Electronic Journal of Differential Equations, Vol. 2024 (2024), No. 57, pp. 1–23.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2024.57

GLOBAL DYNAMICS OF A SPECIAL CLASS OF PLANAR SECTOR-WISE LINEAR SYSTEMS

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Abstract. In this article, we study the global dynamics of a special class of planar sector-wise linear differential systems with two subsystems being the same except for their equilibriums. Taking the position of one equilibrium as the bifurcation parameter, we provide a complete analysis about sliding cycle bifurcation and sliding homoclinic bifurcation. Moreover, we obtain the existence of all important separatrix orbits and their dependence on the bifurcation parameter, including sliding heteroclinic orbit, heteroclinic cycle, limit cycle, sliding homoclinic cycle and sliding cycle.

1. Introduction

Piecewise smooth dynamical systems arise in applications, see [1, 2, 3, 4, 6, 9, 10, 13, 32, 33, 34, 35, 45, 46, 47, 51, 52]. As the simplest piecewise smooth dynamical system, planar piecewise linear systems (hereafter referred as PWLSs) with two pieces separated by a straight line has been deeply studied in recent years [12, 14, 15, 16, 17, 19, 23, 29, 30, 36, 42], mainly due to their use in analyzing the dynamics of complex systems locally, the explicit solvability of each linear subsystem and their own applications in modeling real systems.

The discontinuity of vector field appeared in piecewise smooth systems induces many dynamics that are more complicated and a lot of DIBs having new bifurcation mechanisms, see [18, 36, 50, 53]. Even in planar PWLSs, we can see some kind of sliding cycles, for example, sliding homoclinic cycles [21, 22] and even double homoclinic loops, which can be found in some engineering models such as Valve oscillators [37], Coulomb friction [51] and so on.

In recent years, the main research focus about planar PWLSs with a straight line separation has been transformed from the study of the existence and bifurcation of crossing limit cycles (see [5, 11, 12, 21, 23, 25, 26, 27, 38, 39, 40, 41, 42, 43, 47]) into the study of some critical sliding orbits and their bifurcations. For example, appearance of crossing-sliding bifurcation and a double tangency, pseudo-heteroclinic bifurcations, pseudo-homoclinic bifurcation, pseudo-Hopf bifurcation and so on ([8, 20, 36]).

2020 *Mathematics Subject Classification.* 34C23, 34C60, 34C37.
Key words and phrases. Sector-wise linear systems; sliding cycle bifurcation; sliding homoclinic bifurcation; heteroclinic cycle; non-regular point.

Submitted April 2, 2024. Published September 30, 2024.

In addition to this, the perturbation to discontinuity boundary as an important common phenomenon should also be taken into consideration, since which can lead to many new interesting phenomena and affect both the number and type of crossing limit cycles, see [7, 24, 28, 29, 44, 46, 54, 55, 56].

Therefore, in this article, we study a special family of planar PWLSs with two zones under two considerations, i.e., the discontinuity boundary is given by two rays starting from the same point (which are called planar sector-wise linear systems in [30]) and investigate the global qualitative dynamics of such systems. Particularly, we have obtained explicit dependence on system parameters of the existence, stability and number of all kinds of special sliding points for these planar sector-wise linear systems in [31]. Notice that the discontinuity boundary of planar sector-wise linear systems can be written as $\Sigma_{\pi/2}$ (see [28, Theorem 1]), i.e.,

$$\Sigma_{\pi/2} = \{(x, y) : x \geq 0 \text{ and } y = 0\} \cup \{(x, y) : x = 0 \text{ and } y \geq 0\}.$$

So, here we consider the planar PWLSs

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{F}^+(\mathbf{x}) = (F_1^+(\mathbf{x}), F_2^+(\mathbf{x}))^T = \mathbf{A} \cdot (\mathbf{x} - \mathbf{x}_e^+), & \text{if } \mathbf{x} \in \mathcal{R}_{\pi/2}, \\ \mathbf{F}^-(\mathbf{x}) = (F_1^-(\mathbf{x}), F_2^-(\mathbf{x}))^T = \mathbf{A} \cdot (\mathbf{x} - \mathbf{x}_e^-), & \text{if } \mathbf{x} \in \mathcal{R}_{3\pi/2}, \end{cases} \tag{1.1}$$

where $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$, $\mathbf{x}_e^\pm = (x_e^\pm, y_e^\pm)^T \in \mathbb{R}^2$, $x_e^+ > 0$, $y_e^+ = -y_e^- > 0$, $\mathbf{A} = [a_{ij}]$ are 2×2 real invertible matrices, and

$$\mathcal{R}_{\pi/2} = \{(x, y) : x > 0 \text{ and } y > 0\}, \quad \mathcal{R}_{3\pi/2} = \{(x, y) : xy = 0, \text{ or } x < 0 \text{ or} y < 0\}.$$

We will investigate the global dynamics of system (1.1) as the value of x_e^- changes when both $\mathbf{A}$ have two different real eigenvalues and $a_{12}a_{21} \neq 0$.

For convenience, we refer to the systems $\dot{\mathbf{x}} = \mathbf{A} \cdot (\mathbf{x} - \mathbf{x}_e^+), \mathbf{x} \in \mathcal{R}_{\pi/2}$ and $\dot{\mathbf{x}} = \mathbf{A} \cdot (\mathbf{x} - \mathbf{x}_e^-), \mathbf{x} \in \mathcal{R}_{3\pi/2}$ as the $\oplus$-system and the $\ominus$-system, respectively.

The remainder of this article is organized as follows. Some preliminaries are given in Section 2. The main results can be found in Section 3. The proofs of the main results and some examples can be found in Section 4 and section 5. Our conclusions are provided in Section 6.

2. Preliminaries

Let $\Sigma : H(x, y) = 0$ be the discontinuity boundary of the planar PWLSs with two zones

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{F}^+(\mathbf{x}) = (F_1^+(\mathbf{x}), F_2^+(\mathbf{x}))^T, & \text{if } \mathbf{x} \in \Sigma^+ = \{(x, y) : H(x, y) > 0\}, \\ \mathbf{F}^-(\mathbf{x}) = (F_1^-(\mathbf{x}), F_2^-(\mathbf{x}))^T, & \text{if } \mathbf{x} \in \Sigma^- = \{(x, y) : H(x, y) < 0\}. \end{cases} \tag{2.1}$$

It is often considered that the curve $H(x, y) = 0$ is smooth, although it maybe piecewise smooth. Precisely, the points on Σ can be divided into two kinds: *regular points* of the discontinuity boundary (at which the curve have tangent lines) and *non-regular points* of the discontinuity boundary.

In this article, the discontinuity boundary we considered in system (1.1) has a unique non-regular point $(0, 0)$. So, based on the existent definitions about various peculiar sliding points (see [21, 36, 50, 53]), we first introduce some kinds of regular points of Σ for system (2.1), and then study the non-regular point of Σ for system (1.1) separately.

Definition 2.1. A regular point $\mathbf{p} \in \Sigma$ is called *a crossing point* if

$$\langle \mathbf{F}^+(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle \cdot \langle \mathbf{F}^-(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle > 0,$$

where $H_{\mathbf{x}}(\mathbf{p})$ is the transpose of the gradient of the function $H(x, y)$ at $\mathbf{p}$. Denoted by Σ_c the set of all crossing points. Then Σ_c is the *crossing set* of system (2.1).

Definition 2.2. A regular point $\mathbf{p} \in \Sigma$ is called *a sliding point* if

$$\langle \mathbf{F}^+(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle \cdot \langle \mathbf{F}^-(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle < 0.$$

We denoted by Σ_s the set of all sliding points. Then Σ_s is the *sliding set* of system (2.1). Moreover, Σ_s is *attractive (repulsive)* if

$$\langle \mathbf{F}^+(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle < 0(> 0), \quad \langle \mathbf{F}^-(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle > 0(< 0), \quad \mathbf{p} \in \Sigma_s.$$

In addition to this, the *sliding vector field* $\mathbf{F_s}$ on Σ_s is often defined by using the Filippov convex method ([13, 35]) to be

$$\mathbf{F_s}(\mathbf{p}) = \frac{\langle \mathbf{F}^+(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle \mathbf{F}^-(\mathbf{p}) - \langle \mathbf{F}^-(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle \mathbf{F}^+(\mathbf{p})}{\langle \mathbf{F}^+(\mathbf{p}) - \mathbf{F}^-(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle}, \quad \mathbf{p} \in \Sigma_s, \tag{2.2}$$

and a point $\mathbf{p} \in \Sigma$ satisfying $\mathbf{F_s}(\mathbf{p}) = \mathbf{0}$ is called a *pseudo-equilibrium*.

When $\mathbf{p} \in \Sigma_s$ is a pseudo-equilibrium, $\mathbf{p}$ may show properties of a node, a saddle or a focus according to the stability of $\mathbf{p}$ inside Σ_s and the attractivity of Σ_s, which induces the following definitions.

Definition 2.3. Let $\mathbf{p} \in \Sigma_s$ be an isolated pseudo-equilibrium. Then $\mathbf{p}$ will be an *unstable (a stable) pseudo-node* of system (2.1) if Σ_s is repulsive (attractive) and $\mathbf{p}$ is unstable (stable) inside Σ_s, a *pseudo-saddle* if one of the followings is true: (i) Σ_s is attractive and $\mathbf{p}$ is unstable inside Σ_s; (ii) Σ_s is repulsive and $\mathbf{p}$ is stable inside Σ_s. Moreover, $\mathbf{p}$ will be a pseudo-saddle-node if $\mathbf{p}$ is half-stable inside Σ_s.

Definition 2.4. The point $\mathbf{p} \in \bar{\Sigma}$ is called *a sliding boundary point* if

$$\langle \mathbf{F}^+(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle \cdot \langle \mathbf{F}^-(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle = 0.$$

The set of all sliding boundary points, denoted by Σ_{sb}, is called the *sliding boundary set* of system (1.1).

Definition 2.5. Let $\mathbf{p} \in \Sigma_{sb}$, we call $\mathbf{p}$ a *boundary equilibrium* of system (1.1) if

$$\mathbf{F}^+(\mathbf{p}) = \mathbf{0} \text{ or } \mathbf{F}^-(\mathbf{p}) = \mathbf{0}.$$

More specifically, $\mathbf{p}$ is called a *double boundary equilibrium* when $\mathbf{F}^+(\mathbf{p}) = \mathbf{F}^-(\mathbf{p}) = \mathbf{0}$, and a *unilateral boundary equilibrium* induced by the $\ominus-$system ($\oplus-$system) if

$$\mathbf{F}^-(\mathbf{p}) = \mathbf{0}, \quad \mathbf{F}^+(\mathbf{p}) \neq \mathbf{0},$$
$$(\mathbf{F}^+(\mathbf{p}) = \mathbf{0}, \quad \mathbf{F}^-(\mathbf{p}) \neq \mathbf{0}).$$

Let $\mathbf{p}_0 = (0, 0)$ and $\mathbf{F_s}(\mathbf{p}), \mathbf{p} \in \Sigma_s$ be the unique non-regular point of the discontinuity boundary Σ and the sliding vector field of system (1.1), respectively. Notice that we did not discuss the definition of the sliding field $\mathbf{F_s}$ at $\mathbf{p}_0$. In fact, we even do not care about whether or not this point is a sliding point, which we think is not necessary at least now. However, there is no doubt that the dynamics around $\mathbf{p}_0$ play an important role in studying the global dynamics. Therefore, we need to consider $\mathbf{F_s}$ around this special point.

