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ASYMPTOTIC ANALYSIS OF SIGN-CHANGING TRANSMISSION PROBLEMS WITH RAPIDLY OSCILLATING INTERFACE

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Abstract. We study the asymptotic behavior of a sign-changing transmission problem, stated in a symmetric oscillating domain obtained by gluing together a positive and a negative material, separated by an imperfect and rapidly oscillating interface. The interface separating the two heterogeneous materials has a periodic microstructure and is a small perturbation of a flat interface. The solution of the transmission problem is continuous and its flux has a jump on the oscillating interface. Under certain conditions on the properties of the two materials, we derive the limit problem and we prove the convergence result. The **T**-coercivity method is used to handle the lack of coercivity for both the microscopic and the macroscopic limit problems.

1. Introduction

Metamaterials are artificial composite materials with unusual properties. For instance, electromagnetic metamaterials can exhibit negative dielectric permittivity and magnetic permeability, leading to a negative index of refraction [31, 30]. Among the applications of these materials in optics, one can mention sub-diffraction imaging or sensing and detection technologies. In practice, these negative materials are usually in contact with classical positive materials and this destroys the coercivity of the underlying operators governing the physics of the problem. This leads to several difficulties from both the mathematical (well-posedness) and numerical (convergence analysis) viewpoints. Reference [7] is one of the first papers dealing with the well-posedness issue and this was done using the **T**-coercivity approach. Special interest has been devoted to the particular case of a symmetric indefinite scalar transmission problem through a smooth interface (see [7, Section 3.4]). Inspired by this geometry, our goal in this paper is to investigate a more general situation by considering three modifications with respect to the problem studied in [7, Section 3.4]:

- we consider an interface rapidly oscillating at a speed of ε^{-1}, where ε is a small parameter, and with a small amplitude of order ε^{k+1}, $k \geqslant 0$ being a positive real number (see Figure 1);

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- we assume that the two materials separated by the rapidly oscillating interface, namely the positive and the negative one, are both anisotropic and strongly heterogeneous;
- we impose imperfect transmission conditions across the oscillating interface, with a continuous solution and a discontinuous flux (see problem (2.7)).

Because of the rapid oscillations of the upper and lower boundaries and of the interface, the microscopic problem is difficult to handle numerically. It is natural then to perform an asymptotic analysis, in order to derive an equivalent macroscopic problem, set in a domain with flat boundaries and interface (see Figure 1). We prove that, as $\varepsilon \to 0$, the solution of the microscopic problem (2.7) converges to the unique solution of a well-posed indefinite limit problem (4.6), involving the homogenized matrices associated with each sub-domain.

Owing to the indefinite character of the problem, the well-posedness of the microscopic problem (2.7) needs a careful analysis. This is done by using the **T**-coercivity method (see [7]), which allows us to obtain a well-posedness result and uniform energy estimates, under certain conditions on the coefficients describing the properties of the two heterogeneous materials. Let us emphasize that these well-posedness conditions involve a real number κ (see (3.14)), which can be seen as a generalized contrast between the positive and negative materials.

With the uniform estimates in hand, we can pass to the limit in the bulk terms by adapting techniques from [24]. For the term on the interface, we combine results from [18] with the periodic unfolding method [19]. For obtaining the boundary conditions of the limit solution, we use the zero extension of the microscopic solution and results from [18].

For definite problems in a similar geometrical configuration, presenting an oscillating interface, we refer to [10, 24, 22, 23, 29, 5]. In all these studies, imperfect transmission conditions across the oscillating interface were considered, but in contrast to our case, the flux of the solution was supposed there to be continuous and proportional to the jump of the solution. For the asymptotic analysis of definite problems in two-component periodic composites involving flux jump, we refer, for instance, to [28, 25, 26, 16, 17, 21]. For similar problems stated in domains with oscillating boundaries, we refer to [6, 2, 20, 3, 27, 4].

For the asymptotic analysis of indefinite problems in a different geometrical setting, namely a two-composite medium with periodically distributed negative inclusions, we refer to [12, 9, 13, 11, 14, 15].

This article is organized as follows. In Section 2, we state the problem under study and the notation. In Section 3, we use the **T**-coercivity method to prove the well-posedness of the microscopic problem, under certain conditions on the properties of the materials. Theorem 3.6 provides the main result of this section, namely the energy estimate for the unique solution of the microscopic problem. In Section 4, we pass to the limit in the weak formulation of the microscopic problem and obtain the limit macroscopic problem. This indefinite limit problem is showed to be well-posed and the convergence is proved in Theorem 4.2. Finally, we collect in the Appendix some results on the unfolding method used to prove certain of the convergence results.

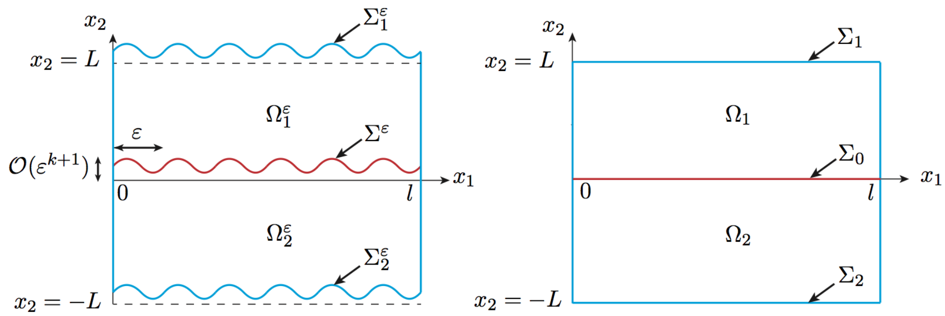

FIGURE 1. Description of the geometry: microscopic case $\varepsilon > 0$ (left) and limit case $\varepsilon = 0$ (right).

