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OPTIMAL SWITCHING OF VACCINATION FOR AN INFECTIOUS DISEASE MODEL

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Abstract. We study a switched SIHR (Susceptible Infected Hospitalized Recovered) model for infectious diseases. The aim is to find the most effective switching signal to manage the disease's effects as effectively as maintaining a continuous vaccination program. A nonlinear, non-convex optimal control problem of the switched system is converted into a linear and convex optimal control problem by applying the theory of moments and semi-definite programming. Finally, some numerical simulations are performed to compare continuous vaccination programs and optimal switching of vaccination control.

1. Introduction

For many decades and even centuries, differential equations have served as powerful tools for modelling a wide range of complex phenomena. These models can be based on either ordinary differential equations (ODEs) or partial differential equations (PDEs). Some examples include modelling the spread of infectious diseases [15], analyzing the impact of criminal behaviour on society [19], studying diseases such as Alzheimer's [2, 9], and applications in engineering and fluid dynamics [22]. In this article, we specifically focus on a mathematical model related to controlling the spread of infectious diseases.

Because of the spread of several infectious diseases, mathematical modelling of these diseases has gained much attention in the past few decades. Modeling infectious diseases and even epidemics helps study all kinds of diseases in different situations. There is a vast amount of literature on this. Numerous mathematical elements can be incorporated to enhance disease modeling. For instance, one can include delay, which can be time-dependent or state-dependent, etc., or even impulse, which can be instantaneous or non-instantaneous, and many more. Some of the articles and books for the references can be [4, 7, 27, 30]. The primary objective of approaching the study of infectious diseases in this manner is to achieve several vital aims: controlling disease transmission through various interventions, forecasting disease spread, or merely comprehending the dynamics of infectious diseases to enhance our readiness for future scenarios. The most prevalent strategies for managing an infectious disease involve vaccination and isolation. The optimal control of these interventions provides an optimum manner in which the disease can be

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controlled under a particular objective function. Since there is much literature on optimal control of infectious diseases, some references specific to optimal control by vaccination are [5, 10, 11, 21].

A switched system is a type of hybrid system in which the system switches between several subsystems depending on some conditions or a switching signal. They have vast applications in engineering [6]. Switched systems can also be employed in the mathematical modeling of infectious diseases. This approach is particularly valuable for emulating real-world scenarios through mathematical models, which can be challenging otherwise, for example, a limited stockpile of drugs and vaccines and the effect of seasonality in the disease transmission [16, 26]. Impulses can also be included in a switched infectious disease model [25]. Optimal control of a switched system can also be an interesting application to disease modeling [28, 31].

In a switched system, determining the optimal control for the system becomes a challenging problem when the sequence of subsystems or the switching signal is not known in advance. This problem can be divided into two main parts, the first of which involves identifying the optimal instants to switch between subsystems, and this in itself is a complex task [29, 8]. Although numerically expensive, some methods like dynamic programming and gradient descent are used to evaluate the optimal switching signal [1]. An innovative approach to this challenge is studied in [17]. This novel method relies on leveraging the theory of moments for global polynomial optimization using semi-definite programming to determine the optimal switching instants. Several research articles discussing this approach include [3, 13, 23].

This article employs the methodology described in [3, 17]. The application of this method to modeling and controlling infectious diseases is also discussed in [20, 23]. Notably, both articles do not consider the population of critically ill or hospitalized individuals. While in [23], the treatment category is treated equivalently to hospitalization; this article assumes that individuals receiving treatment or undergoing isolation are within the infected group, with only critically ill individuals hospitalized. Vaccination is considered the only control intervention. The goal is to minimize the infected and hospitalized populations. To assess the impact of limited vaccination, specifically vaccinations administered at optimal switching instants, the numerical simulations are compared against continuous vaccination scenarios.

