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EXISTENCE OF GLOBAL WEAK SOLUTION TO TUMOR CHEMOTAXIS COMPETITION SYSTEMS WITH LOOP AND SIGNAL DEPENDENT SENSITIVITY

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Abstract. This article examines the weak solution of a fully parabolic chemotaxis-competition system with loop and signal-dependent sensitivity. The system is subject to homogeneous Neumann boundary conditions within an open, bounded domain $\Omega \subset \mathbb{R}^n$, where $n \geq 1$ and $\partial\Omega$ is smooth. We assume that the parameters in the system are positive constants. Additionally, the initial data $(u_{10}, u_{20}, v_{10}, v_{20}) \in L^2(\Omega) \times L^2(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ are non-negative. The existence of a weak solution to the problem is established using energy inequality method.

1. Introduction

This article shows the existence of weak solutions for a chemotaxis-competition system that features loop and signal dependent sensitivity. The system under consideration models the chemotactical communication within the tumor site, specifically the EGF/CSF-1 paracrine invasion loop.

$$\begin{aligned}
u_{1t} &= d_1\Delta u_1 - \nabla\cdot(\chi_1(v_1)u_1\nabla v_1) - \nabla\cdot(\chi_2(v_2)u_1\nabla v_2) \\
&\quad + \delta_1 u_1(1-u_1-a_1u_2), \quad x\in\Omega,\ t>0,\\
u_{2t} &= d_2\Delta u_2 - \nabla\cdot(\xi_1(v_1)u_2\nabla v_1) - \nabla\cdot(\xi_2(v_2)u_2\nabla v_2) \\
&\quad + \delta_2 u_2(1-a_2u_1-u_2), \quad x\in\Omega,\ t>0,\\
v_{1t} &= d_3\Delta v_1 + \alpha_1u_1 + \beta_1u_2 - \gamma_1v_1, \quad x\in\Omega,\ t>0,\\
v_{2t} &= d_4\Delta v_2 + \alpha_2u_1 + \beta_2u_2 - \gamma_2v_2, \quad x\in\Omega,\ t>0,\\
&\frac{\partial u_1}{\partial\nu} = \frac{\partial u_2}{\partial\nu} = \frac{\partial v_1}{\partial\nu} = \frac{\partial v_2}{\partial\nu} = 0, \quad x\in\partial\Omega,\ t>0,\\
u_1(x,0) &= u_{10}, \quad u_2(x,0)=u_{20}, \quad v_1(x,0)=v_{10}, \quad v_2(x,0)=v_{20}, \quad x\in\Omega,
\end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$ is an open bounded domain with smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial\nu}$ indicate differentiation with respect to the outward normal on $\partial\Omega$. The quantities $u_1(x,t)$ and $u_2(x,t)$ represent the densities of macrophages and tumor cells, respectively, while $v_1(x,t)$ and $v_2(x,t)$ represent the concentration of chemical signals secreted by u_1 and u_2, respectively. We assume the all parameters in the

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equation are positive. Here, d_i (for $i = 1, 2, 3, 4$) denotes the diffusion coefficients. Meanwhile, δ_1 and δ_2 represent the growth rates of macrophages and tumor cells, respectively. The coefficient a_1 describes the interaction among macrophages, while a_2 describes the interaction among tumor cells. The parameters α_i, β_i, and γ_i (for $i = 1, 2$) represent the production rate of macrophages and tumor cells, respectively, while γ_i represents the decay of the chemical attractants. The initial conditions u_{10}, u_{20}, v_{10}, and v_{20} satisfy

$$\begin{gathered} u_{10}, u_{20} \in L^2(\Omega), \quad \text{with} \quad u_{10}, u_{20} \geq 0 \quad \text{in } \Omega, \\ v_{10}, v_{20} \in W^{1,2}(\Omega), \quad \text{with} \quad v_{10}, v_{20} \geq 0 \quad \text{in } \Omega. \end{gathered} \tag{1.2}$$

The chemotactic sensitivity functions $\chi_i(v_i)$ and $\xi_i(v_i)$, $i = 1, 2$ satisfy

$$\chi_i(v_i), \xi_i(v_i) \in L^\infty(\Omega), \quad \text{with} \quad \chi_i(v_i), \xi_i(v_i) > 0 \quad \text{in } \Omega. \tag{1.3}$$

The system (1.1) under consideration is a generalized version of the classic Keller-Segel chemotaxis system. Chemotaxis refers to the directional movement of microorganisms in response to a chemical stimulus, and is involved in various biological processes such as disease progression, wound healing, neuron migration, and tumor invasion. Keller and Segel first introduced the original Keller-Segel system in 1970 [13], and since then, the theoretical analysis of Keller-Segel and its variants has been intensively studied due to its numerous applications in biology, medicine, and other sciences. For further insight into the applications of chemotaxis, [22] provides a comprehensive review. Many researchers have been attracted to the study of Keller-Segel chemotaxis systems, as evidenced by the reviews by Bellomo et al. [2], Horstmann [11], Lankeit and Winkler [16], and the references therein. For more information, see [1, 19, 20, 21, 25, 26, 34].

Recently, Wikler [33] discussed the following keller-segel system using some estimates on the Neumann problem

$$\begin{gathered} u_t = \nabla \cdot (D(v)\nabla u) - \chi \nabla \cdot (uS(v)\nabla v) + ru - \mu u^2, \\ v_t = \Delta v - v + u. \end{gathered} \tag{1.4}$$

When $r \in \mathbb{R}$, $D \in C^2([0, \infty))$ and $S \in C^2([0, \infty)) \cap W^{1,\infty}((0, \infty))$, for any $\mu > 0$, the authors provided a result on the global existence of classical solutions in a two dimensional domain.

The system consisting of two species chemotaxis with respect to two chemicals is expressed as follows

$$\begin{gathered} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), \\ \tau v_t = \Delta v - v + w, \\ w_t = \Delta w - \xi \nabla \cdot (w \nabla z), \\ \tau z_t = \Delta z - z + u. \end{gathered} \tag{1.5}$$

Tao and Winkler [24] investigated system (1.5) under the conditions of $\tau = 0$ and $\chi, \xi \in \pm 1$. They established the existence of globally bounded classical solutions to (1.5) for both the attraction-repulsion case ($\chi = 1, \xi = -1$) and double repulsion case ($\chi = \xi = -1$). Furthermore, for the attraction-attraction case ($\chi = \xi = 1$), they proved the global existence and boundedness of solutions to (1.5) if either $m = \int_\Omega u_0 + \int_\Omega w_0$ is less than a certain threshold value in the two-dimensional space ($n = 2$), or if $n \geq 3$ and the values of $|u_0|L^\infty(\Omega)$ and $|w_0|L^\infty(\Omega)$ are sufficiently small. Additionally, they showed that the system (1.5) exhibits a blow-up of solutions in finite time if either $n = 2$ and m is sufficiently large, or if $n \geq 3$ and $m > 0$. Li and Wang [18] extended the results of [24] to the fully parabolic case by presenting the unique global classical solution for the system in two dimensions under the condition that both m_1 and m_2 are small.