Suppose that the sliding field $\mathbf{F_s}$ is well defined on $I_b^y \cup I_a^x$ with

$$I_b^y = \{(0, y) : y \in (0, b)\}, \quad I_a^x = \{(x, 0) : x \in (0, a)\}.$$

Then we call $\mathbf{p}_0$ a *non-regular boundary saddle* of system (1.1) if there exist $a > 0,\ b > 0$ such that

$$\langle \mathbf{F_s}(\mathbf{p}_1), (0, 1)\rangle \cdot \langle \mathbf{F_s}(\mathbf{p}_2), (1, 0)\rangle < 0, \quad \forall \mathbf{p}_1 \in I_b^y,\ \forall \mathbf{p}_2 \in I_a^x,$$

and a *non-regular boundary source (sink)* if there exist $a > 0$ and $b > 0$ such that

$$\langle \mathbf{F_s}(\mathbf{p}_1), (0, 1)\rangle \cdot \langle \mathbf{F_s}(\mathbf{p}_2), (1, 0)\rangle > 0, \quad \langle \mathbf{F_s}(\mathbf{p}_1), (0, 1)\rangle > 0 \quad (< 0),$$

for all $\mathbf{p}_1 \in I_b^y$ and all $\mathbf{p}_2 \in I_a^x$.

In addition to this, according to [36, 48, 49], we obtain that an orbit which has a common segment with the sliding set is called a *sliding orbit*, and a closed orbit with sliding motion is called a *sliding cycle*. Furthermore, a *sliding-zero cycle* is a closed orbit intersecting Σ at an end point of the sliding region. A *sliding heteroclinic orbit (or cycle)* is a hereroclinic one with sliding motion. A *sliding homoclinic orbit (or cycle)* is a homoclinic one with sliding motion.

Moreover, for a piecewise smooth dynamical system with a parameter μ, a *sliding cycle bifurcation* happens at $\mu = \mu_0$ means that there is a sliding-zero cycle when $\mu = \mu_0$, while for $0 < |\mu - \mu_0| \ll 1$ the sliding-zero cycle disappears, and instead there appear a limit cycle and a sliding cycle when μ crosses μ_0 in opposite directions, respectively. And a *sliding homoclinic bifurcation* of the system at $\mu = \mu_0$ means the system has a sliding homoclinic cycle when $\mu = \mu_0$, while for $0 < |\mu - \mu_0| \ll 1$ the sliding homoclinic cycle disappears, instead there appears a sliding cycle when μ varies in one direction and neither sliding homoclinic orbits nor sliding cycles when μ varies in another direction.

Finally, to give our main results, we introduce some notation for the planar linear system

$$\dot{\mathbf{x}} = \mathbf{A} \cdot (\mathbf{x} - \mathbf{x}_e^{\pm}), \tag{2.3}$$

where $\det(\mathbf{A}) < 0$ (i.e., $\mathbf{A}$ has two different real eigenvalues having opposite signs: $\lambda_1 > 0 > \lambda_2$) and $[tr(\mathbf{A})]^2 > 4\det(\mathbf{A})$. Then

$$\lambda_1 = \frac{tr(\mathbf{A}) + \sqrt{[tr(\mathbf{A})]^2 - 4\det(\mathbf{A})}}{2}, \quad \lambda_2 = \frac{tr(\mathbf{A}) - \sqrt{[tr(\mathbf{A})]^2 - 4\det(\mathbf{A})}}{2}.$$

Moreover, denote the invariant manifolds of $\mathbf{x}_e^{\pm}$ by $l_1^{\pm},\ l_2^{\pm}$. Let $k_1^{\pm}, k_2^{\pm}$ be the slopes of $l_1^{\pm}, l_2^{\pm}$, $\mathbf{y}_{m1}^{\pm} = (0, y_{m1}^{\pm})$, $\mathbf{y}_{m2}^{\pm} = (0, y_{m2}^{\pm})$ and $\mathbf{x}_{m1}^{\pm} = (x_{m1}^{\pm}, 0)$, $\mathbf{x}_{m2}^{\pm} = (x_{m2}^{\pm}, 0)$ be the intersections of $l_1^{\pm}$ and $l_2^{\pm}$ with the y-axis and the x-axis, respectively. In addition to this, let $\mathbf{y}_t^{\pm} = (0, y_t^{\pm})$ and $\mathbf{x}_t^{\pm} = (x_t^{\pm}, 0)$ be the contact points at which the trajectories of system (2.3) being tangent to the y-axis and the x-axis, respectively. Then by simple calculations, it follows that

$$k_1^{\pm} = \frac{\lambda_1 - a_{11}}{a_{12}}, \quad k_2^{\pm} = \frac{\lambda_2 - a_{11}}{a_{12}}, \tag{2.4}$$

$$x_t^{\pm} = x_e^{\pm} + \frac{a_{22}}{a_{21}} y_e^{\pm}, \quad y_t^{\pm} = y_e^{\pm} + \frac{a_{11}}{a_{12}} x_e^{\pm}, \tag{2.5}$$

$$
\begin{aligned}
x_{m1}^{\pm} &= x_e^{\pm} - \frac{a_{12}}{\lambda_1 - a_{11}} y_e^{\pm} = x_t^{\pm} - \frac{\lambda_1}{a_{21}} y_e^{\pm}, \\
x_{m2}^{\pm} &= x_e^{\pm} - \frac{a_{12}}{\lambda_2 - a_{11}} y_e^{\pm} = x_t^{\pm} - \frac{\lambda_2}{a_{21}} y_e^{\pm}, \\
y_{m1}^{\pm} &= y_e^{\pm} - \frac{\lambda_1 - a_{11}}{a_{12}} x_e^{\pm} = y_t^{\pm} - \frac{\lambda_1}{a_{12}} x_e^{\pm}, \\
y_{m2}^{\pm} &= y_e^{\pm} - \frac{\lambda_2 - a_{11}}{a_{12}} x_e^{\pm} = y_t^{\pm} - \frac{\lambda_2}{a_{12}} x_e^{\pm}.
\end{aligned} \tag{2.6}
$$

Remark 2.6. It should be noted that, by system (1.1) with $\lambda_1 > 0 > \lambda_2$, we obtain

$$
\begin{aligned}
F_1(0, y_{m1}^{\pm}) &= -\lambda_1 x_e^{\pm}, \quad F_1(0, y_{m2}^{\pm}) = -\lambda_2 x_e^{\pm}, \\
F_2(x_{m1}^{\pm}, 0) &= -\lambda_1 y_e^{\pm}, \quad F_2(x_{m2}^{\pm}, 0) = -\lambda_2 y_e^{\pm},
\end{aligned}
$$

which means $l_1^{\pm}$ and $l_2^{\pm}$ are the unstable and stable invariant manifolds of $\mathbf{x}_e^{\pm}$, respectively.

3. Main results

In this paper, we give the qualitative analysis of system (1.1) under the following set of conditions (H):

$$
\begin{gathered}
\det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21} < 0, \\
a_{12}a_{21} \neq 0, \quad a_{21} > 0, \quad a_{22} > 0, \\
x_e^{+} > 0, \quad y_e^{+} = -y_e^{-} > 0,
\end{gathered} \tag{3.1}
$$

for which, we have the following considerations.

First, in this article we want to investigate the dynamics of system (1.1) related to sliding homoclinic/heteroclinic orbits, so we consider the case $\det(\mathbf{A}) < 0$. The eigenvalues of $\mathbf{A}$ will be denoted by $\lambda_1 > 0 > \lambda_2$. Moreover, since the discontinuity boundary of system (1.1) is given by the positive x-axis and the positive y-axis, we use $a_{21}a_{12} \neq 0$ to guarantee that the section return maps can be well defined.

Notice that the first condition in (3.1) implies $a_{11}a_{22} \neq 0$. Moreover, the signs of a_{12} and a_{21} will be changed at the same time under the transformation $(x, y, t) \to (-x + 2x_e^{+}, y, t)$ with the signs of a_{11} and a_{22} unchanged. Similarly, the signs of a_{11} and a_{22} will be changed simultaneously under the transformation $(x, y, t) \to (-x + 2x_e^{+}, y, -t)$ with the signs of a_{12} and a_{21} unchanged. Therefore, when we need to analyze the section return maps defined on the discontinuity boundary, we only need to discuss the case when $a_{21} > 0$, $a_{22} > 0$. More precisely, we only consider the following two cases:

$$
\begin{aligned}
\Omega_1 &= \{\mathbf{A} : a_{21} > 0, a_{22} > 0, a_{12}a_{11} > 0\}, \\
\Omega_2 &= \{\mathbf{A} : a_{21} > 0, a_{22} > 0, a_{12}a_{11} < 0\}.
\end{aligned}
$$

Finally, because the global dynamics of piecewise smooth systems is so complex that we put $x_e^{+} > 0$, $y_e^{+} = -y_e^{-} > 0$ and only left x_e^{-} as a bifurcation parameter.

To state our main results about the global dynamics of system (1.1) under (3.1) by means of providing the existence of all important separatrix in the state space, we first give the following propositions to clarify the existence of sliding set and pseudo-equilibrium points, and the properties of the unique non-regular $\mathbf{p}_0 = (0, 0)$.

