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MILD SOLUTIONS TO FOURTH-ORDER PARABOLIC EQUATIONS MODELING THIN FILM GROWTH WITH TIME FRACTIONAL DERIVATIVE

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Abstract. In this article, we study initial-boundary problems for fourth-order nonlinear parabolic equations modeling thin film growth with Caputo-type time fractional derivative. By means of the theory of abstract fractional calculus and $L^p - L^q$ estimates, we establish the existence and uniqueness of local mild solutions in the spaces $C([0,T]; L^{\frac{\beta N}{2-\beta}}(\Omega))$ with $1 < \beta < 2$. Moreover, the local solutions can be extended globally if the initial data is sufficiently small.

1. Introduction and main result

Thin film growth processes play a crucial role in various scientific and technological applications, ranging from semiconductor manufacturing to material sciences [14, 17, 23]. Understanding the dynamics of thin film growth is essential for optimizing the quality, stability, and functionality of thin films in these applications. Mathematical modeling provides a powerful tool to capture the intricate dynamics involved in such processes and to develop predictive models that guide experimental design and optimization.

In recent years, there has been a growing interest in utilizing fractional calculus to describe and analyze complex phenomena that exhibit non-local and memory effects. In this article, we focus on boundary value problems of fourth-order parabolic equation modeling thin film growth with time fractional derivative,

$$\begin{gathered} {}_cD_t^\alpha u(t) + \Delta^2 u = \nabla \cdot f(\nabla u), \quad x \in \Omega,\ t > 0, \\ \partial_\nu u|_{\partial\Omega} = \partial_\nu \Delta u|_{\partial\Omega} = 0, \quad t > 0, \\ u(x,0) = \varphi(x), \quad x \in \Omega. \end{gathered} \tag{1.1}$$

Here $\Omega \in \mathbb{R}^N$ with $N \geq 2$ is a bounded smooth domain, $\partial\Omega$ denotes the boundary of Ω and ν is the unit outer vector normal to Ω, ${}^cD_t^\alpha$ is the Caputo derivative with order $\alpha \in (0,1)$, which is defined by

$$ {}_cD_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} (u(s) - u(0))\, ds, $$

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where the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1}e^{-t}dt$. Here the term u represents the scaled film height, and $\Delta^2 u$ denotes the capillarity-driven surface diffusion whereas $\nabla \cdot f(\nabla u)$ denotes the upward hopping of atoms. Throughout this paper, we will assume that $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ with $f(0) = Df(0) = 0$ and for some $\beta > 1$, and f satisfies the following growth condition

$$|f'(\xi_1) - f'(\xi_2)| \le C(|\xi_1|^{\beta-1} + |\xi_2|^{\beta-1})|\xi_1 - \xi_2| \tag{1.2}$$

for any $\xi_1, \xi_2 \in \mathbb{R}^N$. As a simple example of (1.2), we can take $f(\xi) = |\xi|^\beta \xi$.

When $\alpha = 1$, the fractional time Caputo derivative is replaced with the classical integer derivative u_t. The problem (1.1) becomes the usual thin film growth model, which is well studied by many researchers. King et al. [9] proved the existence, uniqueness, regularity and large time behavior of solutions in Sobolev function space. Using Kato's Method, Sandjo et al. [16] established existence, uniqueness and regularity of the solution in spaces of $C^0([0,T]; L^p(\Omega))$ with $p = \frac{n\beta}{2-\beta}$, $1 < \beta < 2$. Furthermore, they illustrated the qualitative behavior of the approximate solution through some numerical simulations. Ishige et al. [6] give sufficient conditions on the existence of global solutions, the maximal existence time and blow up rate for $0 < \beta \le 2$. Y. Feng et al. [5] studied the existence of local mild solutions for any initial data lies in L^2 on the two-dimensional torus with and without advection. We refer the interested reader to the references therein.

Fractional models extend the classical model with non-integer orders differentiation and integration, allowing the incorporation of memory and long-range dependencies into models. There are many numerical simulations for the time fractional thin film growth equations. T. Tang et al. [18] proved the time-fractional molecular beam epitaxy models admit an energy dissipation law as integral cases and proposed a class of finite difference schemes inherited the theoretical energy stability. Chen et al. [4] developed an efficient and accurate, full discrete, linear numerical approximation for the time-fractional thin film model with the classical Caputo fractional derivative of order α and shown the models possessed an energy dissipation law. Wang et al. [20] proposed a variable-step L^1 scheme for the time-fractional molecular beam epitaxy model and also investigated the stability and convergence of the stabilized convex splitting scheme. However, to the best of our knowledge, the well-posedness for the solutions of the time fractional thin film growth equation is not clear, which is the main motivation of the present work. We refer [19] for well posedness of the linear and semilinear time fractional order Cauchy problem with almost sectorial operators, and [22] for well posedness of the time fractional order Cahn-Hilliard equation in $\mathbb{R}^3$.

In this article, we focus on the existence and uniqueness of mild solutions on problem (1.1). Because of the observation that the biharmonic operator can be regarded as a sectorial operator on some spaces, we follow some ideas in [3], properly adapted to our problem.

We denote $\mathcal{A} = \Delta^2$ defined on $L^p(\Omega)$, $1 < p < \infty$ with its domain

$$D(\mathcal{A}) = \{u \in W^{4,p}(\Omega) : \partial_\nu u|_{\partial\Omega} = \partial_\nu \Delta u|_{\partial\Omega} = 0\}.$$

It is clear [15] that the following homogeneous boundary value problem is *normally elliptic* in Ω,

$$\begin{aligned} \partial_t u &= -\mathcal{A}u, \quad x \in \Omega,\ t > 0, \\ u(x,0) &= \varphi(x), \quad x \in \Omega. \end{aligned} \tag{1.3}$$

Hence, the fourth-order operator $-\Delta^2$ with corresponding Neumann boundary $(\frac{\partial}{\partial_\nu}, \frac{\partial \Delta}{\partial_\nu})$ is the infinitesimal generator of an analytic semigroup $T(t) = e^{-t\Delta^2}$ in $L^p(\Omega)$. For a detailed discussion on these results, we refer the reader to [1, 2].

