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EXISTENCE AND BOUNDS FOR KNESER-TYPE SOLUTIONS TO NONCANONICAL THIRD-ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. This article focuses on the existence and asymptotic behavior of Kneser-type solutions to third-order noncanonical differential equations with a delay or advanced argument in the neutral term

$$\Big(r_2(t)\big(r_1(t)z'(t)\big)'\Big)' + g(t)x(t) = 0,$$

where $z(t) = x(t) + p(t)x(\tau(t))$. This equation is transformed into a canonical equation, which reduces the number of classes of positive solutions from 4 to 2. This is done without assuming extra conditions, and greatly simplifies the process of obtaining conditions for the existence of Kneser-type solutions. Also we obtain lower and upper bounds for these solutions, and obtain their rate of convergence to zero. Two examples are provided to illustrate our main results, one with a delay neutral term, and one with an advanced neutral term.

1. Introduction

This article concerns the asymptotic properties of Kneser-type solutions to the third-order neutral differential equation

$$\Big(r_2(t)\big(r_1(t)z'(t)\big)'\Big)' + g(t)x(t) = 0, \quad t \ge t_0 > 0, \tag{1.1}$$

where $z(t) = x(t)+p(t)x(\tau(t))$. During this study, we use the following assumptions:

- (H1) $r_1 \in C^2([t_0,\infty),(0,\infty))$, $r_2 \in C^1([t_0,\infty),(0,\infty))$, $g \in C([t_0,\infty),[0,\infty))$, and $p \in C^3([t_0,\infty),[0,\infty))$ with $0 \le p(t) < 1$;
- (H2) $\tau \in C^3([t_0,\infty),\mathbb{R})$, either $\tau(t) \le t$ or $\tau(t) \ge t$, and $\lim_{t\to\infty} \tau(t) = \infty$;
- (H3) equation (1.1) is in noncanonical form, i.e.,

$$\int_{t_0}^{\infty} \frac{dt}{r_1(t)} < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{dt}{r_2(t)} < \infty.$$

By a solution of (1.1), we mean a function $x \in C([t_x,\infty),\mathbb{R})$ for some $t_x \ge t_0$ such that $z \in C^1([t_x,\infty),\mathbb{R})$, $r_1 z' \in C^1([t_x,\infty),\mathbb{R})$, $r_2(r_1 z')' \in C^1([t_x,\infty),\mathbb{R})$, and x satisfies equation (1.1) on $[t_x,\infty)$. We consider only those solutions of (1.1) that exist on some half-line $[t_x,\infty)$ and such that $\sup\{|x(t)| : T_1 \le t < \infty\} > 0$ for each

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$T_1 \geq t_x$; such solutions are said to be continuable. We tacitly assume that equation (1.1) possesses such solutions. A continuable solution $x(t)$ of (1.1) is said to be *oscillatory* if it has infinitely many zeros; otherwise, it is called *nonoscillatory*. We say that (1.1) has *property A* if any solution $x(t)$ of (1.1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$. Note that if y is a negative solution of (1.1), then $x(t) = -y(t)$ is positive solution of the same equation; statements for positive solutions apply to non-oscillatory solutions.

The qualitative theory of third-order differential equations has seen rapid development recently due to its numerous applications and the mathematical challenges it presents, as indicated by [11, 15, 16, 18]. In particular, the oscillation and asymptotic behavior of third-order functional differential equations have received significant attention, as evidenced by extensive literature; see the references in this article and the references therein. However, a literature review reveals limited findings on the existence and non-existence of Kneser-type solutions for third-order delay and neutral type differential equations.

In [2], the authors studied the existence and estimates for Kneser-type solutions of (1.1) with an advanced neutral term (i.e., $\tau(t) \geq t$) and $r_1(t) = 1$ and in canonical form, that is,

$$\int_{t_0}^{\infty} \frac{1}{r_2(t)} dt = \infty.$$

In [3], the authors generalized the results in [2] for the equation

$$\Big(r_2(t)\big(r_1(t)(z'(t))^{\alpha}\big)'\Big)' + g(t)x^{\alpha}(t) = 0 \tag{1.2}$$

under the condition

$$\int_{t_0}^{\infty} \frac{1}{r_1^{1/\alpha}(t)} dt = \int_{t_0}^{\infty} \frac{1}{r_2(t)} dt = \infty, \tag{1.3}$$

i.e., the authors considered the equation (1.2) in canonical form.

In [7], the authors considered the equation

$$\Big(r_2(t)\big(r_1(t)z'(t)\big)'\Big)' + g(t)x(\sigma(t)) = 0, \tag{1.4}$$

under condition (1.3) and established conditions for the nonexistence of Kneser-type solutions of (1.4).