2. Problem setting

Given $\varepsilon \in (0,1)$, we consider a two-dimensional bounded domain Ω^ε constituted by two sub-domains Ω_1^ε and Ω_2^ε separated by an oscillating interface Σ^ε (see Figure 1). More precisely, given $L, l > 0$, we set

$$\Omega^\varepsilon = \{x = (x_1, x_2) : x_1 \in (0,l),\ -L + H^\varepsilon(x_1) < x_2 < L + H^\varepsilon(x_1)\}, \tag{2.1}$$

$$\begin{gathered}\Omega_1^\varepsilon = \{x = (x_1, x_2) \in \Omega^\varepsilon : x_2 > H^\varepsilon(x_1)\},\\ \Omega_2^\varepsilon = \{x = (x_1, x_2) \in \Omega^\varepsilon : x_2 < H^\varepsilon(x_1)\},\\ \Sigma^\varepsilon = \{x = (x_1, x_2) \in \Omega^\varepsilon : x_2 = H^\varepsilon(x_1)\}.\end{gathered} \tag{2.2}$$

Note that Ω^ε is symmetric with respect to Σ^ε and that

$$\Omega^\varepsilon = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Sigma^\varepsilon.$$

The oscillating interface Σ^ε is described through the one-dimensional function H^ε given by

$$H^\varepsilon(x_1) = \varepsilon^{k+1} h\big(\frac{x_1}{\varepsilon}\big), \tag{2.3}$$

where $h \in \mathcal{C}^1([0,1];\mathbb{R}^+)$ is a 1-periodic function and $k \geqslant 0$. From now on, we assume, for simplicity, that ε is a sequence of strictly positive numbers such that $l/\varepsilon \in \mathbb{N}^*$. We set

$$\overline{h} = \|h\|_{L^\infty(0,1)}, \quad \overline{h'} = \|h'\|_{L^\infty(0,1)}, \tag{2.4}$$

which we suppose to be finite, with $\overline{h} < L$. We also introduce the oscillating upper and lower boundaries of the domain Ω^ε, described by

$$\Sigma_1^\varepsilon = \{x = (x_1, x_2) : x_1 \in (0,l),\ x_2 = L + H^\varepsilon(x_1)\} \tag{2.5}$$

$$\Sigma_2^\varepsilon = \{x = (x_1, x_2) : x_1 \in (0,l),\ x_2 = -L + H^\varepsilon(x_1)\}. \tag{2.6}$$

We denote by n^ε the unit exterior normal to Ω_1^ε. For any function v defined on Ω^ε, we denote by v_1 and v_2 its restrictions to Ω_1^ε and to Ω_2^ε, respectively.

We point out that the results of this paper are still valid in the n-dimensional case ($n \geqslant 3$).

Assume that the sub-domains Ω_1^ε and Ω_2^ε are occupied by a positive and, respectively, by a negative material, described by anisotropic matrix-valued coefficients $A_1^\varepsilon(x)$ and $A_2^\varepsilon(x)$. Given a volume source term f and a prescribed jump flux g^ε, our goal is to analyze the asymptotic behavior, as $\varepsilon \to 0$, of the solution $u^\varepsilon \in H_0^1(\Omega^\varepsilon)$ of the indefinite transmission problem

$$
\begin{gathered}
-\operatorname{div}(A_1^\varepsilon(x)\nabla u_1^\varepsilon) = f \quad \text{in } \Omega_1^\varepsilon \\
-\operatorname{div}(A_2^\varepsilon(x)\nabla u_2^\varepsilon) = f \quad \text{in } \Omega_2^\varepsilon \\
u^\varepsilon = 0 \quad \text{on } \partial\Omega^\varepsilon \\
u_1^\varepsilon - u_2^\varepsilon = 0 \quad \text{on } \Sigma^\varepsilon \\
A_1^\varepsilon(x)\nabla u_1^\varepsilon \cdot n^\varepsilon - A_2^\varepsilon(x)\nabla u_2^\varepsilon \cdot n^\varepsilon = g^\varepsilon \quad \text{on } \Sigma^\varepsilon.
\end{gathered} \tag{2.7}
$$

On the exterior boundary, we prescribed homogeneous Dirichlet boundary conditions. We remark that, across the oscillating interface Σ^ε, the solution u^ε of problem (2.7) is continuous, while its flux exhibits a jump.

Let us now make more precise the hypotheses on the coefficients A_1^ε and A_2^ε and on the data f and g^ε. Let $\mathcal{M}^s$ be the linear space of 2×2 symmetric matrices. Following [1, Chapter 1], given two positive constants $\alpha, \beta > 0$, with $\alpha\beta \leqslant 1$, we define the subspace $\mathcal{M}^s_{\alpha,\beta}$ of coercive matrices with coercive inverses

$$
\mathcal{M}^s_{\alpha,\beta} := \left\{ M \in \mathcal{M}^s : M\xi\cdot\xi \geqslant \alpha|\xi|^2,\, M^{-1}\xi\cdot\xi \geqslant \beta|\xi|^2,\, \forall \xi \in \mathbb{R}^2 \right\}. \tag{2.8}
$$

As pointed out in [1, Remark 1.2.9], it is worth noticing that if $M \in \mathcal{M}^s_{\alpha,\beta}$ (and even if M is not symmetric), then one necessarily has

$$
\alpha|\xi|^2 \leqslant M\xi\cdot\xi \leqslant \beta^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^2. \tag{2.9}
$$

The above relation shows, in particular, why we need to impose the condition $\alpha\beta \leqslant 1$, as soon as $\mathcal{M}^s_{\alpha,\beta}$ is not empty. With these notation, we make the following assumptions.

(P1) We denote by $L^\infty(Y; \mathcal{M}^s_{\alpha,\beta})$ the set of matrix-valued bounded functions defined on Y with values in $\mathcal{M}^s_{\alpha,\beta}$. Let $A_1 = (a^1_{ij})_{1\leqslant i,j\leqslant 2}, A_2 = (a^2_{ij})_{1\leqslant i,j\leqslant 2} \in \mathcal{M}^s$ be 2×2 real symmetric matrix-valued functions defined on $Y = (0,1)^2$ and extended to the whole plane by Y-periodicity. We assume that $A_1 \in L^\infty(Y; \mathcal{M}^s_{\alpha_1,\beta_1})$ and $-A_2 \in L^\infty(Y; \mathcal{M}^s_{\alpha_2,\beta_2})$ for some positive constants $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$. In particular, according to (2.9), we have for every $\xi \in \mathbb{R}^2$ and for almost every $y \in Y$

$$
\alpha_1|\xi|^2 \leqslant A_1(y)\xi\cdot\xi \leqslant \beta_1^{-1}|\xi|^2, \quad \alpha_2|\xi|^2 \leqslant -A_2(y)\xi\cdot\xi \leqslant \beta_2^{-1}|\xi|^2. \tag{2.10}
$$