This article is structured as follows: The model is formulated in section 2. Section 3 covers the evaluation of equilibrium points and the Basic Reproduction Number. Section 4 defines the model as a switched system and then outlines the primary problem, which is the Switched Optimal Control Problem (SOCP). By using the Lagrange polynomial, the SOCP is transformed to its polynomial equivalent of the optimal control problem (PEOCP) in section 5. Finally, in section 6, the moments and moment matrix are discussed briefly, and by using the moments approach, a semi-definite program (SDP) is formulated. Section 7 illustrates the results with a numerical example.

2. Model formulation

The objective of this study is to examine the effect of vaccination as a primary intervention to control the spread of disease. We work on the SIHR model with

critically ill individuals admitted to hospitals.

$$\begin{aligned} S' &= \mu - \beta SI - \mu S - \lambda\epsilon S, \\ I' &= \beta SI - \alpha I - \gamma_1 I + \delta H - (\mu + \mu_1)I, \\ H' &= \alpha I - \delta H - \gamma_2 H - (\mu + \mu_2)H, \\ R' &= \gamma_1 I + \gamma_2 H - \mu R + \lambda\epsilon S, \end{aligned} \tag{2.1}$$

with the initial condition at $t = t_0$ as $S(t_0)$, $I(t_0)$, $H(t_0)$, and $R(t_0)$. The nonlinear SIHR epidemiological model includes four non-negative state variables $S(t)$, $I(t)$, $H(t)$, and $R(t)$ that are defined as Susceptible, Infected, Hospitalized, and Recovered respectively. Here, $S(t)$ represents the number of individuals susceptible to the infection (i.e., they are not infected yet). The individuals who get infected move to the infected compartment at rate β. Individuals in this compartment spread the disease. All infected individuals undergoing treatment or are quarantined are considered in this compartment. From this compartment, only the critically ill humans are shifted to hospitals at rate α. The recovered individuals move to the recovered compartment from the infected compartment at rate γ_1 and from the hospitalized compartment at rate γ_2. Death due to the disease is at rate μ_1 in the infected compartment and μ_2 in the hospitalized compartment. μ is the natural mortality rate. The parameter λ is the vaccination rate, and ϵ is the control parameter, described later in the article. Figure 1 describes the flow diagram of the model (2.1).

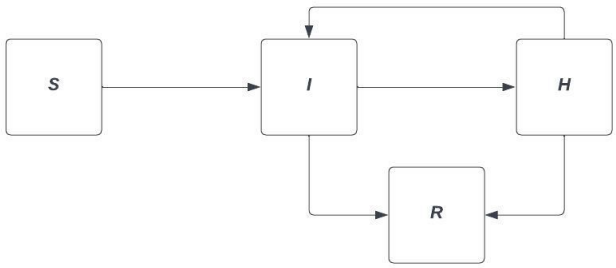

FIGURE 1. Conceptual flow diagram of the SIHR model.

3. Equilibria and basic reproduction number

The disease-free equilibrium point is,

$$E_1 = \Big(\frac{\mu}{\lambda\epsilon + \mu}, 0, 0, \frac{\lambda\epsilon}{\lambda\epsilon + \mu}\Big).$$

The endemic equilibrium is denoted as, $E_2 = (S^*, I^*, H^*, R^*)$. Next, using the next-generation matrix [24], to calculate the basic reproduction number ($\mathcal{R}_0$). Let the population be grouped in k number of compartments. Consider x_i as the population in each compartment.

Then, a general model describing the movement of humans between the compartments can be written as

$$\dot{x}_i = \mathcal{F}_i(x) - \mathcal{V}_i(x), \quad \text{where } x = (x_1, \ldots, x_k)^T.$$

We define $\mathcal{F}_i$ as the influx of infected population, and $\mathcal{V}_i$ denotes the outflow of population from the i^{th} compartment. $\mathcal{V}_i$ can be further written as, $\mathcal{V}_i = \mathcal{V}_i^+ - \mathcal{V}_i^-$. Furthermore, since each function represents the transfer of individuals, it is important to consider $x \geq 0$, $\mathcal{F}_i \geq 0$, $\mathcal{V}_i^+ \geq 0$, $\mathcal{V}_i^- \geq 0$, for $i = 1, \ldots, k$.

