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SOLVING LINEAR DIFFERENTIAL EQUATIONS WITH MIXED ARGUMENTS

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Abstract. We find expressions for solutions to an initial value problem of a linear n-dimensional differential equation. This equation has time deviations of mixed type (advanced and delayed) that have fixed points. The key assumption is that these deviations are odd functions.

1. Introduction

Berezansky, Braverman and Pinelas (2023) [3] studied differential equations of the form

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x(h_k(t)) = 0, \quad t \geq t_0,$$

where the functions b_k and $t \to t - h_k(t)$ are Lebesgue measurable and essentially bounded on an interval of the form $[t_0, +\infty)$ and satisfy $-\sigma_k \leq t - h_k(t) \leq \tau_k$, with σ_k, $\tau_k \geq 0$. They provided conditions for the existence and uniqueness of solutions, with initial value $\phi(t)$, $t \in (t_0 - \max_k \tau_k, t_0) \to \mathbb{R}$, as well as, they studied the asymptotic behaviour. As an application they considered the differential equation

$$\dot{x}(t) + b(t)x(t - \sin t) = 0, \quad t \geq 0 \tag{1.1}$$

and proved the existence of an exponentially decaying solution in $L_\infty(t_0, +\infty)$ [3, Example 4.3]. This is done under the hypothesis that $b(t) \geq b_0 > 0$ and $\sup_{t\geq 0} b(t) < 1$. To illustrate their results, in case of $b(t) = \alpha e^{-\alpha \sin t}$, the authors suggested as a solution the function $x(t) = e^{-\alpha t}$, for a suitable value of the parameter α. Obviously, any function of the form $x(t) = \xi e^{-\alpha t}$, is a solution, for the same values of α and, as we can see, this function is a special solution which does not depend on a certain given initial function at a point $t_0 > 0$, but on the value ξ at 0. Also, in a discussion of the results (no. (2)) at the end of the paper, the authors, among others, posed the following open problem:

> Is it possible to find conditions such that there exists a unique solution (not necessarily bounded) of an initial value problem for some types of equations with mixed arguments?

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In this article we present a type of differential equations and give some answers to these questions. To be more precise we formulate and solve a initial value problem concerning the n-dimensional equation $(DE_{A,B,f})$:

$$\dot{x}(t) = A(t)x(t) + B(t)x(t - \tau(t)) + f(t), \quad t \in (-R, R), \tag{1.2}$$

for some $R \in \mathbb{R} \cup \{+\infty\}$, when the initial values are points in $\mathbb{R}^n$, as the point ξ above. The function f and the $n \times n$-matrix-valued functions A, B are continuous and defined on $(-R, R)$. The argument τ is a real valued odd continuous function and no boundedness restriction on $t - \tau(t)$ (as in [3]) is posed. Thus, if in a set J, the function τ is positive, we have delay argument, while in the symmetric set $-J$ it is negative and so we have advanced argument. Under these characteristics, we give the expression of the (unique) full solution (i.e. it is defined on the whole interval $(-R, R)$) in the form of a series with terms containing iterated integrals.

It is well known that for a differential equation of the form

$$\dot{x}(t) = H(t, x(t), x(g_1(t)), x(g_2(t)), \dots, x(g_m(t))), \quad t \in [a, b],$$

we distinguish the following four cases:

(1) Delay DE: All arguments g_j are delays.
(2) Advanced DE: All arguments g_j are advance.
(3) Deviated DE: Some of the arguments are delays and the rest of them are advance.
(4) Mixed DE: There is a certain argument g_j and subsets J_1, J_2 of the interval $[a, b]$, with non-empty interior, such that $g_j(t) < t$ for $t \in J_1$, and $g_j(t) > t$ for $t \in J_2$.

Recall that an argument g is a delay, if it satisfies $g(t) \leq t$ for all $t \in [a, b]$ and $g(t) < t$ for some t, and it is an advance argument, if $g(t) \geq t$ for all $t \in [a, b]$ and $g(t) > t$ for some t. The most investigated cases are the first three of them, but for the last one, a few results can be found in the literature.

Differential equations with pure delay arguments have studied in many works originated from Volterra (1928) [26] and continued by Minorsky (1942) [22], Myschkis (1949) [23], Hayes (1950) [17], Cunningham (1954) [7], Bellman-Danskin (1954) [5], Razumikhin (1956) [25], Sverkin et al. (1959) [28], to name some of the contributions before 1960. But in the last sixty years a great number of papers and books have been published on the subject. Among them we refer to the classical relative books such as [1, 6, 8, 10, 13, 14, 15, 20].

The problem with the advanced differential equations is, also, too old, originated, probably, by Myschkis (1955) [24]. Contributions to this kind of linear equations, perhaps, start in the work by Kato and McLeod (1971) [19], where an equation of the form

$$y'(t) = ay(\lambda t) + by(t), \quad 0 \leq t < +\infty \tag{1.3}$$

is investigated for various values of the parameter $\lambda > 0$. In [19], for values $0 < \lambda < 1$, equation (1.3) is associated with the boundary condition

$$y(0) = 1, \tag{1.4}$$

but for $\lambda > 1$, the authors claim that no such boundary condition seems to arise. Indeed, they prove that the boundary-value problem (1.3)-(1.4) is well-posed, if $0 < \lambda < 1$, but not, if $\lambda > 1$. Actually, in [19] the authors seek for analytic solutions of the problem. However in [11] a similar approximation of the solutions of equations like (1.3) is obtained.

The main property of the solutions discussed in the literature refers to the asymptotic behaviour and especially to the oscillatory character of them. For instance, such a property of the linear advanced differential equation of the form

$$\dot{x}(t) = a(t)x(t + h(t)) + b(t)x(t + r(t)) = 0, \quad t \geq 0,$$

where $h(t), r(t) \geq 0$, is investigated in [9]. The oscillatory behaviour of solutions of the nth-order nonlinear differential equations with an advanced argument of the form

$$x^{(n)}(t) + \delta q(t)x^{\lambda}(g(t)) = 0,$$

is investigated in [12], where $n \geq 2$ is even, $\delta = \pm 1$ and $g(t) \geq t$, for all $t \geq t_0$. The same property is, also, discussed for the differential equation

$$x'(t) + \delta_1 a(t)x(g(t)) + \delta_2 b(t)x(h(t)) = 0, \quad t \geq t_0,$$

in [4], with coefficients $a(t) \geq 0$, $b(t) \geq 0$, and arguments g delay and h advance.

In [21] the oscillatory behaviour of solutions of the second order functional differential equation with a mixed neutral term of the form

$$\Big(r(t)[(y(t) + p_1(t)y(\rho_1(t)) + p_2(t)y(\rho_2(t)))']^{\gamma}\Big)' + q(t)y^{\gamma}(\sigma(t)) = 0, \quad t \geq t_0$$

is investigated, where $\rho_1(t) \leq t \leq \rho_2(t)$. Similar problems can be found in the references in [21].

On the other hand there are some works, where the main subject is the existence of solutions of differential equations with mixed type arguments. See, for instance, [3] as above. N. Hale et al. (2024), in [16], suggest an absolutely and uniformly convergent series as solutions of the linear equation

$$u'(t) + u(t) = 2u(\alpha t), \quad t > 0, \tag{1.5}$$

where $\alpha > 1$, to approximate solutions to the nonlinear functional equation of the form

$$u'(t) + u(t) = u^2(\alpha t), \quad t > 0.$$

For equation (1.5) it is shown that except of the trivial solution $u = 0$, with initial value $u(0) = 0$, there is a nontrivial solution for some special values of the parameter α. In the sequel we shall give sufficient conditions to guarantee uniqueness of the solutions.

A similar problem was discussed in [27] concerning the problem

$$f'(t) = af(\lambda t), \quad t \in \mathbb{R}, \quad f(0) = 0, \tag{1.6}$$

where nontrivial solutions are constructed for constants $\lambda > 1$ and $a \neq 0$. Notice that this equation is of advanced type for $t > 0$ and delay type for $t < 0$.

More facts about differential equations with mixed arguments can be found in a great number of works in the literature, starting from the work of Kato and MacLeon (1971) [19], mentioned above, as well as in [2, 9, 12]. See, also, the references therein. In all these equations the response function depends on arguments which are either delay, or advance. The problem is what happens when the argument has a mixed type dependence, namely, when in the equation both these situations occur in the same argument.

In this note we deal with a special type of linear n-dimensional functional differential equations with mixed argument of the form (1.2). We intend to give results on the existence of full solutions of such an equation with point-wise initial values.

To obtain our results we use the simple (but very important) fact that the roots of the argument τ are fixed points of the function $t \to t - \tau(t)$. Then the results of [18] play a very important role.

2. Preliminaries

We furnish the space of $n \times n$ matrices with a norm $\| \cdot \|$ induced from the Euclidean norm $| \cdot |$ of the vector space $\mathbb{R}^n$. We start with some facts borrowed from [18]. Let $J := (a, b)$ be an interval of the real line $\mathbb{R}$. Consider the linear n-dimensional delay differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)x(g(t)) + f(t), \quad t \in J, \tag{2.1}$$

where

- (C1) $g : J \to J$ is a continuous function such that $g(t) \le t$, for all $t \in J$ and for some ω with $g(\omega) = \omega$ it holds $g(t) > \omega$ for $t \in (\omega, b)$.
- (C2) A, B are $n \times n$-matrix valued continuous functions defined on J and integrable on (ω, b).
- (C3) $f : J \to \mathbb{R}^n$ is a function continuous and integrable on (ω, b).

