions. Let z_0 be a pole of $f(z)$ and $g(z)$ with multiplicity p and q, respectively. Then $p = q = 2$ follows by (3.1).

(2) Taking the first derivative of the second equation of (3.1), we have

$$3g^2g' = -2f'f''.$$

Taking the first derivative of the first equation of (3.1), we obtain

$$3f^2f' = -2g'g''.$$

Then

$$3f^3f' = 3(1 - g'^2)f' = -2fg'g'',$$

we obtain

$$9(1 - g'^2)^2f'^2 = 4f^2g'^2g''^2.$$

Hence, $\frac{1}{f^2} = \frac{4g'^2g''^2}{9(1-g'^2)^2(1-g^3)}$. Then, we conclude that

$$m(r, g) = m\Big(r, \frac{-2f'f''}{3gg'}\Big) = m\Big(r, \frac{4g'g''f''}{9gg'f^2}\Big)$$

$$
\begin{aligned}
&\le m\Big(r, \frac{f''}{f^2}\Big) + S(r,g) \\
&\le m\Big(r, \frac{1}{f}\Big) + S(r,g) + S(r,f).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
m(r,g^2) &\le m\Big(r, \frac{1}{f^2}\Big) + S(r,f) + S(r,g) \\
&\le m\Big(r, \frac{4g'^2 g''^2}{9(1-g'^2)^2(1-g^3)}\Big) + S(r,f) + S(r,g) \\
&\le m\Big(r, \frac{g'g''}{1-g'^2}\Big) + m\Big(r, \frac{g''}{1-g'}\Big) + m\Big(r, \frac{g''}{1+g'}\Big) \\
&\quad + m\Big(r, \frac{g'}{1-g^3}\Big) + S(r,f) + S(r,g) \\
&\le S(r,f) + S(r,g).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
3T(r,f) &= 2T(r,g') + O(1) \\
&\le 4T(r,g) + S(r,g) = \frac{4}{3}T(r,g^3) + S(r,g) \\
&\le \frac{4}{3}T(r,f'^2) + S(r,g) \\
&\le \frac{16}{3}T(r,f) + S(r,f) + S(r,g).
\end{aligned}
$$

So, we have $m(r,g) = S(r,g)$. In addition, $m(r,f) = S(r,f)$ can be proved similarly. □

Theorem 3.2. *Let $f(z)$ and $g(z)$ be transcendental meromorphic solutions of the complex differential system*

$$
\begin{aligned}
f(z)^4 &= 1 - g'(z)^2, \\
g(z)^4 &= 1 - f'(z)^2.
\end{aligned} \tag{3.2}
$$

Then $m(r,f) = S(r,f)$ and $m(r,g) = S(r,g)$.

Proof. Assume that $f(z)$ and $g(z)$ are transcendental meromorphic solutions of (3.2). We have that $T(r,f) = T(r,g) + S(r,g)$. Taking the first derivative of the first equation of (3.2), we have $4f^3 f' = -2g'g''$. Thus $4f^6 f'^2 = g'^2 g''^2$ and

$$
4f^6(1-g^4) = g'^2 g''^2.
$$

Therefore,

$$
\begin{aligned}
m(r,f) &\le m\Big(r, \frac{g'^2 g''^2}{1-g^4}\Big) \\
&\le m\Big(r, \frac{g'^2}{1-g^2}\Big) + m\Big(r, \frac{g''^2}{1+g^2}\Big) \\
&\le m\Big(r, \frac{g'}{1-g}\Big) + m\Big(r, \frac{g'}{1+g}\Big) + m\Big(r, \frac{g''}{1+ig}\Big) + m\Big(r, \frac{g''}{1-ig}\Big) \\
&\le S(r,g) = S(r,f).
\end{aligned}
$$

And $m(r,g) = S(r,g)$ can be proved in the same way. □

Remark 3.3. If $f = g$ in the system (3.2), then it reduces to the equation $f(z)^4 + f'(z)^2 = 1$. By the result given by Gross [9], we assume that $f(z)^2 = \frac{1-\beta(z)^2}{1+\beta(z)^2}$ and $f'(z) = \frac{2\beta(z)}{1+\beta(z)^2}$, where $\beta(z)$ is any meromorphic function. Furthermore, taking the first derivative of $f(z)^2$ and the basic computations, $\beta(z)$ must satisfy $\beta'(z)^2 = 2\beta(z)(1+\beta(z)^2)$ which is also a Weierstrass differential equation.