If E_1 is the equilibrium point and F and V are the $k \times k$ matrices defined as,

$$F = \left[\frac{\partial \mathcal{F}_i}{\partial x_j}(E_1)\right], \quad V = \left[\frac{\partial \mathcal{V}_i}{\partial x_j}(E_1)\right],$$

then Basic reproduction is defined as

$$\mathcal{R}_0 = \rho(FV^{-1}). \tag{3.1}$$

We calculate the basic reproduction number $\mathcal{R}_0$ for the model (2.1) by using the method given above. We have

$$\mathcal{F} = \begin{bmatrix} \beta SI + \delta H \\ 0 \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} (\alpha + \gamma_1)I + (\mu + \mu_1)I \\ \delta H + \gamma_2 H + (\mu + \mu_2)H - \alpha I \end{bmatrix},$$

where $\mathcal{F}$ and $\mathcal{V}$ are the column vectors of $\mathcal{F}_i$ and $\mathcal{V}_i$, respectively. Then, by considering the notation defined above, we obtain

$$F = \begin{bmatrix} \beta S & \delta \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \alpha + \gamma_1 + \mu + \mu_1 & 0 \\ -\alpha & \delta + \mu + \mu_2 + \gamma_2 \end{bmatrix}.$$

Therefore, the Basic reproduction number is

$$\mathcal{R}_0 = \frac{\mu\beta}{(\lambda\epsilon + \mu)(\alpha + \gamma_1 + \mu + \mu_1)} + \frac{\delta\alpha}{(\alpha + \gamma_1 + \mu + \mu_1)(\delta + \mu + \mu_2 + \gamma_2)}.$$

4. SIHR model as switched system and optimal control problem

Let $y = (S, I, H, R)^\top$. Then we can write model (2.1) as a switched system,

$$\dot{y}(t) = g_{\omega(t)}(y(t)),$$

where $g_i : \mathbb{R}^4 \to \mathbb{R}^4$ is the i^{th} vector field and $\omega : [t_0, T] \to \mathfrak{S}$ is the switching signal. The switching signal $\omega(t)$ is both a time-dependent function and a piece-wise constant. We assume $t_0 = 0$, so the interval becomes $[0, T]$ and initial conditions are $y(0) = (S(0), I(0), H(0), R(0)^\top$. Here ϵ will be the control and switching parameter, i.e., $\epsilon = 0, 1$. Case $\epsilon = 0$ means that there is no vaccination. In real life, it can mean that there is no stock of vaccination doses or that there is no vaccination drive. Case $\epsilon = 1$ signifies the presence of vaccination as an intervention. Each mode in the switching system is associated with a particular subsystem. It is defined as, $\dot{y}(t) = g_i(y(t))$ for $i \in \mathfrak{S} = \{0, 1\}$. The subsystem $i = 0$ corresponds to the case when $\epsilon = 0$ and subsystem $i = 1$ means the case when $\epsilon = 1$. Hence, the system switches between two subsystems according to the presence of vaccination control.

Before further analysis, we shall assume some conditions. Since the switched system consists of a duplet of a finite sequence of nodes and switching time $t_0 = 0 < t_1 < \cdots < t_n = T$, then:

- There are no jump discontinuities in the state.
- There are no infinite switching accumulation points.