From the standard theory of linear functional differential equations (see [14, p. 11]) it is well known, that under these conditions, for any $\sigma \in I$ and any continuous function $\varphi : [\bar{g}(\sigma), \sigma] \to \mathbb{R}^n$, there is a unique solution $x(t) := x(t; (\sigma, \varphi))$ such that $x(t) = \varphi(t)$, for all $t \in [\bar{g}(\sigma), \sigma]$. Here $\bar{g}(\sigma) = \inf_{t \ge \sigma} g(t)$. Let ω be a fixed point of g on I; then $[\bar{g}(\omega), \omega] = [\omega]$ and therefore any initial function φ for the solution $x(\omega, \varphi)$ is actually a point of $\mathbb{R}^n$. Also, if g does not have a fixed point ω in J, (as, for instance, happens in case of constant delay, namely, $g(t) = t - \tau$,) but we know that $\lim_{t \to \omega+} g(t) = \omega \in \mathbb{R} \cup \{-\infty\}$, then we say that ω is a generalised fixed point of g. In this case the solution x, whose limit $\lim_{t \to \omega+} x(t) =: x(\omega) = \xi$ exists, is unique and it is

$$x(t) = E_{A,B}(t, \omega)\xi + z(t), \quad t \in (\omega, b),$$

where $E_{A,B}(t, \omega)$ is the evolution matrix of the homogeneous equation and z is the solution of the non-homogeneous equation such that $z(\omega) = 0$. Here the evolution $E_{A,B}$ is defined by the formula

$$E_{A,B}(t, \omega) := \sum_{m=0}^{+\infty} [Z_\omega^m(A, B) I_{n\times n}](t), \quad \text{if} \quad t \in [\omega, b), \tag{2.2}$$

where $I_{n \times n}$ is the $n \times n$ identity matrix and

$$[Z_\omega^m(A, B)T](t) := \begin{cases} T(t), & \text{if } m = 0, \\ \int_\omega^t \big[A(u)[Z_\omega^{m-1}(A, B)T](u) & \\ +B(u)[Z_\omega^{m-1}(A, B)T](g(u))\big]du, & \text{if } m \ge 1, \end{cases}$$

for all continuous $n \times n$-matrix valued functions T defined at least on the interval (ω, b). Notice that the evolution satisfies the equation

$$\frac{\partial}{\partial t} E_{A,B}(t, \omega) = A(t)E_{A,B}(t, \omega) + B(t)E_{A,B}(g(t), \omega), \quad t \in (\omega, b),$$

with $E_{A,B}(\omega, \omega) = I_{n \times n}$.

The solution z of the non-homogeneous equation with initial value $z(\omega) = 0$, is

$$z(t) = \sum_{m=0}^{+\infty} [Z_\omega^m(A, B)F](t),$$

where $F(t) := \int_\omega^t f(u)du$ for $t \in (\omega, b)$.

As an example, we borrow from [18] the information that the evolution $E(t, -\infty)$ of equation

$$\dot{x}(t) = e^{\lambda t} x(t-1), \quad t \in \mathbb{R},$$

where $\lambda \geq 0$, is

$$E(t, -\infty) = \sum_{m=0}^{+\infty} \frac{\exp[\lambda(mt - \frac{1}{2}m(m-1))]}{m!\lambda^m}, \quad t \in \mathbb{R}.$$

Also, the evolution $E(t, 0)$ of equation

$$\dot{x}(t) = \alpha t^\beta x(\gamma t), \quad t > 0, \tag{2.3}$$

where $\alpha, \beta \geq 0$ and $\gamma \in (0, 1)$, is

$$E(t, 0) = \sum_{m=0}^{+\infty} \frac{\alpha^m \gamma^{\frac{1}{2}m(m-1)(\beta+1)}}{m!(\beta+1)^m} t^{m(\beta+1)}, \quad t \geq 0. \tag{2.4}$$

From the previous arguments, it seems that an initial value problem formed for the differential equation (2.1) is defined on an interval (ω, T) and satisfies $x(\omega) = \xi$. Hence, it behaves like the analogous one for linear ordinary differential equations. And this is true because the initial value is a vector and not a function defined on the "past".

Now we come back to equation (1.2). In this case the basic assumption on the argument τ is the following

(C4) The function $\tau : (-R, R) \to \mathbb{R}$ is continuous and odd and such that in any compact interval there is a finite number of points where it changes its sign.

By assumption (C4), the roots of τ are distinc. We have $\tau(0) = 0 =: r_0$ and let $r_k \in (0, R)$, $k \in \{1, \ldots, N\}$, $(N \in \mathbb{N} \cup \{+\infty\})$, be the (positive) moments where the argument τ changes sign. Notice that, due to oddity of τ, the points $-r_k \in (-R, 0)$, $(k \in \{1, 2, \ldots, N\})$ are the negative zeros of τ. We shall denote by J_{r_k} the interval $[r_k, r_{k+1}]$, if $k = 0, 1, \ldots, N-1$ and $J_{r_N} := [r_N, R)$, if $k = N < +\infty$. Also, we denote by $J_{-r_k} := [-r_k, -r_{k-1}]$, for all $k = 1, \ldots, N$ and in case $k = N < +\infty$, $J_{-R} := (-R, -r_N]$. We let $\mathcal{J}$ be the family of all such intervals. These items are such that, if in some interval J_{r_k}, we have $\tau(t) \geq 0$, then, in the next interval $J_{r_{k+1}}$, it holds $\tau(t) \leq 0$. Moreover, if it holds $\tau(t) \geq 0$ on the interval $J \in \mathcal{J}$, then $\tau(t) \leq 0$ on the dual interval $-J$ and inversely. In other words, in this case, the equation is a delay differential equation on the interval $J \in \mathcal{J}$ and an advanced differential equation on the interval $-J \in \mathcal{J}$. In the sequel we shall denote by $\mathcal{J}_+$ (resp. $\mathcal{J}_-$) the collection of all intervals where it holds $\tau(t) \geq 0, \quad t \in J$ (resp. $\tau(t) \leq 0, \quad t \in J$). Further, we assume the following condition

(C5) The function $g(t) := t - \tau(t)$ maps any $J \in \mathcal{J}$ to itself.

A prototype of such a function τ is the trigonometric function sin appeared in differential equation (1.1).

By using the expression (2.2) we conclude that for any interval $J \in \mathcal{J}_+$, there exists a solution of the problem defined on the whole interval J. The following result can be found in [18], but for completeness of this paper, we shall repeat the proof here.

Lemma 2.1. *Consider equation* (1.2) *defined on an interval* $J := [\alpha, \beta] \in \mathcal{J}_+$ *and assume that conditions* (C2)–(C5) *are satisfied on* J*. Then, for each* $\xi \in \mathbb{R}^n$*, there exists a unique solution defined on* J*, with the initial value* $x(\alpha) = \xi$*. (In case* $\alpha = -\infty$*, or* $\beta = +\infty$ *the values are meant in the limit sense.)*

Proof. Let x_1 and x_2 be two solutions with the same initial value ξ. Then we have

$$x_i(t) = \xi + \int_\alpha^t A(s)x_i(s)ds + \int_\alpha^t B(s)x_i(g(s))ds + F(t),$$

for $i = 1, 2$, where $F(t) := \int_\alpha^t f(s)ds$. Define $p(t) = |x_1(t) - x_2(t)|$ and assume that for some $\alpha', \alpha'' \in J$ it holds $p(t) = 0$, $t \in [\alpha, \alpha']$ and $p(t) > 0$, $t \in (\alpha', \alpha'']$. Then, for any $r \in (\alpha', \beta]$, we have that

$$P_r := \max_{t\in[\alpha',r]} p(t) > 0.$$

Choose a $t_1 \in J$ such that

$$\int_{\alpha'}^{t_1} (\|A(u)\| + \|B(u)\|)du < 1. \tag{2.5}$$

Then, there is a $\bar{t} \in [\alpha', t_1]$, such that

$$\begin{aligned}
P_{t_1} &= p(\bar{t}) \\
&= \Big| \int_{\alpha'}^{\bar{t}} A(s)(x_1(s) - x_2(s))ds + \int_{\alpha'}^{\bar{t}} B(s)(x_1(g(s)) - x_2(g(s)))ds \Big| \\
&\le \int_{\alpha'}^{\bar{t}} \|A(s)\||x_1(s) - x_2(s)|ds + \int_{\alpha'}^{\bar{t}} \|B(s)\||x_1(g(s)) - x_2(g(s))|ds \\
&\le \int_{\alpha'}^{t_1} (\|A(s)\| + \|B(s)\|)p(s)ds \\
&\le P_{t_1} \int_{\alpha'}^{t_1} (\|A(s)\| + \|B(s)\|)ds < P_{t_1},
\end{aligned}$$

because of (2.5). This is a contradiction and so $P_{t_1} = 0$. Now, step by step, we conclude the uniqueness result. □

As we stated above, in the delay situation, our initial value problem behaves like the analogous one for linear ordinary differential equations. Moreover, in the ODE case the corresponding evolution is produced by the transition matrix and it is well known that the evolution is a nonsingular matrix. In the present functional case, this is not true, as it is proved in the scalar case in [18]. The question is under what conditions the evolution is a regular matrix. It is evident that in the real-scalar case, if in (1.2) the coefficients A, B are measurable functions and (nontrivially) nonnegative a.e., then the evolution of the homogeneous part of the equation is positive. In the n-dimensional case we have the following result.

Lemma 2.2. *Consider equation* (1.2) *with* $f = 0$ *and an interval* $J := [\alpha, \beta]$ *in* $\mathcal{J}_+$*. Assume that* A, B *are* $n \times n$*-matrix valued continuous functions defined on* J

and satisfy the condition ($C_{A,B}$)

$$\int_\alpha^\beta (\|A(u)\| + \|B(u)\|)du < \ln 2. \tag{2.6}$$

Then, for each $t \in J$, the evolution matrix $E_{A,B}(t,\alpha)$ is non-singular.