Consider the chemotaxis system with two species and two chemicals, which includes a logistic source term and is described by the following equations

$$\begin{gathered} u_t = d_1\Delta u - \chi_1\nabla\cdot(u\nabla v) + \mu_1 u(1-u-a_1 w),\\ \tau v_t = d_2\Delta v - \lambda_1 v + \alpha_1 w,\\ w_t = d_3\Delta w - \chi_2\nabla\cdot(w\nabla z) + \mu_2 w(1-a_2 u - w),\\ \tau z_t = d_4\Delta z - \lambda_2 z + \alpha_2 u. \end{gathered} \tag{1.6}$$

Zhang et al. [35] studied the global existence and boundedness of solutions to (1.6) with $\tau = 0$ and $d_i = 1$ $(i = 1, 2, 3, 4)$ under smallness assumptions on the initial conditions and appropriate conditions on the strength of the damping death effects. They also established asymptotic stability when $a_1 \geq 0$ and $a_2 < 1$. Tu et al. [30] studied the global bounded classical solution of the same system under the assumption that $\frac{\chi_i}{\mu_i}$ $(i = 1, 2)$ are sufficiently small. They also showed that this solution converges exponentially to the steady state when $a_1, a_2 \in (0, 1)$ and μ_1, μ_2 are sufficiently large. In the case where $a_1 \geq 1 > a_2 > 0$ and μ_2 is sufficiently large, the classical solution of the system converges to $(0, 1, 1, 0)$ as $t \to \infty$. Chunlai et al. [5] established the global-in-time solution of the system (1.6) using the eventual comparison approach and investigated the stability analysis of the system under suitable conditions. Wang and Mu [32] partially improved the results of [35] and [30] under suitable conditions on the parameters χ_i, μ_i, and a_i $(i = 1, 2)$.

Zheng and Mu [36] investigated the global bounded classical solution of (1.6) for $n = 2$ and $\tau = 0$ using a priori estimates and the Moser-Alikakos iteration technique. Meanwhile, when $\tau = 1$ and $n \geq 1$, a globally bounded solution to system (1.6) is shown to exist by the authors, utilizing the maximal Sobolev regularity and semigroup technique. For the three-dimensional case, Li et al. [17] established the global boundedness of the classical solution if $\mu_i \geq \max\{7\chi_i^2 + 1, 51/2\}$, $i = 1, 2$, and proved that the solution exponentially converges to $(1, 1, 1, 1)$ for large time, subject to the conditions that $\mu_1 >$
$chi_2^2/8$ and $\mu_2 >$
$chi_1^2/8$. Additionally, Black [3] examined the global existence of a bounded solution for the two-dimensional Lotka-Volterra competitive system with an additional chemotactic influence, and established the asymptotic behavior of the solution for $n \geq 2$. If μ_i/χ_i^2, $i = 1, 2$, are sufficiently large and $a_1, a_2 < 1$, any global solutions $u \neq 0 \neq w$ of the system converge to the unique positive equilibrium point. Furthermore, the author also demonstrated that the solution of the system converges to $(0, 1, 1, 0)$ as $t \to \infty$ provided $a_1 \geq 1, a_2 < 1$, and $\frac{\mu_2}{\chi_2}$ is sufficiently large. Finally, Pan et al. [23] studied the unique global bounded classical solution of (1.6) for $n = 3$ when μ_i, $i = 1, 2$, are sufficiently large.

The system under consideration is a chemotaxis competition model featuring a loop structure, described by the set of equations

$$\begin{aligned} u_{1t} &= d_1\Delta u_1 - \chi_{11}\nabla\cdot(u_1\nabla v_1) - \chi_{12}\nabla\cdot(u_1\nabla v_2) + \mu_1 u_1(1-u_1-a_1u_2),\\ u_{2t} &= d_2\Delta u_2 - \chi_{21}\nabla\cdot(u_2\nabla v_1) - \chi_{22}\nabla\cdot(u_2\nabla v_2) + \mu_2 u_2(1-a_2u_1-u_2),\\ \tau_1 v_{1t} &= d_3\Delta v_1 + \alpha_{11}u_1 + \alpha_{12}u_2 - \lambda_1 v_1,\\ \tau_2 v_{2t} &= d_4\Delta v_2 + \alpha_{21}u_1 + \alpha_{22}u_2 - \lambda_2 v_2. \end{aligned} \tag{1.7}$$

Espejo et al. [8] investigated the chemotaxis competition system with loop (1.7) in the case where $\mu_i = \tau_i = \lambda_i = 0, i = 1, 2$. They established the necessary and sufficient conditions for global existence and blow-up of solutions by adapting the second moment technique from [6] and [9]. This blow-up behavior of solutions models the aggregation of tumor cells and macrophages. They also demonstrated the system has an energy structure and proved global existence using the logarithmic HLS-inequality. Tu et al. [27] studied the chemotaxis competition system with loop (1.7) with $\tau = 0$ and established sufficient conditions for the existence of global solution, as well as exponential convergence to the unique positive equilibrium point for sufficiently large μ_1 and μ_2. They also showed that if χ_{11}/μ_1, χ_{12}/μ_1, χ_{21}/μ_2, and χ_{22}/μ_2 are sufficiently small for $n \geq 2$, then the system admits a globally bounded classical solution. When $a_1 > 1$ and μ_2 is sufficiently large, the solution converges to a semi-trivial equilibrium point, and this convergence is algebraic when $a_1 = 1$. Tu et al. [28] further examined the global boundedness of the classical solution of the system in two dimensions for the case where $\tau = 1$ and $d_i = 1, i = 1, 2, 3, 4$. They proved that the solution converges exponentially to the same point as in [27]. When μ_1 and μ_2 are sufficiently large, Tu et al. [29] obtained the global bounded classical solution of (1.7) in three dimensions. For more information, refer to [31]. The global existence of classical solutions to 1.7 with chemotaxis sensitivity function was studied by Gurusamy et al. [10].

Inspired by the aforementioned studies and their relevance in biological contexts, we investigate the system (1.1). By utilizing the energy estimates, we establish the global existence of weak solutions subject to suitable conditions on the parameters and non-negative initial data $(u_{10}, u_{20}, v_{10}, v_{20}) \in L^2(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ and $n \geq 1$. Additionally, we present numerical simulations of the system (1.1) in two-dimensional domain.

The structure of our article is as follows: In Section 2, we introduce some fundamental inequalities and a key lemma, and we demonstrate the local existence of classical solutions. In Section 3, we focus on the global existence of solutions to the approximate system. In Section 4, some energy estimates are derived to support the weak solutions analysis. Section 5 discusses the weak solutions to the system (1.1).

First we have the existence of global weak solutions.

Theorem 1.1. *Assume that $\Omega \subset \mathbb{R}^n, n \geq 1$ is an open, bounded domain with smooth boundary. Let the initial data $(u_{10}, u_{20}, v_{10}, v_{20}) \in L^2(\Omega) \times L^2(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ and assume that the functions $\chi_i(v_i)$ and $\xi_i(v_i)$, for $i = 1, 2$ satisfy* (1.3)*. Then, for any positive parameters, the system* (1.1) *admits at least one global weak solution.*

Definition 1.2. Let $(u_{10}, u_{20}, v_{10}, v_{20}) \in L^2(\Omega) \times L^2(\Omega) \times W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ be nonnegative and the functions $\chi_i(v_i)$ and $\xi_i(v_i)$, i=1,2 satisfy (1.3). We say that (u_1, u_2, v_1, v_2) of functions is a global weak solution of (1.1), if

$$u_1 \in L^2_{\rm loc}\big((0,\infty); L^2(\Omega)\big), \quad u_2 \in L^2_{\rm loc}\big((0,\infty); L^2(\Omega)\big),$$
$$v_1 \in L^2_{\rm loc}\big((0,\infty); W^{1,2}(\Omega)\big), \quad v_1 \in L^2_{\rm loc}\big((0,\infty); W^{1,2}(\Omega)\big)$$