The time fractional version of (1.3) with lower order term can be written as

$$\begin{gathered} {}_cD_t^\alpha\, u = -\mathcal{A}u + F(u), \quad x \in \Omega,\ t > 0 \\ u(x,0) = \varphi(x), \quad x \in \Omega. \end{gathered} \tag{1.4}$$

We formally give a mild solution for (1.4), where the rigorous deduction could be found in [3, 8, 10, 19]. We denote by $\mathcal{L}$ Laplace transform operator. By the convolution property of Laplace transform, we have

$$\mathcal{L}(\,{}_cD_t^\alpha\, u) = \lambda^\alpha \mathcal{L}(u) - \lambda^{\alpha-1}\varphi,$$

as in [10]. So taking the Laplace transform to (1.4) gives

$$\mathcal{L}(u) = \lambda^{\alpha-1}(\lambda^\alpha + \mathcal{A})^{-1}\varphi + (\lambda^\alpha + \mathcal{A})^{-1}\mathcal{L}(F(u)).$$

Application of Laplace inversion [3] implies

$$u(t) = E_\alpha\left(-t^\alpha \mathcal{A}\right)\varphi + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-(t-s)^\alpha \mathcal{A}\right) F(u)(s)\mathrm{d}s.$$

where $E_\alpha(-t^\alpha \mathcal{A})$ and $E_{\alpha,\alpha}(-t^\alpha \mathcal{A})$ are the Mittag-Leffler operators (see Section 2). This formal computation then motivates the definition of the mild solution of (1.1) as follows:

Definition 1.1. Let $0 < \alpha < 1$ and $T > 0$.

(i) A function u such that $u \in C([0,T]; L^{\frac{\beta N}{2-\beta}}(\Omega))$ defined by

$$u(x,t) = E_\alpha(-t^\alpha \mathcal{A})\varphi + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A}) \nabla \cdot f(\nabla u)\, \mathrm{d}s, \tag{1.5}$$

is called a local mild solution of (1.1).

(ii) If $T = \infty$, we say that u is a global mild solution of (1.1).

We are now in a position to state the main result of this article.

Theorem 1.2. *Suppose $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with C^4 boundary, $0 < \alpha < 1 < \beta < 2$ and $\varphi \in L^{\frac{\beta N}{2-\beta}}(\Omega)$. Then there exists $T > 0$ such that (1.1) admits a unique mild solution $u \in C([0,T]; L^{\frac{\beta N}{2-\beta}}(\Omega))$ satisfying*

$$\max\Big\{ \sup_{0 \le t \le T} t^{\frac{\alpha}{2\beta} - \frac{\alpha\gamma}{4\beta^2}} \|\nabla u(t)\|_{\frac{\beta^2 N}{\gamma}},\ \sup_{0 \le t \le T} t^{\frac{\alpha}{2}} \|\nabla^2 u(t)\|_{\frac{\beta N}{2-\beta}} \Big\} < \infty,$$

where $0 < \gamma < \min\{2\beta, \beta(N+1) - 2\}$. Furthermore, if $\|\varphi\|_{\frac{\beta N}{2-\beta}}$ is sufficiently small, the solution u can be extended to be global, that is, $T = \infty$.

To explain the meaning of the result, we take $f(\xi) = |\xi|^\beta \xi$ in $(1.1)_1$ as an example. Notice that a smooth function $u(x,t)$ solves the equation in (1.1) for $t > 0$ if and only if

$$u_\lambda(x,t) = \lambda^{\frac{2\alpha}{\beta} - \alpha} u(\lambda^\alpha x, \lambda^4 t)$$

does so too with each given constant λ. In addition, for the initial data φ, under the transformation $\varphi \mapsto \varphi_\lambda$, the $L^{\frac{\beta N}{2-\beta}}(\Omega)$ norm is invariant. Therefore, we expect the global existence and uniqueness of solutions when the initial data is sufficiently small in the critical space $L^{\frac{\beta N}{2-\beta}}(\Omega)$.

It is worth mentioning that the singularity together with strong nonlinearity arising from the time fractional derivative ${}_cD_t^\alpha u$ and the nonlinear term $\nabla \cdot f(\nabla u)$ make its mathematical analysis more difficult in comparison with the fourth-order parabolic equation $(1.3)_1$. For example, the Mittag-Leffler operators $E_\alpha(-t^\alpha \mathcal{A})$ and $E_{\alpha,\alpha}(-t^\alpha \mathcal{A})$ do not satisfy the semigroup properties. So we need to overcome these essential difficulties to get some a priori estimates and extend the local solution to a global one. For more details, one can refer to Section 3.

This article is organized as follows. In the next section, we introduce some elementary properties of Mittag-Leffler operators, which are essential throughout the whole paper and give the main results of this paper.

In Section 3, we establish the existence and uniqueness of mild solutions using appropriate functional spaces and Banach fixed pointed theorem.

2. A priori estimates

Throughout this paper, C stands for a generic positive constant which may vary from line to line. For the analytic semigroup $T(t)$, we have the $L^p - L^q$ estimates, which are state as follows.

Proposition 2.1 ([15, 21])**.** *Let $1 < q \le p < \infty$ and $j = 0, 1, 2, 3$. For $u \in L^q(\Omega)$, we have*

$$\|\nabla^j T(t)u\|_{L^p(\Omega)} \le C\, t^{-\frac{N}{4}(\frac{1}{q}-\frac{1}{p})-\frac{j}{4}} \|u\|_{L^q(\Omega)}, \quad t > 0.$$

Let us recall some properties of Mittag-Leffler operators. For $\alpha \in (0,1)$, we denote the entire function $M_\alpha : \mathbb{C} \to \mathbb{C}$ the Mainardi function by

$$M_\alpha(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1-\alpha(1+n))},$$

which is a particular case of the Wright type function introduced by Mainardi in [12] to characterize the fundamental solutions for some standard boundary value problems in physics. The following classical result gives some essential relations used in this article to obtain the main estimates.

Proposition 2.2 ([19])**.** *Let $0 < \alpha < 1$ and $-1 < \gamma < \infty$. If we restrict M_α to the positive real line, then it holds that*

$$M_\alpha \in \mathcal{S}([0,\infty)), \quad M_\alpha(t) \ge 0 \quad \text{for all } t \ge 0 \text{ and } \int_0^\infty t^\gamma M_\alpha(t)\,\mathrm{d}t = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha\gamma+1)},$$

where $\mathcal{S}([0,\infty))$ is the Schwartz space on $[0,\infty)$.

Now, for each $\alpha \in (0,1)$, we define the Mittag-Leffler families

$$E_\alpha(-t^\alpha \mathcal{A}) = \int_0^\infty M_\alpha(s) T(st^\alpha)\,\mathrm{d}s,$$
$$E_{\alpha,\alpha}(-t^\alpha \mathcal{A}) = \int_0^\infty \alpha s M_\alpha(s) T(st^\alpha)\,\mathrm{d}s.$$

It is interesting to notice that the Mainardi functions act as a bridge between the fractional and the classical abstract theories, for more details see [19, 11, 3].