Proof. Since $E_{A,B}(\alpha,\alpha) = I_{n\times n}$ and the matrix $I_{n\times n}$ is nonsingular, it is enough to discuss it for any fixed $t \in (\alpha,\beta]$. First, by using induction, we shall prove that, for $m \geq 1$, it holds

$$\|Z_\alpha^m(A,B)I_{n\times n}](t)\| \leq \frac{1}{m!}\Big(\int_\alpha^t (\|A(u)\| + \|B(u)\|)du\Big)^m, \quad t \in J. \tag{2.7}$$

Indeed, for $m = 1$, we have

$$\begin{aligned}\|Z_\alpha^m(A,B)I_{n\times n}](t)\| &= \|\int_\alpha^t \big[A(u)I_{n\times n} + B(u)I_{n\times n})\big]du\| \\ &\leq \frac{1}{1!}\Big(\int_\alpha^t (\|A(u)\| + \|B(u)\|)du\Big)^1.\end{aligned}$$

Notice that $g(u) := u - \tau(u) \leq u$, for all $u \in J$. Next, assume that for some $m = r$, the claim is true, namely,

$$\|Z_\alpha^r(A,B)I_{n\times n}](t)\| \leq \frac{1}{r!}\Big(\int_\alpha^t (\|A(u)\| + \|B(u)\|)du\Big)^r.$$

Then, for $m = r+1$, we have

$$\begin{aligned}&\|Z_\alpha^{r+1}(A,B)I_{n\times n}](t)\| \\ &= \|\int_\alpha^t \big[A(u)[Z_\alpha^r(A,B)I_{n\times n}](u) + B(u)[Z_\alpha^r(A,B)I_{n\times n}](g(u))\big]du\| \\ &\leq \frac{1}{r!}\int_\alpha^t \big[\|A(u)\| + \|B(u)\|\big]\Big(\int_\alpha^u (\|A(v)\| + \|B(v)\|)dv\Big)^r du =: \varphi(t).\end{aligned}$$

We shall show that

$$\varphi(t) = \psi(t) := \frac{1}{(r+1)!}\Big(\int_\alpha^t (\|A(v)\| + \|B(v)\|)dv\Big)^{r+1}.$$

To do it we observe that $\varphi(0) = 0 = \psi(0)$. Also,

$$\varphi'(t) = \frac{1}{r!}\big[\|A(t)\| + \|B(t)\|\big]\Big(\int_\alpha^t (\|A(v)\| + \|B(v)\|)dv\Big)^r = \psi'(t).$$

Hence we have $\varphi = \psi$, which proves the inequality (2.7).

Now, assume that the matrix $E_{A,B}(t,\alpha)$ is singular. Then its kernel has dimension greater than or equal to 1. Thus a n-vector w exists such that $|w| \neq 0$ satisfying equation $E_{A,B}(t,\alpha)w = 0$. This fact and (2.7) imply that

$$\begin{aligned}0 &= |E_{A,B}(t,\alpha)w| \\ &= |I_{n\times n}w + \sum_{m=1}^{+\infty}[Z_\omega^m(A,B)I_{n\times n}](t)w| \\ &\geq |w| - \sum_{m=1}^{+\infty}\|[Z_\omega^m(A,B)I_{n\times n}](t)\||w|\end{aligned}$$

$$\begin{aligned}&\geq |w| - \sum_{m=1}^{+\infty} \frac{1}{m!}\Big(\int_{\alpha}^{t}(\|A(u)\| + \|B(u)\|)du\Big)^{m}|w|\\ &= \big[2 - \exp\int_{\alpha}^{\beta}(\|A(u)\| + \|B(u)\|)du\big]|w| > 0,\end{aligned}$$

a contradiction. This proves the result. □

In the sequel we use the transformations

$$s = -t,\ \hat{x}(s) := x(t),\ \hat{A}(s) = -A(-s),\ \hat{B}(s) = -B(-s),\ \hat{f}(s) := -f(-s). \quad (2.8)$$

Remark 2.3. It is clear that if $[\alpha, \beta]$ is an interval in $\mathcal{J}$ and the condition (1.2)with $\hat{A}$ and $\hat{B}$ holds on $[\alpha, \beta]$, then, from the relation

$$\int_{-\beta}^{-\alpha}(\|\hat{A}(s)\| + \|\hat{B}(s)\|)ds = \int_{\alpha}^{\beta}(\|A(t)\| + \|B(t)\|)dt < \ln 2,$$

it follows that with $\hat{A}$ and $\hat{B}$ with $\hat{A}$ and $\hat{B}$ holds on $[-\beta, -\alpha]$.

3. Expression of solutions

Our main result is given using the folloing assumption

(C6) Relation with $\hat{A}$ and $\hat{B}$ holds on all intervals of the form $J_{-r_{2k}}$ for $k = 1, 2, \dots, N$, if $J_0 \in \mathcal{J}_+$; and of the form $J_{-r_{2k+1}}$ for $k = 0, 1, 2, \dots, N$, if $J_0 \in \mathcal{J}_-$.

Theorem 3.1. *Consider the differential equation* (1.2) *and assume that conditions* (C2)–(C6) *are satisfied on any interval $J \in \mathcal{J}$. Then, for each $\xi \in \mathbb{R}^n$, there exists a unique full solution $x(t)$, $t \in (-R, R)$ such that $x(0) = \xi$.*

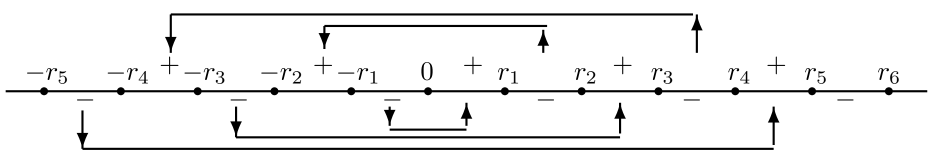
Proof. Our plan is to apply the idea in [18] to each interval of the form J_{r_k} and $J_{-r_{k+1}}$ and obtain successively expressions of the solutions on each such interval. The question is what happens when in an interval of the form (a, b) it holds $t-\tau(t) > t$, namely the differential equation is advanced. To give answer to this question, we use the dual interval $(-b, -a)$, where we have a delay argument, and so the solution can be easily expressed. This justifies our assumption on the oddity of τ. Finally, having found the expression of the solutions on each interval piecewise we "join" them by using the initial values at the fixed points of the argument, namely at the terms of the sequence $\dots, -r_j, -r_{j-1}, \dots, -r_1, 0, r_1, \dots, r_{j-1}, r_j, \dots$, for $j = 1, 2, \dots, N$.

To be more precise, we solve the problem by discussing two cases: case I: $J_0 \in \mathcal{J}_+$ and case II: $J_0 \in \mathcal{J}_-$. In each case we apply induction on successive stages $\mathcal{S}(n)$, $n = 1, 2, \dots$, where the n-th stage refers to extending the solution on the intervals in the order $J_{r_{2n-2}}$, $J_{-r_{2n-1}}$, $J_{-r_{2n}}$, $J_{r_{2n-1}}$, in case I and on the intervals in the order $J_{-r_{2n-1}}$, $J_{r_{2n-2}}$, $J_{r_{2n-1}}$, $J_{-r_{2n}}$, in case II. Making the transformation (2.8) we obtain the equation, $(DE_{\hat{A},\hat{B},\hat{f}})$:

$$\dot{\hat{x}}(s) = \hat{A}(s)\hat{x}(s) + \hat{B}(s)\hat{x}(g(s))) + \hat{f}(s), \quad (3.1)$$

which is associated with (1.2) and has the following property: If (1.2) is an advanced type differential equation on an interval $J \in \mathcal{J}$, because $g(s) = s - \tau(s) = -t - \tau(-t) = -t + \tau(t) = -(t - \tau(t)) \leq -t = s$, equation (3.1) is a delay differential equation and inversely. We call these equations symmetric.

Case I: $J_0 \in \mathcal{J}_+$.
Stage $\mathcal{S}(1)$:



The solution on the intervals where τ gets negative values are obtained from the solution on their dual intervals where we have a delay differential equation.

Step (I.1)$_1$ [*Solution on* J_0]: First we solve the differential equation (1.2), on the interval J_0. Since 0 is a fixed point of the argument $g(t) := t - \tau(t)$, according to [18], the solution of (1.2) with initial value ξ at 0 is uniquely determined by its initial value ξ at 0 and it is

$$x(t) = E_{A,B}(t,0)\xi + \sum_{m=0}^{+\infty}[Z_0^m(A,B)F_0](t), \quad t \in J_0, \tag{3.2}$$

where $F_0(t) := \int_0^t f(u)du$,

$$[Z_0^m(A,B)F_0](t) := \begin{cases} F_0(t), & \text{if } m = 0, \\ \int_0^t \big[A(u)[Z_0^{m-1}(A,B)F_0](u) \\ +B(u)[Z_0^{m-1}(A,B)F_0](g(u))\big]du, & \text{if } m \ge 1 \end{cases}$$

and $E_{A,B}(t,0)$ is the function given by (2.2) for $\omega = 0$.