and satisfies

$$\begin{aligned}
-\int_0^\infty\int_\Omega u_1\phi_t &= d_1\int_\Omega u_{10}\phi_0 - \int_0^\infty\int_\Omega \nabla u_1\cdot\nabla\phi + \int_0^\infty\int_\Omega \chi_1(v_1)u_1\nabla v_1\nabla\phi \\
&\quad + \int_0^\infty\int_\Omega \chi_2(v_2)u_1\nabla v_2\nabla\phi + \delta_1\int_0^\infty\int_\Omega u_1(1-u_1-a_1u_2)\phi, \\
-\int_0^\infty\int_\Omega u_2\phi_t &= d_2\int_\Omega u_{20}\phi_0 - \int_0^\infty\int_\Omega \nabla u_2\cdot\nabla\phi + \int_0^\infty\int_\Omega \xi_1(v_1)u_2\nabla v_1\nabla\phi \\
&\quad + \int_0^\infty\int_\Omega \xi_2(v_2)u_2\nabla v_2\nabla\phi + \delta_2\int_0^\infty\int_\Omega u_2(1-u_2-a_2u_1)\phi, \\
-\int_0^\infty\int_\Omega v_1\phi_t &= d_3\int_\Omega v_{10}\phi_0 - \int_0^\infty\int_\Omega \nabla v_1\cdot\nabla\phi + \alpha_1\int_0^\infty\int_\Omega u_1\phi + \beta_1\int_0^\infty\int_\Omega u_2\phi \\
&\quad - \gamma_1\int_0^\infty\int_\Omega v_1\phi, \\
-\int_0^\infty\int_\Omega v_2\phi_t &= d_4\int_\Omega v_{20}\phi_0 - \int_0^\infty\int_\Omega \nabla v_2\cdot\nabla\phi + \alpha_2\int_0^\infty\int_\Omega u_1\phi + \beta_2\int_0^\infty\int_\Omega u_2\phi \\
&\quad - \gamma_2\int_0^\infty\int_\Omega v_2\phi,
\end{aligned}$$

for all $\phi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$.

2. Preliminaries and local solution

In this section, we establish the existence of local solutions to the approximate system, a standard process that follows the principles outlined in [12].

System (1.1) is approximated for each $\epsilon\in(0,1)$ by the system

$$\begin{aligned}
u_{1\epsilon t} &= d_1\Delta u_{1\epsilon} - \nabla\cdot(\chi_1(v_{1\epsilon})u_{1\epsilon}\nabla v_{1\epsilon}) - \nabla\cdot(\chi_2(v_{2\epsilon})u_{1\epsilon}\nabla v_{2\epsilon}) + f_1(u_{1\epsilon},u_{2\epsilon}), \\
u_{2\epsilon t} &= d_2\Delta u_{2\epsilon} - \nabla\cdot(\xi_1(v_{1\epsilon})u_{2\epsilon}\nabla v_{1\epsilon}) - \nabla\cdot(\xi_2(v_{2\epsilon})u_{2\epsilon}\nabla v_{2\epsilon}) + f_2(u_{1\epsilon},u_{2\epsilon}), \\
v_{1\epsilon t} &= d_3\Delta v_{1\epsilon} + \alpha_1u_{1\epsilon} + \beta_1u_{2\epsilon} - \gamma_1v_{1\epsilon}, \\
v_{2\epsilon t} &= d_4\Delta v_{2\epsilon} + \alpha_2u_{1\epsilon} + \beta_2u_{2\epsilon} - \gamma_2v_{2\epsilon},
\end{aligned}\tag{2.1}$$

with the source

$$\begin{aligned}
f_1(u_{1\epsilon},u_{2\epsilon}) &= \delta_1u_{1\epsilon}(1-u_{1\epsilon}-a_1u_{2\epsilon}) - \epsilon u_{1\epsilon}^q, \\
f_2(u_{1\epsilon},u_{2\epsilon}) &= \delta_2u_{2\epsilon}(1-u_{2\epsilon}-a_2u_{1\epsilon}) - \epsilon u_{2\epsilon}^q.
\end{aligned}$$

We introduce the non-negative approximate initial data

$$\begin{gathered}
u_{10\epsilon},u_{20\epsilon}\in C^0(\overline{\Omega}),\quad\text{with}\quad u_{10\epsilon},u_{20\epsilon}\geq 0\quad\text{in }\Omega, \\
v_{10\epsilon},v_{20\epsilon}\in W^{1,q}(\overline{\Omega}),\quad\text{for some}\quad q>\{2,n\}\text{ with } v_{10\epsilon},v_{20\epsilon}\geq 0\quad\text{in }\Omega
\end{gathered}\tag{2.2}$$

that satisfy the following conditions

$$\begin{aligned}
u_{10\epsilon}&\to u_{10}\quad\text{in } L^2(\Omega), \\
u_{20\epsilon}&\to u_{20}\quad\text{in } L^2(\Omega), \\
v_{10\epsilon}&\to v_{10}\quad\text{in } W^{1,2}(\Omega), \\
v_{20\epsilon}&\to v_{20}\quad\text{in } W^{1,2}(\Omega),
\end{aligned}$$

as $\epsilon \to 0$. Moreover, the chemotactic sensitivity functions $\chi_i(v_{i\epsilon})$ and $\xi_i(v_{i\epsilon}), i = 1, 2$ are positive, non-decreasing and satisfy

$$\begin{gathered} \|\chi_i(v_{i\epsilon})\|_{L^\infty(\Omega)} \le C \quad \text{and} \quad \|\xi_i(v_{i\epsilon})\|_{L^\infty(\Omega)} \le C, \\ \chi_i(v_{i\epsilon}) \to \chi_i(v_i) \quad \text{and} \quad \xi_i(v_{i\epsilon}) \to \xi_i(v_i) \ \text{in} L^\infty(\Omega) \text{ a s} \epsilon \to 0, \end{gathered} \tag{2.3}$$

where $C > 0$.

Lemma 2.1 (Local solution)**.** *Suppose that $\Omega \subset \mathbb{R}^n$, $n \ge 1$ is an open, bounded domain with smooth boundary and $q > \max\{2, n\}$. Assume that the functions $\chi_i(v_{i\epsilon})$ and $\xi_i(v_{i\epsilon}), i = 1, 2$ satisfy* (2.3) *and the initial conditions $(u_{10\epsilon}, u_{20\epsilon}, v_{10\epsilon}, v_{20\epsilon})$ satisfy* (2.2)*. Then there exists $T_{\max} < \infty$ such that* (2.1) *admits a unique solution $(u_{1\epsilon}, u_{2\epsilon}, v_{1\epsilon}, v_{2\epsilon})$ satisfies*

$$\begin{gathered} u_{1\epsilon}, u_{2\epsilon} \in C^0\left(\overline{\Omega} \times [0, T_{\max})\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{\max})\right), \\ v_{1\epsilon}, v_{2\epsilon} \in C^0\left(\overline{\Omega} \times [0, T_{\max})\right) \cap C^{2,1}\left(\overline{\Omega} \times (0, T_{\max})\right) \cap L^\infty_{\rm loc}([0, T_{\max}); W^{1,q}(\Omega)). \end{gathered}$$

Proof. Standard techniques involving the Banach fixed point theorem and parabolic regularity theories can be utilized to derive the proof. For a detailed demonstration, please refer to [12]. Moreover, the non-negativity of the solution in $\Omega \times (0, T_{\max})$ is guaranteed by the maximum principle along with (2.2) . □

3. Global solution

To establish the weak solution of our system, we first demonstrate the existence of global solutions to the approximate system (2.1).