Theorem 3.4. *The complex differential system*

$$\begin{aligned} f(z)^n &= 1 - g'(z)^2, \\ g(z)^n &= 1 - f'(z)^2, \end{aligned} \tag{3.3}$$

has no transcendental meromorphic solutions $f(z)$ and $g(z)$ if $n \neq 2, 3, 4$.

Proof. If $n \geq 5$, by Picard theorem and the completely ramified theorem (A non-constant meromorphic functions $f(z)$ can have at most four completely ramified values), we can get a contradiction from the first equation of (3.3). If $n = 1$, we will consider the existence of meromorphic solutions of

$$\begin{aligned} f(z) &= 1 - g'(z)^2, \\ g(z) &= 1 - f'(z)^2. \end{aligned} \tag{3.4}$$

Obviously, if (3.4) admits transcendental meromorphic solutions $f(z)$ and $g(z)$, then $f(z)$ and $g(z)$ are transcendental entire functions. Taking the first derivative of the first equation of (3.4), we obtain $f'(z) = -2g'(z)g''(z)$, then $g(z) = 1 - 4g'(z)^2g''(z)^2$. Furthermore, taking the derivative of the above equation, we have

$$g'(z) = -8g'(z)g''(z)^3 - 8g'(z)^2g''(z)g'''(z)$$

and

$$-8g''(z)^3 - 8g'(z)g''(z)g'''(z) = 1.$$

For the reason that $g(z)$ is a transcendental entire function and the above equation, then $g''(z)$ has no zeros and $g''(z) = e^{P(z)}$, where $P(z)$ is any entire function. Thus,

$$g''(z)^2 + g'(z)g'''(z) = -\frac{1}{8}e^{-P(z)}. \tag{3.5}$$

By a result given by Mues [25] or see Gundersen and Yang [7, Theorem 2], we obtain that $g'(z) = \alpha_1 e^{\lambda_1 z}$. Thus, $g''(z) = \alpha_1\lambda_1 e^{\lambda_1 z}$ and $g'''(z) = \alpha_1\lambda_1^2 e^{\lambda_1 z}$. Substituting the above into (3.5), we have $2\alpha_1^3\lambda_1^3 e^{3\lambda_1 z} = -\frac{1}{8}$, which is impossible. □

Remark 3.5. (1) All transcendental meromorphic solutions of (3.3) can be expressed when $n = 2$, see Li and Liu [20, Theorem 1.1]. We also have discussed the properties of transcendental meromorphic solutions of (3.3) when $n = 3, 4$. The properties are deserved to considering when the first derivative is replaced with the higher derivatives in (3.3).

(2) Gundersen and Yang [7] obtained many results on the generalizations of (3.5).

Next, we present two results concerning the existence of transcendental meromorphic solutions on two crossed complex differential systems of equations in certain types.

Theorem 3.6. *The complex differential system*

$$\begin{aligned} f(z)f'(z) &= 1 - g'(z), \\ g(z)g'(z) &= 1 - f'(z), \end{aligned} \tag{3.6}$$

has no non-constant transcendental meromorphic solutions.