Step (I.2)$_1$ [*Extension to* J_{-r_1}]: On this interval $\tau(t) \le 0$ and so $g(t) := t - \tau(t) \ge t$, i.e. equation (1.2) is advanced. Observe that for $t \in J_{-r_1}$, we have $s = -t \in J_0$. So, its symmetric differential equation is a delay differential equation on J_0. Then, as in the previous step, we can solve it and obtain its solution

$$\hat{x}(s) = E_{\hat{A},\hat{B}}(s,0)\xi + \sum_{m=0}^{+\infty}[Z_0^m(\hat{A},\hat{B})\hat{F}_0](s), \quad s \in J_0, \tag{3.3}$$

where $\hat{F}_0(t) := \int_0^t \hat{f}(u)du$. Notice that $\hat{x}(0) = x(0) = \xi$. Now, let $w(t) := \hat{x}(-t)$, $t \in J_{-r_1}$. Since $E_{\hat{A},\hat{B}}$ is the evolution of equation (1.2) with $\hat{A}$, $\hat{B}$ and $\hat{f}$ in the place of A, B and f, we can easily obtain that

$$\begin{aligned} \dot{w}(t) &= -\dot{\hat{x}}(-t) \\ &= -\hat{A}(-t)\hat{x}(-t) - \hat{B}(-t)\hat{x}(g(-t)) - \hat{f}(-t) \\ &= A(t)w(t) + B(t)w(g(t)) + f(t). \end{aligned}$$

Setting $x = w$ on J_{-r_1} we find that the unique solution x of the original problem on the interval J_{-r_1} is

$$x(t) = E_{\hat{A},\hat{B}}(-t,0)\xi + \sum_{m=0}^{+\infty}[Z_0^m(\hat{A},\hat{B})\hat{F}_0](-t), \quad t \in J_{-r_1}. \tag{3.4}$$

Step (I.3)$_1$ [*Extension to* J_{-r_2}]: On the interval J_{-r_2} equation (1.2) is a delay differential equation and $-r_2$ is a fixed point of the delay. Hence, its solution with initial value $x(-r_2)$ is

$$x(t) = E_{A,B}(t,-r_2)x(-r_2) + \sum_{m=0}^{+\infty}[Z^m_{-r_2}(A,B)F_{-r_2}](t), \quad t \in J_{-r_2}, \tag{3.5}$$

where $F_{-r_2}(t) := \int_{-r_2}^t f(u)du$ and

$$[Z^m_{-r_2}(A,B)F_{-r_2}](t) := \begin{cases} F_{-r_2}(t), & \text{if } m=0, \\ \int_{-r_2}^t \big[A(u)[Z^{m-1}_{-r_2}(A,B)F_{-r_2}](u)du, & \\ +B(u)[Z^{m-1}_{-r_2}(A,B)F_{-r_2}](g(u))\big]du, & \text{if } m \geq 1. \end{cases} \tag{3.6}$$

Our purpose is to express the value $x(-r_2)$ in terms of the initial value ξ. First we notice that the matrices A, B satisfy condition with $\hat{A}$ and $\hat{B}$ on the interval $-J_{r_1} = J_{-r_2} \in \mathcal{J}_+$. Hence, by Lemma 2.2, the matrix $E_{A,B}(-r_1,-r_2)$ is nonsingular.

Now, from (3.5) we have

$$x(-r_1) = E_{A,B}(-r_1,-r_2)x(-r_2) + \sum_{m=0}^{+\infty}[Z^m_{-r_2}(A,B)F_{-r_2}](-r_1),$$

and from (3.4)

$$x(-r_1) = E_{\hat{A},\hat{B}}(r_1,0)\xi + \sum_{m=0}^{+\infty}[Z^m_0(\hat{A},\hat{B})F_0](r_1).$$

These two equations give

$$\begin{aligned} x(-r_2) = \big(E_{A,B}(-r_1,-r_2)\big)^{-1}\Big[&E_{\hat{A},\hat{B}}(r_1,0)\xi \\ &+ \sum_{m=0}^{+\infty}\big([Z^m_0(\hat{A},\hat{B})F_0](r_1) - [Z^m_{-r_2}(A,B)F_{-r_2}](-r_1)\big)\Big]. \end{aligned}$$

Therefore on the interval J_{-r_2} the solution x has the expression

$$\begin{aligned} x(t) = E_{A,B}(t,-r_2)\big(E_{A,B}(-r_1,-r_2)\big)^{-1}\Big[&E_{\hat{A},\hat{B}}(r_1,0)\xi \\ &+ \sum_{m=0}^{+\infty}\big([Z^m_0(\hat{A},\hat{B})F_0](r_1) - [Z^m_{-r_2}(A,B)F_{-r_2}](-r_1)\big)\Big] \\ &+ \sum_{m=0}^{+\infty}[Z^m_{-r_2}(A,B)F_{-r_2}](t), \quad t \in J_{-r_2}. \end{aligned} \tag{3.7}$$

Step (I.4)$_1$ [*Extension to* J_{r_1}]: On this interval the argument $g(t) = t - \tau(t)$ is of advanced type and for $t \in J_{r_1}$ we have $s = -t \in J_{-r_2}$. So, making the transformation (2.8) we obtain the symmetric equation, which, on the interval J_{-r_2} is a delay differential equation. Hence the equation has the solution expressed as

$$\hat{x}(s) = E_{\hat{A},\hat{B}}(s,-r_2)\hat{x}(-r_2) + \sum_{m=0}^{+\infty}[Z^m_{-r_2}(\hat{A},\hat{B})\hat{F}_{-r_2}](s), \quad s \in J_{-r_2}, \tag{3.8}$$

where, as previously, $\hat{F}_{-r_2}(s) := \int_{-r_2}^{s} \hat{f}(u)du$ and $Z^m_{-r_2}(\hat{A},\hat{B})$ is given as in (3.6), but evaluated with respect to the functions $\hat{A},\hat{B}$. From here we obtain

$$\hat{x}(-r_1) = E_{\hat{A},\hat{B}}(-r_1,-r_2)\hat{x}(-r_2) + \sum_{m=0}^{+\infty}[Z^m_{-r_2}(\hat{A},\hat{B})\hat{F}_{-r_2}](-r_1).$$

On the other hand we know that

$$\hat{x}(-r_1) = x(r_1) = E_{A,B}(r_1,0)\xi + \sum_{m=0}^{+\infty}[Z^m_0(A,B)F_0](r_1).$$

From Remark 2.7 and the fact that the interval $[-r_2,-r_1]$ belongs to $\mathcal{J}_+$, the matrix $E_{\hat{A},\hat{B}}(-r_1,-r_2)$ is non-singular. Then from the system of the last two relations we obtain

$$\begin{aligned}\hat{x}(-r_2) = \big[E_{\hat{A},\hat{B}}(-r_1,-r_2)\big]^{-1}\Big[&E_{A,B}(r_1,0)\xi \\ &+ \sum_{m=0}^{+\infty}\big([Z^m_0(A,B)F_0](r_1) - Z^m_{-r_2}(\hat{A},\hat{B})\hat{F}_{-r_2}](-r_1)\big)\Big].\end{aligned}$$

Therefore, from (3.8) we obtain the expression of the solution $\hat{x}$ as

$$\begin{aligned}\hat{x}(s) = E_{\hat{A},\hat{B}}(s,-r_2)\big[E_{\hat{A},\hat{B}}(-r_1,-r_2)\big]^{-1}\Big[&E_{A,B}(r_1,0)\xi \\ &+ \sum_{m=0}^{+\infty}\big([Z^m_0(A,B)F_0](r_1) - [Z^m_{-r_2}(\hat{A},\hat{B})\hat{F}_{-r_2}](-r_1)\big)\Big] \\ &+ \sum_{m=0}^{+\infty}[Z^m_{-r_2}(\hat{A},\hat{B})\hat{F}_{-r_2}](s), \quad s \in J_{-r_2}.\end{aligned} \tag{3.9}$$

Thus the solution of the original equation on the interval J_{r_1} is

$$\begin{aligned}x(t) &= \hat{x}(-t) \\ &= E_{\hat{A},\hat{B}}(-t,-r_2)\big[E_{\hat{A},\hat{B}}(-r_1,-r_2)\big]^{-1}\Big[E_{A,B}(r_1,0)\xi \\ &\quad + \sum_{m=0}^{+\infty}\big([Z^m_0(A,B)F_0](r_1) - [Z^m_{-r_2}(\hat{A},\hat{B})\hat{F}_{-r_2}](-r_1)\big)\Big] \\ &\quad + \sum_{m=0}^{+\infty}[Z^m_{-r_2}(\hat{A},\hat{B})\hat{F}_{-r_2}](-t), \quad t \in J_{r_1}.\end{aligned} \tag{3.10}$$

At this first stage $\mathcal{S}(1)$ we obtained the solution x of the differential equation (1.2) on the interval $[-r_2,r_2]$ and the solution $\hat{x}$ of the differential equation (3.1) on the same interval. The latter happens, because $\hat{x}$ is obtained to be defined by (3.3) on the interval J_0, by (3.9) on the interval J_{-r_2} and moreover it satisfies $\hat{x}(s) = x(t)$, for $s = -t \in J_{-r_1} \cup J_{r_1}$. Notice that the values of x on the intervals $[-J_{-r_1}] \cup [-J_{r_1}] = J_0 \cup J_{-r_2}$ are already known. The conclusion is that the solutions x and $\hat{x}$ have known values on the interval $[-r_2,r_2]$ and these values are given in terms of the initial value ξ at 0.