For almost every $x \in \mathbb{R}^2$, we define the ε-periodic functions

$$
A_1^\varepsilon(x) := A_1\big(\frac{x}{\varepsilon}\big), \quad A_2^\varepsilon(x) := A_2\big(\frac{x}{\varepsilon}\big). \tag{2.11}
$$

Obviously, it follows from (2.10) that for every $\xi \in \mathbb{R}^2$ and for almost every $x \in \mathbb{R}^2$

$$
\alpha_1|\xi|^2 \leqslant A_1^\varepsilon(x)\xi\cdot\xi \leqslant \beta_1^{-1}|\xi|^2, \quad \alpha_2|\xi|^2 \leqslant -A_2^\varepsilon(x)\xi\cdot\xi \leqslant \beta_2^{-1}|\xi|^2. \tag{2.12}
$$

(P2) The function f belongs to $L^2(\widetilde{\Omega})$, where $\widetilde{\Omega} = (0,l)\times(-L, 2L)$.

(P3) We assume that, for any $\varepsilon > 0$,

$$
g^\varepsilon(x) = g\big(\frac{x_1}{\varepsilon}\big), \quad \forall x = (x_1, H^\varepsilon(x_1)) \in \Sigma^\varepsilon, \tag{2.13}
$$

where $g \in L^\infty(0,1)$ is a 1-periodic function and H^ε is defined in (2.3).

Let $V^\varepsilon = H_0^1(\Omega^\varepsilon)$, endowed with the standard gradient norm. The variational formulation of problem (2.7) is the following one: find $u^\varepsilon \in V^\varepsilon$ such that

$$\mathcal{A}^\varepsilon(u^\varepsilon, v) = \ell^\varepsilon(v), \quad \forall v \in V^\varepsilon, \tag{2.14}$$

where the bilinear form $\mathcal{A}^\varepsilon : V^\varepsilon \times V^\varepsilon \to \mathbb{R}$ and the linear form $\ell^\varepsilon : V^\varepsilon \to \mathbb{R}$ are given by

$$\mathcal{A}^\varepsilon(u, v) = \int_{\Omega_1^\varepsilon} A_1^\varepsilon(x)\nabla u_1(x) \cdot \nabla v(x)\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} A_2^\varepsilon(x)\nabla u_2(x) \cdot \nabla v(x)\,\mathrm{d}x, \tag{2.15}$$

$$\ell^\varepsilon(v) = \int_{\Omega^\varepsilon} f(x)v(x)\,\mathrm{d}x + \int_{\Sigma^\varepsilon} g^\varepsilon(x_1)v(x)\,\mathrm{d}\sigma_x. \tag{2.16}$$

Throughout this article, C will denote a positive constant, independent of ε, whose value can change from line to line.

3. Well-posedness

Since the bilinear form $\mathcal{A}^\varepsilon(\cdot,\cdot)$ given by (2.15) is indefinite (because of (2.10) and (2.11)), one cannot use Lax-Milgram lemma to obtain a well-posedness result for the variational problem (2.14). Thus, for obtaining the well-posedness, we apply the $\mathbf{T}$-coercivity method introduced in [8] and used in [7] to study a large class of sign-changing scalar transmission problems. We start by recalling the definition of $\mathbf{T}$-coercivity.

Definition 3.1. Let $\mathbf{T} \in \mathcal{L}(V)$ be a bounded linear operator on a Hilbert space V. A bilinear form $a(\cdot,\cdot)$ defined on $V \times V$ is $\mathbf{T}$-coercive if there exists $\gamma > 0$ such that

$$a(u, \mathbf{T}u) \geqslant \gamma \|u\|^2, \quad \forall u \in V.$$

For the reader's convenience, we recall a well-posedness result given in [13, Theorem 3.2]) which shows that uniform $\mathbf{T}$-coercivity yields well-posedness and uniform estimates for variational problems involving a parameter.

Theorem 3.2. *Let V be a Hilbert space equipped with the norm $\|\cdot\|$ and let $\mathcal{A}^\varepsilon(\cdot,\cdot)$ be a bilinear form on V satisfying the following conditions.*

1. *$\mathcal{A}^\varepsilon(\cdot,\cdot)$ is symmetric: $\mathcal{A}^\varepsilon(u, v) = \mathcal{A}^\varepsilon(v, u)$, for all $u, v \in V$.*
2. *$\mathcal{A}^\varepsilon(\cdot,\cdot)$ is uniformly continuous: there exists $M > 0$ such that*
$$\mathcal{A}^\varepsilon(u, v) \leqslant M\|u\|\|v\|, \quad \forall u, v \in V. \tag{3.1}$$
3. *$\mathcal{A}^\varepsilon(\cdot,\cdot)$ is uniformly $\mathbf{T}$-coercive: there exists a family $(\mathbf{T}^\varepsilon)_{\varepsilon>0}$ of uniformly bounded linear operators on V and $\gamma > 0$ such that*
$$\mathcal{A}^\varepsilon(u, \mathbf{T}^\varepsilon u) \geqslant \gamma\|u\|^2, \quad \forall u \in V. \tag{3.2}$$

Then, given a uniformly bounded family $(\ell^\varepsilon)_{\varepsilon>0}$ in V', the space of linear forms on V, the variational problem

$$\text{find } u^\varepsilon \in V \text{ such that } \mathcal{A}^\varepsilon(u^\varepsilon, v) = \ell^\varepsilon(v), \quad \forall v \in V \tag{3.3}$$

admits a unique solution $u^\varepsilon \in V$ for all $\varepsilon > 0$ and there exists $C > 0$ independent of ε such that

$$\|u^\varepsilon\| \leqslant C. \tag{3.4}$$

We are going to use the above abstract result to investigate the well-posedness of the sign-changing transmission problem (2.7) set in Ω^ε. Our objective is to construct two families of uniformly $\mathbf{T}$-coercive operators and this will be done by using suitably chosen lifting (or extension) operators $\mathbf{R}_1^\varepsilon$ and $\mathbf{R}_2^\varepsilon$ for one sub-domain to another. More precisely, we first adapt [7, Theorem 2.1] to the anisotropic case studied here (see Proposition 3.3 below). This result shows that $\mathbf{T}$-coercivity holds provided the "maximal contrasts" between the positive and negative materials (measured through the positive numbers $\alpha_1\beta_2$ and $\alpha_2\beta_1$) are large enough compared to the norms of the lifting operators $\mathbf{R}_1^\varepsilon$ and $\mathbf{R}_2^\varepsilon$. Next, we obtain upper bounds for these lifting operators, by extending [7, Theorem 3.10] to the case of highly oscillating interface, paying a special attention to the dependence on ε of the involved constants (see Proposition 3.4 below).