The main goal of this work is to study the impact of optimal switching instants with respect to an objective function. Let us assume the objective functional to be

$$J = \int_{t_0=0}^{T} \{I(t) + H(t)\} dt.$$

Let the running cost be denoted by $\mathcal{L}_{\omega(t)}$,i.e., $\mathcal{L}_{\omega(t)}(t, y(t)) = I(t) + H(t)$. Hence, $\mathcal{L}_0 = I(t) + H(t)$ and $\mathcal{L}_1(t, y(t)) = I(t) + H(t)$. Therefore, the switched optimal control problem (SOCP) can be written as

$$\min_{\omega(t)} J(t_0, T, y(t), \omega(t)), \tag{4.1}$$

subject to

$$\dot{y}(t) = g_{\omega(t)}(y(t)), \quad \text{where } \omega(t) \in \mathfrak{S} = \{0, 1\}. \tag{4.2}$$

5. Polynomial equivalent optimal control problem (PEOCP)

We will first define the above SOCP (4.1) and (4.2) in the form of a continuous system, which is not a switched system with a control variable. A polynomial expression is built, which can mimic the behaviour of the switched system. Like the approach used in [17], Lagrange polynomials transform the switched system into a continuous one. The control variable shall be denoted by ξ, and $\xi \in \Delta = \{\xi \in \mathbb{R} |\ \phi(\xi) = 0\}$ where,

$$\phi(\xi) = \xi(1 - \xi). \tag{5.1}$$

According to the kth Lagrange polynomial $p_k(\xi)$, if $k = 0, 1$, then

$$p_0(\xi) = (1 - \xi),\ \ p_1(\xi) = \xi. \tag{5.2}$$

From the results in [17], we can write system (4.2) as

$$\dot{y}(t) = \mathcal{G}(y, \xi) = g_0(y)p_0(\xi) + g_1(y)p_1(\xi). \tag{5.3}$$

Similarly, the running cost can be written as

$$\mathcal{L}(y(t), \xi) = \mathcal{L}_0(t, y(t))p_0(\xi) + \mathcal{L}_1(t, y(t))p_1(\xi). \tag{5.4}$$

Therefore (4.1) becomes

$$J = \int_{t_0=0}^{T} \mathcal{L}(y(t), \xi)dt.$$

Therefore, we can formulate the PEOCP as

$$\min_{\xi \in \Delta}\ J(t_0, T, y(t), \xi), \tag{5.5}$$

subject to

$$\dot{y}(t) = \sum_{k=0}^{1} g_k(y)p_k(\xi). \tag{5.6}$$

The problem becomes non-convex because the set Δ is non-convex. To overcome it, we shall redefine the PEOCP using the moments technique.

6. Semi-definite relaxation using moments approach

In this section, we will first discuss the moments and localizing matrices and then move on to the semi-definite program for the PEOCP.

We will cover the topic of moments and moment matrices briefly. For more details, one can refer to [20, 13, 14] and the references within. The smallest closed set $\mathcal{Q} \subset \mathbb{R}^n$ such that $\nu(\mathbb{R}^n \backslash \mathcal{Q}) = 0$ for a measure ν on $\mathbb{R}^n$ is defined as the support of ν, expressed as $\text{supp}(\nu)$, and the measure ν is said to be supported by $\mathcal{Q}$ if $\text{supp}(\nu) \subset \mathcal{Q}$.

If ν is a probability measure supported by $\mathcal{Q} \subset \mathbb{R}$ and $\mathbb{R}[x]_n$ is the space of polynomials of degree at most n and in variable x, then the ith moment of ν is defined as

$$z_i = \int_{\mathcal{Q}} x^i \, \nu(dx).$$

It means for a polynomial $l(x) \in \mathbb{R}[x]_n$ written as $l(x) = \sum_{i=0}^{n} l_i x^i$,

$$\int_{\mathcal{Q}} l(x) \, \nu(dx) = \sum_{i=0}^{n} l_i z_i.$$

Now for probability measure ν, if $z = \{z_i\}_{i=0}^{2s}$ is a sequence of moments, then moment matrix $\mathcal{M}_s(z)$ is defined as a $(s+1) \times (s+1)$ matrix with (i,j) the entry as $z_{i+j}, 0 \le i, j \le s$. Hence, the matrix becomes,

$$\mathcal{M}_s(z) = \begin{bmatrix} z_0 & z_1 & z_2 & \cdots & z_s \\ z_1 & z_2 & z_3 & \cdots & z_{(s+1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ z_s & z_{(s+1)} & z_{(s+2)} & \cdots & z_{2s} \end{bmatrix}.$$