The nonexistence of Kneser-type solutions was discussed in [21] for the equation

$$\Big(r_2(t)\big(z''(t)\big)^{\alpha}\Big)' + g(t)x^{\beta}(\sigma(t)) = 0,$$

under the condition

$$\int_{t_0}^{\infty} \frac{1}{r_2^{1/\alpha}(t)} dt = \infty.$$

Recently in [23], the authors discussed the existence and estimates for the Kneser-type solutions for the semi-canonical equation with an advanced argument in the neutral term

$$\Big(r_2(t)\big(r_1(t)(z'(t))^{\alpha}\big)'\Big)' + g(t)x^{\alpha}\big(\sigma(t)\big) = 0 \tag{1.5}$$

under the conditions

$$\int_{t_0}^{\infty} \frac{1}{r_2(t)} dt < \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{1}{r_1^{1/\alpha}(t)} dt = \infty,$$

by transforming (1.5) into a canonical type equation.

On the other hand, in [4], the authors discussed the existence and estimates for Kneser-type solutions of the third-order canonical type delay differential equation (i.e., $\sigma(t) < t$)

$$x'''(t) + g(t)x(\sigma(t)) = 0,$$

and in [20] the authors studied the convergence to zero of Kneser solutions of general canonical type third-order delay differential equations.

From the above discussion, it is clear that the existence and estimates for the Kneser-type solutions have been investigated for third-order delay differential equations or advanced type neutral differential equations in canonical or semi-canonical form. To the best of the authors knowledge, no results exist on the existence of Kneser-type solutions for noncanonical type differential equations with a delay or advanced argument in the neutral term.

This article aims to address this question by taking (1.1) in noncanonical form with a delay or advanced argument in the neutral term. To achieve this, we transform the noncanonical equation (1.1) into a canonical form. This reduces the number of classes of positive solutions to two instead of the usual four classes for noncanonical equations. Hence, this significantly simplifies finding conditions for the existence and finding estimates of Kneser-type solutions. Two examples are provided to illustrate our main results.

2. Preliminary results

In this section, we introduce results necessary to demonstrate our main findings. In view of (H3), it is possible to define the following functions for $t \ge t_0$:

$$R_{12}(t) = \int_t^\infty \frac{1}{r_1(s)} \int_s^\infty \frac{1}{r_2(w)}\,dw\,ds, \quad R_{21}(t) = \int_t^\infty \frac{1}{r_2(s)} \int_s^\infty \frac{1}{r_1(w)}\,dw\,ds,$$

$$q_1(t) = \frac{r_1(t)R_{12}^2(t)}{R_{21}(t)}, \quad q_2(t) = \frac{r_2(t)R_{21}^2(t)}{R_{12}(t)}.$$

Note that R_{12} and R_{21} are positive and decreasing. We begin with the following theorem whose proof can be found in [1, Theorem 2.1].

Theorem 2.1. *The noncanonical equation* (1.1) *can be expressed in an equivalent canonical form as*

$$\Big(q_2(t)\Big(q_1(t)\big(\frac{z(t)}{R_{12}(t)}\big)'\Big)'\Big)' + R_{21}(t)g(t)x(t) = 0, \tag{2.1}$$

with

$$\int_{t_0}^\infty \frac{1}{q_1(t)}dt = \int_{t_0}^\infty \frac{1}{q_2(t)}dt = \infty\,. \tag{2.2}$$

Setting $\psi(t) = z(t)/R_{12}(t)$ in (2.1), we have the following 2 lemmas.

Lemma 2.2. *A function x is a solution of the noncanonical equation* (1.1) *if and only if x is solution of the canonical equation*

$$\Big(q_2(t)\big(q_1(t)\psi'(t)'\Big)' + R_{21}(t)g(t)x(t) = 0, \quad t \ge t_0\,. \tag{2.3}$$

Lemma 2.3. *The noncanonical neutral differential equation* (1.1) *has an eventually positive solution if and only if the canonical equation* (2.3) *has an eventually positive solution.*

It has been established, see for example [5, 19], that if x is an eventually positive solution of (1.1), then there exists $t_1 \geq t_0$ such that for all $t \geq t_1$, the corresponding function $z(t) = x(t) + p(t)x(\tau(t))$ belongs to one of the following four classes:

$$\begin{aligned}
\mathcal{S}_1 &= \{z : z > 0,\ L_1 z < 0,\ L_2 z < 0,\ L_3 z < 0\},\\
\mathcal{S}_2 &= \{z : z > 0,\ L_1 z < 0,\ L_2 z > 0,\ L_3 z < 0\},\\
\mathcal{S}_3 &= \{z : z > 0,\ L_1 z > 0,\ L_2 z > 0,\ L_3 z < 0\},\\
\mathcal{S}_4 &= \{z : z > 0,\ L_1 z > 0,\ L_2 z < 0,\ L_3 z < 0\},
\end{aligned}$$

where

$$L_1 z = r_1 z', \quad L_2 z = r_2 (r_1 z')', \quad L_3 z = \big(r_2 (r_1 z')'\big)'.$$

From the above classification, we see that (1.1) has two types of monotonically increasing solutions and two types of monotonically decreasing solutions. Lemma 2.3 simplifies the study of (1.1), since (2.3) has only of two types of solutions: one eventually decreasing and the other eventually increasing, as stated in the following lemma. This lemma follows from a generalization of the well-known Kiguradze lemma [13, Lemma 1.1] applied to (1.4).