The next result comprises the main assertions about the theory of abstract fractional calculus.

Proposition 2.3 ([19]). *$E_\alpha(-t^\alpha\mathcal{A})$ and $E_{\alpha,\alpha}(-t^\alpha\mathcal{A})$ are well defined from $L^p(\Omega)$ to $L^p(\Omega)$, $p\in(1,\infty)$. Moreover, for $t\geq 0$, $E_\alpha(-t^\alpha\mathcal{A})$ and $E_{\alpha,\alpha}(-t^\alpha\mathcal{A})$ are uniformly continuous in the uniform operator topology on $L^p(\Omega)$.*

Our proof relies on the well-known Weierstrass M-test in Banach space and L^p-L^q estimates for Mittag-Leffler operators.

Lemma 2.4 (Weierstrass M-test [7]). *Let X denote a Banach space equipped with the norm $\|\cdot\|_X$. Suppose $\{\omega_j\}_{j\geq 0}$ is a sequence of continuous functions from $[0,T]$ to X such that*

$$\sup_{0\leq t\leq T}\|\omega_j(t)\|_X\leq M_j,\quad j=0,1,2,\dots$$

where $0<T\leq\infty$ and the sequence $\{M_j\}_{j\geq 0}$ satisfies $\sum_{j=0}^\infty M_j<\infty$, then the sequence $\{\omega_j\}_{j\geq 0}$ converges uniformly on $[0,T]$, that is,

$$\sum_{j=0}^\infty \omega_j\in C([0,T];X).$$

Using Proposition 2.1, we can obtain similar L^p-L^q estimates for both families of Mittag-Leffler operators, which also depend on the exponent of differentiation α.

Proposition 2.5. *Let $1<q\leq p<\infty$ and $\frac{1}{q}-\frac{1}{p}<\frac{4-j}{N}$ with any nonnegative integer $j<4$. For $u\in L^q(\Omega)$, we have*

$$\|\nabla^j E_\alpha(-t^\alpha\mathcal{A})u\|_{L^p(\Omega)}\leq Ct^{-\frac{N\alpha}{4}(\frac{1}{q}-\frac{1}{p})-\frac{j\alpha}{4}}\|u\|_{L^q(\Omega)},\quad t>0. \tag{2.1}$$

Proof. Noting that if $0\leq\frac{1}{q}-\frac{1}{p}<\frac{4-j}{N}$ with $j<4$, we have

$$-\frac{N}{4}(\frac{1}{q}-\frac{1}{p})-\frac{j}{4}>-1.$$

By Proposition 2.1 and 2.2, we obtain

$$\begin{aligned}
\|\nabla^j E_\alpha(-t^\alpha\mathcal{A})u\|_{L^p(\Omega)} &\leq \int_0^\infty M_\alpha(s)\|\nabla^j T(st^\alpha)u\|_{L^p(\Omega)}\,\mathrm{d}s\\
&\leq C\Big(\int_0^\infty M_\alpha(s)s^{-\frac{N}{4}(\frac{1}{q}-\frac{1}{p})-\frac{j}{4}}\,\mathrm{d}s\Big)\Big(t^{-\frac{N\alpha}{4}(\frac{1}{q}-\frac{1}{p})-\frac{j\alpha}{4}}\|u\|_{L^q(\Omega)}\Big)\\
&\leq Ct^{-\frac{N\alpha}{4}(\frac{1}{q}-\frac{1}{p})-\frac{j\alpha}{4}}\|u\|_{L^q(\Omega)}.
\end{aligned}$$

The proof is complete. □

Proposition 2.6. *Let $1<q\leq p<\infty$ and $\frac{1}{q}-\frac{1}{p}<\frac{8-j}{N}$ with a nonnegative integer $j<4$. Then we have*

$$\|\nabla^j E_{\alpha,\alpha}(-t^\alpha\mathcal{A})u\|_{L^p(\Omega)}\leq Ct^{-\frac{N\alpha}{4}(\frac{1}{q}-\frac{1}{p})-\frac{j\alpha}{4}}\|u\|_{L^q(\Omega)},\quad t>0.$$

The proof of the above propostion is similar to that of Proposition 2.5, so we omit it.

3. Proof of Theorem 1.2

In this section, based on the L^p-L^q estimates in section 2, we will prove Theorem 1.2 by using the method of successive approximation. To simplify the notation, we denote the norm $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_p$. Recalling the integral equation (1.5), we define a sequence $\{u_j\}_{j\geq 0}$ as follows,

$$\begin{gathered} u_0(x,t) = E_\alpha(-t^\alpha \mathcal{A})\varphi, \\ u_j(x,t) = E_\alpha(-t^\alpha \mathcal{A})\varphi + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\nabla\cdot f(\nabla u_{j-1})\,\mathrm{d}s \end{gathered} \tag{3.1}$$

for positive integers j and $t>0$. In additional, we define

$$R(t)_j = \max\Big\{ \sup_{0<s\leq t} s^{\frac{\alpha}{2}} \|\nabla^2 u_j(s)\|_{\frac{\beta N}{2-\beta}},\ \sup_{0<s\leq t} s^{\frac{\alpha}{2\beta}-\frac{\alpha\gamma}{4\beta^2}} \|\nabla u_j(s)\|_{\frac{\beta^2 N}{\gamma}} \Big\} \tag{3.2}$$

for $t>0$ and $j\geq 0$, where $0<\gamma<\min\{2\beta, \beta(N+1)-2\}$.

Next, we show $u_j(t)$ belongs $L^{\frac{\beta N}{2-\beta}}(\Omega)$ and is continuous under some smallness conditions. The proofs require the following iteration lemma.