Now we assume that the process up to the k-stage $\mathcal{S}(k)$ gave the solution of (1.2) and of its symmetric equation (3.1) defined on the interval $[-r_{2k},r_{2k}]$, as functions depending on the initial value ξ at 0. Let $X^{(k)}(\cdot;(0,\xi))$ and $\hat{X}^{(k)}(\cdot;(0,\xi))$, respectively, be these solutions. Hence, the values $X^{(k)}(r_{2k};(0,\xi))$ and $X^{(k)}(-r_{2k};(0,\xi))$

are known and they will be used in the $S(k+1)$ stage. Notice that it holds $\hat{X}^{(k)}(s;(0,\xi)) = X^{(k)}(-s;(0,\xi))$, for all $s \in [-r_{2k}, r_{2k}]$. To proceed to the $(k+1)$-stage, $\mathcal{S}(k+1)$, namely, to extend the solution to the interval $[-r_{2k+2}, r_{2k+2}]$, we work as follows:

Stage $\mathcal{S}(k+1)$:

Step (I.1)$_{k+1}$ [*Extension to* $J_{r_{2k}}$]: On the interval $J_{r_{2k}}$ the argument is a delay, so the solution with initial value $x(r_{2k})$ at r_{2k} is

$$x(t) = E_{A,B}(t, r_{2k})x(r_{2k}) + \sum_{m=0}^{+\infty}[Z^m_{r_{2k}}(A,B)F_{r_{2k}}](t), \quad t \in J_{2k}, \tag{3.11}$$

where $F_{r_{2k}}(t) := \int_{r_{2k}}^t f(u)du$, and

$$[Z^m_{r_{2k}}(A,B)F_{r_{2k}}](t) := \begin{cases} F_{r_{2k}}(t), & \text{if } m=0,\\ \int_{r_{2n}}^t \big[A(u)[Z^{m-1}_{r_{2k}}(A,B)F_{r_{2k}}](u) & \\ +B(u)[Z^{m-1}_{r_{2k}}(A,B)F_{r_{2k}}](g(u))\big]du, & \text{if } m \ge 1. \end{cases}$$

The value $x(r_{2k})$ can be found from the k-th stage for $t = r_{2k}$, which is equal to $X^{(k)}(r_{2k};(0,\xi))$. Hence the solution on $J_{r_{2k}}$ is

$$x(t) = E_{A,B}(t, r_{2k})X^{(k)}(r_{2k};(0,\xi)) + \sum_{m=0}^{+\infty}[Z^m_{r_{2k}}(A,B)F_{r_{2k}}](t), \quad t \in J_{r_{2k}}. \tag{3.12}$$

Step (I.2)$_{k+1}$ [*Extension to* $J_{-r_{2k+1}}$]: On this interval the equation is of advanced type. Then, as above, making the transformation (2.8) we obtain the symmetric equation defined on $J_{r_{2k}}$. Then, by the previous step, we obtain that its solution is

$$\hat{x}(s) = E_{\hat{A},\hat{B}}(s, r_{2k})\hat{X}^{(k)}(r_{2k};(0,\xi)) + \sum_{m=0}^{+\infty}[Z^m_{r_{2k}}(\hat{A},\hat{B})\hat{F}_{r_{2k}}](s), \quad s \in J_{r_{2k}}, \tag{3.13}$$

where, we used that $\hat{X}^{(k)}(r_{2k};(0,\xi)) = X^{(k)}(-r_{2k};(0,\xi)))$. This implies that the solution x on the interval $J_{-r_{2k+1}}$, with initial value $x(0) = \xi$, is

$$\begin{aligned} x(t) = \hat{x}(-t) &= E_{\hat{A},\hat{B}}(-t, r_{2k})X^{(k)}(-r_{2k};(0,\xi)) \\ &+ \sum_{m=0}^{+\infty}[Z^m_{r_{2k}}(\hat{A},\hat{B})\hat{F}_{r_{2k}}](-t), \quad t \in J_{-r_{2k+1}}. \end{aligned} \tag{3.14}$$

Step (I.3)$_{k+1}$ [*Extension to* $J_{-r_{2k+2}}$]: On this interval equation (1.2) is a delay differential equation and $-r_{2k+2}$ is a fixed point of the argument. So, its solution x with initial value $x(-r_{2k+2})$ is

$$x(t) = E_{A,B}(t, -r_{2k+2})x(-r_{2k+2}) + \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+2}}(A,B)F_{-r_{2k+2}}](t), + \tag{3.15}$$

for $t \in J_{-r_{2k+2}}$, where $F_{-r_{2k+2}}(t) := \int_{-r_{2k+2}}^t f(u)du$ and

$$\begin{aligned} &[Z^m_{-r_{2k+2}}(A,B)F_{-r_{2k+2}}](t) \\ &:= \begin{cases} F_{-r_{2k+2}}(t), & \text{if } m=0,\\ \int_{-r_{2k+2}}^t \big[A(u)[Z^{m-1}_{-r_{2k+2}}(A,B)F_{-r_{2k+2}}](u)du, & \\ +B(u)[Z^{m-1}_{-r_{2k+2}}(A,B)F_{-r_{2k+2}}](g(u))\big]du, & \text{if } m \ge 1. \end{cases} \end{aligned} \tag{3.16}$$

We shall express the value $x(-r_{2k+2})$ in terms of the initial value ξ. First we notice that the matrices A, B satisfy condition (2.6) with $\hat{A}$ and $\hat{B}$ on the interval $J_{-r_{2k+2}} \in \mathcal{J}_+$. To proceed we observe that from (3.5) we have

$$\begin{aligned} x(-r_{2k+1}) = E_{A,B}&(-r_{2k+1}, -r_{2k+2})x(-r_{2k+2}) \\ &+ \sum_{m=0}^{+\infty} [Z^m_{-r_{2k+2}}(A,B)F_{-r_{2k+2}}](-r_{2k+1}), \end{aligned}$$

and from (3.14),

$$x(-r_{2k+1}) = E_{\hat{A},\hat{B}}(r_{2k+1}, r_{2k})X^{(k)}(-r_{2k}; (0,\xi)) + \sum_{m=0}^{+\infty} [Z^m_{r_{2k}}(\hat{A},\hat{B})\hat{F}_{r_{2k}}](r_{2k+1}),$$

for $t \in J_{-r_{2k+1}}$. These two equations give

$$\begin{aligned} x(-r_{2k+2}) = \big(E_{A,B}(-r_{2k+1}, -r_{2k+2})\big)^{-1} \Big[& E_{\hat{A},\hat{B}}(r_{2k+1}, r_{2n})X^{(k)}(-r_{2k}; (0,\xi)) \\ + \sum_{m=0}^{+\infty} \big[& [Z^m_{r_{2k}}(\hat{A},\hat{B})\hat{F}_{2k}](r_{2k+1}) - [Z^m_{-r_{2k+2}}(A,B)F_{-r_{2k+2}}](-r_{2n+1})\big]\Big]. \end{aligned}$$

because, by Lemma 2.2, the matrix $E_{A,B}(-r_{2k+1}, -r_{2k+2})$ is non-singular. Therefore on the interval $J_{-r_{2k+2}}$ the solution x has the expression

$$\begin{aligned} x(t) = {} & E_{A,B}(t, -r_{2k+2})\big(E_{A,B}(-r_{2k+1}, -r_{2k+2})\big)^{-1} \\ & \times \Big[E_{\hat{A},\hat{B}}(r_{2k+1}, r_{2k})X^{(k)}(-r_{2k}; (0,\xi)) \\ & + \sum_{m=0}^{+\infty} \big[Z^m_{r_{2k}}(\hat{A},\hat{B})\hat{F}_{2k}](r_{2k+1}) - [Z^m_{-r_{2k+2}}(A,B)F_{-r_{2k+2}}](-r_{2k+1})\big)\Big] \\ & + \sum_{m=0}^{+\infty} [Z^m_{-r_{2k+2}}(A,B)F_{-r_{2k+2}}](t), \quad t \in J_{-r_{2k+2}}. \end{aligned} \tag{3.17}$$

Step (I.4)$_{k+1}$ [*Extension to* $J_{r_{2k+1}}$]: On this interval the argument $g(t) = t - \tau(t)$ is of advanced type and, for $t \in J_{r_{2k+1}}$, we have $s = -t \in J_{-r_{2n+2}}$. So, making the transformation (2.8) we obtain its symmetric equation which is a delay differential equation. This equation has the solution

$$\hat{x}(s) = E_{\hat{A},\hat{B}}(s, -r_{2k+2})\hat{x}(-r_{2k+2}) + \sum_{m=0}^{+\infty} [Z^m_{-r_{2k+2}}(\hat{A},\hat{B})\hat{F}_{-r_{2k+2}}](s), \tag{3.18}$$

for $s \in J_{-r_{2k+2}}$, where, as before, $\hat{F}_{-r_{2k+2}}(s) := \int_{-r_{2k+2}}^{s} \hat{f}(u)du$ and $Z^m_{-r_{2k+2}}(\hat{A},\hat{B})$ is given as in (3.6), but evaluated with respect to the functions $\hat{A}, \hat{B}, \hat{f}$ defined on the interval $J_{-r_{2k+2}}$. From here we obtain

$$\begin{aligned} \hat{x}(-r_{2k+1}) = E_{\hat{A},\hat{B}}&(-r_{2k+1}, -r_{2k+2})\hat{x}(-r_{2k+2}) \\ &+ \sum_{m=0}^{+\infty} [Z^m_{-r_{2k+2}}(\hat{A},\hat{B})\hat{F}_{-r_{2k+2}}](-r_{2k+1}). \end{aligned}$$

On the other hand, from (3.12), we know that

$$\hat{x}(-r_{2k+1}) = x(r_{2k+1})$$

$$= E_{A,B}(r_{2k+1}, r_{2k})X^{(n)}(r_{2k}; (0, \xi)) + \sum_{m=0}^{+\infty} [Z^m_{r_{2k}}(A, B)F_{r_{2k}}](r_{2k+1}).$$

Since the functions $\hat{A}, \hat{B}$ satisfy condition (2.6) with $\hat{A}$ and $\hat{B}$ on the interval $[-r_{2k+2}, -r_{2k+1}]$, from Remark 2.7, it follows that the matrix $E_{\hat{A},\hat{B}}(-r_{2k+1}, -r_{2k+2})$ is nonsingular. The system of these two relations give the value

$$\begin{aligned} \hat{x}(-r_{2k+2}) = \big[E_{\hat{A},\hat{B}}(-r_{2k+1}, -r_{2k+2})\big]^{-1} \Big\{ E_{A,B}(r_{2k+1}, r_{2k})X^{(n)}(r_{2k}; (0, \xi)) \\ + \sum_{m=0}^{+\infty} \big([Z^m_{r_{2n}}(A, B)F_{r_{2k}}](r_{2k+1}) - [Z^m_{-r_{2k+2}}(\hat{A}, \hat{B})\hat{F}_{-r_{2k+2}}](-r_{2k+1}) \big) \Big\}. \end{aligned}$$