Lemma 2.4. *Assume that x is an eventually positive solution of* (2.3)*. Then the corresponding function ψ belongs to one of the following two classes:*

$$\begin{aligned}
\mathcal{N}_0 &= \{\psi : \psi > 0,\ \mathcal{L}_1 \psi < 0,\ \mathcal{L}_2 \psi > 0,\ \mathcal{L}_3 \psi < 0\},\\
\mathcal{N}_2 &= \{\psi : \psi > 0,\ \mathcal{L}_1 \psi > 0,\ \mathcal{L}_2 \psi > 0,\ \mathcal{L}_3 \psi < 0\},
\end{aligned}$$

where

$$\mathcal{L}_1 \psi = q_1 \psi', \quad \mathcal{L}_2 \psi = q_2 (q_1 \psi')', \quad \mathcal{L}_3 \psi = \big(q_2 (q_1 \psi')'\big)'.$$

Definition 2.5. A solution x is called Kneser solution, if its corresponding function ψ belongs to $\mathcal{N}_0$.

3. Main results

In this section, first we derive conditions for making the class $\mathcal{N}_2$ empty, so the positive solutions of (2.3) belong $\mathcal{N}_0$. To simplify notation we define:

$$\begin{gathered}
Q_1(t) = \int_{t_1}^{t} \frac{1}{q_1(s)}\, ds, \quad Q_2(t) = \int_{t_1}^{t} \frac{1}{q_2(s)}\, ds,\\
Q_{12}(t) = \int_{t_1}^{t} \frac{1}{q_1(s)} \int_{t_1}^{s} \frac{1}{q_2(w)}\, dw\, ds,\\
\Delta(t) = \int_{t}^{\infty} \frac{1}{q_2(s)} \int_{s}^{\infty} g(w) R_{21}(w) R_{12}(w)\, dw\, ds,\\
\phi(t) = \exp\Big(\int_{t_1}^{t} \frac{\Delta(s)}{q_1(s)}\, ds\Big),
\end{gathered}$$

where $t \geq t_1 \geq t_0$.

Now we introduce 2 additional conditions on the coefficient $p(t)$. There exits $t_2 \geq t_1$ such that for all $t \geq t_2$, we have

$$p(t) \frac{R_{12}(\tau(t))}{R_{12}(t)} \frac{\phi(t)}{\phi(\tau(t))} < 1 \quad \text{if } \tau(t) \leq t, \tag{3.1}$$

$$p(t) \frac{R_{12}(\tau(t))}{R_{12}(t)} \frac{Q_{12}(\tau(t))}{Q_{12}(t)} < 1 \quad \text{if } \tau(t) \geq t, \tag{3.2}$$

Since R_{12} and Q_{12} and ϕ are positive, the inequality $0 \le p(t)$ is preserved. Since ϕ is increasing, when $\tau(t) \le t$, we have $\phi(t)/\phi(\tau(t)) \ge 1$. Therefore (3.1) implies

$$p(t)\frac{R_{12}(\tau(t))}{R_{12}(t)} < 1\,.$$

Assuming (3.1) and (3.2), we define G such that for $t \ge t_2$,

$$0 < G(t) = \begin{cases} 1 - p(t)\frac{R_{12}(\tau(t))}{R_{12}(t)} & \text{if } \tau(t) \le t, \\ 1 - p(t)\frac{R_{12}(\tau(t))}{R_{12}(t)}\frac{Q_{12}(\tau(t))}{Q_{12}(t)} & \text{if } \tau(t) \ge t, \end{cases} \tag{3.3}$$

Assuming (3.1), we define G_1 such that for $t \ge t_2$,

$$0 < G_1(t) = \begin{cases} 1 - p(t)\frac{R_{12}(\tau(t))}{R_{12}(t)} & \text{if } \tau(t) \ge t, \\ 1 - p(t)\frac{R_{12}(\tau(t))}{R_{12}(t)}\frac{\phi(t)}{\phi(\tau(t))} & \text{if } \tau(t) \le t. \end{cases} \tag{3.4}$$

Lemma 3.1. *Let x be an eventually positive solution of* (2.3) *with the corresponding function $\psi \in \mathcal{N}_2$. Then*