Proposition 3.1. *Suppose that condition* (3.1) *holds and* $\mathbf{A} \in \Omega_1 \cup \Omega_2$. *Let*

$$\mu_1 := \frac{a_{22}}{a_{21}} y_e^+, \quad \mu_2 := \frac{a_{12}}{a_{11}} y_e^+.$$

Then the sliding set Σ_s *of system* (1.1) *is* $\Sigma_s = \Sigma_s^x \cup \Sigma_s^y$ *with*

$$\begin{aligned} \Sigma_s^x &= \{(x,0) : x > 0, x \in (\min\{x_t^-, x_t^+\}, \max\{x_t^-, x_t^+\})\}, \\ &= \begin{cases} \{(x,0) : 0 \le x < x_t^+\}, & \text{if } x_e^- < \mu_1, \\ \{(x,0) : x_t^- < x < x_t^+\}, & \text{if } \mu_1 \le x_e^- < x_e^+ + 2\mu_1, \\ \emptyset, & \text{if } x_e^- = x_e^+ + 2\mu_1, \\ \{(x,0) : x_t^+ < x < x_t^-\}, & \text{if } x_e^- > x_e^+ + 2\mu_1, \end{cases} \end{aligned} \tag{3.2}$$

and

$$\Sigma_s^y = \{(0,y) : y > 0, y \in (\min\{y_t^-, y_t^+\}, \max\{y_t^-, y_t^+\})\}. \tag{3.3}$$

About Σ_s^y, *we have the following statements:*

(a1) *When* $\mathbf{A} \in \Omega_1$, *we have*

$$\Sigma_s^y = \begin{cases} \{(0,y) : 0 \le y < y_t^+\}, & \text{if } x_e^- < \mu_2, \\ \{(0,y) : y_t^- < y < y_t^+\}, & \text{if } \mu_2 \le x_e^- < x_e^+ + 2\mu_2, \\ \emptyset, & \text{if } x_e^- = x_e^+ + 2\mu_2, \\ \{(0,y) : y_t^+ < y < y_t^-\}, & \text{if } x_e^- > x_e^+ + 2\mu_2. \end{cases} \tag{3.4}$$

(a2) *When* $\mathbf{A} \in \Omega_2$ *with* $x_e^+ + \mu_2 > 0$, *we have*

$$\Sigma_s^y = \begin{cases} \{(0,y) : 0 \le y < y_t^-\}, & \text{if } x_e^- < \mu_2, \\ \emptyset, & \text{if } x_e^- \ge \mu_2. \end{cases} \tag{3.5}$$

(a3) *When* $\mathbf{A} \in \Omega_2$ *with* $x_e^+ + \mu_2 \le 0$, *we have*

$$\Sigma_s^y = \begin{cases} \{(0,y) : y_t^+ < y < y_t^-\}, & \text{if } x_e^- < x_e^+ + 2\mu_2, \\ \emptyset, & \text{if } x_e^- = x_e^+ + 2\mu_2, \\ \{(0,y) : y_t^- < y < y_t^+\}, & \text{if } x_e^+ + 2\mu_2 < x_e^- \le \mu_2, \\ \{(0,y) : 0 \le y < y_t^+\}, & \text{if } x_e^- > \mu_2. \end{cases} \tag{3.6}$$

Proof. Note that the discontinuity boundary of system (1.1) is composed of the origin, the positive x-axis and the positive y-axis. So according to Definition 2.2, when $\mathbf{x} = (x, 0)$ is on the positive x-axis, we have

$$\langle \mathbf{F}^+(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle \cdot \langle \mathbf{F}^-(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle = F_2^+(x,0)F_2^-(x,0) = a_{21}^2(x - x_t^+)(x - x_t^-),$$

and when $\mathbf{x} = (0, y)$ is on the positive y-axis, we have

$$\langle \mathbf{F}^+(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle \cdot \langle \mathbf{F}^-(\mathbf{p}), H_{\mathbf{x}}(\mathbf{p})\rangle = F_1^+(0,y)F_1^-(0,y) = a_{12}^2(y - y_t^+)(y - y_t^-).$$

Moreover, when $\mathbf{A} \in \Omega_1 \cup \Omega_2$, we have $x_t^+ > 0$ and $0 < \mu_1 < 2\mu_1 + x_e^+$ under the condition (3.1). And by easy computations, it follows that

$$x_e^- = \mu_1 \Leftrightarrow x_t^- = 0, \quad x_e^- = x_e^+ + 2\mu_1 \Leftrightarrow x_t^- = x_t^+. \tag{3.7}$$

It is obvious that x_t^- increases with respect to x_e^- by (2.5). Then (3.2) and (3.3) can be obtained directly.

In addition, by simple calculations, we obtain

$$x_e^- = \mu_2 \Leftrightarrow y_t^- = 0, \quad x_e^- = x_e^+ + 2\mu_2 \Leftrightarrow y_t^- = y_t^+. \tag{3.8}$$

On the one hand, when $\mathbf{A} \in \Omega_1$, by (2.5), we have $y_t^+ > 0$, and y_t^- is a linear monotone increasing function with respect to x_e^-. Then (3.4) can be obtained easily. On the other hand, when $\mathbf{A} \in \Omega_2$, by (2.5), it follows that

$$y_t^+ \begin{cases} < 0, & \text{if } x_e^+ + \mu_2 > 0, \\ \geq 0, & \text{if } x_e^+ + \mu_2 \leq 0. \end{cases} \tag{3.9}$$

And y_t^- is a linear monotone decreasing function with respect to x_e^-. Then (3.5) and (3.6) can be proved directly. □

Based on the results about sliding sets given in Proposition 3.1, we can now write out the sliding vector field of system (1.1) defined by using the Filippov convex method as follows:

$$\mathbf{F}_s(\mathbf{p}) = \begin{cases} \frac{1}{F_2^+(\mathbf{p}) - F_2^-(\mathbf{p})} \begin{pmatrix} F_2^+(\mathbf{p})F_1^-(\mathbf{p}) - F_2^-(\mathbf{p})F_1^+(\mathbf{p}) \\ 0 \end{pmatrix}, & \text{if } \mathbf{p} \in \Sigma_s^x; \\ \frac{1}{F_1^+(\mathbf{p}) - F_1^-(\mathbf{p})} \begin{pmatrix} 0 \\ F_1^+(\mathbf{p})F_2^-(\mathbf{p}) - F_1^-(\mathbf{p})F_2^+(\mathbf{p}) \end{pmatrix}, & \text{if } \mathbf{p} \in \Sigma_s^y. \end{cases} \tag{3.10}$$

Then the existence, number and stability of pseudo-equilibrium points (i.e., the equilibrium points of $\mathbf{F}_s$) of system (1.1) are provided in the next proposition.

Proposition 3.2. *Suppose that condition* (3.1) *holds and* $\mathbf{A} \in \Omega_1 \cup \Omega_2$. *There exists at most one pseudo-equilibrium point in system* (1.1). *More precisely, about the pseudo-equilibrium point in* Σ_s^x, *we have:*

- (a1) *If* $x_e^- \geq -x_e^+$ *and* $x_e^- \neq x_e^+ + 2\mu_1$, *there exists a unique pseudo-equilibrium point* $(x^*, 0)$, *which is a stable (an unstable) pseudo-node when* $x_e^- < x_e^+ + 2\mu_1$ $(x_e^- > x_e^+ + 2\mu_1)$;
- (a2) *If* $x_e^- < -x_e^+$ *or* $x_e^- = x_e^+ + 2\mu_1$, *there exists no pseudo-equilibrium point in* Σ_s^x.

About the pseudo-equilibrium point in Σ_s^y, *we have:*

- (b1) *If* $x_e^- \leq -x_e^+$ *and* $x_e^- \neq x_e^+ + 2\mu_2$, *there exists a unique pseudo-equilibrium point* $(0, y^*)$, *which is a stable (an unstable) pseudo-node when* $a_{11}[x_e^- - (x_e^+ + 2\mu_2)] < 0$ $(a_{11}[x_e^- - (x_e^+ + 2\mu_2)] > 0)$;
- (b2) *If* $x_e^- > -x_e^+$ *or* $x_e^- = x_e^+ + 2\mu_2$, *there exists no pseudo-equilibrium point in* Σ_s^y.

Proof. By the definition of a pseudo-equilibrium given in Definition 2.2 and (3.10), we know that $\mathbf{p}_{ex} = (x^*, 0)$ is a pseudo-equilibrium of system (1.1) if and only if $\mathbf{p}_{ex} \in \Sigma_s^x$ and

$$\begin{aligned} &F_2^+(x^*, 0)F_1^-(x^*, 0) - F_2^-(x^*, 0)F_1^+(x^*, 0) \\ &= -2\det(\mathbf{A}) \cdot y_e^+ \cdot (x^* - \frac{x_e^- + x_e^+}{2}) = 0. \end{aligned} \tag{3.11}$$

Similarly, $\mathbf{p}_{ey} = (0, y^*)$ is a pseudo-equilibrium of system (1.1) if and only if $\mathbf{p}_{ey} \in \Sigma_s^y$ and

$$\begin{aligned} &F_1^+(0, y^*)F_2^-(0, y^*) - F_1^-(0, y^*)F_2^+(0, y^*) \\ &= -\det(\mathbf{A}) \left[(x_e^+ - x_e^-)y^* + (x_e^+ + x_e^-)y_e^+\right] = 0. \end{aligned} \tag{3.12}$$

Then we obtain $x^* = \frac{x_e^- + x_e^+}{2}$ and $y^* = \frac{x_e^- + x_e^+}{x_e^- - x_e^+} y_e^+$ directly from (3.11) and (3.12). In addition to this, we have

$$x^* = \frac{x_t^+ + x_t^-}{2} = \frac{x_{m1}^+ + x_{m1}^-}{2} = \frac{x_{m2}^+ + x_{m2}^-}{2},$$

which means that $\mathbf{p}_{ex}$ is the midpoint of $\mathbf{x}_e^\pm$, $\mathbf{x}_t^\pm$, $\mathbf{x}_{m1}^\pm$ and $\mathbf{x}_{m2}^\pm$ at the same time. And

$$\lim_{x_e^- \to -\infty} y^* = y_e^+, \quad \lim_{x_e^- \to (x_e^+)^-} y^* = -\infty, \quad \lim_{x_e^- \to (x_e^+)^+} y^* = +\infty, \quad \lim_{x_e^- \to +\infty} y^* = y_e^+,$$

this implies that y^* decreases with respect to x_e^-, which is illustrated in Figure 1.

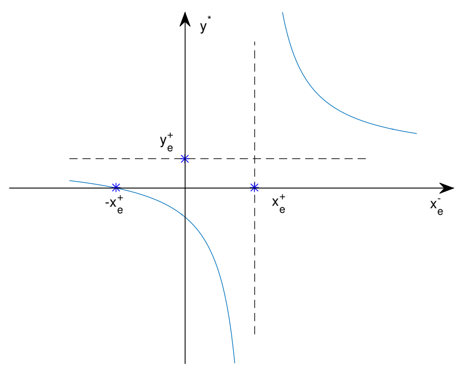

FIGURE 1. The changing of y^* with respect to x_e^-

On the one hand, it is easy to see that $x^* \geq 0$ when $x_e^- \geq -x_e^+$, and $x^* = 0$ if and only if $x_e^- = -x_e^+$. Since $x_e^- = -x_e^+ < 0 < \mu_1$, which means that $\mathbf{p}_{ex} = (0,0) \in \Sigma_s^x$ is a pseudo-equilibrium point by (3.2). Furthermore, to prove whether $\mathbf{p}_{ex} = (x^*, 0) \in \Sigma_s^x$ as $x^* > 0$, we need the following calculations:

$$x^* - x_t^- = \frac{1}{2}(x_e^+ + 2\mu_1 - x_e^-), \quad x^* - x_t^+ = \frac{1}{2}(x_e^- - 2\mu_1 - x_e^+),$$

which implies that

$$\begin{aligned} x_t^- < x^* < x_t^+, &\quad \text{if } x_e^- < x_e^+ + 2\mu_1; \\ x_t^+ < x^* < x_t^-, &\quad \text{if } x_e^- > x_e^+ + 2\mu_1. \end{aligned}$$

Then by (3.2), we obtain $\mathbf{p}_{ex} = (x^*, 0) \in \Sigma_s^x$, that is $\mathbf{p}_{ex} = (x^*, 0)$ is a pseudo-equilibrium point of system (1.1) only as $x_e^- > -x_e^+$ and $x_e^- \neq x_e^+ + 2\mu_1$.