Proof. If (3.6) admits transcendental meromorphic solutions $f(z)$ and $g(z)$, then $f(z)$ and $g(z)$ must be transcendental entire functions by checking the poles multiplicities of $f(z)$ and $g(z)$. By the addition and subtraction of two equations in (3.6), we have $ff' + gg' = 2 - f' - g'$ and $ff' - gg' = f' - g'$. Integrating the above two equations, we have

$$\frac{1}{2}f^2 - \frac{1}{2}g^2 = f - g + A_1,$$

and

$$\frac{1}{2}f^2 + \frac{1}{2}g^2 = 2z - f - g + A_2,$$

thus we have $T(r,f) = T(r,g) + S(r,f)$ by Valiron-Mohon'ko theorem [16, Theorem 2.2.5]. However, we obtain $f^2 = 2z - 2g + A_1 + A_2$ by adding the above two equations, which implies that $2T(r,f) = T(r,g) + S(r,f)$, which is a contradiction. □

Theorem 3.7. *The complex differential system*

$$\begin{aligned} f(z)^n g'(z) &= 1, \\ g(z)^n f'(z) &= 1, \end{aligned} \tag{3.7}$$

has no meromorphic solutions.

Proof. We assume that z_0 is a pole of $f(z)$ with multiplicity p, then z_0 is a pole of $f'(z)$ with multiplicity $p+1$. From (3.7), we obtain that z_0 is a zero of $g'(z)$ with multiplicity np and z_0 is a zero of $g(z)$ with multiplicity $\frac{p+1}{n}$. Thus, $np + 1 = \frac{p+1}{n}$. Therefore, we have that either $n = 1$ or $f(z)$ and $g(z)$ are all entire functions with no zeros for avoiding a contradiction. We discuss two cases below.

Case 1. If $n = 1$, then

$$\begin{aligned} f(z)g'(z) &= 1, \\ g(z)f'(z) &= 1. \end{aligned} \tag{3.8}$$

Subtracting the second from the equation equations of (3.8), we have $\left(\frac{g}{f}\right)' = 0$ and $g = cf$ follows; however, $cff' = 1$ has no any meromorphic solutions, where c is a constant.

Case 2. If $f(z)$ and $g(z)$ are all entire functions with no zeros, then we assume that $f(z) = e^{P(z)}$ and $g(z) = e^{Q(z)}$, where $P(z)$ and $Q(z)$ are entire functions. From (3.7), we have

$$\begin{aligned} Q'(z)e^{nP(z)+Q(z)} &= 1, \\ P'(z)e^{nQ(z)+P(z)} &= 1. \end{aligned} \tag{3.9}$$

Case 2.1. If $f(z)$ and $g(z)$ are entire functions of finite order, that is $P(z)$ and $Q(z)$ are polynomials. From (3.9), we have $P(z) = Az + a$, $Q(z) = Bz + b$ $(AB \neq 0)$, substituting $P(z)$ and $Q(z)$ into (3.9), we have $A = -B$, $n = 1$. However, $e^{a+b} = \frac{1}{A}$ and $e^{a+b} = \frac{1}{B}$, thus $A = B$ follows, which is impossible.

Case 2.2. If $f(z)$ and $g(z)$ are entire functions of infinite order, that is $P(z)$ and $Q(z)$ are transcendental entire functions. From the equations of (3.7), we have $S(r,f) = S(r,g)$. From the equations of (3.9), by letting $P'(z) = e^{s(z)}$ and $Q'(z) = e^{t(z)}$, we have

$$\begin{aligned} e^{nP(z)+Q(z)} &= e^{-t(z)}, \\ e^{nQ(z)+P(z)} &= e^{-s(z)}. \end{aligned} \tag{3.10}$$

From [36, Theorem 1.47] and the first main theorem of Nevanlinna theory, we obtain

$$T(r, Q'(z)) = T(r, e^{-t(z)}) + O(1) = S(r, e^{Q(z)}) = S(r, g(z)) = S(r, f(z))$$

and

$$T(r, P'(z)) = T(r, e^{-s(z)}) + O(1) = S(r, e^{P(z)}) = S(r, f(z)) = S(r, g(z)).$$

The system (3.10) implies that $e^{(n^2-1)P(z)} = e^{-nt(z)+s(z)}$ and $n = 1$ follows for avoiding a contradiction, this is the Case 1. □

4. Crossed version of Clunie lemma

The Clunie lemma is an efficient tool to study the properties of meromorphic solutions of complex differential equations and has many variants with numerous applications in the bibliography. The original lemma is due to Clunie [2, Lemma 1] and the following version can be found in [16, Lemma 2.4.2].