Lemma 3.1. *The solution $(u_{1\epsilon}, u_{2\epsilon}, v_{1\epsilon}, v_{2\epsilon})$ of* (2.1) *for every $\epsilon \in (0, 1)$ satisifes the following conditions*

$$\int_\Omega u_{1\epsilon} \le C, \quad \forall t \in (0, T_{\max,\epsilon}), \tag{3.1}$$

$$\int_\Omega u_{2\epsilon} \le C, \quad \forall t \in (0, T_{\max,\epsilon}), \tag{3.2}$$

$$\int_\Omega v_{1\epsilon} \le C, \quad \forall t \in (0, T_{\max,\epsilon}), \tag{3.3}$$

$$\int_\Omega v_{2\epsilon} \le C, \quad \forall t \in (0, T_{\max,\epsilon}), \tag{3.4}$$

$$\int_\Omega |\nabla v_{1\epsilon}|^2 \le C, \quad \forall t \in (0, T_{\max,\epsilon}), \tag{3.5}$$

$$\int_\Omega |\nabla v_{2\epsilon}|^2 \le C, \quad \forall t \in (0, T_{\max,\epsilon}). \tag{3.6}$$

Moreover, we have

$$\int_0^T \int_\Omega u_{1\epsilon}^2 + \frac{\epsilon}{\delta_1} \int_0^T \int_\Omega u_{1\epsilon}^q \le C(T+1), \quad \forall t \in (0, T_{\max,\epsilon}), \tag{3.7}$$

$$\int_0^T \int_\Omega u_{2\epsilon}^2 + \frac{\epsilon}{\delta_2} \int_0^T \int_\Omega u_{2\epsilon}^q \le C(T+1), \quad \forall t \in (0, T_{\max,\epsilon}), \tag{3.8}$$

where the constant $C > 0$.

Proof. Upon integrating the first equation in (2.1), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_{1\epsilon} = \delta_1\int_\Omega u_{1\epsilon}(1-u_{1\epsilon}-a_1u_{2\epsilon}) - \epsilon\int_\Omega u_{1\epsilon}^q, \tag{3.9}$$

$$\le \delta_1\int_\Omega u_{1\epsilon} - \delta_1\int_\Omega u_{1\epsilon}^2, \tag{3.10}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_{2\epsilon} \le \delta_2\int_\Omega u_{2\epsilon} - \delta_2\int_\Omega u_{2\epsilon}^2. \tag{3.11}$$

The proof follows a similar approach as outlined in [10, lemma 1]. Next, we integrate the third equation in (2.1) over Ω, yielding

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_{1\epsilon} = \alpha_1\int_\Omega u_{1\epsilon} + \beta_1\int_\Omega u_{2\epsilon} - \gamma_1\int_\Omega v_{1\epsilon}.$$

By utilizing (3.1) and (3.2), we can derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_{1\epsilon} = -\gamma_1\int_\Omega v_{1\epsilon} + C,$$

Now, applying ODE arguments, we deduce that

$$\int_\Omega v_{1\epsilon} \le \max\Big\{\int_\Omega v_{10\epsilon}, C\Big\}.$$

Employing a similar procedure, we can derive equation (3.4). We then proceed by multiplying the third equation in (2.1) with $-\Delta v_{1\epsilon}$ and integrating it over Ω, gives

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v_{1\epsilon}|^2 + d_3\int_\Omega |\Delta v_{1\epsilon}|^2 + \gamma_1\int_\Omega |\nabla v_{1\epsilon}|^2 \\ &= -\int_\Omega (\alpha_1 u_{1\epsilon} + \beta_1 u_{2\epsilon})\,\Delta v_{1\epsilon}, \end{aligned} \tag{3.12}$$

for all $t\in(0,T_{\max})$. Using Young's inequality, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v_{1\epsilon}|^2 + d_3\int_\Omega |\Delta v_{1\epsilon}|^2 + \gamma_1\int_\Omega |\nabla v_{1\epsilon}|^2 \\ &\le d_3\int_\Omega |\Delta v_{1\epsilon}|^2 + \frac{\alpha_1^2}{2d_3}\int_\Omega u_{1\epsilon}^2 + \frac{\beta_1^2}{2d_3}\int_\Omega u_{2\epsilon}^2, \end{aligned}$$

thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v_{1\epsilon}|^2 + \gamma_1\int_\Omega |\nabla v_{1\epsilon}|^2 \le \frac{\alpha_1^2}{2d_3}\int_\Omega u_{1\epsilon}^2 + \frac{\beta_1^2}{2d_3}\int_\Omega u_{2\epsilon}^2.$$

For each $\epsilon\in(0,1)$, we set

$$y_\epsilon(t) = \frac{\alpha_1^2}{2d_3\delta_1}\int_\Omega u_{1\epsilon} + \frac{\beta_1^2}{2d_3\delta_2}\int_\Omega u_{2\epsilon} + \frac{1}{2}\int_\Omega |\nabla v_{1\epsilon}|^2, \quad \forall t\in(0,T_{\max,\epsilon}).$$

Therefore,

$$\begin{aligned} y_\epsilon(t)' + 2\gamma_1 y_\epsilon(t) &\le \frac{\alpha_1^2}{2d_3\delta_1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_{1\epsilon} + \frac{\beta_1^2}{2d_3\delta_2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_{2\epsilon} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v_{1\epsilon}|^2 \\ &\quad + \frac{\gamma_1\alpha_1^2}{d_3\delta_1}\int_\Omega u_{1\epsilon} + \frac{\gamma_1\beta_1^2}{d_3\delta_2}\int_\Omega u_{2\epsilon} + \gamma_1\int_\Omega |\nabla v_{1\epsilon}|^2. \end{aligned}$$

Using (3.10) and (3.11), we obtain

$$y_\epsilon'(t)' + 2\gamma_1 y_\epsilon(t) \le \frac{\alpha_1^2}{d_3}\Big(\frac{1}{2}+\frac{\gamma_1}{\delta_1}\Big)\int_\Omega u_{1\epsilon} + \frac{\beta_1^2}{d_3}\Big(\frac{1}{2}+\frac{\gamma_1}{\delta_2}\Big)\int_\Omega u_{2\epsilon}.$$

Now, using (3.1) and (3.2), one has

$$y_\epsilon(t)' + 2\gamma_1 y_\epsilon(t) \le C, \quad \forall t \in (0, T_{\max}).$$

applying the ODE argument, yields

$$y_\epsilon(t) \le \max\Big\{\sup_{\epsilon\in(0,1)} y_\epsilon(0), \frac{C}{2\gamma_1}\Big\}, \quad \forall t \in (0, T_{\max,\epsilon}),$$

this proves (3.5). The proof for (3.6) follows a similar approach. We then proceed by integrating (3.9) with respect to time over $(0,T)$ and utilizing (3.1), which leads to

$$\int_\Omega u_{1\epsilon}(\cdot,T) + \delta_1\int_0^T\int_\Omega u_{1\epsilon}^2 + \epsilon\int_0^T\int_\Omega u_{1\epsilon}^q \le \int_\Omega u_{1\epsilon}(\cdot,0) + \delta_1 CT, \quad \forall T\in(0,T_{\max,\epsilon}).$$

Hence, we obtain (3.7) from the above equation. Following the same argument, we can also obtain (3.8). This completes the proof. □

Lemma 3.2 (Global solution)**.** *Assume that the functions $\chi_i(v_{i\epsilon})$ and $\xi_i(v_{i\epsilon}), i = 1,2$ satisfy* (2.3) *and the initial conditions $(u_{10\epsilon}, u_{20\epsilon}, v_{10\epsilon}, v_{20\epsilon})$ satisfy* (2.2)*. Then the solution of* (2.1) *remains global in time for any $\epsilon \in (0,1)$.*

The proof of the above lemma is similar to [10, Theorem 1] and it is omited here.

4. Energy estimates

We present a priori estimates that are necessary to establish the main results.