On the other hand, according to Figure 1, it is easy to see that $y^* > 0$ when $x_e^- \in (-\infty, -x_e^+) \cup (x_e^+, +\infty)$, and $y^* = 0$ if and only if $x_e^- = -x_e^+$. Firstly, when $x_e^- = -x_e^+$, simple computation shows that $x_e^- - \mu_2 = -x_e^+ - \mu_2 = -\frac{a_{12}}{a_{11}} y_t^+$. Then we obtain that $\mathbf{p}_{ey} = (0,0) \in \Sigma_s^y$ as long as $x_e^+ + \mu_2 \neq 0$ by (3.4)-(3.6). Furthermore, to prove whether $\mathbf{p}_{ey} = (0, y^*) \in \Sigma_s^y$ as $x_e^- \in (-\infty, -x_e^+) \cup (x_e^+, +\infty)$, we need the following statements:

$$\begin{gathered}
y^* - y_t^- = \frac{x_e^-}{x_e^- - x_e^+} \frac{a_{11}}{a_{12}} (x_e^+ + 2\mu_2 - x_e^-), \\
y^* - y_t^+ = \frac{x_e^+}{x_e^- - x_e^+} \frac{a_{11}}{a_{12}} (x_e^+ + 2\mu_2 - x_e^-), \\
x_e^+ + 2\mu_2 > 0, \quad \text{if } \mathbf{A} \in \Omega_1, \\
x_e^+ + 2\mu_2 < 0, \quad \text{if } \mathbf{A} \in \Omega_2,
\end{gathered}$$

which implies that as $\mathbf{A} \in \Omega_1$ and $x_e^- < x_e^+$ ($\mathbf{A} \in \Omega_2$ and $x_e^- > x_e^+$), we have $x_e^- < x_e^+ + 2\mu_2$ ($x_e^- > x_e^+ + 2\mu_2$), then we obtain that

$$y_t^- < y^* < y_t^+ \quad (y^* > \max\{y_t^-, y_t^+\}).$$

As $\mathbf{A} \in \Omega_1$ and $x_e^- > x_e^+$ ($\mathbf{A} \in \Omega_2$ and $x_e^- < -x_e^+$), we obtain that

$$\begin{gathered}
y^* < \min\{y_t^-, y_t^+\} (y_t^- < y^* < y_t^+), \quad \text{if } x_e^- > x_e^+ + 2\mu_2; \\
y^* > \max\{y_t^-, y_t^+\} (y_t^+ < y^* < y_t^-), \quad \text{if } x_e^- < x_e^+ + 2\mu_2.
\end{gathered}$$

Then by (3.4)-(3.6), it follows that $\mathbf{p}_{ey} = (0, y^*) \in \Sigma_s^y$, that is $\mathbf{p}_{ey} = (0, y^*)$ is a pseudo-equilibrium point of system (1.1) only as $x_e^- < -x_e^+$ and $x_e^- \neq x_e^+ + 2\mu_2$.

Finally, we discuss the stability of $\mathbf{p}_{ex}$ and $\mathbf{p}_{ey}$. Let $f(x) = 2\det(\mathbf{A}) \cdot y_e^+ \cdot (x - \frac{x_e^- + x_e^+}{2})$ and $g(x) = \det(\mathbf{A}) \cdot [(x_e^+ - x_e^-)y + (x_e^+ + x_e^-)y_e^+]$. Then x^* and y^* are the isolated root of the equation $f(x) = 0$ and $g(x) = 0$, respectively. We first discuss the stability of $\mathbf{p}_{ex}$.

By Definition 2.3, on the one hand, $\Sigma_s^x = \{(x, 0) : F_2^+(x, 0) \cdot F_2^-(x, 0) < 0\}$. So the sliding set Σ_s^x is escaping if $F_2^-(x, 0) < 0$ and attracting if $F_2^-(x, 0) > 0$. On the other hand, the sliding field (3.10) at $(x, 0) \in \Sigma_s^x$ is

$$\mathbf{F_{sx}}(x, 0) = \begin{pmatrix} F_{sx}^1(x, 0) \\ 0 \end{pmatrix}, \quad F_{sx}^1(x, 0) = \frac{f(x)}{F_2^-(x, 0) - F_2^+(x, 0)}. \tag{3.13}$$

Then from (3.13) and the fact that $F_2^+(x, 0) \cdot F_2^-(x, 0) < 0$, we have

$$\operatorname{sign}(F_{sx}^1(x, 0)) = \operatorname{sign}(F_2^-(x, 0) f(x)). \tag{3.14}$$

And it is easy to see $f'(x^*) = 2\det(\mathbf{A}) \cdot y_e^+ < 0$. So we must have $f(x) > 0$ on $(x^* - \varepsilon, x^*)$ and $f(x) < 0$ on $(x^*, x^* + \varepsilon)$ for some small $\varepsilon > 0$. From (3.14), if the sliding set is attracting (escaping), i.e., $F_2^-(x, 0) > 0 (< 0)$, $\mathbf{p}_{ex}$ will be stable (unstable). That is $\mathbf{p}_{ex}$ will always be a pseudo-node of system (1.1) with $\mathbf{A} \in \Omega_1 \cup \Omega_2$, which is a stable (an unstable) pseudo-node when $x_e^- < x_e^+ + 2\mu_1 (x_e^- > x_e^+ + 2\mu_1)$. Similarly, we can also get the stability of $\mathbf{p}_{ey}$. The proof is complete. ☐

Next, we give the properties of the non-regular point $\mathbf{p}_0$.

Proposition 3.3. *Suppose that condition* (3.1) *holds and* $\mathbf{A} \in \Omega_1 \cup \Omega_2$. *We have the following statements about the non-regular point* $\mathbf{p}_0$.
(a) When $\mathbf{A} \in \Omega_1$, *about the non-regular point* $\mathbf{p}_0$, *we have*

- (a1) *As* $x_e^- = \mu_1$ *or* $x_e^- = \mu_2$, $\mathbf{p}_0$ *is a boundary equilibrium point;*
- (a2) *As* $\min\{\mu_1, \mu_2\} < x_e^- < \max\{\mu_1, \mu_2\}$, $\mathbf{p}_0$ *is a sliding boundary point;*
- (a3) *As* $x_e^- > \max\{\mu_1, \mu_2\}$, $\mathbf{p}_0$ *is a crossing point;*

(a4) *When* $a_{12} > 0$ *(*$a_{12} < 0$*),* $\mathbf{p}_0$ *is a non-regular boundary saddle (a non-regular boundary sink), a stable pseudo-node (a pseudo-saddle-node), a non-regular boundary saddle (a non-regular boundary source) for* $x_e^- < -x_e^+$*,* $x_e^- = -x_e^+$ *and* $-x_e^+ < x_e^- < \min\{\mu_1, \mu_2\}$*, respectively.*

(b) When $\mathbf{A} \in \Omega_2$*, about the non-regular point* $\mathbf{p}_0$*, we have*

(b1) *As* $x_e^- = \mu_1$ *or* $x_e^- = \mu_2$*,* $\mathbf{p}_0$ *is a boundary equilibrium point;*
(b2) *As* $x_e^- > \mu_1$*,* $\mathbf{p}_0$ *is a crossing point;*
(b3) *When* $x_e^+ + \mu_2 > 0$ *(*$x_e^+ + \mu_2 < 0$*),* $\mathbf{p}_0$ *is a non-regular boundary sink (a sliding boundary point), a pseudo-saddle-node (a stable pseudo-node), a non-regular boundary source (a non-regular boundary saddle), a sliding boundary point (a non-regular boundary saddle) for* $x_e^- < \min\{-x_e^+, \mu_2\}$*,* $x_e^- = -x_e^+$*,* $\min\{-x_e^+, \mu_2\} < x_e^- < \max\{-x_e^+, \mu_2\}$ *and* $\max\{-x_e^+, \mu_2\} < x_e^- < \mu_1$*, respectively.*

Proof. To discuss the non-regular point $\mathbf{p}_0$, we need the following statements:

$$\mu_2 - \mu_1 = -\frac{\det(\mathbf{A})}{a_{11}a_{21}} y_e^+, \quad \mu_2 - (-x_e^+) = \frac{a_{12}}{a_{11}} y_t^+, \quad \mu_1 - (-x_e^+) = x_t^+ > 0.$$

Then it is easy to see that when $\mathbf{A} \in \Omega_1$, we obtain

$$\begin{gathered} -x_e^+ < 0 < \mu_1 < \mu_2, \quad \text{for } a_{12} > 0; \\ -x_e^+ < 0 < \mu_2 < \mu_1, \quad \text{for } a_{12} < 0. \end{gathered}$$

And when $\mathbf{A} \in \Omega_2$, we obtain

$$-x_e^+ < \mu_2 < 0 < \mu_1, \quad \text{for } x_e^+ + \mu_2 > 0; \tag{3.15}$$

$$\mu_2 < -x_e^+ < 0 < \mu_1, \quad \text{for } x_e^+ + \mu_2 \le 0. \tag{3.16}$$

And for $x_e^- = -x_e^+$, it is easy to see $\mathbf{p}_{ex} = \mathbf{p}_{ey} = \mathbf{p}_0$. Thus by Proposition 3.1 and Proposition 3.2, the statements in this proposition can be obtained directly. □

Before we give the global qualitative dynamics of system (1.1), we denote

$$\begin{gathered} \mu_3 := x_e^+ + \frac{a_{22} - a_{11}}{a_{21}} y_e^+, \quad \mu_4 := x_e^+ + \frac{a_{22}(\lambda_2 - a_{11}) - a_{12}a_{21}}{a_{21}(\lambda_2 - a_{11})} y_e^+, \\ \mu_5 := x_e^+ - \frac{2a_{12}}{\lambda_2 - a_{11}} y_e^+, \quad \mu_6 := \frac{(\lambda_2 - a_{11})x_e^+ - 2a_{12}y_e^+}{\lambda_1 - a_{11}}, \\ \mu_7 := -\frac{a_{11}x_e^+ + 2a_{12}y_e^+}{\lambda_1 - a_{11}}, \quad \mu_8 := \frac{2a_{12}y_e^+ - (\lambda_2 - a_{11})}{a_{11}}, \\ \mu_9 := -\frac{a_{12}}{\lambda_1 - a_{11}} y_e^+. \end{gathered} \tag{3.17}$$

And by simple computations, it follows that

$$\begin{gathered} x_e^- = \mu_3 \Leftrightarrow x_{m2}^+ = x_{m1}^-, \quad x_e^- = \mu_4 \Leftrightarrow x_{m2}^+ = x_t^-, \\ x_e^- = \mu_5 \Leftrightarrow x_{m2}^+ = x_{m2}^-, \quad x_e^- = \mu_6 \Leftrightarrow y_{m1}^- = y_{m2}^+, \\ x_e^- = \mu_7 \Leftrightarrow y_{m1}^- = y_t^+, \quad x_e^- = \mu_8 \Leftrightarrow y_t^- = y_{m2}^+, \\ x_e^- = \mu_9 \Leftrightarrow y_{m1}^- = 0. \end{gathered} \tag{3.18}$$

Secondly, to make our analysis of the dynamics more concrete, without loss of generality, we assume that

$$x_{m1}^{+} > 0 \Leftrightarrow 0 < y_e^{+} < \frac{\lambda_1 - a_{11}}{a_{12}} x_e^{+}. \tag{3.19}$$

Since the discussion with $x_{m1}^{+} \le 0$ is similar, so we omit it.