Proposition 3.3. *We introduce the sub-spaces*

$$V_1^\varepsilon := \left\{ v_1 = v_{|\Omega_1^\varepsilon} : v \in H_0^1(\Omega^\varepsilon) \right\}, \quad V_2^\varepsilon := \left\{ v_2 = v_{|\Omega_2^\varepsilon} : v \in H_0^1(\Omega^\varepsilon) \right\},$$

endowed with the norms

$$\|v_1\|_{V_1^\varepsilon} = \|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}, \quad \|v_2\|_{V_2^\varepsilon} = \|\nabla v_1\|_{L^2(\Omega_2^\varepsilon)}.$$

Let $\mathbf{R}_1^\varepsilon \in \mathcal{L}(V_1^\varepsilon, V_2^\varepsilon)$ and $\mathbf{R}_2^\varepsilon \in \mathcal{L}(V_2^\varepsilon, V_1^\varepsilon)$ be two lifting operators:

- *$\mathbf{R}_1^\varepsilon \in \mathcal{L}(V_1^\varepsilon, V_2^\varepsilon)$ such that $(\mathbf{R}_1^\varepsilon u_1)_{|\Sigma^\varepsilon} = u_{1|\Sigma^\varepsilon}$ for all $u_1 \in V_1^\varepsilon$,*
- *$\mathbf{R}_2^\varepsilon \in \mathcal{L}(V_2^\varepsilon, V_1^\varepsilon)$ such that $(\mathbf{R}_2^\varepsilon u_2)_{|\Sigma^\varepsilon} = u_{2|\Sigma^\varepsilon}$ for all $u_2 \in V_2^\varepsilon$.*

We associate with these operators the two operators $\mathbf{T}_1^\varepsilon, \mathbf{T}_2^\varepsilon \in \mathcal{L}\left(H_0^1(\Omega_\varepsilon)\right)$ defined by:

$$\mathbf{T}_1^\varepsilon u := \begin{cases} u_1 & \text{in } \Omega_1^\varepsilon \\ -u_2 + 2\mathbf{R}_1^\varepsilon u_1 & \text{in } \Omega_2^\varepsilon, \end{cases} \quad \mathbf{T}_2^\varepsilon u := \begin{cases} u_1 - 2\mathbf{R}_2^\varepsilon u_2 & \text{in } \Omega_1^\varepsilon \\ -u_2 & \text{in } \Omega_2^\varepsilon. \end{cases} \tag{3.5}$$

Finally, assume that there exist $\rho_1^\star, \rho_2^\star > 0$ such that, for all $\varepsilon > 0$,

$$\|\mathbf{R}_1^\varepsilon\|^2 \leqslant \rho_1^\star, \quad \|\mathbf{R}_2^\varepsilon\|^2 \leqslant \rho_2^\star. \tag{3.6}$$

Then, under conditions (2.9), the following uniform $\mathbf{T}$-coercivity results for the bilinear form $\mathcal{A}^\varepsilon(\cdot,\cdot)$ defined by (2.15), hold.

- *If $\alpha_1\beta_2 \geqslant \rho_1^\star$, then $\mathcal{A}^\varepsilon(\cdot,\cdot)$ is uniformly $\mathbf{T}_1^\varepsilon$-coercive.*
- *If $\alpha_2\beta_1 \geqslant \rho_2^\star$, then $\mathcal{A}^\varepsilon(\cdot,\cdot)$ is uniformly $\mathbf{T}_2^\varepsilon$-coercive.*

Proof. Assume that $\alpha_1\beta_2 > \rho_1^\star$ and let us choose η_1 such that $\frac{\rho_1^\star}{\alpha_1\beta_2} < \eta_1 < 1$. Then, using Cauchy-Schwarz and Young inequalities together with (2.10), (3.5) and (3.6), we have that for every $u \in H_0^1(\Omega^\varepsilon)$,

$$\begin{aligned}
&\mathcal{A}^\varepsilon(u, \mathbf{T}_1^\varepsilon u) \\
&= \int_{\Omega_1^\varepsilon} A_1^\varepsilon \nabla u_1 \cdot \nabla u_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} \big(-A_2^\varepsilon\big) \nabla u_2 \cdot \nabla u_2 \,\mathrm{d}x + 2\int_{\Omega_2^\varepsilon} A_2^\varepsilon \nabla u_2 \cdot \nabla(\mathbf{R}_1^\varepsilon u_1) \,\mathrm{d}x \\
&\geqslant \int_{\Omega_1^\varepsilon} A_1^\varepsilon \nabla u_1 \cdot \nabla u_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} \big(-A_2^\varepsilon\big) \nabla u_2 \cdot \nabla u_2 \,\mathrm{d}x - \eta_1 \int_{\Omega_2^\varepsilon} \big(-A_2^\varepsilon\big) \nabla u_2 \cdot \nabla u_2 \,\mathrm{d}x \\
&\quad - \frac{1}{\eta_1} \int_{\Omega_2^\varepsilon} \big(-A_2^\varepsilon\big) \nabla(\mathbf{R}_1^\varepsilon u_1) \cdot \nabla(\mathbf{R}_1^\varepsilon u_1) \,\mathrm{d}x \\
&\geqslant \alpha_1 \|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)} + \alpha_2(1-\eta_1)\|\nabla u_2\|^2_{L^2(\Omega_2^\varepsilon)} - \frac{1}{\beta_2\eta_1} \|\nabla(\mathbf{R}_1^\varepsilon u_1)\|^2_{L^2(\Omega_2^\varepsilon)}
\end{aligned}$$

$$\geqslant \Big(\alpha_1 - \frac{\|\mathbf{R}_1^\varepsilon\|^2}{\beta_2\eta_1}\Big)\|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)} + \alpha_2(1-\eta_1)\|\nabla u_2\|^2_{L^2(\Omega_2^\varepsilon)}$$
$$\geqslant \Big(\alpha_1 - \frac{\rho_1^\star}{\beta_2\eta_1}\Big)\|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)} + \alpha_2(1-\eta_1)\|\nabla u_2\|^2_{L^2(\Omega_2^\varepsilon)}.$$