(i) $\frac{q_1(t)\psi'(t)}{Q_2(t)}$ *is decreasing, and*
(ii) $\frac{\psi(t)}{Q_{12}(t)}$ *is decreasing.*

Proof. The proof of (i) is the same proof as in [1, Lemma 3.1]. Let x be an eventually positive solution of (2.3) with the corresponding function $\psi \in \mathcal{N}_2$. Since $x(t) > 0$, by (2.3), we have that $q_2(t)(q_1(t)\psi'(t))'$ is decreasing, and

$$q_1(t)\psi'(t) \ge \int_{t_1}^{t} \frac{q_2(s)(q_1(s)\psi'(s))'}{q_2(s)}\,ds \ge Q_2(t)q_2(t)(q_1(t)\psi'(t))'.$$

Then

$$\Big(\frac{q_1(t)\psi'(t)}{Q_2(t)}\Big)' = \frac{Q_2(t)q_2(t)(q_1(t)\psi'(t))' - q_1(t)\psi'(t)}{Q_2^2(t)q_2(t)} \le 0\,.$$

Hence, $q_1(t)\psi'(t)/Q_2(t)$ is decreasing, which proves (i).

(ii) Using that $q_1(t)\psi'(t)/Q_2(t)$ is decreasing, we obtain

$$\psi(t) \ge \int_{t_1}^{t} \frac{q_1(s)\psi'(s)}{q_1(s)}\frac{Q_2(s)}{Q_2(s)}\,ds \ge \frac{Q_{12}(t)q_1(t)\psi'(t)}{Q_2(t)}. \tag{3.5}$$

Therefore,

$$\Big(\frac{\psi(t)}{Q_{12}(t)}\Big)' = \frac{q_1(t)\psi'(t)Q_{12}(t) - \psi(t)Q_2(t)}{q_1(t)Q_{12}^2(t)} \le 0,$$

which implies $\psi(t)/Q_{12}(t)$ is decreasing and so (ii) is proved. □

Lemma 3.2. *Let* (3.1) *and* (3.2) *hold and let x be an eventually positive solution of* (2.3) *with the corresponding function $\psi \in \mathcal{N}_2$. Then*

$$G(t)\psi(t) \le \frac{x(t)}{R_{12}(t)} \le \psi(t), \quad \textit{for all } t \ge t_2\,. \tag{3.6}$$

Proof. Let $x(t)$ be an eventually positive solution of (2.3) with the corresponding function $\psi(t) \in \mathcal{N}_2$ for $t \ge t_1 \ge t_0$. Then, from the definition of $\psi(t)$, it is easy to see that

$$\psi(t) = \frac{z(t)}{R_{12}(t)} \ge \frac{x(t)}{R_{12}(t)}. \tag{3.7}$$

Moreover,

$$\begin{aligned} x(t) &= R_{12}(t)\psi(t) - p(t)x(\tau(t)) \\ &\geq R_{12}(t)\psi(t) - p(t)R_{12}(\tau(t))\psi(\tau(t)) \\ &= R_{12}(t)\Big(\psi(t) - p(t)\frac{R_{12}(\tau(t))}{R_{12}(t)}\psi(\tau(t))\Big). \end{aligned}$$

When $\tau(t) \leq t$, since ψ is increasing, $\psi(\tau(t)) \leq \psi(t)$ which is used for obtaining $G(t)$ in the above inequality. Also when $\tau(t) \geq t$, since ψ/Q_{12} is decreasing, $\psi(\tau(t))/Q_{12}(\tau(t)) \leq \psi(t)/Q_{12}(t)$, which is used for obtaining $G(t)$ in the above inequality. In both cases we have

$$x(t) \geq R_{12}(t)G(t)\psi(t). \tag{3.8}$$

Combining this inequality and (3.7) we obtain (3.6). □

Theorem 3.3. *Let* (3.1) *and* (3.2) *hold and let x be an eventually positive solution of* (1.1). *If*

$$\begin{aligned} &\limsup_{t\to\infty}\Big(\frac{1}{Q_2(t)}\int_{t_2}^{t} g(s)R_{21}(s)R_{12}(s)G(s)Q_2(s)Q_{12}(s)\,ds \\ &+ Q_{12}(t)\int_t^\infty g(s)R_{21}(s)R_{12}(s)G(s)\,ds\Big) > 1\,, \end{aligned} \tag{3.9}$$

then the class $\mathcal{N}_2$ is empty.