Theorem 3.4. *Suppose that condition* (3.1) *holds and* $\mathbf{A} \in \Omega_1 \cup \Omega_2$. *We have*

$$\max\{2\mu_1 + x_e^{+}, \mu_3\} < \mu_4 < \mu_5$$

and the following statements hold.

- (a) *When* $x_e^{-} = \mu_3$, *there exists one heteroclinic cycle;*
- (b) *When* $\min\{2\mu_1 + x_e^{+}, \mu_3\} < x_e^{-} < \max\{2\mu_1 + x_e^{+}, \mu_3\}$, *there exists one limit cycle, which is repulsive (attractive) if* $\mu_3 < 2\mu_1 + x_e^{+}$ ($\mu_3 > 2\mu_1 + x_e^{+}$)*;*
- (c) *When* $x_e^{-} = \mu_4$ *and* $x_e^{-} = \mu_5$, *there both exist two sliding heteroclinic orbits.*

Theorem 3.5. *Suppose that condition* (3.1) *holds and* $\mathbf{A} \in \Omega_1$, *the following statements are true.*

(a) When $a_{11} > 0$, *we have*

$$\mu_6 < \mu_7 < 0 < \mu_1 < \mu_3 < 2\mu_1 + x_e^{+} < \mu_4 < \mu_5$$

and the following statements hold.

- (a1) *When* $x_e^{-} < \min\{-x_e^{+}, \mu_6\}$, *there exists one sliding heteroclinic orbit;*
- (a2) *When* $x_e^{-} = \mu_6$, *there exist one heteroclinic orbit and one sliding heteroclinic orbit;*
- (a3) *When* $x_e^{-} = -x_e^{+}$, *if* $x_e^{-} < \mu_6$, *there exists one sliding heteroclinic orbit; Or there exist two sliding heteroclinic orbits;*
- (a4) *When* $\min\{-x_e^{+}, \mu_6\} < x_e^{-} < \mu_3$, *there exist two sliding heteroclinic orbits.*

(b) When $a_{11} < 0$, *we have*

$$-x_e^{+} < 0 < \mu_9 < \min\{2\mu_1 + x_e^{+}, \mu_3\} < \max\{2\mu_1 + x_e^{+}, \mu_3\} < \mu_4 < \mu_5$$

and the following statements.

- (b1) *When* $x_e^{-} < -x_e^{+}$, *there exists one sliding orbit containing* $\mathbf{p}_{ey}$, $\mathbf{p}_0$ *and* $\mathbf{x}_e^{+}$*;*
- (b2) *When* $-x_e^{+} \le x_e^{-} < \mu_9$, *there exists one sliding heteroclinic orbit;*
- (b3) *When* $\mu_9 \le x_e^{-} < \min\{2\mu_1 + x_e^{+}, \mu_3\}$, *there exist two sliding heteroclinic orbits.*

Theorem 3.6. *Suppose that condition* (3.1) *holds and* $\mathbf{A} \in \Omega_2$, *the following statements are true.*

(a) When $a_{11} < 0$ *and* $0 < y_e^{+} < -\frac{a_{11}}{a_{12}} x_e^{+}$, *we have*

$$\mu_8 < \min\{\mu_6, \mu_2\} < 0 < \mu_1 < \max\{2\mu_1 + x_e^{+}, \mu_3\} < \mu_4 < \mu_5$$

and the following statements hold.

- (a1) *When* $x_e^{-} < \min\{\mu_8, -x_e^{+}\}$, *there exists one sliding orbit containing* $\mathbf{x}_e^{+}$, $\mathbf{p}_0$ *and* $\mathbf{p}_{ey}$*;*
- (a2) *When* $x_e^{-} = \mu_8$, *there exists one sliding cycle;*
- (a3) *When* $x_e^{-} = \mu_6$, *there exists one sliding heteroclinic orbit;*
- (a4) *If* $\mu_8 < -x_e^{+}$, *we obtain*
  - (a41) *When* $\mu_8 < x_e^{-} < -x_e^{+}$, *there exists one sliding cycle;*
  - (a42) *When* $x_e^{-} = -x_e^{+}$, *there exist one sliding homoclinic cycle and one sliding heteroclinic orbit;*

(a5) *When* $\min\{-x_e^+, \mu_6, \mu_2\} < x_e^- < \min\{2\mu_1 + x_e^+, \mu_3\}$*, there exist two sliding heteroclinic orbits and at most one sliding cycle.*

(b) When $a_{11} < 0$ *and* $-\frac{a_{11}}{a_{12}} x_e^+ < y_e^+ < \frac{\lambda_1 - a_{11}}{a_{12}} x_e^+$*, we have*

$$\mu_8 < \mu_6 < \mu_7, \quad \mu_8 < 2\mu_2 + x_e^+ < \mu_2 < -x_e^+ < 0 < \max\{2\mu_1 + x_e^+, \mu_3\} < \mu_4 < \mu_5$$

and the following statements hold.

(b1) *When* $x_e^- = \mu_8$*, there exists one sliding heteroclinic orbit;*

(b2) *When* $x_e^- = \mu_6$*, there exists one heteroclinic orbit;*

(b3) *If* $2\mu_2 + x_e^+ < \mu_6 < \mu_2$*, when* $\mu_8 < x_e^- < 2\mu_2 + x_e^+$*, there exists one sliding homoclinic cycle;*

(b4) *If* $\mu_6 < 2\mu_2 + x_e^+$*, when* $\mu_8 < x_e^- < 2\mu_2 + x_e^+$*, there exist a sliding cycle bifurcation and a sliding homoclinic bifurcation;*

(b5) *When* $2\mu_2 + x_e^+ < x_e^- < \min\{2\mu_1 + x_e^+, \mu_3\}$*, there exist two sliding heteroclinic orbits.*

There is no other separatrix of system (1.1) except for the separatrixes given in the above theorems.

4. Proofs of main results

Proof of Theorem 3.4. By (2.5) and (2.6), it is easy to see that x_{m1}^- x_{m2}^- and x_t^- increase with respect to x_e^-. And by (3.19) and simple calculation, it follows that for $\mathbf{A} \in \Omega_1 \cup \Omega_2$, we have

$$x_{m2}^- < x_t^- < x_{m1}^-, \ 0 < x_{m1}^+ < x_t^+ < x_{m2}^+. \tag{4.1}$$

Then $\mu_3 < \mu_4 < \mu_5$ and $2\mu_1 + x_e^+ < \mu_4$ can be obtained directly since (3.7) and (3.18), which means $\max\{\mu_3, 2\mu_1 + x_e^+\} < \mu_4 < \mu_5$.

As $x_e^- = \mu_3$, we know $x_{m1}^{\mp} = x_{m2}^{\pm}$, it is also easy to see $x_{m1}^+ < \min\{x_t^+, x_t^-\} < \max\{x_t^+, x_t^-\} < x_{m2}^+$ by (3.19). So there is a heteroclinic cycle connecting $\mathbf{x}_e^+$ with $\mathbf{x}_e^-$.

In addition, by simple calculation, it follows that

$$\mu_3 - (2\mu_1 + x_e^+) = -\frac{y_e^+}{a_{21}}(a_{11} + a_{22}).$$

Then for $\mathbf{A} \in \Omega_1 \cup \Omega_2$, we obtain that

$$\begin{aligned} &\mu_3 > 2\mu_1 + x_e^+, \quad \text{if } a_{11} + a_{22} < 0; \\ &\mu_3 < 2\mu_1 + x_e^+, \quad \text{if } a_{11} + a_{22} > 0. \end{aligned} \tag{4.2}$$

For $\mu_3 < x_e^- < 2\mu_1 + x_e^+$, we have $x_{m1}^+ < x_{m2}^- < x_t^- < x_t^+ < x_{m2}^+ < x_{m1}^-$. And by Proposition 3.2, $\mathbf{p}_{ex} \in \Sigma_s^x$ is stable. Then there must exist a repulsive limit cycle $\mathcal{L}_1$. To prove this, we need to construct the Poincaré map as follows:

$$\mathcal{P}_1 : D_{\mathcal{P}_1} \mapsto D_{\mathcal{P}_1}, \quad \mathcal{P}_1(x_0, 0) = (x_1, 0),$$

where $D_{\mathcal{P}_1} = \{(x, 0) : x \in [x_t^+, x_{m2}^+]\}$, and $(x_1, 0)$ is the first arriving point at $D_{\mathcal{P}_1}$ of the orbit of system (1.1) with the initial point $(x_0, 0) \in D_{\mathcal{P}_1}$. Since $x_t^- < x_t^+ < x_{m2}^+$, there must exist $(x_1^\star, 0)$, $(x_2^\star, 0) \in D_{\mathcal{P}_1}$ and $x_t^+ < x_2^\star < x_1^\star < x_{m2}^+$, such that

$$\mathcal{P}_1(x_1^\star, 0) = (x_{m2}^+, 0), \quad \mathcal{P}_1(x_2^\star, 0) = (x_t^+, 0).$$

This means there must be $(x^\star, 0) \in D_{\mathcal{P}_1}$, such that

$$\mathcal{P}_1(x^\star, 0) = (x^\star, 0),$$

which implies the existence of the repulsive limit cycle $\mathcal{L}_1$. Similarly, for $\mu_3 < x_e^- < 2\mu_1 + x_e^+$, we can prove the existence of an attractive limit cycle $\mathcal{L}_2$.