Lemma 4.1. *For each $\epsilon \in (0,1)$, there exists a constant $C > 0$ such that for all $T > 0$,*

$$\int_\Omega v_{1\epsilon}^2(\cdot,T) + \gamma_1\int_0^T\int_\Omega v_{1\epsilon}^2 \le C(T+1), \tag{4.1}$$

$$\int_\Omega v_{2\epsilon}^2(\cdot,T) + \gamma_2\int_0^T\int_\Omega v_{2\epsilon}^2 \le C(T+1). \tag{4.2}$$

Proof. We multiply the third equation in (2.1) by $v_{1\epsilon}$ and integrate over Ω to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_{1\epsilon}^2 + d_3\int_\Omega |\nabla v_{1\epsilon}|^2 + \gamma_1\int_\Omega v_{1\epsilon}^2 \\
&\le \alpha_1\int_\Omega u_{1\epsilon}v_{1\epsilon} + \beta_1\int_\Omega u_{2\epsilon}v_{1\epsilon}, \\
&\le \frac{\gamma_1}{4}\int_\Omega v_{1\epsilon}^2 + \frac{\alpha_1^2}{\gamma_1}\int_\Omega u_{1\epsilon}^2 + \frac{\gamma_1}{4}\int_\Omega v_{1\epsilon}^2 + \frac{\beta_1^2}{\gamma_1}\int_\Omega u_{2\epsilon}^2,
\end{aligned}$$

this gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_{1\epsilon}^2 + 2d_3\int_\Omega |\nabla v_{1\epsilon}|^2 + \gamma_1\int_\Omega v_{1\epsilon}^2 \le \frac{2\alpha_1^2}{\gamma_1}\int_\Omega u_{1\epsilon}^2 + \frac{2\beta_1^2}{\gamma_1}\int_\Omega u_{2\epsilon}^2,$$

for all $t > 0$. Integrating with respect to time, yields

$$\int_\Omega v_{1\epsilon}^2(\cdot,T) + 2d_3\int_0^T\int_\Omega |\nabla v_{1\epsilon}|^2 + \gamma_1\int_0^T\int_\Omega v_{1\epsilon}^2$$

$$\leq \int_\Omega v_{1\epsilon}^2(\cdot,0) + \frac{2\alpha_1^2}{\gamma_1}\int_0^T\int_\Omega u_{1\epsilon}^2 + \frac{2\beta_1^2}{\gamma_1}\int_0^T\int_\Omega u_{2\epsilon}^2.$$

Using (3.7) and (3.8), we obtain

$$\int_\Omega v_{1\epsilon}^2(\cdot,T) + 2d_3\int_0^T\int_\Omega |\nabla v_{1\epsilon}|^2 + \gamma_1\int_0^T\int_\Omega v_{1\epsilon}^2 \leq C(T+1), \quad \forall T>0.$$

This completes the proof. □

Lemma 4.2. *For each $\epsilon \in (0,1)$, there exists a constant $C>0$ such that for all $T>0$,*

$$\int_0^T\int_\Omega |\Delta v_{1\epsilon}|^2 \leq C(T+1), \tag{4.3}$$

$$\int_0^T\int_\Omega |\Delta v_{2\epsilon}|^2 \leq C(T+1), \tag{4.4}$$

Proof. From (3.12), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v_{1\epsilon}|^2 + d_3\int_\Omega |\Delta v_{1\epsilon}|^2 + \gamma_1\int_\Omega |\nabla v_{1\epsilon}|^2 \\
&= -\int_\Omega (\alpha_1 u_{1\epsilon} + \beta_1 u_{2\epsilon})\,\Delta v_{1\epsilon}, \\
&\leq \frac{d_3}{2}\int_\Omega |\Delta v_{1\epsilon}|^2 + \frac{\alpha_1^2}{d_3}\int_\Omega u_{1\epsilon}^2 + \frac{\beta_1^2}{d_3}\int_\Omega u_{2\epsilon}^2,
\end{aligned}$$

for all $t>0$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v_{1\epsilon}|^2 + d_3\int_\Omega |\Delta v_{1\epsilon}|^2 + 2\gamma_1\int_\Omega |\nabla v_{1\epsilon}|^2 \leq \frac{2\alpha_1^2}{d_3}\int_\Omega u_{1\epsilon}^2 + \frac{2\beta_1^2}{d_3}\int_\Omega u_{2\epsilon}^2.$$

Integrating with respect to time, we infer that

$$\begin{aligned}
&\int_\Omega |\nabla v_{1\epsilon}(\cdot,T)|^2 + d_3\int_0^T\int_\Omega |\Delta v_{1\epsilon}|^2 + 2\gamma_1\int_0^T\int_\Omega |\nabla v_{1\epsilon}|^2 \\
&\leq \int_\Omega |\nabla v_{1\epsilon}(\cdot,0)|^2 + \frac{2\alpha_1^2}{d_3}\int_0^T\int_\Omega u_{1\epsilon}^2 + \frac{2\beta_1^2}{d_3}\int_0^T\int_\Omega u_{2\epsilon}^2.
\end{aligned}$$

Using (3.7) and (3.8), we attain

$$\int_\Omega |\nabla v_{1\epsilon}(\cdot,T)|^2 + d_3\int_0^T\int_\Omega |\Delta v_{1\epsilon}|^2 + \gamma_1\int_0^T\int_\Omega |\nabla v_{1\epsilon}|^2 \leq C(T+1),$$

for all $T>0$. We can apply the same procedure as above to prove (4.4). This completes the proof. □

Lemma 4.3. *For each $\epsilon \in (0,1)$, there exists a constant $C>0$ such that*

$$\begin{aligned}
&\int_0^T\int_\Omega \frac{|\nabla u_{1\epsilon}|^2}{1+u_{1\epsilon}} + \delta_1\int_0^T\int_\Omega u_{1\epsilon}^2\ln(1+u_{1\epsilon}) + \epsilon\int_0^T\int_\Omega u_{1\epsilon}^q\ln(1+u_{1\epsilon}) \\
&\leq C(T+1)
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
&\int_0^T\int_\Omega \frac{|\nabla u_{2\epsilon}|^2}{1+u_{2\epsilon}} + \delta_2\int_0^T\int_\Omega u_{2\epsilon}^2\ln(1+u_{2\epsilon}) + \epsilon\int_0^T\int_\Omega u_{2\epsilon}^q\ln(1+u_{2\epsilon}) \\
&\leq C(T+1),
\end{aligned} \tag{4.6}$$

for all $T > 0$.

Proof. By using $\ln(1+u_{1\epsilon})$ as a test function and applying the first equation in (2.1), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big((1+u_{1\epsilon})\ln(1+u_{1\epsilon}) - u_{1\epsilon}\Big) \\
&\le -d_1 \int_\Omega \frac{|\nabla u_{1\epsilon}|^2}{1+u_{1\epsilon}} + C\int_\Omega \frac{u_{1\epsilon}}{1+u_{1\epsilon}} \nabla u_{1\epsilon}\cdot\nabla v_{1\epsilon} + C\int_\Omega \frac{u_{1\epsilon}}{1+u_{1\epsilon}} \nabla u_{1\epsilon}\cdot\nabla v_{2\epsilon} \\
&\quad + \delta_1 \int_\Omega u_{1\epsilon}(1-u_{1\epsilon}-a_1 u_{2\epsilon})\ln(1+u_{1\epsilon}) - \epsilon\int_\Omega u_{1\epsilon}^q \ln(1+u_{1\epsilon}),
\end{aligned}$$

for all $t > 0$. It known that for all values of $u_{1\epsilon}$ greater than 0, the inequality $0 \le \ln(1+u_{1\epsilon}) \le u_{1\epsilon}$ holds and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big((1+u_{1\epsilon})\ln(1+u_{1\epsilon}) - u_{1\epsilon}\Big) \\
&\le -d_1 \int_\Omega \frac{|\nabla u_{1\epsilon}|^2}{1+u_{1\epsilon}} + C\int_\Omega \big(\ln(1+u_{1\epsilon}) - u_{1\epsilon}\big)\Delta v_{1\epsilon} + C\int_\Omega \big(\ln(1+u_{1\epsilon}) - u_{1\epsilon}\big)\Delta v_{2\epsilon} \\
&\quad + \delta_1\int_\Omega u_{1\epsilon}^2 - \delta_1 \int_\Omega u_{1\epsilon}^2 \ln(1+u_{1\epsilon}) - \epsilon\int_\Omega u_{1\epsilon}^q \ln(1+u_{1\epsilon}).
\end{aligned}$$