Lemma 3.1 ([13]). *Let $\lambda,\beta>0$ and b_j be a nonnegative sequence such that*

$$b_j \leq b_0 + \lambda b_{j-1}^{1+\beta}$$

for all positive integer j. If $2\lambda(2b_0)^\beta<1$, then for each nonnegative integer j, we have

$$b_j \leq \frac{b_0}{1-\lambda(2b_0)^\beta}.$$

Lemma 3.2. *For any $T>0$, there exists a constant $\varepsilon_0>0$ independent of T such that if $R(T)_0\leq\varepsilon_0$, each $u_j(t)$ is well defined as an element of $L^{\frac{\beta N}{2-\beta}}(\Omega)$ for any $t>0$, and*

$$R(T)_j \leq 2R(T)_0, \quad j\geq 0. \tag{3.3}$$

Proof. Because $f'(\xi)$ behaves like $|\xi|^\beta$, we have

$$\|f'(\nabla u)\|_s \leq C\|\nabla u\|_s^\beta.$$

By Hölder's inequality, it is easy to verify that

$$\begin{aligned} &\|\nabla\cdot f(\nabla u_j(s))\|_{\frac{\beta N}{2-\beta+\gamma}} \\ &= \|\nabla^2 u_j(s)\cdot f'(\nabla u_j(s))\|_{\frac{\beta N}{2-\beta+\gamma}} \\ &\leq \|\nabla^2 u_j(s)\|_{\frac{\beta N}{2-\beta}} \|f'(\nabla u_j(s))\|_{\frac{\beta N}{\gamma}} \\ &= s^{-\alpha+\frac{\alpha\gamma}{4\beta}} \Big(s^{\frac{\alpha}{2}}\|\nabla^2 u_j(s)\|_{\frac{\beta N}{2-\beta}}\Big)\Big(s^{\frac{\alpha}{2}-\frac{\alpha\gamma}{4\beta}}\|\nabla u_j(s)\|^\beta_{\frac{\beta^2 N}{\gamma}}\Big) \\ &\leq s^{-\alpha+\frac{\alpha\gamma}{4\beta}} \Big(\sup_{0<s\leq t} s^{\frac{\alpha}{2}}\|\nabla^2 u_j(s)\|_{\frac{\beta N}{2-\beta}}\Big)\Big(\sup_{0<s\leq t} s^{\frac{\alpha}{2\beta}-\frac{\alpha\gamma}{4\beta^2}}\|\nabla u_j(s)\|_{\frac{\beta^2 N}{\gamma}}\Big)^\beta \\ &\leq s^{-\alpha+\frac{\alpha\gamma}{4\beta}} R(t)_j^{1+\beta}. \end{aligned} \tag{3.4}$$

Applying the differential operator ∇ to (3.1), and applying Propositions 2.5 and 2.6, and (3.4), we obtain that

$\|\nabla u_{j+1}\|_{\frac{\beta^2 N}{\gamma}}$

$$
\begin{aligned}
&\le \|\nabla E_\alpha(-t^\alpha\mathcal{A})\varphi\|_{\frac{\beta^2N}{\gamma}} + \int_0^t (t-s)^{\alpha-1}\|\nabla E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\nabla\cdot f(\nabla u_j)\|_{\frac{\beta^2N}{\gamma}}\,\mathrm{d}s \\
&\le \|\nabla u_0\|_{\frac{\beta^2N}{\gamma}} + C\int_0^t (t-s)^{\alpha-1+\frac{\alpha\gamma}{4\beta^2}-\frac{\alpha}{2\beta}-\frac{\alpha\gamma}{4\beta}}\|\nabla\cdot f(\nabla u_j)\|_{\frac{\beta N}{2-\beta+\gamma}}\,\mathrm{d}s \qquad (3.5)\\
&\le \|\nabla u_0\|_{\frac{\beta^2N}{\gamma}} + CR(t)_j^{1+\beta}\int_0^t (t-s)^{\alpha-1+\frac{\alpha\gamma}{4\beta^2}-\frac{\alpha}{2\beta}-\frac{\alpha\gamma}{4\beta}} s^{-\alpha+\frac{\alpha\gamma}{4\beta}}\,\mathrm{d}s \\
&\le \|\nabla u_0\|_{\frac{\beta^2N}{\gamma}} + Ct^{\frac{\alpha\gamma}{4\beta^2}-\frac{\alpha}{2\beta}}R(t)_j^{1+\beta},
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\|\nabla^2 u_{j+1}\|_{\frac{\beta N}{2-\beta}} \\
&\le \|\nabla^2 E_\alpha(-t^\alpha\mathcal{A})\varphi\|_{\frac{\beta N}{2-\beta}} + \int_0^t (t-s)^{\alpha-1}\|\nabla^2 E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\nabla\cdot f(\nabla u_j)\|_{\frac{\beta N}{2-\beta}}\,\mathrm{d}s \\
&\le \|\nabla^2 u_0\|_{\frac{\beta N}{2-\beta}} + C\int_0^t (t-s)^{\frac{\alpha}{2}-1-\frac{\alpha\gamma}{4\beta}}\|\nabla\cdot f(\nabla u_j)\|_{\frac{\beta N}{2-\beta+\gamma}}\,\mathrm{d}s \\
&\le \|\nabla^2 u_0\|_{\frac{\beta N}{2-\beta}} + CR(t)_j^{1+\beta}\int_0^t (t-s)^{\frac{\alpha}{2}-1-\frac{\alpha\gamma}{4\beta}} s^{-\alpha+\frac{\alpha\gamma}{4\beta}}\,\mathrm{d}s \qquad (3.6)\\
&\le \|\nabla^2 u_0\|_{\frac{\beta N}{2-\beta}} + Ct^{-\frac{\alpha}{2}}R(t)_j^{1+\beta}.
\end{aligned}
$$

Combining (3.5) and (3.6), for any fixed $T>0$, we have

$$
R(T)_{j+1} \le R(T)_0 + CR(T)_j^{1+\beta},
$$

where $C>0$ is independent of T. According to Lemma 3.1, there exists a constant $\varepsilon_0>0$ such that $R(T)_0\le\varepsilon_0$, and $2C(2R(T)_0)^\beta<1$, which yields

$$
R(T)_j \le 2R(T)_0, \quad j\ge 0.
$$

At last, we obtain

$$
\begin{aligned}
\|u_{j+1}\|_{\frac{\beta N}{2-\beta}} &\le \|E_\alpha(-t^\alpha\mathcal{A})\varphi\|_{\frac{\beta N}{2-\beta}} + \int_0^t (t-s)^{\alpha-1}\|E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\nabla\cdot f(\nabla u_j)\|_{\frac{\beta N}{2-\beta}}\,\mathrm{d}s \\
&\le R(T)_0 + C\int_0^T (t-s)^{\alpha-1-\frac{\alpha\gamma}{4\beta}}\|\nabla\cdot f(\nabla u_j)\|_{\frac{\beta N}{2-\beta+\gamma}}\,\mathrm{d}s \\
&\le R(T)_0 + CR(T)_j^{1+\beta}\int_0^T (t-s)^{\alpha-1-\frac{\alpha\gamma}{4\beta}} s^{-\alpha+\frac{\alpha\gamma}{4\beta}}\,\mathrm{d}s \\
&\le R(T)_0 + CR(T)_j^{1+\beta} \\
&\le CR(T)_0,
\end{aligned}
$$

which means $u_j(t)\in L^p(\Omega)$ for any $0<t\le T$ and $j\ge 0$. □

Next, we shall show the strong continuity of $u_j(t)$ for $t\in[0,T]$.