Proof. Let x be an eventually positive solution of (1.1). Since $\lim_{t\to\infty}\tau(t) = \infty$, we can assume $x(\tau(t)) > 0$. By Lemma 2.3, x is also a positive solution to (2.3). To obtain a contradiction assume that the corresponding function $\psi(t)$ belongs to the class $\mathcal{N}_2$. Applying (3.6) to (2.3), one obtains

$$\Big(q_2(t)\big(q_1(t)\psi'(t)\big)'\Big)' + g(t)R_{21}(t)R_{12}(t)G(t)\psi(t) \leq 0, \quad t \geq t_2.$$

Integrating the latter inequality from t to ∞, we obtain

$$\big(q_1(t)\psi'(t)\big)' \geq \frac{1}{q_2(t)}\int_t^\infty g(s)R_{21}(s)R_{12}(s)G(s)\psi(s)\,ds.$$

Again integrating from t_2 to t, we obtain

$$\begin{aligned} q_1(t)\psi'(t) &\geq \int_{t_2}^{t}\frac{1}{q_2(s)}\Big(\int_s^\infty g(v)R_{21}(v)R_{12}(v)G(v)\psi(v)\,dv\Big)\,ds \\ &= \int_{t_2}^{t}\frac{1}{q_2(s)}\Big(\int_s^t g(v)R_{21}(v)R_{12}(v)G(v)\psi(v)\,dv\Big)\,ds \\ &\quad + \int_{t_2}^{t}\frac{1}{q_2(s)}\Big(\int_t^\infty g(v)R_{21}(v)R_{12}(v)G(v)\psi(v)\,dv\Big)\,ds \\ &\geq \int_{t_2}^{t} g(s)R_{21}(s)R_{12}(s)G(s)Q_2(s)\psi(s)\,ds \\ &\quad + Q_2(t)\int_t^\infty g(s)R_{21}(s)R_{12}(s)G(s)\psi(s)\,ds. \end{aligned}$$

Using (3.5) in the latter inequality, we obtain

$$\begin{aligned}\frac{Q_2(t)\psi(t)}{Q_{12}(t)} \geq &\int_{t_2}^{t} g(s)R_{21}(s)R_{12}(s)G(s)Q_2(s)\psi(s)\,ds \\ &+ Q_2(t)\int_{t}^{\infty} g(s)R_{21}(s)R_{12}(s)G(s)\psi(s)\,ds.\end{aligned} \tag{3.10}$$

Using that $\psi(t)$ is increasing and $\psi(t)/Q_{12}(t)$ is decreasing in (3.10), we obtain

$$\begin{aligned}1 \geq &\frac{1}{Q_2(t)}\int_{t_2}^{t} g(s)R_{21}(s)R_{12}(s)G(s)Q_2(s)Q_{12}(s)\,ds \\ &+ Q_{12}(t)\int_{t}^{\infty} g(s)R_{21}(s)R_{12}(s)G(s)\,ds.\end{aligned}$$

Computing the lim sup as $t \to \infty$ on both sides of the inequality, we arrive at a contradiction to (3.9). The proof is complete. □

Next, we present lower and upper bounds for the positive solutions of (1.1). For simplicity, we define:

$$F(t) = \frac{\phi(t)}{q_1(t)}\int_{t}^{\infty}\frac{1}{q_2(v)}\int_{v}^{\infty}\frac{g(s)R_{21}(s)R_{12}(s)G_1(s)}{\phi(s)}\,ds\,dv.$$

Lemma 3.4. *Let* (3.1) *hold and let x be an eventually positive solution of* (2.3) *with the corresponding function $\psi \in \mathcal{N}_0$. Then*

(i) *$\psi(t)\phi(t)$ is increasing, and*
(ii) *$G_1(t)\psi(t) \leq \frac{x(t)}{R_{12}(t)} \leq \psi(t)$.*

Proof. Since $\psi \in \mathcal{N}_0$, we have

$$\psi(t) > 0, \quad q_1(t)\psi'(t) < 0, \quad q_2(t)(q_1(t)\psi'(t))' > 0, \quad \big(q_2(t)(q_1(t)\psi'(t))'\big)' < 0,$$

for all $t \geq t_2$. Since $q_2(t)(q_1(t)\psi'(t))'$ is positive and decreasing, there exists a constant ℓ such that

$$\lim_{t\to\infty} q_2(t)\big(q_1(t)\psi'(t)\big)' = \ell \geq 0.$$

We claim that $\ell = 0$. If not, then $(q_1(t)\psi'(t))' \geq \frac{\ell}{2q_2(t)} > 0$ and therefore

$$q_1(t)\psi'(t) \geq q_1(t_2)\psi'(t_2) + \frac{\ell}{2}\int_{t_2}^{t}\frac{1}{q_2(s)}\,ds \to \infty \quad \text{as } t \to \infty,$$

which contradicts that $q_1(t)\psi'(t) < 0$ for all $t \geq t_2$. Thus,

$$\lim_{t\to\infty} q_2(t)(q_1(t)\psi'(t))' = 0.$$

Since $q_1(t)\psi'(t)$ is negative and increasing, there exists a constant m such that