Thus, from (3.18) we obtain the expression of the solution $\hat{x}$ as

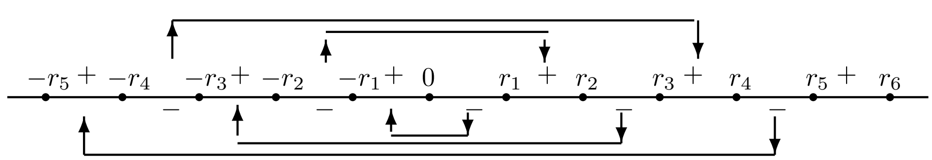
$$\begin{aligned} \hat{x}(s) &= E_{\hat{A},\hat{B}}(s, -r_{2k+2})\big[E_{\hat{A},\hat{B}}(-r_{2k+1}, -r_{2k+2})\big]^{-1} \\ &\quad \times \Big\{ E_{A,B}(r_{2k+1}, r_{2k})X^{(k)}(r_{2k}; (0, \xi)) + \sum_{m=0}^{+\infty} \big([Z^m_{r_{2k}}(A, B)F_{r_{2k}}](r_{2k+1}) \\ &\quad - [Z^m_{-r_{2k+2}}(\hat{A}, \hat{B})\hat{F}_{-r_{2k+2}}](-r_{2k+1}) \big) \Big\} \\ &\quad + \sum_{m=0}^{+\infty} [Z^m_{-r_{2k+2}}(\hat{A}, \hat{B})\hat{F}_{-r_{2k+2}}](s), \quad s \in J_{-r_{2k+2}} \end{aligned}$$

and so the solution of the original equation on the interval $J_{r_{2k+1}}$ is

$$\begin{aligned} x(t) &= \hat{x}(-t) \\ &= E_{\hat{A},\hat{B}}(-t, -r_{2n+2})\big[E_{\hat{A},\hat{B}}(-r_{2k+1}, -r_{2k+2})\big]^{-1} \\ &\quad \times \Big\{ E_{A,B}(r_{2k+1}, r_{2k})X^{(k)}(r_{2k}; (0, \xi)) \\ &\quad + \sum_{m=0}^{+\infty} \big([Z^m_{r_{2k}}(A, B)F_{r_{2k}}](r_{2k+1}) - [Z^m_{-r_{2k+2}}(\hat{A}, \hat{B})\hat{F}_{-r_{2k+2}}](-r_{2k+1}) \big) \Big\} \\ &\quad + \sum_{m=0}^{+\infty} [Z^m_{-r_{2k+2}}(\hat{A}, \hat{B})\hat{F}_{-r_{2k+2}}](-t), \quad t \in J_{r_{2k+1}}. \end{aligned} \tag{3.19}$$

So far we have arrived to stage $S(k+1)$ and we extended the solution x of (1.2) on the interval $J_{r_{2k}} \cup J_{-r_{2k+1}} \cup J_{-r_{2k+2}} \cup J_{r_{2k+1}}$, by the types (3.12), (3.14), (3.17), (3.19). At the same time we obtained the solution of its symmetric on the same intervals. Thus we obtained the expression of the solution x on the entire interval $[-r_{2k+2}, r_{2k+2}]$, for all positive integers k.

Case II: $J_0 \in \mathcal{J}_-$.
Stage $\mathcal{S}(1)$:



Step (II.1)$_1$ [*Solution on* J_{-r_1}]: Consider the delay differential equation (1.2) defined on the interval J_{-r_1}. Since $-r_1$ is a fixed point of the argument $g(t) := t-\tau(t)$, according to [18], it admits a unique solution x with initial value $x(-r_1)$ at $-r_1$, which is uniquely determined by its initial value and it is

$$x(t) = E_{A,B}(t,-r_1)x(-r_1) + \sum_{m=0}^{+\infty}[Z^m_{-r_1}(A,B)F_{-r_1}](t), \quad t \in J_{-r_1}, \tag{3.20}$$

where $F_{-r_1}(t) := \int_{-r_1}^t f(u)du$ and, for $t \in [-r_1, 0]$,

$$[Z^m_{-r_1}(A,B)F_{-r_1}](t) := \begin{cases} F_{-r_1}(t), & \text{if } m=0,\\ \int_{-r_1}^t \big[A(u)[Z^{m-1}_{-r_1}(A,B)F_{-r_1}](u) & \\ +B(u)[Z^{m-1}_{-r_1}(A,B)F_{-r_1}](g(u))\big]du, & \text{if } m \ge 1. \end{cases}$$

Keeping in mind that $x(0)=\xi$, we have

$$x(-r_1) = [E_{A,B}(0,-r_1)]^{-1}\big[\xi - \sum_{m=0}^{+\infty}[Z^m_{-r_1}(A,B)F_{-r_1}](0)\big],$$

since, by (C6), condition (2.6) with $\hat{A}$ and $\hat{B}$ is satisfied on the interval $J_{-r_1} = -J_0 \in \mathcal{I}_+$. And so, by Lemma 2.2, the matrix $E_{A,B}(0,-r_1)$ is nonsingular. Hence the solution x on the interval J_{-r_1} is

$$\begin{aligned} x(t) &= E_{A,B}(t,-r_1)[E_{A,B}(0,-r_1)]^{-1}\big[\xi - \sum_{m=0}^{+\infty}[Z^m_{-r_1}(A,B)F_{-r_1}](0)\big] \\ &\quad + \sum_{m=0}^{+\infty}[Z^m_{-r_1}(A,B)F_{-r_1}](t), \quad t \in J_{-r_1}. \end{aligned}$$

Step (II.2)$_1$ [*Extension to* J_0]: On this interval it happens $\tau(t) \le 0$, and so its symmetric is a delay differential equation on J_{-r_1}. Then, as in Step (II.1)$_1$, we can solve it and obtain its solution

$$\begin{aligned} \hat{x}(s) &= E_{\hat{A},\hat{B}}(s,-r_1)[E_{\hat{A},\hat{B}}(0,-r_1)]^{-1}\big[\xi - \sum_{m=0}^{+\infty}[Z^m_{-r_1}(\hat{A},\hat{B})\hat{F}_{-r_1}](0)\big] \\ &\quad + \sum_{m=0}^{+\infty}[Z^m_{-r_1}(\hat{A},\hat{B})\hat{F}_{-r_1}](s), \quad s \in J_{-r_1}, \end{aligned}$$

where $\hat{F}_{-r_1}(t) := \int_0^t \hat{f}(u)du$. Notice that $\hat{x}(0)=\xi$. Notice that the matrix $E_{\hat{A},\hat{B}}(0,-r_1)$ is nonsingular, due to Remark 2.7 and condition (2.6) with $\hat{A}$ and $\hat{B}$ is satisfied on the interval $J_{-r_1} \in \mathcal{J}_+$. Now, let $w(t) := \hat{x}(-t)$, $t \in J_0$. Since $E_{\hat{A},\hat{B}}$ is the evolution of equation (1.2) with $\hat{A}$, $\hat{B}$ and $\hat{f}$ in the place of A, B and f, we obtain

$$\begin{aligned} \dot{w}(t) &= -\dot{\hat{x}}(-t) \\ &= -\hat{A}(-t)\hat{x}(-t) - \hat{B}(-t)\hat{x}(g(-t)) - \hat{f}(-t) \\ &= A(t)w(t) + B(t)w(g(t)) + f(t). \end{aligned}$$

Put $x = w$ on J_0 and so, the unique solution x of the original problem on the interval J_0 is

$$\begin{aligned} x(t) &= \hat{x}(-t) \\ &= E_{\hat{A},\hat{B}}(-t,-r_1)\big[E_{\hat{A},\hat{B}}(0,-r_1)\big]^{-1}\big[\xi - \sum_{m=0}^{+\infty}[Z^m_{-r_1}(\hat{A},\hat{B})\hat{F}_{-r_1}](0)\big] \\ &\quad + \sum_{m=0}^{+\infty}[Z^m_{-r_1}(\hat{A},\hat{B})\hat{F}_{-r_1}](-t), \quad t \in J_0. \end{aligned} \tag{3.21}$$

Step (II.3)$_1$ [*Extension to* J_{r_1}]: On the interval J_{r_1} equation (1.2) is a delay differential equation and r_1 is a fixed point of the delay. Hence, its solution with initial value $x(r_1)$ is

$$x(t) = E_{A,B}(t,r_1)x(r_1) + \sum_{m=0}^{+\infty}[Z^m_{r_1}(A,B)F_{r_1}](t), \quad t \in J_{r_1}, \tag{3.22}$$

where $F_{r_1}(t) := \int_{r_1}^t f(u)du$ and

$$[Z^m_{r_1}(A,B)F_{r_1}](t) := \begin{cases} F_{r_1}(t), & \text{if } m = 0, \\ \int_{r_1}^t \big[A(u)[Z^{m-1}_{r_1}(A,B)F_{r_1}](u)du & \\ +B(u)[Z^{m-1}_{r_1}(A,B)F_{r_1}](g(u))\big]du, & \text{if } m \geq 1. \end{cases} \tag{3.23}$$