Lemma 3.3. *Under the assumption $R(T)_0\le\varepsilon_0$ in Lemma 3.2, for any $j\ge 0$, we have*

$$
u_j \in C([0,T]; L^{\frac{\beta N}{2-\beta}}(\Omega)), \quad T>0,\ j=0,1,2,\cdots.
$$

Proof. Fixing $t_0 \in (0,T)$, we have the estimate

$$
\begin{aligned}
&\|u_{j+1}(t) - u_{j+1}(t_0)\|_{\frac{\beta N}{2-\beta}} \\
&\le \|(E_\alpha(-t^\alpha \mathcal{A}) - E_\alpha(-t_0^\alpha \mathcal{A}))\varphi\|_{\frac{\beta N}{2-\beta}} \\
&\quad + \int_{t_0}^{t} (t-s)^{\alpha-1} \|E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\nabla\cdot f(\nabla u_j)\|_{\frac{\beta N}{2-\beta}} \,\mathrm{d}s \\
&\quad + \int_0^{t_0} \big\|(t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\nabla\cdot f(\nabla u_j) \\
&\quad - (t_0-s)^{\alpha-1} E_{\alpha,\alpha}(-(t_0-s)^\alpha \mathcal{A})\nabla\cdot f(\nabla u_j)\big\|_{\frac{\beta N}{2-\beta}} \,\mathrm{d}s \\
&= I_1 + I_2 + I_3,
\end{aligned}
\tag{3.7}
$$

where $t_0 < t \le T$.

Using the strong continuity of $E_\alpha(-t^\alpha \mathcal{A})$ on $L^p(\Omega)$ for $t \in [0,\infty)$, we deduce easily that the first term I_1 goes to zero as $t \to t_0^+$. By Proposition 2.6, the estimate of the second term yields

$$
\begin{aligned}
I_2 &= \int_{t_0}^{t} (t-s)^{\alpha-1} \|E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\nabla\cdot f(\nabla u_j)\|_{\frac{\beta N}{2-\beta}} \,\mathrm{d}s \\
&\le C \int_{t_0}^{t} (t-s)^{\alpha-1-\frac{\alpha\gamma}{4\beta}} \|\nabla\cdot f(\nabla u_j)\|_{\frac{\beta N}{2-\beta+\gamma}} \,\mathrm{d}s \\
&\le C R(T)_0^{1+\beta} \int_{t_0}^{t} (t-s)^{\alpha-1-\frac{\alpha\gamma}{4\beta}} s^{-\alpha+\frac{\alpha\gamma}{4\beta}} \,\mathrm{d}s \\
&\le C R(T)_0^{1+\beta} \Big(\frac{t-t_0}{t_0}\Big)^{1-\alpha+\frac{\alpha\gamma}{4\beta}}.
\end{aligned}
$$

Obviously, this term vanishes as $t \to t_0^+$. For the estimate of the third term, we first denote

$$
\begin{aligned}
g(t,s) = \big\|&(t-s)^{\alpha-1} E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\nabla\cdot f(\nabla u_j) \\
&- (t_0-s)^{\alpha-1} E_{\alpha,\alpha}(-(t_0-s)^\alpha \mathcal{A})\nabla\cdot f(\nabla u_j)\big\|_{\frac{\beta N}{2-\beta}},
\end{aligned}
$$

where $0 < s < t_0 < t$. Combining this with (3.4), it is easy to see that

$$
\begin{aligned}
g(t,s) &\le |(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}| \cdot \big\|E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\nabla\cdot f(\nabla u_j(s))\big\|_{\frac{\beta N}{2-\beta}} \\
&\quad + (t_0-s)^{\alpha-1}\big\|E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\nabla\cdot f(\nabla u_j(s)) \\
&\quad - E_{\alpha,\alpha}(-(t_0-s)^\alpha \mathcal{A})\nabla\cdot f(\nabla u_j(s))\big\|_{\frac{\beta N}{2-\beta}} \\
&\le C|(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}|(t-s)^{-\frac{\alpha\gamma}{4\beta}} \cdot \|\nabla\cdot f(\nabla u_j(s))\|_{\frac{\beta N}{2-\beta+\gamma}} \\
&\quad + (t_0-s)^{\alpha-1}\big\|E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\nabla\cdot f(\nabla u_j(s)) \\
&\quad - E_{\alpha,\alpha}(-(t_0-s)^\alpha \mathcal{A})\nabla\cdot f(\nabla u_j(s))\big\|_{\frac{\beta N}{2-\beta}} \\
&\le C|(t-s)^{\alpha-1} - (t_0-s)^{\alpha-1}|(t-s)^{-\frac{\alpha\gamma}{4\beta}} s^{-\alpha+\frac{\alpha\gamma}{4\beta}} R(T)_0 \\
&\quad + (t_0-s)^{\alpha-1}\big\|E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\nabla\cdot f(\nabla u_j(s)) \\
&\quad - E_{\alpha,\alpha}(-(t_0-s)^\alpha \mathcal{A})\nabla\cdot f(\nabla u_j(s))\big\|_{\frac{\beta N}{2-\beta}}.
\end{aligned}
\tag{3.8}
$$

Then combining with the strong continuity of $E_{\alpha,\alpha}(-t^\alpha\mathcal{A})$ on $L^{\frac{\beta N}{2-\beta}}(\Omega)$ for $t \in (0,\infty)$, we have

$$\lim_{t\to t_0^+} g(t,s) = 0$$

for each fixed $s \in (0,t_0)$. In addition, the last term of (3.8) is integrable. Applying dominated convergence theorem yields the third term I_3 also tends to zero as $t \to t_0^+$. Indeed, for $0 < s < t_0 < t$, we have

$$\begin{aligned}
I_3 &= \int_0^{t_0} g(t,s)\,\mathrm{d}s \\
&\le C\int_0^{t_0} (t_0-s)^{\alpha-1-\frac{\alpha\gamma}{4\beta}} \|\nabla\cdot f(\nabla u_j)\|_{\frac{\beta N}{2-\beta+\gamma}}\,\mathrm{d}s, \\
&\le CR(T)_j^{1+\beta}\int_0^{t_0} (t_0-s)^{\alpha-1-\frac{\alpha\gamma}{4\beta}} s^{-\alpha+\frac{\alpha\gamma}{4\beta}}\,\mathrm{d}s \\
&\le CR(T)_0^{1+\beta}.
\end{aligned}$$