$$\lim_{t\to\infty} q_1(t)\psi'(t) = m \leq 0.$$

We claim that $m = 0$. If not, then $\psi'(t) \leq \frac{m}{q_1(t)} < 0$ and we have

$$\psi(t) \leq \psi(t_2) + m\int_{t_2}^{t}\frac{1}{q_1(s)}\,ds \to -\infty \quad \text{as } t \to \infty,$$

which contradicts that $\psi(t)$ is positive. Therefore, $\lim_{t\to\infty} q_1(t)\psi'(t) = 0$. Now, an integration of (2.3) from t to ∞ yields

$$\begin{aligned}\big(q_1(t)\psi'(t)\big)' &= \frac{1}{q_2(t)}\int_t^\infty g(s)R_{21}(s)x(s)\,ds\\ &\le \frac{1}{q_2(t)}\int_t^\infty g(s)R_{21}(s)R_{12}(s)\psi(s)\,ds \qquad (3.11)\\ &\le \frac{\psi(t)}{q_2(t)}\int_t^\infty g(s)R_{21}(s)R_{12}(s)\,ds,\end{aligned}$$

where we have used $\psi(t) \ge \frac{x(t)}{R_{12}(t)}$ and $\psi(t)$ is decreasing. Again integrating (3.11), we obtain

$$\psi'(t) \ge -\frac{\psi(t)}{q_1(t)}\int_t^\infty \frac{1}{q_2(s)}\Big(\int_s^\infty g(v)R_{21}(v)R_{12}(v)\,dv\Big)\,ds = -\psi(t)\frac{\Delta(t)}{q_1(t)}.$$

Hence,

$$(\psi(t)\phi(t))' = \psi'(t)\phi(t) + \psi(t)\phi'(t) \ge \psi(t)\Big(\phi'(t) - \frac{\Delta(t)}{q_1(t)}\phi(t)\Big).$$

Since $\phi(t)$ is a solution of the differential equation $\phi'(t) - \frac{\Delta(t)}{q_1(t)}\phi(t) = 0$, we conclude that $\psi(t)\phi(t)$ is increasing. From the definition of $\psi(t)$, and $0 \le p(t)$, we see that

$$\psi(t) = \frac{z(t)}{R_{12}(t)} = \frac{1}{R_{12}(t)}\big(x(t) + p(t)x(\tau(t))\big).$$

So, $\psi(t) \ge \frac{x(t)}{R_{12}(t)}$ and

$$\begin{aligned}x(t) &\ge R_{12}(t)\psi(t) - p(t)R_{12}(\tau(t))\psi(\tau(t))\\ &\ge R_{12}(t)\Big(\psi(t) - p(t)\frac{R_{12}(\tau(t))}{R_{12}(t)}\psi(\tau(t))\Big)\\ &\ge R_{12}(t)G_1(t)\psi(t).\end{aligned}$$

Hence,

$$G_1(t)\psi(t) \le \frac{x(t)}{R_{12}(t)} \le \psi(t),$$

and the proof is complete. □

Theorem 3.5. *Let conditions* (3.1), (3.2) *and* (3.9) *be satisfied. Then there exist positive constants* α_1 *and* α_2 *such that every positive solution* x *of* (1.1) *satisfies*

$$\alpha_1\frac{G_1(t)}{\phi(t)} \le \frac{x(t)}{R_{12}(t)} \le \alpha_2\exp\Big(-\int_{t_2}^t F(s)\,ds\Big), \quad \textit{for } t \ge t_2\,. \qquad (3.12)$$

Proof. Assume that x is a positive solution of (1.1). Then, by Lemma 2.3, x is also a positive solution of (2.3). By Theorem 3.3, $\psi(t)$ belongs to $\mathcal{N}_0$ for $t \ge t_2$. From (i) and (ii) of Lemma 3.4 we have

$$\frac{x(t)}{R_{12}(t)} \ge \frac{G_1(t)}{\phi(t)}\phi(t)\psi(t) \ge \frac{G_1(t)}{\phi(t)}\phi(t_2)\psi(t_2). \qquad (3.13)$$

On the other hand, integrating (2.3) from t to ∞ and taking into account Lemma 3.4(ii), we obtain

$$(q_1(t)\psi'(t))' = \frac{1}{q_2(t)}\int_t^\infty g(s)R_{21}(s)x(s)\,ds$$

$$\begin{aligned} &\geq \frac{1}{q_2(t)} \int_t^\infty g(s) R_{21}(s) R_{12}(s) G_1(s) \psi(s)\, ds \\ &\geq \frac{\psi(t)\phi(t)}{q_2(t)} \int_t^\infty \frac{g(s) R_{21}(s) R_{12}(s) G_1(s)}{\phi(s)}\, ds. \end{aligned}$$