Thus, we proved that

$$\mathcal{A}^\varepsilon(u, \mathbf{T}_1^\varepsilon u) \geqslant \gamma_1\|\nabla u\|^2_{L^2(\Omega^\varepsilon)},$$

with

$$\gamma_1 = \min\Big(\alpha_1 - \frac{\rho_1^\star}{\beta_2\eta_1}, \alpha_2(1-\eta_1)\Big) > 0.$$

Similarly, when $\alpha_2\beta_1 > \rho_2^\star$, one can prove that

$$\mathcal{A}^\varepsilon(u, \mathbf{T}_2^\varepsilon u) \geqslant \gamma_2\|\nabla u\|^2_{L^2(\Omega^\varepsilon)},$$

with

$$\gamma_2 = \min\Big(\alpha_2 - \frac{\rho_2^\star}{\beta_1\eta_2}, \alpha_1(1-\eta_2)\Big) > 0,$$

for some η_2 such that $\frac{\rho_2^\star}{\alpha_2\beta_1} < \eta_2 < 1$. We have thus proved the uniform **T**-coercivity of the bilinear form $\mathcal{A}^\varepsilon$. □

Taking advantage of the symmetric geometry of our problem, let us now construct particular lifting operators whose norms can be explicitly estimated.

Proposition 3.4. *Let us introduce the two lifting operators* $\mathbf{R}_1^\varepsilon \in \mathcal{L}(V_1^\varepsilon, V_2^\varepsilon)$ *and* $\mathbf{R}_2^\varepsilon \in \mathcal{L}(V_2^\varepsilon, V_1^\varepsilon)$ *obtained by symmetry with respect to the interface* Σ^ε *defined by* (2.2)*:*

$$\begin{aligned}\forall u_1 \in V_1^\varepsilon: \quad (\mathbf{R}_1^\varepsilon u_1)(x_1,x_2) = u_1\left(x_1, 2H^\varepsilon(x_1) - x_2\right)\\ \forall u_2 \in V_2^\varepsilon: \quad (\mathbf{R}_2^\varepsilon u_2)(x_1,x_2) = u_2\left(x_1, 2H^\varepsilon(x_1) - x_2\right).\end{aligned} \tag{3.7}$$

Then, we have the estimate

$$\|\mathbf{R}_1^\varepsilon\|^2 = \|\mathbf{R}_2^\varepsilon\|^2 \leqslant \rho^\varepsilon := 1 + 2\varepsilon^k\overline{h'} + 4\varepsilon^{2k}\overline{h'}^2, \tag{3.8}$$

where $\overline{h'}$ *is defined in* (2.4)*.*

Moreover, in the particular case where h vanishes identically (i.e. for flat interface and flat upper and lower boundaries), we have

$$\|\mathbf{R}_1^\varepsilon\| = \|\mathbf{R}_2^\varepsilon\| = 1. \tag{3.9}$$

Proof. Following the proof of [7, Theorem 3.10], the change of variables

$$x_1 = \xi_1, \quad x_2 = 2H^\varepsilon(\xi_1) - \xi_2$$

shows that

$$\begin{aligned}\|\nabla(\mathbf{R}_1^\varepsilon u_1)\|^2_{L^2(\Omega_2^\varepsilon)} &= \int_{\Omega_2^\varepsilon}\Big(|\frac{\partial(\mathbf{R}_1^\varepsilon u_1)}{\partial\xi_1}|^2 + |\frac{\partial(\mathbf{R}_1^\varepsilon u_1)}{\partial\xi_2}|^2\Big)\,\mathrm{d}\xi\\ &= \int_{\Omega_1^\varepsilon}\Big(|\frac{\partial u_1}{\partial x_1} + 2\frac{\mathrm{d}H^\varepsilon}{\mathrm{d}x_1}\frac{\partial u_1}{\partial x_2}|^2 + |\frac{\partial u_1}{\partial x_2}|^2\Big)\,\mathrm{d}x\\ &= \int_{\Omega_1^\varepsilon}\Big(|\nabla u_1|^2 + 4\frac{\mathrm{d}H^\varepsilon}{\mathrm{d}x_1}\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2} + 4|\frac{\mathrm{d}H^\varepsilon}{\mathrm{d}x_1}|^2|\frac{\partial u_1}{\partial x_2}|^2\Big)\,\mathrm{d}x\\ &\leqslant \int_{\Omega_1^\varepsilon}\Big(|\nabla u_1|^2 + 2\frac{\mathrm{d}H^\varepsilon}{\mathrm{d}x_1}|\nabla u_1|^2 + 4|\frac{\mathrm{d}H^\varepsilon}{\mathrm{d}x_1}|^2|\frac{\partial u_1}{\partial x_2}|^2\Big)\,\mathrm{d}x\\ &\leqslant \Big(1 + 2\|\frac{\mathrm{d}H^\varepsilon}{\mathrm{d}x_1}\|_{L^\infty(0,l)} + 4\|\frac{\mathrm{d}H^\varepsilon}{\mathrm{d}x_1}\|^2_{L^\infty(0,l)}\Big)\|\nabla u_1\|^2_{L^2(\Omega_1^\varepsilon)}.\end{aligned}$$

Hence,

$$\|\mathbf{R}_1^\varepsilon u_1\|_{V_2^\varepsilon}^2 \leqslant \left(1 + 2\|\frac{\mathrm{d}H^\varepsilon}{\mathrm{d}x_1}\|_{L^\infty(0,l)} + 4\|\frac{\mathrm{d}H^\varepsilon}{\mathrm{d}x_1}\|_{L^\infty(0,l)}^2\right)\|u_1\|_{V_1^\varepsilon}^2.$$

Since $\frac{\mathrm{d}H^\varepsilon}{\mathrm{d}x_1} = \varepsilon^k h'\big(\frac{x_1}{\varepsilon}\big)$, we have $\|\frac{\mathrm{d}H^\varepsilon}{\mathrm{d}x_1}\|_{L^\infty(0,l)} = \varepsilon^k \overline{h'}$. The same calculations hold for $\mathbf{R}_2^\varepsilon$ and this shows that (3.8) holds true.