We shall express the value $x(r_1)$ in terms of the initial value ξ. From relation (3.21) we have

$$x(r_1) = [E_{\hat{A},\hat{B}}(0,-r_1)]^{-1}\big[\xi - \sum_{m=0}^{+\infty}[Z^m_{-r_1}(\hat{A},\hat{B})\hat{F}_{-r_1}](0)\big] + \sum_{m=0}^{+\infty}[Z^m_{-r_1}(\hat{A},\hat{B})\hat{F}_{-r_1}](-r_1),$$

because $E_{\hat{A},\hat{B}}(-r_1,-r_1) = I_{n\times n}$. Hence on the interval J_{r_1} the solution x has the expression

$$\begin{aligned} x(t) &= E_{A,B}(t,r_1)\Big([E_{\hat{A},\hat{B}}(0,-r_1)]^{-1}\big[\xi - \sum_{m=0}^{+\infty}[Z^m_{-r_1}(\hat{A},\hat{B})\hat{F}_{-r_1}](0)\big] \\ &\quad + \sum_{m=0}^{+\infty}[Z^m_{-r_1}(\hat{A},\hat{B})\hat{F}_{-r_1}](-r_1)\Big) + \sum_{m=0}^{+\infty}[Z^m_{r_1}(A,B)F_{r_1}](t). \end{aligned}$$

Step (II.4)$_1$ [*Extension to* J_{-r_2}]: On this interval the argument $g(t) = t-\tau(t)$ is of advanced type and for $t \in J_{-r_2}$, we have $s = -t \in J_{r_1}$. Making the transformation (2.8) we obtain its symmetric which is a delay differential equation on the interval J_{r_1}. Hence, by the previous step, this equation has the solution expressed as

$$\begin{aligned} \hat{x}(s) &= E_{\hat{A},\hat{B}}(s,r_1)\Big([E_{A,B}(0,-r_1)]^{-1}\big[\xi - \sum_{m=0}^{+\infty}[Z^m_{-r_1}(A,B)F_{-r_1}](0)\big] \\ &\quad + \sum_{m=0}^{+\infty}[Z^m_{-r_1}(A,B)F_{-r_1}](-r_1)\Big) + \sum_{m=0}^{+\infty}[Z^m_{r_1}(\hat{A},\hat{B})\hat{F}_{r_1}](s), \end{aligned}$$

where, recall that, $\hat{F}_{r_1}(s) := \int_{r_1}^s \hat{f}(u)du$ and $Z^m_{r_1}(\hat{A},\hat{B})$ is given as in (3.23), but with $\hat{A},\hat{B}$ in the place of A,B, respectively.. Thus, the expression of the solution

$\hat{x}$ is

$$\begin{aligned}
x(t) &= \hat{x}(-t) \\
&= E_{\hat{A},\hat{B}}(-t,r_1)\Big([E_{A,B}(0,-r_1)]^{-1}\big[\xi - \sum_{m=0}^{+\infty}[Z^m_{-r_1}(A,B)\hat{F}_{-r_1}](0)\big] \\
&\quad + \sum_{m=0}^{+\infty}[Z^m_{-r_1}(A,B)F_{-r_1}](-r_1)\Big) + \sum_{m=0}^{+\infty}[Z^m_{r_1}(\hat{A},\hat{B})\hat{F}_{r_1}](-t), \quad t \in J_{-r_2}.
\end{aligned}$$

Now, assume that we have applied the stages $S(1), S(2), \dots, S(k)$ and we found the solutions $X^{(k)}(\cdot,(0,\xi))$ and $\hat{X}^{(k)}(\cdot,(0,\xi))$ on the interval $[-r_{2k}, r_{2k}]$. Recall that it holds $X^{(k)}(t,(0,\xi)) = \hat{X}^{(k)}(-t,(0,\xi))$, $t \in [-r_{2k}, r_{2k}]$. We shall extend it to $[-r_{2k+2}, r_{2k+2}]$ by proving stage $S(k+1)$.

Stage $\mathcal{S}(k+1)$:
Step (II.1)$_{k+1}$ [*Extension to* $J_{-r_{2k+1}}$]: On the interval $J_{-r_{2k+1}}$ the argument is a delay, so the solution with initial value $x(-r_{2k+1})$ at $-r_{2k+1}$ is

$$x(t) = E_{A,B}(t,-r_{2k+1})x(-r_{2k+1}) + \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+1}}(A,B)F_{-r_{2k+1}}](t), \tag{3.24}$$

for $t \in J_{-r_{2k+1}}$, where $F_{-r_{2k+1}}(t) := \int_{-r_{2k+1}}^t f(u)du$, and

$$[Z^m_{-r_{2k+1}}(A,B)F_{-r_{2k+1}}](t) := \begin{cases} F_{-r_{2k+1}}(t), & \text{if } m=0, \\ \int_{-r_{2k+1}}^t \big[A(u)[Z^{m-1}_{-r_{2k+1}}(A,B)F_{-r_{2k+1}}](u) & \\ +B(u)[Z^{m-1}_{-r_{2k+1}}(A,B)F_{-r_{2k+1}}](g(u))\big]du, & \text{if } m \geq 1. \end{cases}$$

The value $x(-r_{2k+1})$ can be found from the fact that $X^{(k)}(-r_{2k};(0,\xi))$ is already known. Then we have

$$\begin{aligned}
&X^{(k)}(-r_{2k};(0,\xi)) \\
&= x(-r_{2k}) \\
&= E_{A,B}(-r_{2k},-r_{2k+1})x(-r_{2k+1}) + \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+1}}(A,B)F_{-r_{2k+1}}](-r_{2k})
\end{aligned}$$

and so

$$\begin{aligned}
x(-r_{2k+1}) &= [E_{A,B}(-r_{2k},-r_{2k+1})]^{-1}\Big[X^{(k)}(-r_{2k};(0,\xi)) \\
&\quad - \sum_{m=0}^{+\infty}[Z^m_{-r_{2n+1}}(A,B)F_{-r_{2k+1}}](-r_{2k})\Big].
\end{aligned}$$

Notice that the matrix $E_{A,B}(-r_{2k},-r_{2k+1})$ is nonsingular, since, by our assumption, condition $C_{A,B}$ is satisfied by the matrices A, B on the interval $J_{-r_{2k+1}} \in \mathcal{J}_+$. Now substitute the value $x(-r_{2k+1})$ into (3.24) and obtain the expression of x on the desired interval as follows.

$$\begin{aligned} x(t) = E_{A,B}(t,-r_{2k+1})[E_{A,B}(-r_{2k},-r_{2k+1})]^{-1}\Big[X^{(k)}(-r_{2k};(0,\xi)) \\ - \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+1}}(A,B)F_{-r_{2k+1}}](-r_{2k})\Big] \\ + \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+1}}(A,B)F_{-r_{2k+1}}](t), \quad t\in J_{-r_{2k+1}}. \end{aligned} \tag{3.25}$$

Step (II.2)$_{k+1}$ [*Extension to* $J_{r_{2k}}$]: On this interval it happens $\tau(t)\le 0$, and so we have $s=-t\in J_{-r_{2k+1}}$. So, making the transformation (2.8), it is enough to seek for solutions of (3.1) defined on $J_{-r_{2k+1}}$, which is a delay differential equation on it. Then, as in Step (II.1)$_{k+1}$, we can solve it and obtain its solution

$$\begin{aligned} \hat{x}(s) &= E_{\hat{A},\hat{B}}(s,-r_{2k+1})[E_{\hat{A},\hat{B}}(-r_{2k},-r_{2k+1})]^{-1} \\ &\times\Big[\hat{X}^{(k)}(-r_{2k};(0,\xi)) - \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+1}}(\hat{A},\hat{B})F_{-r_{2k+1}}](-r_{2k})\Big] \\ &+ \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+1}}(\hat{A},\hat{B})\hat{F}_{-r_{2k+1}}](s), \quad s\in J_{-r_{2k+1}}, \end{aligned}$$

where $\hat{F}_{-r_{2k+1}}(s):=\int_{-r_{2k+1}}^{s}\hat{f}(u)du$. The matrix $E_{\hat{A},\hat{B}}(-r_{2k},-r_{2k+1})$ is nonsingular, due to Remark 2.7 and condition $(C_{\hat{A},\hat{B}})$ is satisfied on the interval $[-r_{2k+1},-r_{2k}]$. Now, let $w(t):=\hat{x}(-t)$, $t\in J_{r_{2k}}$. Since $E_{\hat{A},\hat{B}}$ is the evolution of equation (3.1), we have

$$\begin{aligned} \dot{w}(t) &= -\dot{\hat{x}}(-t) \\ &= -\hat{A}(-t)\hat{x}(-t) - \hat{B}(-t)\hat{x}(g(-t))) - \hat{f}(-t) \\ &= A(t)w(t) + B(t)w(g(t))) + f(t), \quad t\in J_{r_{2k}}, \end{aligned}$$

where, notice that $\hat{x}(-r_{2k}) = X^{(k)}(-r_{2k};(0,\xi))$. Now we set $x=w$ on $I_{r_{2k}}$ and so we see that the unique solution x of the original problem on the interval $J_{r_{2k}}$ is

$$\begin{aligned} x(t) &= \hat{x}(-t) \\ &= E_{\hat{A},\hat{B}}(-t,-r_{2k+1})[E_{\hat{A},\hat{B}}(-r_{2k},-r_{2k+1})]^{-1} \\ &\quad\times\Big[\hat{X}^{(k)}(-r_{2k};(0,\xi)) - \sum_{m=0}^{+\infty}[Z^m_{-r_{2n+1}}(\hat{A},\hat{B})F_{-r_{2k+1}}](-r_{2k})\Big] \\ &\quad+ \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+1}}(\hat{A},\hat{B})\hat{F}_{-r_{2k+1}}](-t), \quad t\in J_{r_{2k}}. \end{aligned} \tag{3.26}$$