Using Young's inequality we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big((1+u_{1\epsilon})\ln(1+u_{1\epsilon}) - u_{1\epsilon}\Big) \\
&\le -d_1 \int_\Omega \frac{|\nabla u_{1\epsilon}|^2}{1+u_{1\epsilon}} + \frac{C}{2}\int_\Omega |\Delta v_{1\epsilon}|^2 + \frac{C}{2}\int_\Omega \big(\ln(1+u_{1\epsilon}) - u_{1\epsilon}\big)^2 + \frac{C}{2}\int_\Omega |\Delta v_{2\epsilon}|^2 \\
&\quad + \frac{C}{2}\int_\Omega \big(\ln(1+u_{1\epsilon}) - u_{1\epsilon}\big)^2 + \delta_1\int_\Omega u_{1\epsilon}^2 - \delta_1\int_\Omega u_{1\epsilon}^2\ln(1+u_{1\epsilon}) \\
&\quad - \epsilon\int_\Omega u_{1\epsilon}^k \ln(1+u_{1\epsilon}).
\end{aligned}$$

From the inequality $(a-b)^2 \le a^2 + b^2$, we conclude that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big((1+u_{1\epsilon})\ln(1+u_{1\epsilon}) - u_{1\epsilon}\Big) \\
&\le -d_1 \int_\Omega \frac{|\nabla u_{1\epsilon}|^2}{1+u_{1\epsilon}} + \frac{C}{2}\int_\Omega |\Delta v_{1\epsilon}|^2 + \frac{C}{2}\int_\Omega |\Delta v_{2\epsilon}|^2 + \frac{C}{2}\int_\Omega \ln(1+u_{1\epsilon})^2 + \frac{C}{2}\int_\Omega u_{1\epsilon}^2 \\
&\quad + \frac{C}{2}\int_\Omega \ln(1+u_{1\epsilon})^2 + \frac{C}{2}\int_\Omega u_{1\epsilon}^2 + \delta_1\int_\Omega u_{1\epsilon}^2 - \delta_1\int_\Omega u_{1\epsilon}^2\ln(1+u_{1\epsilon}) \\
&\quad - \epsilon\int_\Omega u_{1\epsilon}^q \ln(1+u_{1\epsilon}),
\end{aligned}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big((1+u_{1\epsilon})\ln(1+u_{1\epsilon}) - u_{1\epsilon}\Big) \\
&\le -d_1 \int_\Omega \frac{|\nabla u_{1\epsilon}|^2}{1+u_{1\epsilon}} + \frac{C}{2}\int_\Omega |\Delta v_{1\epsilon}|^2 + \frac{C}{2}\int_\Omega |\Delta v_{2\epsilon}|^2 + (C+C+\delta_1)\int_\Omega u_{1\epsilon}^2 \\
&\quad - \delta_1\int_\Omega u_{1\epsilon}^2\ln(1+u_{1\epsilon}) - \epsilon\int_\Omega u_{1\epsilon}^q \ln(1+u_{1\epsilon}).
\end{aligned}$$

By integrating with respect to time and utilizing the fact that $(1+u_{1\epsilon})\ln(1+u_{1\epsilon})-u_{1\epsilon}>0$, it is possible to derive

$$\begin{aligned} &d_1\int_0^T\int_\Omega\frac{|\nabla u_{1\epsilon}|^2}{1+u_{1\epsilon}}+\delta_1\int_0^T\int_\Omega u_{1\epsilon}^2\ln(1+u_{1\epsilon})+\epsilon\int_0^T\int_\Omega u_{1\epsilon}^q\ln(1+u_{1\epsilon})\\ &\le\int_\Omega\Big((1+u_{1\epsilon0})\ln(1+u_{1\epsilon0})-u_{1\epsilon0}\Big)+\frac{C}{2}\int_0^T\int_\Omega|\Delta v_{1\epsilon}|^2\\ &\quad+\frac{C}{2}\int_0^T\int_\Omega|\Delta v_{2\epsilon}|^2+(C+C+\delta_1)\int_0^T\int_\Omega u_{1\epsilon}^2. \end{aligned}$$

Using the previous Lemmas, we obtain

$$d_1\int_0^T\int_\Omega\frac{|\nabla u_{1\epsilon}|^2}{1+u_{1\epsilon}}+\delta_1\int_0^T\int_\Omega u_{1\epsilon}^2\ln(1+u_{1\epsilon})+\epsilon\int_0^T\int_\Omega u_{1\epsilon}^q\ln(1+u_{1\epsilon})\le C(T+1),$$

for all $T>0$. The same argument gives us that

$$d_2\int_0^T\int_\Omega\frac{|\nabla u_{2\epsilon}|^2}{1+u_{2\epsilon}}+\delta_2\int_0^T\int_\Omega u_{2\epsilon}^2\ln(1+u_{2\epsilon})+\epsilon\int_0^T\int_\Omega u_{2\epsilon}^q\ln(1+u_{2\epsilon})\le C(T+1),$$

for all $T>0$. This completes the proof. □

Lemma 4.4. *For all values of $\epsilon\in(0,1)$, there exists a positive constant C such that*

$$\|u_{1\epsilon}\|_{L^{4/3}((0,T);W^{1,\frac43}(\Omega))}\le C(T+1),\quad \|u_{2\epsilon}\|_{L^{4/3}((0,T);W^{1,\frac43}(\Omega))}\le C(T+1),\quad(4.7)$$

for all $T>0$.

Proof. Let

$$\int_0^T\int_\Omega|\nabla u_{1\epsilon}|^{4/3}=\int_0^T\int_\Omega\frac{|\nabla u_{1\epsilon}|^{4/3}}{(1+u_{1\epsilon})^{2/3}}(1+u_{1\epsilon})^{2/3}.$$

Using the Young's inequality, then (3.7) and (4.5) the above estimate yields

$$\begin{aligned} \int_0^T\int_\Omega|\nabla u_{1\epsilon}|^{4/3}&\le\int_0^T\int_\Omega\Big(\frac{|\nabla u_{1\epsilon}|^{4/3}}{(1+u_{1\epsilon})^{2/3}}\Big)^{3/2}+\frac14\int_0^T\int_\Omega(1+u_{1\epsilon})^2,\\ &\le\int_0^T\int_\Omega\frac{|\nabla u_{1\epsilon}|^2}{(1+u_{1\epsilon})}+\frac14\int_0^T\int_\Omega(1+u_{1\epsilon})^2\le C(T+1) \end{aligned}.\quad(4.8)$$

Again, using the Young's inequality, one obtains

$$\int_0^T\int_\Omega u_{1\epsilon}^{4/3}\le\int_0^T\int_\Omega u_{1\epsilon}^2+\frac14|\Omega|T\le C(T+1).$$

Combining the preceding two estimates, we have established the proof for all $T>0$. The same reasoning can be applied for $u_{2\epsilon}$. □

The following lemma is used for showing the strong compactness properties of the solution $(u_{1\epsilon},u_{2\epsilon},v_{1\epsilon},v_{2\epsilon})$.