Similarly, we can also prove the same limit as $t \to t_0^-$ with $t_0 \in (0,T]$. Thus,

$$\lim_{t\to t_0} \|u_{j+1}(t) - u_{j+1}(t_0)\|_{\frac{\beta N}{2-\beta}} = 0, \quad t_0 \in (0,T].$$

As for the continuity up to $t = 0$ of u_{j+1}, Observe that

$$\begin{aligned}
&\|u_{j+1}(t) - u_{j+1}(0)\|_{\frac{\beta N}{2-\beta}} \\
&\le \|E_\alpha(-t^\alpha\mathcal{A})\varphi - \varphi\|_{\frac{\beta N}{2-\beta}} + \int_0^t (t-s)^{\alpha-1}\|E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\nabla\cdot f(\nabla u_j(s))\|_{\frac{\beta N}{2-\beta}}\,\mathrm{d}s \\
&\le \|E_\alpha(-t^\alpha\mathcal{A})\varphi - \varphi\|_{\frac{\beta N}{2-\beta}} + C\int_0^t (t-s)^{\alpha-1-\frac{\alpha\gamma}{4\beta}}\|\nabla\cdot f(\nabla u_j)\|_{\frac{\beta N}{2-\beta+\gamma}}\,\mathrm{d}sr \\
&\le \|E_\alpha(-t^\alpha\mathcal{A})\varphi - \varphi\|_{\frac{\beta N}{2-\beta}} + CR(t)_j^{1+\beta}\int_0^t (t-s)^{\alpha-1-\frac{\alpha\gamma}{4\beta}} s^{-\alpha+\frac{\alpha\gamma}{4\beta}}\,\mathrm{d}s \\
&\le \|E_\alpha(-t^\alpha\mathcal{A})\varphi - \varphi\|_{\frac{\beta N}{2-\beta}} + CR(t)_0^{1+\beta}.
\end{aligned}$$

Obviously, if

$$\lim_{t\to 0^+} R(t)_0 = 0, \tag{3.9}$$

then combining with the strong continuity of $E_\alpha(-t^\alpha\mathcal{A})$, we obtain that

$$\lim_{t\to 0^+} \|u_{j+1}(t) - u_{j+1}(0)\|_{\frac{\beta N}{2-\beta}} = 0.$$

To prove (3.9), we notice that since $\varphi \in L^{\frac{\beta N}{2-\beta}}(\Omega)$, for any $\varepsilon > 0$, there exists $\widetilde{\varphi} \in C_0^\infty(\Omega)$ such that

$$\|\varphi - \widetilde{\varphi}\|_{\frac{\beta N}{2-\beta}} < \varepsilon.$$

Then we have

$$\begin{aligned}
&t^{\frac{\alpha}{2\beta}-\frac{\alpha\gamma}{4\beta^2}}\|\nabla u_0(t)\|_{\frac{\beta^2 N}{\gamma}} \\
&= t^{\frac{\alpha}{2\beta}-\frac{\alpha\gamma}{4\beta^2}}\|\nabla E_\alpha(-t^\alpha\mathcal{A})\varphi\|_{\frac{\beta^2 N}{\gamma}} \\
&\le t^{\frac{\alpha}{2\beta}-\frac{\alpha\gamma}{4\beta^2}}\|\nabla E_\alpha(-t^\alpha\mathcal{A})(\varphi-\widetilde{\varphi})\|_{\frac{\beta^2 N}{\gamma}} + t^{\frac{\alpha}{2\beta}-\frac{\alpha\gamma}{4\beta^2}}\|\nabla E_\alpha(-t^\alpha\mathcal{A})\widetilde{\varphi}\|_{\frac{\beta^2 N}{\gamma}} \\
&\le \|\varphi-\widetilde{\varphi}\|_{\frac{\beta N}{2-\beta}} + t^{\frac{\alpha}{4}}\|\nabla\widetilde{\varphi}\|_{\frac{\beta N}{2-\beta}} \le \varepsilon
\end{aligned}$$

for small $t>0$, which implies

$$\lim_{t\to 0^+} t^{\frac{\alpha}{2\beta}-\frac{\alpha\gamma}{4\beta^2}}\|\nabla u_0(t)\|_{\frac{\beta^2 N}{\gamma}}=0. \tag{3.10}$$

Similarly, we can obtain

$$\lim_{t\to 0^+} t^{\frac{\alpha}{2}}\|\nabla^2 u_0(t)\|_{\frac{\beta N}{2-\beta}}=0. \tag{3.11}$$

Combining (3.10) with (3.11), we derive (3.9). Summing up, we see that

$$u_j\in C([0,T];L^{\frac{\beta N}{2-\beta}}(\Omega))$$

for any $T>0$ and $j\geq 0$. □

Proof of Theorem 1.2. To prove the main result, we only need to show the uniform convergence of the sequence $\{u_j\}_{j\geq 0}$ under the assumption that $R(T)_0\leq\varepsilon_0$. By Lemma 2.4, we define the sequence

$$\begin{gathered}\omega_0(x,t)=u_0(x,t),\\ \omega_j(x,t)=u_j(x,t)-u_{j-1}(x,t),\quad j\geq 1.\end{gathered} \tag{3.12}$$

Obviously, $\omega_j\in C([0,T];L^{\frac{\beta N}{2-\beta}}(\Omega))$ for any $T>0$ and $j\geq 0$. In addition, we have the identity

$$\begin{aligned}
&\omega_{j+1}(x,t)\\
&=u_{j+1}(x,t)-u_j(x,t)\\
&=\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\Big(\nabla\cdot f(\nabla u_j)-\nabla\cdot f(\nabla u_{j-1})\Big)\,\mathrm{d}s\\
&=\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\Big(\nabla^2 u_j\cdot f'(\nabla u_j)-\nabla^2 u_{j-1}\cdot f'(\nabla u_{j-1})\Big)\,\mathrm{d}s\\
&=\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\nabla^2\omega_j\cdot f'(\nabla u_j)\,\mathrm{d}s\\
&\quad+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\nabla^2 u_{j-1}\cdot\Big(f'(\nabla u_j)-f'(\nabla u_{j-1})\Big)\,\mathrm{d}s.
\end{aligned} \tag{3.13}$$

As in Lemma 3.2, to estimate ω_j, we to derive a priori estimates for $\nabla\omega_j$ and $\nabla^2\omega_j$. So we define

$$\widetilde{R(t)}_j=\max\Big\{\sup_{0<s\leq t}s^{\frac{\alpha}{2}}\|\nabla^2\omega_j(s)\|_{\frac{\beta N}{2-\beta}},\ \sup_{0<s\leq t}s^{\frac{\alpha}{2\beta}-\frac{\alpha\gamma}{4\beta^2}}\|\nabla\omega_j(s)\|_{\frac{\beta^2 N}{\gamma}}\Big\} \tag{3.14}$$

for $t>0$ and $j\geq 0$, where $0<\gamma<\min\{2\beta,\beta(N+1)-2\}$. Obviously, $\widetilde{R(t)}_0=R(t)_0$.