Step (II.3)$_{k+1}$ [*Extension to* $J_{r_{2k+1}}$]: On the interval $J_{r_{2k+1}}$ equation (1.2) is a delay differential equation and r_{2k+1} is a fixed point of the delay. Hence, its solution with initial value $x(r_{2k+1})$ is

$$x(t) = E_{A,B}(t,r_{2k+1})x(r_{2k+1}) + \sum_{m=0}^{+\infty}[Z^m_{r_{2k+1}}(A,B)F_{r_{2k+1}}](t), \quad t\in I_{r_{2k+1}}, \tag{3.27}$$

where $F_{r_{2k+1}}(t) := \int_{r_{2k+1}}^t f(u)du$ and

$$[Z^m_{r_{2k+1}}(A,B)F_{r_{2k+1}}](t) := \begin{cases} F_{r_{2k+1}}(t), & \text{if } m=0, \\ \int_{r_{2k+1}}^t \big[A(u)[Z^{m-1}_{r_{2k+1}}(A,B)F_{r_{2k+1}}](u)du \\ +B(u)[Z^{m-1}_{r_{2k+1}}(A,B)F_{r_{2k+1}}](g(u))\big]du, & \text{if } m \geq 1. \end{cases}$$

We shall express the value $x(r_{2k+1})$ in terms of the initial value ξ. From (3.26) we have

$$\begin{aligned} x(r_{2k+1}) &= [E_{\hat{A},\hat{B}}(-r_{2k},-r_{2k+1})]^{-1}\Big[\hat{X}^{(k)}(-r_{2n};(0,\xi)) \\ &\quad - \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+1}}(\hat{A},\hat{B})\hat{F}_{-r_{2k+1}}](-r_{2k})\Big] \\ &\quad + \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+1}}(\hat{A},\hat{B})\hat{F}_{-r_{2k+1}}](-r_{2k+1}), \quad t \in J_{r_{2k}}. \end{aligned}$$

because $E_{\hat{A},\hat{B}}(-r_{2k+1},-r_{2k+1}) = I_{n\times n}$. Hence on the interval $J_{r_{2k+1}}$ the solution x has the expression

$$\begin{aligned} x(t) &= E_{A,B}(t,r_{2n+1})\Big\{[E_{\hat{A},\hat{B}}(-r_{2k},-r_{2k+1})]^{-1} \\ &\quad \times \Big[\hat{X}^{(k)}(-r_{2k};(0,\xi)) - \sum_{m=0}^{+\infty}[Z^m_{-r_{2n+1}}(\hat{A},\hat{B})\hat{F}_{-r_{2k+1}}](-r_{2k})\Big] \\ &\quad + \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+1}}(\hat{A},\hat{B})\hat{F}_{-r_{2k+1}}](-r_{2k+1})\Big\} \\ &\quad + \sum_{m=0}^{+\infty}[Z^m_{r_{2k+1}}(A,B)F_{r_{2k+1}}](t), \quad t \in J_{r_{2k+1}}. \end{aligned} \tag{3.28}$$

Step (II.4)$_{k+1}$ *Extension to* $J_{-r_{2k+2}}$: On this interval the argument $g(t) = t-\tau(t)$ is of advanced type and for $t \in J_{-r_{2k+2}}$, we have $s = -t \in J_{r_{2k+1}}$. So, by the transformation (2.8), we obtain its symmetric on the interval $J_{r_{2k+1}}$, which is a delay differential equation. By the previous step, this equation has the solution $\hat{x}$ expressed as

$$\begin{aligned} \hat{x}(s) &= E_{\hat{A},\hat{B}}(s,r_{2k+1})\Big\{[E_{A,B}(-r_{2k},-r_{2k+1})]^{-1} \\ &\quad \times \Big[X^{(k)}(-r_{2k};(0.\xi)) - \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+1}}(A,B)F_{-r_{2k+1}}](-r_{2k})\Big] \\ &\quad + \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+1}}(A,B)F_{-r_{2k+1}}](-r_{2k+1})\Big\} + \sum_{m=0}^{+\infty}[Z^m_{r_{2k+1}}(\hat{A},\hat{B})\hat{F}_{r_{2k+1}}](s). \end{aligned}$$

where, recall that, $\hat{F}_{r_{2k+1}}(s) := \int_{r_{2k+1}}^{s} \hat{f}(u)du$ and $Z^m_{r_{2k+1}}(\hat{A}, \hat{B})$ is given as in (3.23). Thus, the expression of the solution x is

$$\begin{aligned}
x(t) &= \hat{x}(-t)\\
&= E_{\hat{A},\hat{B}}(-t, r_{2k+1})\Big\{[E_{A,B}(-r_{2k}, -r_{2k+1})]^{-1}\\
&\quad \times \Big[X^{(k)}(-r_{2k}; (0,\xi)) - \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+1}}(A,B)F_{-r_{2k+1}}](-r_{2k})\Big]\\
&\quad + \sum_{m=0}^{+\infty}[Z^m_{-r_{2k+1}}(A,B)F_{-r_{2k+1}}](-r_{2k+1})\Big\}\\
&\quad + \sum_{m=0}^{+\infty}[Z^m_{r_{2k+1}}(\hat{A},\hat{B})\hat{F}_{r_{2k+1}}](-t), \quad t \in J_{-r_{2k+2}}\,.
\end{aligned} \tag{3.29}$$

By arriving to stage $S(k+1)$ we extended the solution x of equation (1.2) with $\hat{A}$, $\hat{B}$ and $\hat{f}$ in the place of A, B and f, on the interval $J_{-r_{2k+1}} \cup J_{r_{2k}} \cup J_{r_{2k+1}} \cup J_{-r_{2k+2}}$, by the types (3.25), (3.26), (3.28) and (3.29) and its symmetric on the same interval $[-r_{2k+2}, r_{2k+2}]$, for all positive integers k. The proof is complete. □

4. Applications

(1) Consider the differential equation

$$\dot{x}(t) + b(t)x(t - \sin t) = f(t), \quad t \geq 0 \tag{4.1}$$

with mixed argument, where the function $f : \mathbb{R} \to \mathbb{R}$ is continuous. Write it in the form (**??**) as

$$\dot{x}(t) = -b(t)x(t - \sin t) + f(t), \quad t \in \mathbb{R}.$$

The points $r_k = k\pi$, $k = 0, 1, \dots$ are the fixed points of the argument $t - \sin t$. Moreover all conditions (C1)–(C4) are satisfied. Also, condition (C5) is satisfied, since, for all integers j and $t \in [j\pi, (j+1)\pi]$ it holds

$$j\pi \leq t - \sin t \leq (j+1)\pi.$$

Notice that the function $t \to t - \sin t$ is increasing. Now, if the function b satisfies

$$\int_{-\infty}^{+\infty} |b(u)|du < +\infty, \quad \int_{(2j+1)\pi}^{(2j+2)\pi} |b(u)|du < \ln 2, \quad \int_{(2j+1)\pi}^{(2j+2)\pi} |b(-u)|du < \ln 2,$$

for all integers $j = 1, 2, \dots$, then condition (2.6) with $A = 0$ and $B = -b$ is satisfied and so, by Theorem 3.1, it follows that, for any $\xi \in \mathbb{R}$, there is a full solution x of equation (4.1) such that $x(0) = \xi$.

(2) Consider equation (1.6) for which we assume that the coefficient a is not constant, but it depends continuously on time and it satisfies

$$\int_{-\infty}^{0} |a(s)|ds < \ln 2 \quad \text{and} \quad \int_{0}^{+\infty} |a(s)|ds < \ln 2.$$

Then it is easy to see that all conditions of Theorem 3.1 are satisfied. The collection $\mathcal{J}$ consists of two intervals $J_{-\infty} := (-\infty, 0] \in \mathcal{J}_+$ and $J_0 := [0, +\infty) \in \mathcal{J}_-$. Notice that $\lambda > 1$. Thus Theorem 3.1 is applied and the solutions exist and are defined by

$$x(t; (0,\xi)) = \begin{cases} E_{0,a}(t, -\infty)\xi + \sum_{m=0}^{+\infty}[Z^m_{-\infty}(0,a)](t)\xi, & t \leq 0,\\ E_{0,\hat{a}}(-t, -\infty)\xi + \sum_{m=0}^{+\infty}[Z^m_{-\infty}(0,\hat{a})](-t)\xi, & t > 0,\end{cases}$$

where

$$[Z^m_{-\infty}(0,a)](t) = \begin{cases} 1, & \text{if } m = 0, \\ \int_{-\infty}^t a(u)[Z^{m-1}_{-\infty}(0,a)](\lambda u)du, & \text{if } m \geq 1, \end{cases}$$

for $t \leq 0$ and $\hat{a}(s) := -a(-s)$.

(3) Consider equation

$$\dot{x}(t) = a(t)x(t) + b(t)x(t^{2k+1}), \quad t \in \mathbb{R},$$

where a, b are continuous real valued functions that satisfy

$$\int_{-1}^{1} (|a(s)| + |b(s)|)ds < +\infty, \quad \int_{-\infty}^{-1} (|a(s)| + |b(s)|)ds < \ln 2,$$

$$\int_{1}^{+\infty} (|a(s)| + |b(s)|)ds < \ln 2.$$

In this case the collection $\mathcal{J}$ consists of the intervals $J_{-\infty} := (-\infty, -1] \in \mathcal{J}_+$, $J_{-1} := [-1, 0] \in \mathcal{J}_-$, $J_0 := [0, 1] \in \mathcal{J}_+$ and $J_1 := [1, +\infty) \in \mathcal{J}_-$. Then Theorem 3.1 is applied and the existence of a unique full solution with initial value $\xi \in \mathbb{R}$ at 0 is guaranteed.

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