As $x_e^- > \max\{2\mu_1 + x_e^+, \mu_3\}$, we have $x_t^+ < x_t^-$ and $x_{m1}^- > x_{m2}^+$. Then according to Proposition 3.1 and Proposition 3.2, it follows that $\Sigma_s^x = \{(x, 0) : x_t^+ < x < x_t^-\}$ is repulsive and $\mathbf{p}_{ex}$ is unstable. So there is no other special dynamics expect for two sliding heteroclinic orbits $\mathcal{L}_3$ and $\mathcal{L}_4$, both of which connect $\mathbf{x}_e^\pm$ and $\mathbf{p}_{ex}$ as $x_e^- = \mu_4$ and $x_e^- = \mu_5$, respectively. The proof is complete. □

Proof of Theorem 3.5. (a) In this case, since $a_{11} > 0$ and (3.19), we obtain that

$$y_{m2}^+ > y_t^+ > y_e^+ > 0 > y_{m1}^+, \quad 0 < x_{m1}^+ < x_e^+ < x_t^+ < x_{m2}^+.$$

By (2.6), it is obvious that y_{m1}^- decreases with respect to x_e^-. So we obtain $\mu_6 < \mu_7 < 0$ directly because of (3.18). Moreover, when $x_e^- = \mu_5$, we have $x_{m1}^\mp = x_{m2}^\pm$, which means $x_t^- > 0$. So we obtain that $0 < \mu_1 < \mu_3$. Since (4.2), it is easy to see $\mu_3 < 2\mu_1 + x_e^+$. Then by Theorem 3.4, we obtain that

$$\mu_6 < \mu_7 < 0 < \mu_1 < \mu_3 < 2\mu_1 + x_e^+ < \mu_4 < \mu_5.$$

In addition, by simple calculation, it follows that

$$\mu_6 - (-x_e^+) = \frac{(a_{22} - a_{11})x_e^+ - 2a_{12}y_e^+}{\lambda_1 - a_{11}}, \quad \mu_7 - (-x_e^+) = \frac{\lambda_2 - a_{11} - 2a_{12}y_e^+}{\lambda_1 - a_{11}}, \tag{4.3}$$

the signs of which are not sure. Then $-x_e^+ < \mu_6 < \mu_7 < 0$, $\mu_6 < -x_e^+ < \mu_7 < 0$ or $\mu_6 < \mu_7 < -x_e^+ < 0$ could happen.

Firstly, according to Proposition 3.1, Proposition 3.2 and Remark 2.6, it is obvious to see that there is a sliding heteroclinic orbit $\mathcal{M}_1$ from $\mathbf{x}_e^+$ to $\mathbf{p}_{ey}$ for $x_e^- \le -x_e^+$, and a sliding heteroclinic orbit $\mathcal{M}_2$ from $\mathbf{x}_e^+$ to $\mathbf{p}_{ex}$ for $x_e^- > -x_e^+$. Moreover, a heteroclinic orbit $\mathcal{M}_3$ from $\mathbf{x}_e^-$ to $\mathbf{x}_e^+$ appears only as $x_e^- = \mu_6$ obviously. For $\mu_6 < x_e^- < \mu_7$, the orbit of system (1.1) starting from $(0, y_{m1}^-)$ will arrive at the sliding set Σ_s. And for $\mu_7 \le x_e^- < \mu_3$, the invariant manifold l_1^- of $\mathbf{x}_e^-$ will intersects with Σ_s. So there is a sliding heteroclinic orbit $\mathcal{M}_4$ from $\mathbf{x}_e^-$ to the unique pseudo-equilibrium point $\mathbf{p}_{ex}$ or $\mathbf{p}_{ey}$ for $\mu_6 < x_e^- < \mu_3$.

Thus we obtain that $\mathcal{M}_1$ exists for $x_e^- < \min\{-x_e^+, \mu_6\}$. When $\min\{-x_e^+, \mu_6\} < x_e^- < \mu_3$, there exist two sliding heteroclinic orbits $\mathcal{M}_2$ and $\mathcal{M}_4$. As $x_e^- = \mu_6$, there exists the heteroclinic orbit $\mathcal{M}_3$ and the sliding heteroclinic orbit $\mathcal{M}_4$.

(b) Since $a_{11} < 0$, we obtain that

$$x_{m2}^- < x_t^- < x_{m1}^- < x_e^-, \quad 0 < x_e^+ < x_{m1}^+ < x_t^+ < x_{m2}^+.$$

By (2.6) and (3.18), we know $x_{m1}^- = y_{m1}^- = 0$ as $x_e^- = \mu_9$, then $0 < \mu_9 < \min\{2\mu_1 + x_e^+, \mu_3\}$ can be obtained directly. And by Theorem 3.4, we have

$$-x_e^+ < 0 < \mu_9 < \min\{2\mu_1 + x_e^+, \mu_3\} < \max\{2\mu_1 + x_e^+, \mu_3\} < \mu_4 < \mu_5.$$

According to Propositions 3.1, 3.2 and 3.3, it is easy to see there exists a sliding orbit connecting $\mathbf{x}_e^+$, $\mathbf{p}_{ey}$ and the non-regular boundary sink $\mathbf{p}_0$ for $x_e^- < -x_e^+$, which will become a sliding heteroclinic orbit from $\mathbf{x}_e^+$, $\mathbf{p}_{ex}$ for $-x_e^+ \le x_e^- < \min\{2\mu_1 + x_e^+, \mu_3\}$. Once $x_e^- = \mu_9$, there appears a sliding heteroclinic orbit from $\mathbf{x}_e^-$ to $\mathbf{p}_{ex}$, which will persist until $x_e^- = \min\{2\mu_1 + x_e^+, \mu_3\}$. The proof is complete. □

Proof of Theorem 3.6. When $\mathbf{A} \in \Omega_2$, it is not difficult to verify that the case of matrix $\mathbf{A}$ with $a_{21} > 0$, $a_{22} > 0$, $a_{12} < 0$, $a_{11} > 0$ is invalid since $\det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21} > 0$, which contradicts that $\det(\mathbf{A}) < 0$ in the condition (3.1). So we just need to give the discussion of system (1.1) with $\mathbf{A} \in \Omega_2$ and $a_{11} < 0$ under assumption (3.19) as follows.

(a) When $a_{11} < 0$ and $0 < y_e^+ < -\frac{a_{11}}{a_{12}} x_e^+$, we know that $y_t^+ < 0$ by (3.9). In addition, we obtain that

$$y_{m2}^+ > y_e^+ > 0 > y_t^+ > y_{m1}^+, \quad 0 < x_{m1}^+ < x_e^+ < x_t^+ < x_{m2}^+.$$

Note that $y_{m1}^- > y_t^-$ and both of them decrease with respect to x_e^- by (2.5) and (2.6). Then as $x_e^- = \mu_8$, we must have $y_{m1}^- > y_t^- = y_{m2}^+ > 0$, which means $\mu_8 < \mu_6 < 0$ and $\mu_8 < \mu_2 < 0$ since (3.8) and (3.18). Moreover, by simple calculation, it follows that

$$\begin{aligned} \mu_8 - (-x_e^+) &= \frac{2a_{11}y_e^+ - (\lambda_2 - 2a_{11})x_e^+}{a_{11}}, \\ \mu_2 - \mu_6 &= \frac{(\lambda_1 + a_{11})a_{12}y_e^+ - a_{11}(\lambda_2 - a_{11})x_e^+}{(\lambda_1 - a_{11})a_{11}}. \end{aligned} \tag{4.4}$$

And by (4.3), we know the signs of $\mu_8 - (-x_e^+)$, $\mu_2 - \mu_6$ and $\mu_6 - (-x_e^+)$ are not sure. Then by (3.15) and Theorem 3.4, we know

$$\mu_8 < \min\{\mu_6, \mu_2\} < 0 < \mu_1 < \max\{2\mu_1 + x_e^+, \mu_3\} < \mu_4 < \mu_5.$$

Meanwhile, we obtain that $-x_e^+ < \mu_8 < \min\{\mu_6, \mu_2\}$, $\mu_8 < -x_e^+ < \min\{\mu_6, \mu_2\}$ or $\mu_8 < \mu_6 < -x_e^+ < \mu_2$ could take place.

By Propositions 3.1, 3.2 and 3.3, and Remark 2.6, for $x_e^- < \min\{\mu_8, -x_e^+\}$, there is a sliding orbit $\mathcal{J}_1$ containing $\mathbf{p}_{ey}$, $\mathbf{x}_e^+$ and $\mathbf{p}_0$, which is a non-regular boundary sink. As $x_e^- = \mu_8$, if $\mu_8 < -x_e^+$, there is a sliding cycle $\mathcal{J}_2$ connecting $\mathbf{p}_{ey}$, $\mathbf{x}_e^+$ and $\mathbf{p}_0$, which is a non-regular boundary sink; If $\mu_8 > -x_e^+$, there is a sliding cycle $\mathcal{J}_3$ connecting $\mathbf{p}_{ex}$, $\mathbf{x}_e^+$ and $\mathbf{p}_0$, which is a non-regular boundary source. Moreover, as $x_e^- = \mu_6$, there is a heteroclnic orbit $\mathcal{J}_4$ from $\mathbf{x}_e^-$ to $\mathbf{x}_e^+$.

As $\mu_8 < x_e^- < \mu_3$, the orbits of $\oplus$-system starting from $(0, y_t^-)$ will arrive at Σ_s^x since $y_t^+ < 0$ and Σ_s^x is attractive before $x_e^- = \mu_3$. Then on the one hand, if $\mu_8 < -x_e^+$, for $\mu_8 < x_e^- < -x_e^+$, $\mathcal{J}_2$ will become a sliding cycle $\mathcal{J}_{21}$ connecting $\mathbf{p}_{ey}$ and the non-regular boundary sink $\mathbf{p}_0$. Then $\mathcal{J}_{21}$ will turn into a sliding homoclinic cycle $\mathcal{J}_{22}$ as $x_e^- = -x_e^+$, which containing the pseudo-saddle-node $\mathbf{p}_{ex} = \mathbf{p}_{ey} = \mathbf{p}_0$, meanwhile, a sliding heteroclinic orbit $\mathcal{J}_5$ from $\mathbf{x}_e^+$ to $\mathbf{p}_{ex}$ occurs, which will exist until $x_e^- = \min\{2\mu_1 + x_e^+, \mu_3\}$. And $\mathcal{J}_{22}$ will turn into a sliding cycle $\mathcal{J}_{23}$ connecting $\mathbf{p}_{ex}$ and the non-regular boundary source $\mathbf{p}_0$ as long as the first arriving point on Σ_s^x of the orbit of $\oplus$-system starting from $(0, y_t^-)$ is on the right of $\mathbf{p}_{ex}$. However, if $\mu_8 > -x_e^+$, $\mathcal{J}_3$ can only becomes $\mathcal{J}_{23}$ before $x_e^- = \mu_3$.