For the polynomial $l(x)$ and sequence of moments z defined above the Localising matrix can be defined as an $(s+1) \times (s+1)$ symmetric matrix with its (i,j)th entry as

$$\mathcal{M}_s(lz)(i,j) = \sum_{k=0}^{n} l_k z_{i+j+k}.$$

One can read [23, 12] for examples and other details. From the results discussed in [20, 23] for control variable $\xi \in \Delta$, and the constraint $\phi(\xi) = 0$ or $\phi(\xi) \ge 0$ and $-\phi(\xi) \ge 0$. We will denote

$$\phi_1 = \phi(\xi) \quad \text{and} \quad \phi_2 = -\phi(\xi).$$

The following two lemmas help formulate the semi-definite program.

Lemma 6.1. *If $z = \{z_i\}_{i=0}^{2s}$ is a sequence of moments of probability measure ν supported by $\mathcal{Q}$, then $\mathcal{M}_s(z) \succeq 0$.*

Lemma 6.2. *Assume that the polynomials $\phi_1(\xi)$ and $\phi_2(\xi)$ satisfies $d_\phi^1 = \lceil deg(\phi_1)/2 \rceil = 1$ and $d_\phi^2 = \lceil deg(\phi_2)/2 \rceil = 1$. Consider the sequence $z = \{z_i\}_{i=0}^{2s}$ of moments of probability measure ν supported by set $\mathcal{Q}_1 = \{\nu \in \mathbb{R} | \phi_1 \ge 0\}$. Then $M_{s-d_\phi^1}(\phi_1 z) \succeq 0$, i.e, $M_{s-1}(\phi_1 z) \succeq 0$ and similarly $M_{s-1}(\phi_2 z) \succeq 0$.*

Proof. Letting $h \in \mathbb{R}[\xi]_{(s-1)}$, that is $h(\xi) = \sum_{j=0}^{s-1} h_j \xi^j$, then

$$\begin{aligned} h^\top \mathcal{M}_{(s-1)}(\phi_1 z) h &= \sum_{\alpha=0}^{s-1} \sum_{\beta=0}^{s-1} h_\alpha h_\beta \mathcal{M}_{(s-1)}(\phi_1 z)(\alpha, \beta) \\ &= \sum_{\alpha=0}^{s-1} \sum_{\beta=0}^{s-1} h_\alpha h_\beta \big[0 + z_{\alpha+\beta+1} - z_{\alpha+\beta+2}\big] \\ &= \sum_{\alpha=0}^{s-1} \sum_{\beta=0}^{s-1} h_\alpha h_\beta \big[\int_{\mathcal{Q}_1} \xi^{\alpha+\beta+1} \, \nu(d\xi) - \int_{\mathcal{Q}_1} \xi^{\alpha+\beta+2} \, \nu(d\xi) \big] \end{aligned}$$

$$= \int_{\mathcal{Q}_1} (h(\xi))^2 \phi_1(\xi) \nu(d\xi) \geq 0.$$

Hence $\mathcal{M}_{(s-1)}(\phi_1 z) \succeq 0$. Similarly, we can consider measure μ supported by set $\mathcal{Q}_2 = \{\mu \in \mathbb{R} | \phi_2 \geq 0\}$ along with its corresponding sequence of moments and proceeding as above, it is not difficult to prove that $\mathcal{M}_{(s-1)}(\phi_2 z) \succeq 0$. □

Suppose that Δ is a Borel subset of $\mathbb{R}^n$ and $P(\Delta)$ is the space of probability measures ν which has support contained in Γ. Then we can say that

$$\min_{\xi \in \Delta} J = \min_{\nu \in P(\Delta)} \int_{\Delta} J \nu(d\xi). \tag{6.1}$$