Lemma 4.5. *There exists $C>0$ and let $\epsilon\in(0,1)$ and $p>1+\frac{n}{2}$, such that*

$$\|\frac{\partial u_{1\epsilon}}{\partial t}\|_{L^1\big((0,T);(W_0^{p,2}(\Omega))'\big)}\le C(T+1),\quad(4.9)$$

$$\|\frac{\partial u_{2\epsilon}}{\partial t}\|_{L^1\big((0,T);(W_0^{p,2}(\Omega))'\big)}\le C(T+1),\quad(4.10)$$

$$\|\frac{\partial v_{1\epsilon}}{\partial t}\|_{L^2\big((0,T);(W^{1,2}(\Omega))'\big)} \le C(T+1), \tag{4.11}$$

$$\|\frac{\partial v_{2\epsilon}}{\partial t}\|_{L^2\big((0,T);(W^{1,2}(\Omega))'\big)} \le C(T+1), \tag{4.12}$$

for all $T > 0$.

Proof. Multiply the first equation in (2.1) by $\phi \in C_0^\infty(\Omega)$, and then integrate by parts to obtain

$$\begin{aligned}
|\int_\Omega u_{1\epsilon t}\phi| \le \Big(\|\nabla\phi\|_{L^\infty(\Omega)} + \|\phi\|_{L^\infty(\Omega)}\Big)\Big(d_1\int_\Omega \nabla u_{1\epsilon} + M_1\int_\Omega u_{1\epsilon}\nabla v_{1\epsilon}\\
+ M_2\int_\Omega u_{1\epsilon}\nabla v_{2\epsilon} + \delta_1\int_\Omega u_{1\epsilon} + \delta_1\int_\Omega u_{1\epsilon}^2 + \delta_1 a_1\int_\Omega u_{1\epsilon}u_{2\epsilon} + \epsilon\int_\Omega u_{1\epsilon}^q\Big).
\end{aligned}$$

Basic inequalities imply

$$\begin{aligned}
|\int_\Omega u_{1\epsilon t}\phi| \le \|\phi\|_{W^{1,\infty}(\Omega)}\Big(\frac{3}{4}\int_\Omega |\nabla u_{1\epsilon}|^{4/3} + \frac{d_1^4|\Omega|}{4} + \frac{M_1}{2}\int_\Omega u_{1\epsilon}^2 + \frac{M_1}{2}\int_\Omega |\nabla v_{1\epsilon}|^2\\
+ \frac{M_2}{2}\int_\Omega u_{1\epsilon}^2 + \frac{M_2}{2}\int_\Omega |\nabla v_{2\epsilon}|^2 + \delta_1\int_\Omega u_{1\epsilon} + \delta_1\int_\Omega u_{1\epsilon}^2\\
+ \frac{\delta_1 a_1}{2}\int_\Omega u_{1\epsilon}^2 + \frac{\delta_1 a_1}{2}\int_\Omega u_{2\epsilon}^2 + \epsilon\int_\Omega u_{1\epsilon}^q\Big).
\end{aligned}$$

As a consequence of the embedding $W_0^{p,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ for $p > 1 + \frac{n}{2}$, there exists a positive constant C such that $\|\phi\|_{W^{1,\infty}(\Omega)} = \|\nabla\phi\|_{L^\infty(\Omega)} + \|\phi\|_{L^\infty(\Omega)} \le C\|\phi\|_{W_0^{p,2}(\Omega)}$. By utilizing the previous lemmas, we can apply the aforementioned inequality to obtain

$$\begin{aligned}
&\int_0^T \|u_{1\epsilon t}(\cdot,t)\|_{(W_0^{p,2}(\Omega))'}\\
&\le \frac{3}{4}\int_0^T\int_\Omega |\nabla u_{1\epsilon}|^{4/3} + \frac{d_1^4|\Omega|T}{4} + \frac{M_1}{2}\int_0^T\int_\Omega u_{1\epsilon}^2 + \frac{M_1}{2}\int_0^T\int_\Omega |\nabla v_{1\epsilon}|^2\\
&\quad + \frac{M_2}{2}\int_0^T\int_\Omega u_{1\epsilon}^2 + \frac{M_2}{2}\int_0^T\int_\Omega |\nabla v_{2\epsilon}|^2 + \delta_1\int_0^T\int_\Omega u_{1\epsilon}\\
&\quad + \delta_1\int_0^T\int_\Omega u_{1\epsilon}^2 + \frac{\delta_1 a_1}{2}\int_0^T\int_\Omega u_{1\epsilon}^2 + \frac{\delta_1 a_1}{2}\int_0^T\int_\Omega u_{2\epsilon}^2 + \epsilon\int_0^T\int_\Omega u_{1\epsilon}^q\\
&\le C(T+1).
\end{aligned}$$

Similarly, we obtain

$$\int_0^T \|u_{2\epsilon t}(\cdot,t)\|_{(W_0^{p,2}(\Omega))'} \le C(T+1).$$

Choose $\phi \in W^{1,2}(\Omega)$, test the third equation in (2.1) and applying the Hölder's inequality infer that

$$\begin{aligned}
\int_\Omega v_{1\epsilon t}\phi &\le d_3\int_\Omega \nabla v_{1\epsilon}\cdot\nabla\phi - \gamma_1\int_\Omega v_{1\epsilon}\phi + \alpha_1\int_\Omega u_{1\epsilon}\phi + \beta_1\int_\Omega u_{2\epsilon}\phi\\
&\le d_3\Big(\int_\Omega |\nabla v_{1\epsilon}|^2\Big)^{1/2}\Big(\int_\Omega |\nabla\phi|^2\Big)^{1/2} + \gamma_1\Big(\int_\Omega v_{1\epsilon}^2\Big)^{1/2}\Big(\int_\Omega \phi^2\Big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+\alpha_1\Big(\int_\Omega u_{1\epsilon}^2\Big)^{1/2}\Big(\int_\Omega \phi^2\Big)^{1/2}+\beta_1\Big(\int_\Omega u_{2\epsilon}^2\Big)^{1/2}\Big(\int_\Omega \phi^2\Big)^{1/2}\\
\le{}& \Big(d_3\Big(\int_\Omega |\nabla v_{1\epsilon}|^2\Big)^{1/2}+\gamma_1\Big(\int_\Omega v_{1\epsilon}^2\Big)^{1/2}+\alpha_1\Big(\int_\Omega u_{1\epsilon}^2\Big)^{1/2}\\
&+\beta_1\Big(\int_\Omega u_{2\epsilon}^2\Big)^{1/2}\Big)\|\phi\|_{W^{1,2}(\Omega)}.
\end{aligned}
$$

This implies

$$
\|v_{1\epsilon t}(\cdot,t)\|^2_{\left(W^{1,2}(\Omega)\right)'}\le C\int_\Omega |\nabla v_{1\epsilon}|^2+C\int_\Omega v_{1\epsilon}^2+C\int_\Omega u_{1\epsilon}^2+C\int_\Omega u_{2\epsilon}^2.
$$

Integrating with respect to time, one obtains

$$
\begin{aligned}
&\int_0^T \|v_{1\epsilon t}(\cdot,t)\|^2_{\left(W^{1,2}(\Omega)\right)'}\\
&\le C\int_0^T\int_\Omega |\nabla v_{1\epsilon}|^2+C\int_0^T\int_\Omega v_{1\epsilon}^2+C\int_0^T\int_\Omega u_{1\epsilon}^2+C\int_0^T\int_\Omega u_{2\epsilon}^2,\\
&\le C(T+1),
\end{aligned}
$$

for all $T>0$. Similarly, we can show that

$$
\int_0^T \|v_{2\epsilon t}(\cdot,t)\|^2_{\left(W^{1,2}(\Omega)\right)'}\le C(T+1),
$$

for all $T>0$. This completes the proof. □

5. Existence of weak solutions

Next, as $\epsilon\to 0$, we proceed to passing the limits in order to construct a weak solution of (1.1).