In the particular case of a flat interface, the lifting operators $\mathbf{R}_1^\varepsilon$ and $\mathbf{R}_2^\varepsilon$ are simply given by the symmetry with respect to the x_1-axis:

$$\forall u_1 \in V_1^\varepsilon: \quad (\mathbf{R}_1^\varepsilon u_1)(x_1,x_2) = u_1(x_1,-x_2)$$
$$\forall u_2 \in V_2^\varepsilon: \quad (\mathbf{R}_2^\varepsilon u_2)(x_1,x_2) = u_2(x_1,-x_2).$$

Consequently, $\|\mathbf{R}_1^\varepsilon\| = \|\mathbf{R}_2^\varepsilon\| = 1$. □

We collect in the next lemma some results needed in the sequel.

Lemma 3.5. *The following estimates hold true for every $v \in H_0^1(\Omega^\varepsilon)$:*

$$\|v\|_{L^2(\Omega^\varepsilon)} \leqslant C\|\nabla v\|_{L^2(\Omega^\varepsilon)}, \tag{3.10}$$

$$\Big|\int_{\Sigma^\varepsilon} v(x)\,\mathrm{d}\sigma_x\Big| \leqslant C\|\nabla v\|_{L^2(\Omega^\varepsilon)}. \tag{3.11}$$

Proof. The first inequality states that Poincaré's inequality holds in Ω^ε with a constant independent of ε. Indeed, since the zero-extension $\widetilde{v}$ of v to $\widetilde{\Omega} = (0,l)\times(-L,2L)$ belongs to $H_0^1(\widetilde{\Omega})$, we have by Poincaré's inequality in the fixed domain $\widetilde{\Omega}$

$$\|v\|_{L^2(\Omega^\varepsilon)} = \|\widetilde{v}\|_{L^2(\widetilde{\Omega})} \leqslant C\|\nabla\widetilde{v}\|_{L^2(\widetilde{\Omega})} = C\|\nabla v\|_{L^2(\Omega^\varepsilon)},$$

where we used for the last equality the identity $\nabla\widetilde{v} = \widetilde{\nabla v}$. Hence, (3.10) holds.

To prove the second estimate, we first rewrite the integral in the left-hand side as a one-dimensional integral in the coordinate x_1, namely

$$\int_{\Sigma^\varepsilon} v(x)\,\mathrm{d}\sigma_x = \int_0^l v\left(x_1, \varepsilon^{k+1}h\big(\frac{x_1}{\varepsilon}\big)\right)\sqrt{1+\varepsilon^{2k}|h'\left(\frac{x_1}{\varepsilon}\right)|^2}\,\mathrm{d}x_1.$$

Since $k \geqslant 0$ and h' satisfies (2.4), one has

$$\begin{aligned}
&\Big|\int_{\Sigma^\varepsilon} v(x)\,\mathrm{d}\sigma_x\Big| \\
&\leqslant C\int_0^l \Big|v\Big(x_1, \varepsilon^{k+1}h\big(\frac{x_1}{\varepsilon}\big)\Big)\Big|\,\mathrm{d}x_1 \\
&\leqslant C\Big(\int_0^l \Big|v\Big(x_1, \varepsilon^{k+1}h\big(\frac{x_1}{\varepsilon}\big)\Big) - v(x_1,0)\Big|\,\mathrm{d}x_1 + \int_0^l |v(x_1,0)|\,\mathrm{d}x_1\Big).
\end{aligned} \tag{3.12}$$

According to [24, Proposition 3.2] (which is an adaptation of [18, Lemma 1]) written with our notation, one has

$$\|v\big(x_1, \varepsilon^{k+1}h\big(\frac{x_1}{\varepsilon}\big)\big) - v(x_1,0)\|_{L^2(0,l)} \leqslant C\sqrt{\varepsilon^{k+1}}\|\nabla v\|_{L^2(\Omega^\varepsilon)}. \tag{3.13}$$

Then, using in (3.12) the Cauchy-Schwarz inequality and estimate (3.13) for the first term, and the classical trace inequality and the Poincaré inequality (3.10) for the second one, we obtain (since $k \geqslant 0$)

$$\Big|\int_{\Sigma^\varepsilon} v(x)\,\mathrm{d}\sigma_x\Big| \leqslant C\|\nabla v\|_{L^2(\Omega^\varepsilon)}$$

and the proof is complete. □

We are now in position to prove a well-posedness result for the microscopic problem (2.7).

Theorem 3.6. *Assume that*

$$\kappa := \max(\alpha_1\beta_2, \alpha_2\beta_1) > \kappa^\star, \tag{3.14}$$

where

$$\kappa^\star := \begin{cases} 1 & \text{if } k > 0 \\ 1 + 2\overline{h'} + 4\overline{h'}^2 & \text{if } k = 0. \end{cases}$$

Then, there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*)$, the variational formulation (2.14) of problem (2.7) admits a unique solution $u^\varepsilon \in V^\varepsilon$. Moreover, there exists a positive constant C independent of ε such that the following a priori *estimates hold true for all $\varepsilon \in (0, \varepsilon^*)$,*

$$\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leqslant C, \quad \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \leqslant C. \tag{3.15}$$

Proof. To apply Theorem 3.2, we first prove the uniform **T**-coercivity of the bilinear form $\mathcal{A}^\varepsilon(\cdot,\cdot)$. According to (3.8) in Proposition 3.4, we have

$$\|\mathbf{R}_1^\varepsilon\|^2 = \|\mathbf{R}_2^\varepsilon\|^2 \leqslant \rho^\varepsilon := 1 + 2\varepsilon^k\overline{h'} + 4\varepsilon^{2k}\overline{h'}^2.$$

Now, we need to distinguish between the two cases $k > 0$ and $k = 0$.