The proof of (6.1) can be found in [13] and [12]. Hence the minimization problem (5.5) becomes

$$\begin{aligned} \min_{\xi \in \Delta} J(t_0, T, y(t), \xi) &= \min_{\nu \in P(\Delta)} \int_{\Delta} J \, \nu(d\xi) \\ &= \min_{\nu \in P(\Delta)} \int_{\Delta} \int_{t_0=0}^{T} \mathcal{L}(y, \xi) \, dt \, \nu(d\xi). \end{aligned}$$

Then $\min_{\nu \in P(\Delta)} \int_{\Delta} \int_{t_0=0}^{T} \mathcal{L}(y, \xi) \, dt \, \nu(d\xi)$ is equal to

$$\begin{aligned} &\min_{\nu \in P(\Delta)} \int_{\Delta} \int_{t_0=0}^{T} \mathcal{L}_0(t, y(t)) p_0(\xi) + \mathcal{L}_1(t, y(t)) p_1(\xi) \, dt \, \nu(d\xi) \\ &= \min_{\nu \in P(\Delta)} \int_{\Delta} \int_{t_0=0}^{T} \{(\mathcal{L}_0 + \mathcal{L}_1)\xi + \mathcal{L}_0\} \, dt \, \nu(d\xi) \\ &= \min_{z \in \mathcal{Z}} \int_{t_0=0}^{T} \{(\mathcal{L}_0 + \mathcal{L}_1) z_1 + \mathcal{L}_0 z_0\} \, dt \\ &= \min_{z \in \mathcal{Z}} \int_{t_0=0}^{T} \sum_{k=0}^{1} \sum_{i=0}^{1} \mathcal{L}_k(t, y(t)) c_{ki} z_i dt, \end{aligned}$$

where $\mathcal{Z}$ is the space of moments i.e., $\mathcal{Z} = \{z = \{z_i\} | z_i = \int_{\Delta} \xi^i \nu(d\xi), \nu \in P(\Delta)\}$. Also, $c_{00} = 1$, $c_{01} = -1$, $c_{10} = 0$, and $c_{11} = 1$.

Now, by using the approach in [17], a semi-definite program (SDP) of relaxation order $s \geq 1$ can be formulated as SDP_s:

$$\begin{aligned} \min_{z \in \mathcal{Z}} \int_{t_0=0}^{T} \sum_{k=0}^{1} \sum_{i=0}^{1} \mathcal{L}_k(t, y(t)) c_{ki} z_i \, dt, \\ M_s(z) \succeq 0, \\ M_{s-1}(\phi_1 z) \succeq 0, \\ M_{s-1}(\phi_2 z) \succeq 0, \\ \dot{y}(t) = \sum_{k=0}^{1} \sum_{i=0}^{1} g_k(y) c_{ki} z_i. \end{aligned} \tag{6.2}$$

So, the SDP for the lowest order of relaxation is for $s = 1$. which is SDP_1:

$$\begin{aligned} \min_{z\in\mathcal{Z}} \int_{t_0=0}^{T} \sum_{k=0}^{1}\sum_{i=0}^{1} \mathcal{L}_k(t, y(t)) c_{ki} z_i \, dt, \\ M_1(z) \succeq 0, \\ M_0(\phi_1 z) \succeq 0, \\ M_0(\phi_2 z) \succeq 0, \\ \dot{y}(t) = \sum_{k=0}^{1}\sum_{i=0}^{1} g_k(y) c_{ki} z_i. \end{aligned} \tag{6.3}$$

7. Numerical simulations

We perform some numerical simulations to support the analysis presented above. The parameter values under consideration are not specific to any particular disease (table 1), and research articles that influence them are [23, 18]. The reproduction number is calculated to be $\mathcal{R}_0 = 2.05$ when there are no vaccination strategies, i.e., $\epsilon = 0$ and $\mathcal{R}_0 = 0.24$ when there is continuous vaccination, i.e., $\epsilon = 1$. The method followed for the numerical simulation is similar to [23].