Let $R(T)_0\leq\varepsilon_0$. By Hölder's inequality, we obtain that for $0<s\leq t\leq T$,

$$\begin{aligned}
\|\nabla^2\omega_j(s)\cdot f'(\nabla u_j(s))\|_{\frac{\beta N}{2-\beta+\gamma}}&\leq\|\nabla^2\omega_j(s)\|_{\frac{\beta N}{2-\beta}}\|f'(\nabla u_j(s))\|_{\frac{\beta N}{\gamma}}\\
&=s^{-\alpha+\frac{\alpha\gamma}{4\beta}}\Big(s^{\frac{\alpha}{2}}\|\nabla^2\omega_j(s)\|_{\frac{\beta N}{2-\beta}}\Big)\Big(s^{\frac{\alpha}{2}-\frac{\alpha\gamma}{4\beta}}\|\nabla u_j(s)\|^{\beta}_{\frac{\beta^2 N}{\gamma}}\Big)\\
&\leq s^{-\alpha+\frac{\alpha\gamma}{4\beta}}\widetilde{R(T)}_j R(T)_j^{\beta}\\
&\leq Cs^{-\alpha+\frac{\alpha\gamma}{4\beta}}\widetilde{R(T)}_j R(T)_0^{\beta}.
\end{aligned} \tag{3.15}$$

In addition, the growth condition on f' and Hölder's inequality yield, for $0 < s \leq t \leq T$,

$$
\begin{aligned}
&\|f'(\nabla u_j(s)) - f'(\nabla u_{j-1}(s))\|_{\frac{\beta N}{\gamma}} \\
&\leq \Big\|\nabla\omega_j(s)\big(|\nabla u_j(s)|^{\beta-1} + |\nabla u_{j-1}(s)|^{\beta-1}\big)\Big\|_{\frac{\beta N}{\gamma}} \\
&\leq \|\nabla\omega_j(s)\|_{\frac{\beta^2 N}{\gamma}} \Big\||\nabla u_j(s)|^{\beta-1} + |\nabla u_{j-1}(s)|^{\beta-1}\Big\|_{\frac{\beta^2 N}{\gamma(\beta-1)}} \\
&\leq \|\nabla\omega_j(s)\|_{\frac{\beta^2 N}{\gamma}} \Big(\|\nabla u_j(s)\|^{\beta-1}_{\frac{\beta^2 N}{\gamma}} + \|\nabla u_{j-1}(s)\|^{\beta-1}_{\frac{\beta^2 N}{\gamma}}\Big) \\
&\leq s^{-\frac{\alpha}{2}+\frac{\alpha\gamma}{4\beta}} \widetilde{R(T)}_j \Big(R(T)_j^{\beta-1} + R(T)_{j-1}^{\beta-1}\Big).
\end{aligned}
$$

Then for $R(T)_0 \leq \varepsilon_0$, we have

$$
\begin{aligned}
&\|\nabla^2 u_{j-1}(s) \cdot \big(f'(\nabla u_j(s)) - f'(\nabla u_{j-1}(s))\big)\|_{\frac{\beta N}{2-\beta+\gamma}} \\
&\leq \|\nabla^2 u_{j-1}(s)\|_{\frac{\beta N}{2-\beta}} \|f'(\nabla u_j(s)) - f'(\nabla u_{j-1}(s))\|_{\frac{\beta N}{\gamma}} \\
&\leq s^{-\alpha+\frac{\alpha\gamma}{4\beta}} \widetilde{R(T)}_j R(T)_{j-1} \Big(R(T)_j^{\beta-1} + R(T)_{j-1}^{\beta-1}\Big) \\
&\leq C s^{-\alpha+\frac{\alpha\gamma}{4\beta}} \widetilde{R(T)}_j R(T)_0^{\beta}.
\end{aligned}
\tag{3.16}
$$

Applying the differential operator ∇ to (3.13), and combining this with (3.15)–(3.16), we have

$$
\begin{aligned}
&\|\nabla\omega_{j+1}\|_{\frac{\beta^2 N}{\gamma}} \\
&\leq \int_0^t (t-s)^{\alpha-1} \|\nabla E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\nabla^2\omega_j \cdot f'(\nabla u_j)\|_{\frac{\beta^2 N}{\gamma}}\,\mathrm{d}s \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|\nabla E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\nabla^2 u_{j-1} \cdot \big(f'(\nabla u_j) - f'(\nabla u_{j-1})\big)\|_{\frac{\beta^2 N}{\gamma}}\,\mathrm{d}s \\
&\leq C\int_0^t (t-s)^{\alpha-1+\frac{\alpha\gamma}{4\beta^2}-\frac{\alpha}{2\beta}-\frac{\alpha\gamma}{4\beta}} \|\nabla^2\omega_j \cdot f'(\nabla u_j)\|_{\frac{\beta N}{2-\beta+\gamma}}\,\mathrm{d}s \\
&\quad + C\int_0^t (t-s)^{\alpha-1+\frac{\alpha\gamma}{4\beta^2}-\frac{\alpha}{2\beta}-\frac{\alpha\gamma}{4\beta}} \|\nabla^2 u_{j-1} \cdot \big(f'(\nabla u_j) - f'(\nabla u_{j-1})\big)\|_{\frac{\beta N}{2-\beta+\gamma}}\,\mathrm{d}s \\
&\leq C\widetilde{R(T)}_j R(T)_0^\beta \int_0^t (t-s)^{\alpha-1+\frac{\alpha\gamma}{4\beta^2}-\frac{\alpha}{2\beta}-\frac{\alpha\gamma}{4\beta}} s^{-\alpha+\frac{\alpha\gamma}{4\beta}}\,\mathrm{d}s \\
&\leq C t^{\frac{\alpha\gamma}{4\beta^2}-\frac{\alpha}{2\beta}} \widetilde{R(T)}_j R(T)_0^\beta, \quad j \geq 1.
\end{aligned}
\tag{3.17}
$$