Case $k > 0$. In this case, we have $\lim_{\varepsilon\to 0} \rho^\varepsilon = 1$ and, by assumption, $\kappa := \max(\alpha_1\beta_2, \alpha_2\beta_1) > 1 = \kappa^\star$. Let us choose ε^* such that

$$1 \leqslant \rho^\star := \rho^{\varepsilon^*} < \kappa.$$

Therefore,

$$\|\mathbf{R}_1^\varepsilon\|^2 = \|\mathbf{R}_2^\varepsilon\|^2 \leqslant \rho^\varepsilon \leqslant \rho^{\varepsilon^*} = \rho^\star < \kappa, \quad \forall \varepsilon \in (0, \varepsilon^*). \tag{3.16}$$

Case $k = 0$. In this case, ρ^ε is independent of ε since $\rho^\varepsilon = 1 + 2\overline{h'} + 4\overline{h'} := \rho^\star$, for all $\varepsilon > 0$. Hence, for all $\varepsilon > 0$, we have by (3.8) and by using the assumption (3.14) on κ,

$$\|\mathbf{R}_1^\varepsilon\|^2 = \|\mathbf{R}_2^\varepsilon\|^2 \leqslant \rho^\varepsilon = \rho^\star < \kappa, \quad \forall \varepsilon > 0. \tag{3.17}$$

Thanks to (3.16) and (3.17), we can apply Proposition 3.3 with $\rho_1^\star = \rho_2^\star = \rho^\star$ and we obtain the following alternative to hold.

- If $\kappa = \alpha_1\beta_2$, then we have $\kappa \geqslant \rho^\star$, and thus $\mathcal{A}^\varepsilon(\cdot,\cdot)$ is uniformly $\mathbf{T}_1^\varepsilon$-coercive (for all $\varepsilon \in (0, \varepsilon^*)$ when $k > 0$, and for all $\varepsilon > 0$ when $k = 0$).
- If $\kappa = \alpha_2\beta_1$, then we have $\kappa \geqslant \rho^\star$, and thus $\mathcal{A}^\varepsilon(\cdot,\cdot)$ is uniformly $\mathbf{T}_2^\varepsilon$-coercive (for all $\varepsilon \in (0, \varepsilon^*)$ when $k > 0$ and for all $\varepsilon > 0$ when $k = 0$).

To conclude the proof, the only assumption in Theorem 3.2 that needs to be checked is the uniform continuity of the linear form ℓ^ε given by (2.16). Since $f \in L^2(\widetilde{\Omega})$, the first term (the volume term) of ℓ^ε is clearly continuous on $H_0^1(\Omega^\varepsilon)$ by Poincaré's inequality (3.10). For the second term (the surface term) of ℓ^ε, we first use the boundedness of the function g to obtain

$$\Big|\int_{\Sigma^\varepsilon} g^\varepsilon(x)v(x)\,\mathrm{d}\sigma_x\Big| \leqslant C\Big|\int_{\Sigma^\varepsilon} v(x)\,\mathrm{d}\sigma_x\Big|. \tag{3.18}$$

By combining (3.18) and (3.11), we obtain the continuity of the second term in the linear form ℓ^ε, and the proof is now complete. □

Remark 3.7. Let us consider in (2.10) the particular case of matrices of the form $A_1(y) = a_1(y)\mathrm{Id}$ and $A_2(y) = a_2(y)\,\mathrm{Id}$, where a_1 and a_2 are in $L^\infty(Y)$, $Y-$ periodic, taking positive and, respectively, negative values. More precisely, assume that there exist positive constants $a_1^-, a_1^+, a_2^-, a_2^+$ such that, for almost every $y \in Y$,

$$0 < a_1^- \leqslant a_1(y) \leqslant a_1^+, \quad 0 < a_2^- \leqslant -a_2(y) \leqslant a_2^+.$$

In this case, the constant $\kappa = \max(\alpha_1\beta_2, \alpha_2\beta_1)$, defined in (3.14), becomes

$$\kappa = \max(a_1^-/a_2^+, a_2^-/a_1^+).$$

As expected, this is exactly the constant obtained in [7, Theorem 3.10] for a scalar sign-changing transmission problem through a $\mathcal{C}^1-$interface. Moreover, if the functions a_1 and a_2 are constant, then

$$\kappa = \max(a_1/|a_2|, |a_2|/a_1).$$

In particular, one has:

- if $k > 0$, then the well-posedness condition $\kappa > 1 = \kappa^\star$ reads $a_1 \neq |a_2|$, *i.e.* the constants a_1 and a_2 should not be opposite;
- if $k = 0$, then the well-posedness condition reads $a_1/|a_2| \notin [1/\kappa^\star, \kappa^\star]$, with $\kappa^\star = 1 + 2\overline{h'} + 4\overline{h'}^2$, i.e. the contrasts should be large or small enough.

4. Convergence analysis

We remark that the dependence on ε of the domain Ω^ε is due to the oscillations of its upper and lower boundaries. Since the solution u^ε of problem (2.7) is defined in Ω^ε, general compactness results do not apply and, hence, we shall extend u^ε to a fixed domain. Taking into account the homogeneous Dirichlet boundary conditions, we extend u^ε by zero to a fixed domain, namely $\widetilde{\Omega} = (0, l) \times (-L, 2L)$.

For any function v defined on Ω^ε, $\widetilde{v}$ stands for its extension with zero to $\widetilde{\Omega}$. We remark that $\nabla\widetilde{v} = \widetilde{\nabla v}$. For this particular extension, classical compactness arguments allow us to state the following result.