On the other hand, if $\mu_6 < -x_e^+ < \mu_2$, we have as $\mu_6 < x_e^- < \mu_2$, $\mathcal{J}_4$ will firstly become a sliding heteroclinic orbit $\mathcal{J}_{41}$ from $\mathbf{x}_e^-$ to $\mathbf{p}_{ey}$ for $x_e^- < -x_e^+$, then $\mathcal{J}_{41}$ will become a sliding heteroclinic orbit $\mathcal{J}_{42}$ from $\mathbf{x}_e^-$ to $\mathbf{p}_{ex}$ for $x_e^- \geq -x_e^+$, which will persist until $x_e^- = \min\{2\mu_1 + x_e^+, \mu_3\}$; If $-x_e^+ < \min\{\mu_6, \mu_2\}$, we obtain that $\mathcal{J}_4$ will only become $\mathcal{J}_{42}$.

(b) In this case, we know $y_t^+ > 0$ by (3.9). In addition, we obtain that

$$y_{m2}^+ > y_e^+ > y_t^+ > 0 > y_{m1}^+, \quad y_{m1}^- > y_t^- < y_{m2}^-.$$

Then since y_t^- decreases with respect to x_e^-, we obtain $\mu_8 < 2\mu_2 + x_e^+ < \mu_2$ directly by (3.8) and (3.18). Note that $\mu_2 < -x_e^+$ by (3.16). So by Theorem 3.4, we obtain

$$\mu_8 < 2\mu_2 + x_e^+ < \mu_2 < -x_e^+ < 0 < \max\{2\mu_1 + x_e^+, \mu_3\} < \mu_4 < \mu_5.$$

By (3.8) and (3.18), it easy to see that $\mu_8 < \mu_6 < \mu_7$ and $2\mu_3 + x_e^+ < \mu_7$. However, the signs of $\mu_7 - (-x_e^+)$ and $\mu_2 - \mu_6$ in this case are not sure by (4.3) and (4.4), respectively. And by easy computation, we obtain that

$$2\mu_2 + x_e^+ - \mu_6 = \frac{\lambda_1 - \lambda_2}{\lambda_1 - a_{11}} x_e^+ + \frac{2a_{12}\lambda_1}{a_{11}(\lambda_1 - a_{11})} y_e^+,$$

which implies $2\mu_2 + x_e^+ < \mu_6$ and $2\mu_2 + x_e^+ > \mu_6$ both may be true.

By Propositions 3.1, 3.2 and 3.3, and Remark 2.6, we see that there is no special separatrixes as $x_e^- < \mu_8$. As $x_e^- = \mu_8$, there exists a sliding heteroclinic orbit from $\mathbf{p}_{ey}$ to $\mathbf{x}_e^+$. In addition, it is obvious that a heteroclinic orbit from $\mathbf{x}_e^-$ to $\mathbf{x}_e^+$ appears as $x_e^- = \mu_6$. Furthermore, as $x_e^- > \max\{\mu_6, \mu_2\}$, the orbit of $\oplus$-system starting from $\mathbf{y}_{m1}^-$ will arrive Σ_s. Thus there will exist a sliding heteroclinic orbit from $\mathbf{x}_e^-$ to $\mathbf{p}_{ey}$ if $x_e^- < -x_e^+$, or from $\mathbf{x}_e^-$ to $\mathbf{p}_{ex}$ if $x_e^- \geq -x_e^+$ until $x_e^- = \min\{2\mu_1 + x_e^+, \mu_3\}$.

In addition to this, when $x_e^- < 2\mu_2 + x_e^+$, we have $y_t^- > y_t^+ > 0$, then let $\mathbf{y}^1 = (0, y^1)$ and $\mathbf{y}^2 = (0, y^2)$ be the first arriving point on the positive y-axis of the forward orbits of $\ominus$-system starting from $\mathbf{y}_t^+$ and $\mathbf{p}_{ey}$, respectively. And when $x_e^- < \mu_2$, denote by $\mathbf{y}^- = (0, y^-)$ and $\mathbf{y}^+ = (0, y^+)$ the first arriving point on the positive y-axis of the forward and backward orbits of $\ominus$-system and $\oplus$-system with initial condition $\mathbf{p}_0$, respectively. Then when $x_e^- < 2\mu_2 + x_e^+ < \mu_2$, we must have $y_{m1}^- > y^- > y^1 > y^2$. Since $\mathbf{y}^+ = (0, y^+)$ and $\mathbf{y}_{m2}^+ = (0, y_{m2}^+)$ will not change with the value of x_e^-, moreover, $y_{m2}^+ > y^+$, so there must be $\xi_1 < \xi_2$, $\xi_1 < \mu_6$, $\xi_2 < \mu_7$ such that

$$\begin{aligned} y^- &= y_{m2}^+, \quad \text{if } x_e^- = \xi_1\,; \\ y^- &= y^+, \quad \text{if } x_e^- = \xi_2\,. \end{aligned}$$

This mean there exist a sliding homoclinic cycle containing $\mathbf{x}_e^+$ called $\mathcal{H}_1$ and a sliding-zero cycle containing $\mathbf{p}_0$ called $\mathcal{H}_2$ as $x_e^- = \xi_1$ and $x_e^- = \xi_2$, respectively.

Apart from this, one the one hand, if $\mu_6 < 2\mu_2 + x_e^+ < \mu_2$, there must be $\xi_{11} < \xi_{12} < \xi_1$, such that

$$\begin{aligned} y^1 &= y_{m2}^+, \quad \text{if } x_e^- = \xi_{11}; \\ y^2 &= y_{m2}^+, \quad \text{if } x_e^- = \xi_{12}, \end{aligned}$$

which mean there both appears a sliding heteroclinc orbit $\mathcal{H}_3$ from $\mathbf{p}_{ey}$ to $\mathbf{x}_e^+$ as $x_e^- = \xi_{11}$ and $x_e^- = \xi_{12}$, that is $\mathcal{H}_3$ and $\mathcal{H}_1$ will appear in turn before $x_e^- = \mu_6$; If $2\mu_2 + x_e^+ < \mu_6 < \mu_2$, it is easy to see only $\mathcal{H}_1$ will appear before $x_e^- = \mu_6$; If $2\mu_2 + x_e^+ < \mu_2 < \mu_6$, neither $\mathcal{H}_3$ nor $\mathcal{H}_1$ will appear before $x_e^- = \mu_6$ since $\mathbf{p}_0 \in \Sigma_s$. On the other hand, because of the sign of $2\mu_2 + x_e^+ - \mu_6$ is not sure, so the size relationship of ξ_2 with $2\mu_2 + x_e^+$ is uncertain. Then there may exist some special separatrixes, which will be discussed as follows:

If $y^- > y^+$ is true for $x_e^- < 2\mu_2 + x_e^+$, there must be a sliding cycle $\mathcal{H}_4$ containing $\mathbf{p}_0$ before $y^- = y^+$ since the orbit of $\oplus$-system starting from $\mathbf{y}^-$ will arrive at Σ_s^x. If $y^- > y^+$ is true for $2\mu_2 + x_e^+ < x_e^- < \mu_2$, $\mathcal{H}_4$ must exist, meanwhile, there must exist a repulsive limit cycle $\mathcal{H}_5$ before $x_e^- = \xi_2$. To prove this, we need constructing the Poincaré map as follows:

$$\mathcal{P}_2 : D_{\mathcal{P}_2} \mapsto D_{\mathcal{P}_2}, \quad \mathcal{P}_2(0, y_0) = (0, y_1),$$

where $D_{\mathcal{P}_2} = \{(0, y) : y \in [y_t^-, y^-]\}$, and $(0, y_1)$ is the first arriving point at $D_{\mathcal{P}_2}$ of the forward orbit of system (1.1) starting from $(0, y_0)$. Then we must have $\mathcal{P}_2((0, y^+)) = (0, y^-)$. And there must be $(0, y_1^*)$ and $(0, y_2^*)$, which satisfy

$y_t^+ < y_2^* < y_1^* < y^+$, such that $\mathcal{P}_2(0, y_1^*) = (0, y_2^*)$ since $\mathbf{p}_{ey}$ is atable. So there must exist $y_1^* < y_p^* < y^+$, such that

$$\mathcal{P}_2(0, y_p^*) = (0, y_p^*).$$

which implies the existence of the repulsive $\mathcal{H}_5$.

Whether $y^- = y^+$ is true for $x_e^- < 2\mu_2 + x_e^+$ or $2\mu_2 + x_e^+ < x_e^- < \mu_2$, it is obvious that there must arise a sliding-zero cycle $\mathcal{H}_2$.

If $y^- < y^+$ is true for $x_e^- < 2\mu_2 + x_e^+$, we obtain a attractive limit cycle $\mathcal{H}_6$ before $x_e^- = 2\mu_2 + x_e^+$. To prove this, we need constructing the Poincar$\acute{e}$ map as follows:

$$\mathcal{P}_3 : D_{\mathcal{P}_3} \mapsto D_{\mathcal{P}_3}, \quad \mathcal{P}_3(0, y_0) = (0, y_1),$$

where $D_{\mathcal{P}_3} = \{(0, y) : y \in [0, y_t^+]\}$, and $(0, y_1)$ is the first arriving point at $D_{\mathcal{P}_3}$ of the forward orbit of system (1.1) starting from $(0, y_0)$. Thus there must be $(0, y_1^\diamond)$ and $(0, y_2^\diamond)$, $0 < y_1^\diamond < y_2^\diamond < y_t^+$, such that

$$\mathcal{P}_3(0, 0) = (0, y_1^\diamond), \quad \mathcal{P}_3((0, y_t^+)) = (0, y_2^\diamond),$$

since $y^- < y^+$ and $0 < y_t^+ < y_t^-$, respectively. So there must be $(0, y_p^*)$, such that $y_1^\diamond < y_p^\diamond < y_2^\diamond$

$$\mathcal{P}_3(0, y_p^\diamond) = (0, y_p^\diamond).$$

which implies the existence of the attractive $\mathcal{H}_6$. If $y^- < y^+$ is true for $2\mu_2 + x_e^+ < x_e^- < \mu_2$, there must exist a sliding heteroclinic orbit $\mathcal{H}_{71}$ from $\mathbf{x}_e^+$ to $\mathbf{p}_{ey}$ until $x_e^- = -x_e^+$ since Σ_s^y is attractive. Then $\mathcal{H}_{71}$ turns into a sliding heteroclinic orbit $\mathcal{H}_{72}$ from $\mathbf{x}_e^+$ to $\mathbf{p}_{ex}$ until $x_e^- = \min\{2\mu_1 + x_e^+, \mu_3\}$.

Thus according to the above analysis, we obtain $\mathcal{H}_4$, $\mathcal{H}_2$ and $\mathcal{H}_6$ will appears in turn if $y^- < y^+$ is true for $x_e^- < 2\mu_2 + x_e^+$, which implies that there is a sliding cycle bifurcation as $x_e^- = \xi_2$. In addition, $\mathcal{H}_1$ and $\mathcal{H}_4$ will appear in turn if $y^- > y^+$ is true for $x_e^- < 2\mu_2 + x_e^+$, which implies that there is a sliding homoclinic bifurcation as $x_e^- = \xi_1$. The proof is complete. □

5. Examples

In this section, three examples are provided to illustrate some separatrixes in the theorems in Section 3. It should be noted that the red and the blue curves in the following figures represent the orbits of the ⊕-system and the ⊖-system, respectively. And the arrows represents the directions of the orbits on the forward time.