Lemma 5.1. *There exist u_1,u_2,v_1,v_2 on $\Omega\times(0,\infty)$ and a sequence $\{\epsilon_j\}_{j\in\mathbb{N}}\subset(0,1)$, with $\epsilon_j\to 0$ as $j\to\infty$, such that*

$$u_{1\epsilon}\to u_1 \quad \text{in } L^2_{\rm loc}(\overline{\Omega}\times[0,\infty)) \text{ and a.e in } \Omega\times(0,\infty), \tag{5.1}$$

$$\nabla u_{1\epsilon}\rightharpoonup \nabla u_1 \quad \text{in } L^{4/3}_{\rm loc}(\overline{\Omega}\times[0,\infty)), \tag{5.2}$$

$$\epsilon u_{1\epsilon}^q\rightharpoonup 0 \quad \text{in } L^1_{\rm loc}(\overline{\Omega}\times[0,\infty)), \tag{5.3}$$

$$u_{1\epsilon}^2\rightharpoonup u_1^2 \quad \text{in } L^1_{\rm loc}(\overline{\Omega}\times[0,\infty)), \tag{5.4}$$

$$v_{1\epsilon}\to v_1 \quad \text{in } L^2_{\rm loc}(\overline{\Omega}\times[0,\infty)) \text{ and a.e in } \Omega\times(0,\infty), \tag{5.5}$$

$$\nabla v_{1\epsilon}\rightharpoonup \nabla v_1 \quad \text{in } L^2_{\rm loc}(\overline{\Omega}\times[0,\infty)), \tag{5.6}$$

$$\chi_1(v_{1\epsilon})u_{1\epsilon}\nabla v_{1\epsilon}\rightharpoonup \chi_1(v_1)u_1\nabla v_1 \quad \text{in } L^1_{\rm loc}(\overline{\Omega}\times[0,\infty)) \tag{5.7}$$

$$u_{2\epsilon}\to u_2 \quad \text{in } L^2_{\rm loc}(\overline{\Omega}\times[0,\infty)) \text{ and a.e in } \Omega\times(0,\infty), \tag{5.8}$$

$$\nabla u_{2\epsilon}\rightharpoonup \nabla u_2 \quad \text{in } L^{4/3}_{\rm loc}(\overline{\Omega}\times[0,\infty)), \tag{5.9}$$

$$\epsilon u_{2\epsilon}^q\rightharpoonup 0 \quad \text{in } L^1_{\rm loc}(\overline{\Omega}\times[0,\infty)), \tag{5.10}$$

$$u_{2\epsilon}^2\rightharpoonup u_2^2 \quad \text{in } L^1_{\rm loc}(\overline{\Omega}\times[0,\infty)), \tag{5.11}$$

$$u_{1\epsilon}u_{2\epsilon}\to u_1u_2 \quad \text{in } L^1_{\rm loc}(\overline{\Omega}\times[0,\infty)), \tag{5.12}$$

$$v_{2\epsilon}\to v_2 \quad \text{in } L^2_{\rm loc}(\overline{\Omega}\times[0,\infty)) \text{ and a.e in } \Omega\times(0,\infty), \tag{5.13}$$

$$\nabla v_{2\epsilon} \rightharpoonup \nabla v_2 \quad in\ L^2_{\rm loc}(\overline{\Omega} \times [0,\infty)), \tag{5.14}$$

$$\chi_2(v_{2\epsilon})u_{1\epsilon}\nabla v_{2\epsilon} \rightharpoonup \chi_2(v_2)u_1\nabla v_2 \quad in\ L^1_{\rm loc}(\overline{\Omega} \times [0,\infty)), \tag{5.15}$$

$$\xi_1(v_{1\epsilon})u_{2\epsilon}\nabla v_{1\epsilon} \rightharpoonup \xi_1(v_1)u_2\nabla v_1 \quad in\ L^1_{\rm loc}(\overline{\Omega} \times [0,\infty)), \tag{5.16}$$

$$\xi_2(v_{2\epsilon})u_{2\epsilon}\nabla v_{2\epsilon} \rightharpoonup \xi_2(v_2)u_2\nabla v_2 \quad in\ L^1_{\rm loc}(\overline{\Omega} \times [0,\infty)) \tag{5.17}$$

Proof. To prove the results, we simply take the subsequence $\epsilon := \epsilon_j$. Lemma 4.4 and 4.5 show the boundedness of $\{u_{1\epsilon}\}$ in $L^{4/3}\big((0,T);W^{1,\frac{4}{3}}(\Omega)\big)$ and $\{u_{1\epsilon t}\}$ in $L^1\big((0,T);(W^{p,2}_0(\Omega))'\big)$. Because of the embedding $W^{1,\frac{4}{3}}(\Omega) \hookrightarrow\hookrightarrow L^{4/3}(\Omega) \hookrightarrow \big(W^{p,2}_0(\Omega)\big)'$, the Aubin-Lion's lemma [4] yields a subsequence such that $u_{1\epsilon} \to u_1$ in $L^{4/3}(\Omega\times[0,\infty))$ as $\epsilon \to 0$ and this convergence is almost everywhere in $\Omega\times(0,\infty)$ for some $u_1 \in L^{4/3}(\Omega \times [0,\infty))$.

Furthermore, (3.7) and Egorov's theorem yield a subsequence along which $u_{1\epsilon} \to u_1$, ensuring the boundedness of $\{u_{1\epsilon}\}$ in $L^2_{\rm loc}(\Omega \times [0,\infty))$ and allowing us to conclude (5.2) from (4.8). Additionally, the sequence $\{u^2_{1\epsilon}\}$ being equi-bounded and equi-integrable follows from (3.7) and (4.5). Applying the Dunford-Pettis theorem [7], a subsequence of $\{u^2_{1\epsilon}\}$ is weakly convergent in $L^1_{\rm loc}(\Omega \times [0,\infty))$ and hence

$$\|u_{1\epsilon}\|_{L^2_{\rm loc}(\overline{\Omega}\times[0,\infty))} \to \|u_1\|_{L^2_{\rm loc}(\overline{\Omega}\times[0,\infty))} \quad \text{as } \epsilon \to 0$$

by taking constant as test function. This result, along with the convergence $u_{1\epsilon} \to u_1$ in $L^2_{\rm loc}(\Omega \times [0,\infty))$, allow us to achieve (5.1).

Since the sequence $\{\epsilon u^q_{1\epsilon}\}$ is equi-integrable from (4.5), $u^q_{1\epsilon}$ weakly converges to 0 by applying the Dunford-Pettis theorem, which yields (5.3). Repeating the same arguments, we obtain (5.8)-(5.11) and the result (5.12) follows from the combination of (5.1) and (5.8).

Furthermore, combining (3.5) and (4.1), we obtain that $\|v_{1\epsilon}\|_{L^2((0,T);W^{1,2}(\Omega))}$ is bounded for all $T > 0$. Along with a subsequence, (5.5) follows from previous argument and (4.12) using Aubin-Lion's Lemma. At the same time, we can conclude (5.6). The same arguments are used to prove (5.13) and (5.14).

Finally, the combination of (5.1) and (5.6), gives (5.7). The same arguments are used to prove the remaining results (5.15) - (5.17). This completes the proof. □

Lemma 5.2. (u_1, u_2, v_1, v_2) *is a global weak solution to* (1.1) *in the sense of Definition 1.2.*

Proof. Let $\phi \in C^\infty_0(\overline{\Omega} \times [0,\infty))$ and test it in the approximate problem (2.1). Applying the convergence properties from Lemma 5.1 and passing the limits, we obtain the proof. □

Proof of Theorem 1.1. The proof follows by the combination of Lemma 5.1 and Lemma 5.2. □

6. Conclusion

This study provides the global existence and boundedness of weak solutions to the chemotaxis competition system with loop and signal dependent sensitivity based on the energy inequality method.

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