Table 1. Parameters used

Parameter	Value	Parameter	Value
β	0.99	δ	0.1
μ	0.033	γ_1	0.2
$\mu_1 = \mu_2$	0.0833	γ_2	0.1
α	0.2	λ	0.5
$S(0)$	0.79	$I(0)$	0.16
$H(0)$	0.05	$R(0)$	0

The primary aim of this research study is to reduce the number of people in both the infected and hospitalized compartments. This objective has been chosen because it is more cost-effective to promote recovery through vaccination than treating individuals with specific medications after infection. Furthermore, hospitalization costs significantly exceed the expense of a vaccination dose. Here, we minimize the objective function by finding the optimal number of switches of vaccination so that we get similar behavior to continuous vaccination and the disease is in control even when $\mathcal{R}_0 > 1$.

In Figure 2, it is illustrated that the susceptible population is the least in the presence of continuous vaccination. We can conclude that most of the population recovers from vaccination and will never be infected. Conversely, in case of no vaccination, the susceptible population is maximum, which means that all may get infected in the future. For the optimal switching of vaccination control, the susceptible population is close to that in the case of continuous vaccination.

In Figures 3 and 4, the time series of the population in the infected and hospitalized compartments are the same for both continuous vaccination and vaccinating optimally. Since $\mathcal{R}_0 > 1$ in case of no vaccination control, the infected and hospitalized compartment population is stable at the endemic equilibrium. Finally, in Figure 5, the recovered population is maximum for the case of constant vaccination

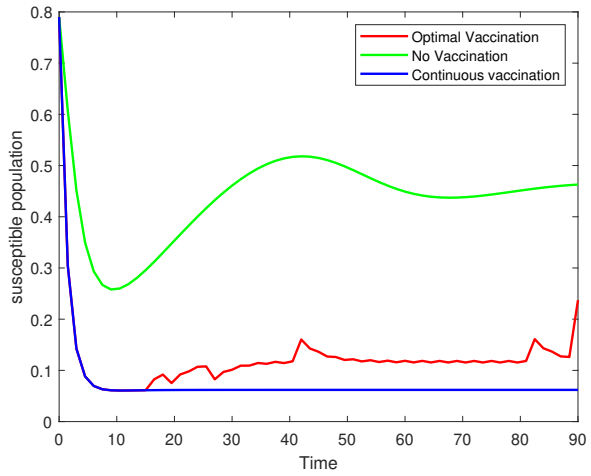

FIGURE 2. Time evolution of model (2.1) when $\mathcal{R}_0 > 1$.

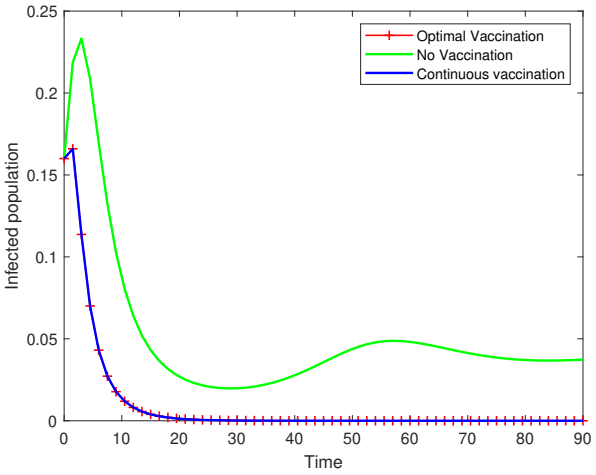

FIGURE 3. Time evolution of model (2.1) when $\mathcal{R}_0 > 1$.

and minimum for no vaccination. Finally, Figure 6 depicts the optimal switching signal. The optimal switching signal indicates that initiating vaccination strategies at the onset of an epidemic is critical for controlling the disease, and the availability of vaccines (duration of vaccination campaigns) can be gradually reduced over time. Based on all the figures related to the numerical simulation, it can be concluded that even when the reproduction number exceeds one and the availability of vaccines is limited, the disease can still be effectively controlled through the optimal use of vaccination. The effect of switching vaccination strategies optimally, while not identical, is nearly as effective as continuous vaccination of the population.