Once more integration yields

$$-\psi'(t) \geq \frac{\psi(t)\phi(t)}{q_1(t)} \int_t^\infty \frac{1}{q_2(v)} \int_v^\infty \frac{g(s) R_{21}(s) R_{12}(s) G_1(s)}{\phi(s)}\, ds\, dv,$$

or equivalently

$$\frac{\psi'(t)}{\psi(t)} \leq -F(t).$$

Integrating the latter inequality from t_2 to t yields

$$\frac{x(t)}{R_{12}(t)} \leq \psi(t) \leq \psi(t_2) \exp\Big(- \int_{t_2}^t F(s)\, ds \Big). \tag{3.14}$$

Combining (3.13) and (3.14) gives the desired result (3.12). ☐

4. Examples

In this section, we present two examples to illustrate the main results.

Example 4.1. We consider the non-canonical differential equation with an advanced neutral term,

$$\Big(t^2 \big(t^2 (x(t) + p_0 x(\lambda t))' \big)' \Big)' + a t x(t) = 0, \quad t \geq 1, \tag{4.1}$$

where $a > 0$, $\lambda > 1$, and $0 < p_0 < 1/(2 - \lambda^{-1})^2$. Here $r_1(t) = r_2(t) = t^2$, $z(t) = x(t) + p_0 x(\tau(t))$, $p(t) = p_0$, $\tau(t) = \lambda t > t$, $g(t) = at$, $t_0 = 1$. Simple calculations show that

$$\begin{gathered} R_{12}(t) = R_{21}(t) = \frac{1}{2t^2}, \quad q_1(t) = q_2(t) = \frac{1}{2}, \\ Q_1(t) = Q_2(t) = \frac{(t-1)}{2}, \quad Q_{12}(t) = 2(t-1)^2. \end{gathered}$$

The transformation $\psi(t) = z(t)/R_{12}(t)$ yields the canonical equation

$$\psi'''(t) + \frac{2a}{t} x(t) = 0.$$

First we check the conditions for applying Theorem 3.3. Condition (3.2) becomes

$$p_0 \Big(\frac{t - \lambda^{-1}}{t-1} \Big)^2 < 1, \quad \forall t \geq t_2\,.$$

Since $\lambda > 1$, for $t > 1$, it follows that $t - \lambda^{-1} > 0$ and $\frac{t-\lambda^{-1}}{t-1}$ is decreasing. Therefore we select $t_2 = 2$. Then (3.2) is implied by $p_0(2 - \lambda^{-1})^2 < 1$, which follows from the choice of p_0 in (4.1). Then for $\tau(t) \geq t$ and $t \geq t_2 = 2$, we have

$$G(t) = 1 - p_0 \Big(\frac{t - \lambda^{-1}}{t-1} \Big)^2 \geq 1 - p_0 (2 - \lambda^{-1})^2 > 0$$

To check condition (3.9), we label the 2 integrals as I and II. Then

$$I \geq \frac{a\big(1 - p_0(2 - \lambda^{-1})^2\big)}{2} \frac{1}{(t-1)} \int_2^t (1 - s^{-1})^3\, ds$$

and
$$II \geq \frac{a\big(1 - p_0(2-\lambda^{-1})^2\big)}{2}(t-1)^2 \int_t^\infty s^{-3}\, ds$$
Adding the above inequalities, and using L'Hopital's Rule to compute the limits as $t \to \infty$, we have
$$\lim_{t\to\infty} I + II \geq \frac{3a}{4}\big(1 - p_0(2-\lambda^{-1})^2\big)$$
which will be greater than 1 if we choose
$$a > \frac{4}{3\big(1 - p_0(2-\lambda^{-1})^2\big)}\,.$$
Under this condition all positive solutions of (4.1) are Kneser-type solutions.

To check the conditions for Theorem 3.5, we observe that $G_1(t) = 1 - p_0\lambda^{-2}$. Since $p_0 < 1$ and $\lambda > 1$, we have $G_1(t) > 0$ for all $t \geq 1$. Also we have $\phi(t) = t^{a/2}$,
$$\Delta(t) = \int_t^\infty 2 \int_s^\infty aw \frac{1}{2w^2}\frac{1}{2w^2}\, dw\, ds = \frac{a}{2}\int_t^\infty \int_s^\infty \frac{1}{w^3}\, dw\, ds = \frac{a}{4t},$$
$$F(t) = 2t^{a/2}\int_t^\infty 2\int_s^\infty aw\frac{1}{2w^2}\frac{1}{2w^2}\big(1 - p_0\lambda^{-2}\big)/w^{a/2}\, dw\, ds = \frac{4a\big(1-p_0\lambda^{-2}\big)}{(4+a)(2+a)}t^{-1},$$
$$\exp\Big(-\int_1^t F(s)\, ds\Big) = t^{-\delta}, \quad \text{where } \delta = \frac{4a\big(1-p_0\lambda^{-2}\big)}{(4+a)(2+a)}\,.$$