Lemma 4.1. *Let $f(z)$ be a transcendental meromorphic solution of*

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f), Q(z, f)$ are differential polynomials with small coefficients, say $\{a_\lambda : \lambda \in I\}$, such that $m(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. If $\deg(Q(z, f)) \le n$, then

$$m(r, P(z, f)) = S(r, f),$$

where $S(r, f) = o(T(r, f))$ for all r outside of an exceptional set with finite linear measure.

Obviously, the Clunie lemma can be applied to estimate the counting function of poles of $P(z, f)$, namely $N(r, P(z, f))$, to determine the existence of transcendental meromorphic or entire solutions of complex differential equations. For instance,

$$f'(z) = a_0(z) + a_1(z)f(z) + f(z)^2 \tag{4.1}$$

has no transcendental entire solutions, where $a_0(z)$, $a_1(z)$ are small functions with respect to $f(z)$, otherwise $m(r, f) = S(r, f)$ follows immediately by the Clunie lemma. Equation (4.1) is called Riccati differential equation and admits actually transcendental meromorphic solutions, more details on the existence of meromorphic solutions of Riccati differential equation can be found in [16, Chapter 9]. Doeringer [3], He-Xiao [14], Korhonen [11], Yang and Ye [35, Theorem 1] also derived the different estimate on the proximity functions. The difference version of Clunie lemma is given by Halburd and Korhonen [12], Laine and Yang [17]. The delay-differential version of Clunie lemmas can be found in [21, Chapter 1]. These different versions of Clunie lemma have many applications on complex differential equations, complex difference equations and complex delay-differential equations. Theorems 2.1, 3.1, 3.2 inspire us to consider a corresponding version of Clunie lemma for the crossed complex differential systems of equations.

Theorem 4.2. *Let $f(z)$ and $g(z)$ be transcendental meromorphic solutions of the system*

$$\begin{aligned} f(z)^n L_1(z,f) &= L_2(z,g), \\ g(z)^n L_1(z,g) &= L_2(z,f), \end{aligned} \tag{4.2}$$

where $L_1(z,f)$ and $L_2(z,f)$ are differential polynomials in $f(z)$ with small coefficients, $L_1(z,g)$ and $L_2(z,g)$ are differential polynomials in $g(z)$ with small coefficients. If $\gamma_{L_2} \le n$ and $|L_1(z,f)| \ge 1$ when $|f| \ge 1$, then $m(r, L_1(z,f)) = S_1(r)$; if $\gamma_{L_2} \le n$ and $|L_1(z,g)| \ge 1$ when $|g| \ge 1$, then $m(r, L_1(z,g)) = S_1(r)$, where $S_1(r) = S(r,f) + S(r,g)$.

Proof. Assume that E_1 and E_2 satisfy

$$E_1 = \{\varphi \in [0,2\pi] | |f(re^{i\varphi})| < 1\},$$
$$E_2 = [0,2\pi] \setminus E_1.$$

According to the definition of $m(r, L_1(z,f))$, we have

$$m(r, L_1(z,f)) = \frac{1}{2\pi}\int_{E_1} \log^+ |L_1(z,f)| d\varphi + \frac{1}{2\pi}\int_{E_2} \log^+ |L_1(z,f)| d\varphi := I_1 + I_2.$$

To estimate the first part I_1, we rewrite $L_1(z,f)$ as

$$\begin{aligned} |L_1(z,f)| &= \big|\sum_{i\in I} M_i(z,f)\big| \\ &= \big|\sum_{i\in I} a_i f^{l_{0i}} (f')^{l_{1i}} \cdots (f^{(\nu)})^{l_{\nu i}}\big| \\ &\le \sum_{i\in I} a_\lambda \big|\frac{f'}{f}\big|^{l_{1i}} \cdots \big|\frac{f^{(\nu)}}{f}\big|^{l_{\nu i}}, \end{aligned}$$

by the lemma of the logarithmic derivative, we conclude that

$$I_1 = \frac{1}{2\pi}\int_{E_1} \log^+ |L_1(z,f)| d\varphi = S(r,f).$$

To consider the second part I_2, we see that

$$\begin{aligned} |L_1(z,f)| &= \big|\frac{L_2(z,g)}{f^n}\big| \\ &= \big|\frac{L_2(z,f)}{f^n}\big| \big|\frac{L_2(z,g)}{L_2(z,f)}\big| \big|\frac{g^n}{g^n}\big| \\ &= \big|\frac{L_2(z,f)}{f^n}\big| \big|\frac{1}{L_1(z,g)}\big| \big|\frac{L_2(z,g)}{g^n}\big|. \end{aligned} \tag{4.3}$$