Similarly, we have

$$
\begin{aligned}
&\|\nabla^2\omega_{j+1}\|_{\frac{\beta N}{2-\beta}} \\
&\leq \int_0^t (t-s)^{\alpha-1} \|\nabla^2 E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\nabla^2\omega_j \cdot f'(\nabla u_j)\|_{\frac{\beta N}{2-\beta}}\,\mathrm{d}s \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|\nabla^2 E_{\alpha,\alpha}(-(t-s)^\alpha \mathcal{A})\nabla^2 u_{j-1} \cdot \big(f'(\nabla u_j) - f'(\nabla u_{j-1})\big)\|_{\frac{\beta N}{2-\beta}}\,\mathrm{d}s \\
&\leq C\int_0^t (t-s)^{\frac{\alpha}{2}-1-\frac{\alpha\gamma}{4\beta}} \|\nabla^2\omega_j \cdot f'(\nabla u_j)\|_{\frac{\beta N}{2-\beta+\gamma}}\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\quad + C\int_0^t (t-s)^{\frac{\alpha}{2}-1-\frac{\alpha\gamma}{4\beta}}\|\nabla^2 u_{j-1}\cdot\big(f'(\nabla u_j)-f'(\nabla u_{j-1})\big)\|_{\frac{\beta N}{2-\beta+\gamma}}\,\mathrm{d}s\\
&\le C\widetilde{R(T)}_j R(T)_0^\beta\int_0^t (t-s)^{\frac{\alpha}{2}-1-\frac{\alpha\gamma}{4\beta}} s^{-\alpha+\frac{\alpha\gamma}{4\beta}}\,\mathrm{d}s\\
&\le Ct^{-\frac{\alpha}{2}}\widetilde{R(T)}_j R(T)_0^\beta, \quad j\ge 1.
\end{aligned} \tag{3.18}
$$

Combining (3.14), (3.17) with (3.18), for any fixed $T>0$, we have

$$
\widetilde{R(T)}_{j+1}\le C\widetilde{R(T)}_j R(T)_0^\beta,\quad j=1,2,\dots \tag{3.19}
$$

where $C>0$ is independent of T. Noting that

$$
\omega_1=\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\nabla\cdot f(\nabla u_0)\,\mathrm{d}s,
$$

following the same idea in the proof (3.17) and (3.18), we obtain

$$
\widetilde{R(T)}_1\le C\widetilde{R(T)}_0 R(T)_0^\beta = CR(T)_0^{1+\beta}. \tag{3.20}
$$

Combining (3.19) with (3.20), we finally derive

$$
\widetilde{R(T)}_j\le R(T)_0\Big(CR(T)_0^\beta\Big)^j,\quad j=0,1,2,\dots
$$

provided that $R(T)_0\le\varepsilon_0$.

With the help of a priori estimates above, now we can estimate ω_j,

$$
\begin{aligned}
&\|\omega_{j+1}\|_{\frac{\beta N}{2-\beta}}\\
&\le \int_0^t (t-s)^{\alpha-1}\|E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\nabla^2\omega_j\cdot f'(\nabla u_j)\|_{\frac{\beta N}{2-\beta}}\,\mathrm{d}s\\
&\quad+\int_0^t (t-s)^{\alpha-1}\|E_{\alpha,\alpha}(-(t-s)^\alpha\mathcal{A})\nabla^2 u_{j-1}\cdot\Big(f'(\nabla u_j)-f'(\nabla u_{j-1})\Big)\|_{\frac{\beta N}{2-\beta}}\,\mathrm{d}s\\
&\le C\int_0^t (t-s)^{\alpha-1-\frac{\alpha\gamma}{4\beta}}\|\nabla^2\omega_j\cdot f'(\nabla u_j)\|_{\frac{\beta N}{2-\beta+\gamma}}\,\mathrm{d}s\\
&\quad+C\int_0^t (t-s)^{\alpha-1-\frac{\alpha\gamma}{4\beta}}\|\nabla^2 u_{j-1}\cdot\big(f'(\nabla u_j)-f'(\nabla u_{j-1})\big)\|_{\frac{\beta N}{2-\beta+\gamma}}\,\mathrm{d}s\\
&\le C\widetilde{R(T)}_j R(T)_0^\beta\int_0^t (t-s)^{\alpha-1-\frac{\alpha\gamma}{4\beta}} s^{-\alpha+\frac{\alpha\gamma}{4\beta}}\,\mathrm{d}s\\
&\le R(T)_0\Big(CR(T)_0^\beta\Big)^{j+1},\quad j\ge 1.
\end{aligned} \tag{3.21}
$$

Similarly, we obtain that

$$
\|\omega_1\|_{\frac{\beta N}{2-\beta}}\le CR(T)_0^{1+\beta}.
$$

Therefore,

$$
M_j:=\sup_{0\le t\le T}\|\omega_j(t)\|_{\frac{\beta N}{2-\beta}}\le R(T)_0\Big(CR(T)_0^\beta\Big)^j,\quad j\ge 0
$$

and the sequence $\{M_j\}_{j\ge 0}$ is summable provided that

$$
R(T)_0<\min\{\varepsilon_0, C^{-\frac{1}{\beta}}\}. \tag{3.22}
$$

Since $\lim_{T\to 0^+} R(T)_0 = 0$, we can choose $T > 0$ small enough such that (3.22) holds. Then Lemma 2.4 implies that $\{u_j\}_{j\geq 0}$ is a Cauchy sequence in $C([0,T]; L^{\frac{\beta N}{2-\beta}}(\Omega))$, and converges to a unique solution $u \in C([0,T]; L^{\frac{\beta N}{2-\beta}}(\Omega))$ of the integral equation (1.5).

Moreover, recalling that

$$u_0(x,t) = E_\alpha(-t^\alpha \mathcal{A})\varphi,$$

and combining this with Proposition 2.5, we obtain

$$R(T)_0 \leq \max\Big\{ \sup_{0\leq t\leq T} t^{\frac{\alpha}{2\beta}-\frac{\alpha\gamma}{4\beta^2}} \|\nabla u_0(t)\|_{\frac{\beta^2 N}{\gamma}}, \sup_{0\leq t\leq T} t^{\frac{\alpha}{2}} \|\nabla^2 u_0(t)\|_{\frac{\beta N}{2-\beta}} \Big\} \leq C\|\varphi\|_{\frac{\beta N}{2-\beta}}.$$

So if $\|\varphi\|_{\frac{\beta N}{2-\beta}}$ is sufficiently small, the solution u can be extended to be global. The proof of Theorem 1.2 is complete. $\square$

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