By Theorem 3.5 there are positive constants α_1 and α_2 such that all positive solutions to (4.1) satisfy
$$\alpha_1 t^{-2-\frac{a}{2}} \leq x(t) \leq \alpha_2 t^{-2-\delta}\,.$$
As a particular case $p_0 = 1/18$, $\lambda = 2$, and $a = 2$ yield the bounds
$$\alpha_1 t^{-3} \leq x(t) \leq \alpha_2 t^{-503/216}\,.$$
When $p_0 = 0$, Theorem 3.5 yields an estimate for ordinary differential equation
$$\Big(t^2\big(t^2 x'(t)\big)'\Big)' + atx(t) = 0, \quad a > 0,\ t \geq 1;$$
namely
$$\alpha_1 t^{-2-\frac{a}{2}} \leq x(t) \leq \alpha_2 t^{-2-\frac{4a}{(4+a)(2+a)}}\,.$$

Example 4.2. We consider the non-canonical differential equation with an delayed neutral term,
$$\Big(t^2\big(t^2(x(t) + p_0 x(\lambda t))'\big)'\Big)' + atx(t) = 0, \quad t \geq 1, \tag{4.2}$$
where $a > 0$, $\tau(t) = \lambda t$ with $\lambda < 1$, $0 < p_0 < \lambda^{2+\frac{a}{2}}$. The expression for $g, q, q, r, R, \phi, \Delta$ are the same as in Example 4.1.

First we check the conditions for applying Theorem 3.3. Condition (3.1) becomes
$$p_0\frac{1}{\lambda^2} < 1, \quad \forall t \geq t_2\,,$$
which is implied by the assumption $0 < p_0 < \lambda^{2+\frac{a}{2}}$. Setting $t_2 = 1$ we have
$$G(t) = 1 - p_0\lambda^{-2} > 0, \quad \forall t \geq t_2 = 1\,.$$

To check condition (3.9), we label the 2 integrals as I and II. Then

$$I = \frac{a}{2}\big(1 - p_0\lambda^{-2}\big)\frac{1}{(t-1)}\int_1^t \big(1 - s^{-1}\big)^3\, ds$$

and

$$II = \frac{a}{2}\big(1 - p_0\lambda^{-2}\big)(t-1)^2\int_t^\infty s^{-3}\, ds\,.$$

Adding the above equalities, and using L'Hopital's Rule to compute the limits as $t \to \infty$, we have

$$\lim_{t\to\infty} I + II = \frac{3a}{4}\big(1 - p_0\lambda^{-2}\big),$$

which will be greater than 1 if we choose

$$a > \frac{4}{3\big(1 - p_0\lambda^{-2}\big)}\,.$$

Under this condition all positive solutions of (4.2) are Kneser-type solutions.

To check the conditions for Theorem 3.5, we compute the following expressions:

$$G_1(t) = 1 - p_0\lambda^{-2-\frac{a}{2}},$$

which is positive by the assumption $0 < p_0 < \lambda^{2+\frac{a}{2}}$. Then we compute

$$F(t) = \frac{4a\big(1 - p_0\lambda^{-2-\frac{a}{2}}\big)}{(4+a)(2+a)}t^{-1},$$

$$\exp\Big(-\int_1^t F(s)\, ds\Big) = t^{-\gamma}, \quad \text{where } \gamma = \frac{4a\big(1 - p_0\lambda^{-2-\frac{a}{2}}\big)}{(4+a)(2+a)}\,.$$

By Theorem 3.5 there are positive constants α_1 and α_2 such that all positive solutions to (4.2) satisfy

$$\alpha_1 t^{-2-\frac{a}{2}} \le x(t) \le \alpha_2 t^{-2-\gamma}\,.$$

As a particular case $p_0 = 1/16$, $\lambda = 1/2$, and $a = 2$, we have the bounds

$$\alpha_1 t^{-3} \le x(t) \le \alpha_2 t^{-13/6}\,.$$

5. Conclusion

In this study, we derived conditions for the existence and estimates of Kneser-type solutions of the noncanonical third-order differential equations with a delay or advanced argument in the neutral term. This was achieved by transforming the noncanonical equation (1.1) into a canonical form without adding any new conditions. We obtained estimates for the Kneser-type solutions of (1.1), which are new contribution to the literature. These estimates for Kneser-type solutions are not easily obtained for the noncanonical equation (1.1) without such a transformation. Furthermore, the results from references [2, 3, 4, 7, 20, 21, 23] do not apply to equations (4.1) and (4.2) as they are noncanonical. It is interesting to study similar properties of (1.1) when the neutral term is of mixed type.

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