Furthermore, if $\varphi \in E_2$ and $|g(re^{i\varphi})| < 1$, by the first equation of (4.2), we obtain

$$|L_1(z,f)| = \big|\frac{L_2(z,g)}{f^n}\big|.$$

Hence,

$$\frac{1}{2\pi}\int_{E_2} \log^+ |L_1(z,f)| d\varphi = \frac{1}{2\pi}\int_{E_2} \log^+ \big|\frac{L_2(z,g)}{f^n}\big| d\varphi = S(r,g),$$

then $m(r, L_1(z,f)) = S(r,f) + S(r,g)$ follows.

We proceed to assume that $\varphi \in E_2$ and $|g(re^{i\varphi})| \geq 1$. Since $\gamma_{L_2} \leq n$, we can conclude that

$$\begin{aligned}|\frac{L_2(z,f)}{f^n}| &= |\frac{\sum_{j\in J} N_j(z,f)}{f^n}| \\ &= |\frac{\sum_{j\in J} b_j f^{s_{0j}}(f')^{s_{1j}}\dots(f^{(\mu)})^{s_{\mu j}}}{f^n}| \\ &\leq \sum_{j\in J} b_j |\frac{f'}{f}|^{s_{1j}}\dots|\frac{f^{(\mu)}}{f}|^{s_{\mu j}}.\end{aligned}$$

By the lemma of the logarithmic derivative again, we obtain

$$\frac{1}{2\pi}\int_{E_2} \log^+ |\frac{L_2(z,f)}{f^n}| d\varphi = S(r,f).$$

Similarly, for $\varphi \in E_2$ and $|g(re^{i\varphi})| \geq 1$, we have

$$\frac{1}{2\pi}\int_{E_2} \log^+ |\frac{L_2(z,g)}{g^n}| d\varphi = S(r,g).$$

By

$$|L_1(z,f)| = |\frac{L_2(z,f)}{f^n}|\,|\frac{1}{L_1(z,g)}|\,|\frac{L_2(z,g)}{g^n}|,$$

we have

$$I_2 = \frac{1}{2\pi}\int_{E_2} \log^+ |L_1(z,f)| d\varphi = S(r,f) + S(r,g)$$

with the condition that $|L_1(z,g)| \geq 1$ when $|g| \geq 1$. In addition, $m(r, L_1(z,g)) = S_1(r)$ can be proved in the same way. □

Let us see the condition that $|L_1(z,f)| \geq 1$ when $|f| \geq 1$, $|L_1(z,g)| \geq 1$ when $|g| \geq 1$. In fact, this condition is satisfied in Theorems 2.1, 3.1, 3.2.

5. Discussion

Some properties on of crossed differential systems of certain types have been considered in Sections 2 and 3. We pose the following question for the further studies.

Question 5.1. Can we include all transcendental meromorphic solutions of the crossed differential systems in Theorems 2.1, 3.1, and 3.2? Furthermore, how can we describe the uniqueness, growth or pseudo-primeness of transcendental meromorphic solutions of the crossed differential system (1.2)?

The complex differential polynomials $f(z)^2 - f'(z)$ and $f(z)^n - f'(z)^2 (n = 3, 4)$ are considered in Theorems 2.1, 3.1, and 3.2, however $f(z)^2 - f'(z)$ is not a UDPM. It remains an open question whether $f(z)^n - f'(z)^2$ $(n \geq 3)$ are UDPM or not. The following question may be also worth considering.

Question 5.2. Can we obtain conditions to describe completely the UDPM?

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