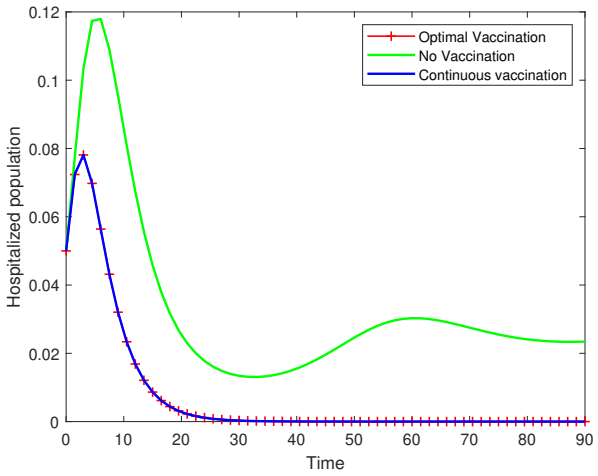

FIGURE 4. Time evolution of model (2.1) when $\mathcal{R}_0 > 1$.

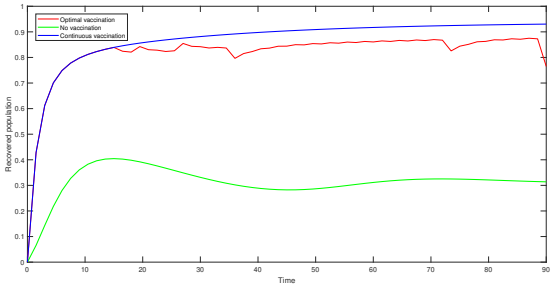

FIGURE 5. Time evolution of model (2.1) when $\mathcal{R}_0 > 1$.

8. Conclusion

This research article studies a simple SIR model with hospitalization as a different compartment. In light of the global experience with a pandemic and the lessons learned from being ill-prepared, it is imperative to take proactive measures for the future. Various interventions have been devised to manage the disease, but in this article, we focus solely on vaccination control. It is important to note that vaccination doses can be limited based on the available stockpile within a city or country, making them potentially unavailable at times. Hence, we considered vaccination as a switched intervention. By using the strategy in [17], we solve the switched control problem. Implementing vaccination at the initial stages of disease spread is crucial, as this is when infections escalate rapidly. As time progresses, the period of effective control diminishes, and the intervals between vaccination drives can be increased. To conclude, optimal switching and Optimal control of the control interventions can help control the spread of disease even when its reproduction number is greater than 1.

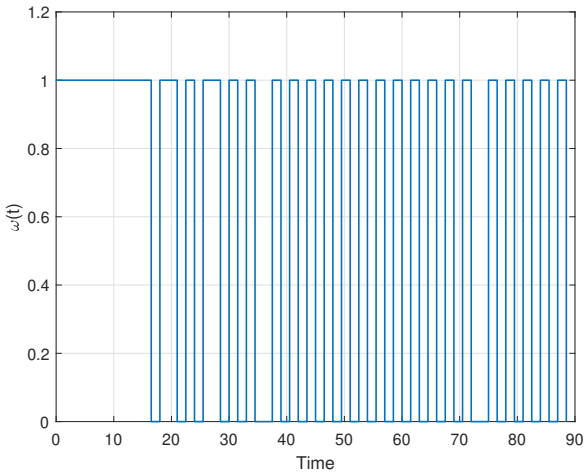

FIGURE 6. Time evolution of model (2.1) when $\mathcal{R}_0 > 1$.

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