Example 5.1. Let

$$A = \begin{pmatrix} -4 & 2 \\ 3 & 1 \end{pmatrix}, \quad X_e^- = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, \quad X_e^+ = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \tag{5.1}$$

By simple calculations, we have

$$x_t^+ = 3, \quad x_{m1}^+ = 1, \quad x_{m2}^+ = 8, \quad x_t^- = 4, \quad x_{m1}^- = 6, \quad x_{m2}^- = -1,$$
$$2\mu_1 + x_e^+ = 4 < \mu_3 = 7.$$

Through numerical simulation, there exists an attractive limit cycle illustrated in Figure 2, which supported the conclusion about $\mathcal{L}_2$ in Theorem 3.4.

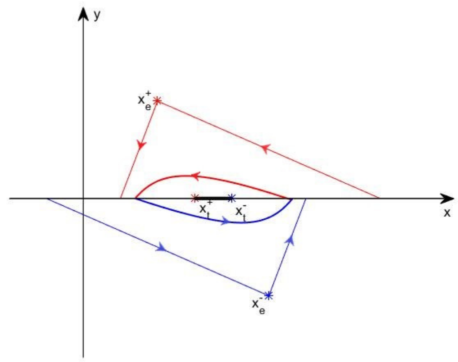

FIGURE 2. Existence of the attractive limit cycle of equation (5.1)

Example 5.2. Let

$$A = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}, \quad X_e^- = \begin{pmatrix} 9 \\ -3 \end{pmatrix}, \quad X_e^+ = \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \tag{5.2}$$

By simple calculations, we have

$$x_t^+ = 6, \quad x_{m1}^+ = 1, \quad x_{m2}^+ = 8, \quad x_t^- = 5, \quad x_{m1}^- = 10, \quad x_{m2}^- = 3,$$
$$2\mu_1 + x_e^+ = 10 > \mu_3 = 7.$$

Through numerical simulation, there exists a repulsive limit cycle illustrated in Figure 3, which supported the conclusion about $\mathcal{L}_1$ in Theorem 3.4.

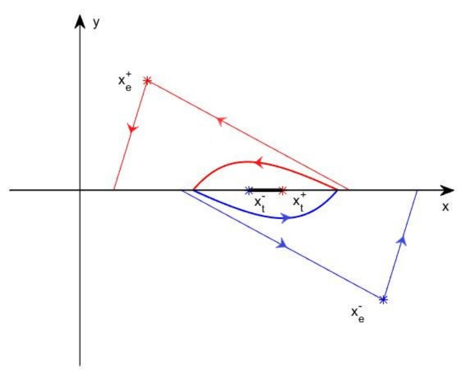

FIGURE 3. Existence of the repulsive limit cycle of equation (5.2)

Example 5.3. Let

$$A = \begin{pmatrix} -4 & 2 \\ 3 & 1 \end{pmatrix}, \quad X_e^- = \begin{pmatrix} \mu \\ -10 \end{pmatrix}, \quad X_e^+ = \begin{pmatrix} 4 \\ 10 \end{pmatrix}. \tag{5.3}$$

For $-9 < \mu < -6$, we will show there exist a sliding homoclinic bifurcation and a sliding cycle bifurcation. By easy calculations, it follows that

$$x_t^+ = 22/3, \quad y_t^+ = 2, \quad y_{m1}^+ = -2, \quad y_{m2}^+ = 12.$$

In addition, we obtain the following statements:

(1) As $\mu = -8.4$, we have

$$x_t^- = -176/15, \quad y_t^- = 6.8, \quad y_{m1}^- = 15.2, \quad y_{m2}^- = -14.2.$$

By numerical simulation, we have $y^- = y_{m2}^+$, and there exists a sliding homoclinic cycle illustrated in Figure 4, which is in agreement with the orbit $\mathcal{H}_1$ in Theorem 3.6.

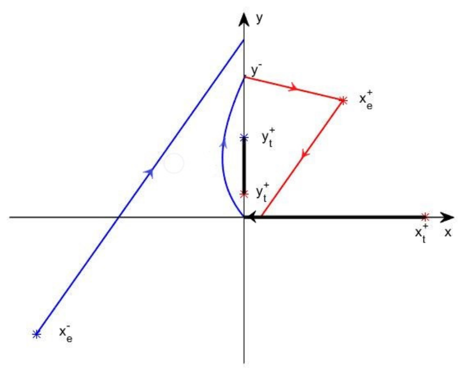

FIGURE 4. Existence of the sliding homoclinic cycle of equation (5.3) with $\mu = -8.4$

(2) As $\mu = -7$, we have

$$x_t^- = -31/3, \quad y_t^- = 4, \quad y_{m1}^- = 11, \quad y_{m2}^- = -27/2.$$

By numerical simulation, we have $y^- < y_{m2}^+$, and there exists a sliding cycle illustrated in Figure 5, which is in agreement with the orbit $\mathcal{H}_4$ in Theorem 3.6.

(3) As $\mu = -6.2$, we have

$$x_t^- = -143/15, \quad y_t^- = 2.4, \quad y_{m1}^- = 8.6, \quad y_{m2}^- = -13.1.$$

By numerical simulation, we have $y^- = y^+$, and there exists a sliding-zero cycle illustrated in Figures 6 and 7, which is in agreement with the orbit $\mathcal{H}_2$ in Theorem 3.6.

(4) As $\mu = -6.15$, we have

$$x_t^- = -123/20, \quad y_t^- = 2.3, \quad y_{m1}^- = 8.45, \quad y_{m2}^- = -523/40.$$

By numerical simulation, we have $y^- < y^+$, and there exists the attractive limit cycle illustrated in Figures 8 and 9, which is in agreement with the orbit $\mathcal{H}_6$ in Theorem 3.6.

It follows that there is a sliding homoclinic bifurcation at $\mu = -8.4$ and a sliding cycle bifurcation at $\mu = -6.2$.

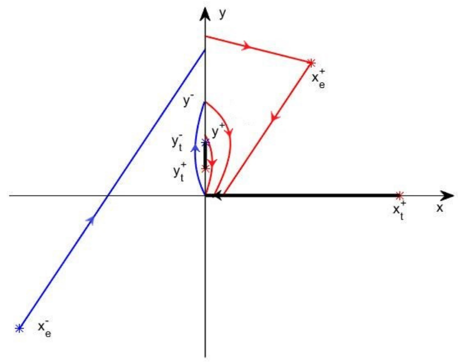

FIGURE 5. Existence of the sliding cycle of equation (5.3) with $\mu = -7$

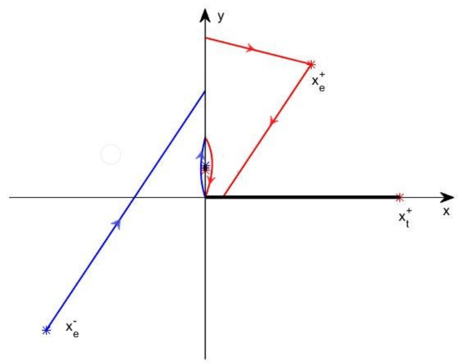

FIGURE 6. Existence of the sliding-zero cycle of equation (5.3) with $\mu = -6.2$

6. Conclusions

In this article, taking the perturbations to the separation boundary into consideration, we studied a special class of planar sector-wise linear systems, which are separated by two rays starting from the same point. More precisely, the two subsystems of this planar sector-wise linear systems are the same except for the positions of $\mathbf{x}_e^{\pm}$. We mainly discussed the global qualitative dynamics of the system above with $\mathbf{x}_e^{\pm}$ being saddles, since the analysis of dynamics with $\mathbf{x}_e^{\pm}$ being sink or source points is similar. Because the global dynamics of piecewise smooth systems is so complex we put $x_e^+ > 0$, $y_e^+ = -y_e^- > 0$ and only left x_e^- as a bifurcation parameter. Based on our results of the existence of sliding set and pseudo-equilibrium points, and the properties of the unique non-regular point $\mathbf{p}_0 = (0, 0)$, we obtained that there exist sliding cycle bifurcation and sliding homoclinic bifurcation in the above system. Moreover, we also got the existence of all important separatrixes

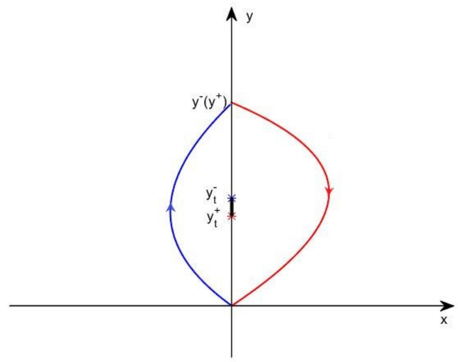

FIGURE 7. Expansion of the sliding-zero cycle in Figure 6

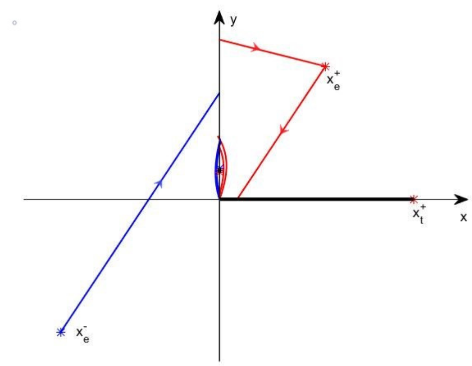

FIGURE 8. Existence of the attractive limit cycle of equation (5.3) with $\mu = -6.15$

and their dependence on the bifurcation parameter x_e^-. For example, there is a sliding heteroclinic orbit, which first turns a heteroclinic cycle and then a limit cycle, meanwhile, there exists a sliding heteroclinic orbit which first turns a limit cycle and then a heteroclinic cycle. In addition, we found the coexistence of a sliding homoclinic cycle and a sliding heteroclinic orbit, while we also found the coexistence of a sliding cycle and a limit cycle.

Acknowledgments. This work is supported by National Natural Science Foundation of China (11301196) and by the Fundamental Research Funds for the Central Universities, HUST(2015QN128).

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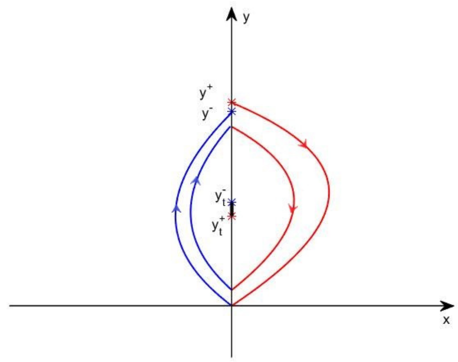

FIGURE 9. Eexpansion of the attractive limit cycle in Figure 8

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