Proposition 4.1. *Let u^ε be the unique solution of problem* (2.7)*, which satisfies the* a priori *estimates* (3.15)*. Then, its extension $\widetilde{u}^\varepsilon$ satisfies the* a priori *estimates*

$$\|\widetilde{u}^\varepsilon\|_{L^2(\widetilde{\Omega})} \leqslant C, \quad \|\nabla\widetilde{u}^\varepsilon\|_{L^2(\widetilde{\Omega})} \leqslant C. \tag{4.1}$$

These estimates obviously imply that there exists a function $\widetilde{u} \in H_0^1(\widetilde{\Omega})$ such that, up to a subsequence, one has

$$\widetilde{u}^\varepsilon \to \widetilde{u} \quad \text{strongly in } L^2(\widetilde{\Omega}), \quad \nabla\widetilde{u}^\varepsilon \rightharpoonup \nabla\widetilde{u} \quad \text{weakly in } L^2(\widetilde{\Omega}). \tag{4.2}$$

To state the main convergence result of the paper, we introduce the limit domain (see Figure 1, right)

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma_0,$$

where

$$\Omega_1 = (0, l) \times (0, L), \quad \Omega_2 = (0, l) \times (-L, 0), \quad \Sigma_0 = (0, l) \times \{0\}.$$

In what follows, we denote by u the restriction of the limit function $\widetilde{u}$ to the domain Ω:

$$u = \widetilde{u}|_\Omega. \tag{4.3}$$

Also we associate to any matrix $A \in L^\infty(Y;\mathcal{M}^s_{\alpha,\beta})$, $\mathcal{M}^s_{\alpha,\beta}$ being defined in (2.8), a corresponding 2×2 homogenized matrix defined by

$$A^{\rm hom} = (A^{\rm hom}_{ij})_{1\leqslant i,j\leqslant 2} \quad A^{\rm hom}_{ij} = \frac{1}{|Y|}\int_Y A(y)\nabla(\chi_i + y_i)\cdot\nabla(\chi_j + y_j)\,\mathrm{d}y, \tag{4.4}$$

where the functions $\chi_1, \chi_2 \in H^1(Y)$ are Y-periodic solutions, defined up to an additive constant, of the cell problems

$$-\operatorname{div}\left[A(y)\nabla\left(\chi_i + y_i\right)\right] = 0 \quad \text{in } Y,\ i = 1,2. \tag{4.5}$$

Classically (see, for instance, [1, Remark 1.3.12]), the homogenized matrix also belongs to the set $\mathcal{M}^s_{\alpha,\beta}$ and, hence,

$$\alpha|\xi|^2 \leqslant A^{\rm hom}\xi\cdot\xi \leqslant \beta^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^2.$$

Theorem 4.2. *Assume that* $\kappa := \max(\alpha_1\beta_2, \alpha_2\beta_1) > \kappa^\star$, *where*

$$\kappa^\star := \begin{cases} 1 & \text{if } k > 0 \\ 1 + 2\overline{h'} + 4\overline{h'}^2 & \text{if } k = 0. \end{cases}$$

Then, the unique solution u^ε *of the variational problem* (2.14) *converges, in the sense of* (4.2)-(4.3)*, to the unique solution* $u \in H^1_0(\Omega)$ *of the sign-changing limit transmission problem*

$$\begin{gathered} -\operatorname{div}\left(A_1^{\rm hom}\nabla u_1\right) = f \quad \text{in } \Omega_1 \\ -\operatorname{div}\left(A_2^{\rm hom}\nabla u_2\right) = f \quad \text{in } \Omega_2 \\ u = 0 \quad \text{on } \partial\Omega \\ u_1 - u_2 = 0 \quad \text{on } \Sigma_0 \\ A_1^{\rm hom}\nabla u_1\cdot n - A_2^{\rm hom}\nabla u_2\cdot n = G \quad \text{on } \Sigma_0, \end{gathered} \tag{4.6}$$

where the constant homogenized symmetric matrices $A_1^{\rm hom}$ *and* $A_2^{\rm hom}$*, associated to* A_1 *and, respectively, to* A_2*, are defined by* (4.4)-(4.5) *and*

$$G := \begin{cases} \int_0^1 g(y_1)\,\mathrm{d}y_1 & \text{if } k > 0 \\ \int_0^1 g(y_1)\sqrt{1 + |h'(y_1)|^2}\,\mathrm{d}y_1 & \text{if } k = 0. \end{cases} \tag{4.7}$$

Proof. We remark that the homogenized symmetric matrices $A_1^{\rm hom}$ and $-A_2^{\rm hom}$ are positive definite, due to (2.10). We start by proving the well-posedness of the sign-changing problem (4.6) by using the **T**-coercivity method. The proof goes along the same lines as those detailed in the proof of Theorem 3.6. Since the interface is flat, the corresponding lifting operators are of norm one in this case. Consequently, adapting the proof of Proposition 3.3 to the flat case and in the limit domain Ω, we obtain the **T**-coercivity of the bilinear form associated with the limit problem (4.6) for $\kappa > 1$. This last inequality holds true since, by assumption, $\kappa > \kappa^\star \geqslant 1$.

We prove now the convergence result for the solution u^ε of (2.7) to the solution u of (4.6). According to (2.14), the variational formulation of problem (2.7) reads as follows: find $u^\varepsilon \in V^\varepsilon$ such that

$$\begin{aligned} &\int_{\Omega_1^\varepsilon} A_1^\varepsilon(x)\nabla u_1^\varepsilon(x)\cdot\nabla v(x)\,\mathrm{d}x + \int_{\Omega_2^\varepsilon} A_2^\varepsilon(x)\nabla u_2^\varepsilon(x)\cdot\nabla v(x)\,\mathrm{d}x \\ &= \int_{\Omega^\varepsilon} f(x)v(x)\,\mathrm{d}x + \int_{\Sigma^\varepsilon} g^\varepsilon(x)v(x)\,\mathrm{d}\sigma_x, \end{aligned} \tag{4.8}$$

for all $v \in V^\varepsilon$. From (4.8), we are led to

$$\begin{aligned} &\int_{\widetilde{\Omega}} \chi_{\Omega_1^\varepsilon}(x) A_1^\varepsilon(x) \nabla \widetilde{u}_1^\varepsilon(x) \cdot \nabla v(x)\,\mathrm{d}x + \int_{\widetilde{\Omega}} \chi_{\Omega_2^\varepsilon}(x) A_2^\varepsilon(x) \nabla \widetilde{u}_2^\varepsilon(x) \cdot \nabla v(x)\,\mathrm{d}x \\ &= \int_{\widetilde{\Omega}} \chi_{\Omega^\varepsilon}(x) f(x) v(x)\,\mathrm{d}x + \int_{\Sigma^\varepsilon} g^\varepsilon(x) v(x)\,\mathrm{d}\sigma_x, \end{aligned} \tag{4.9}$$

for any $v \in \mathcal{D}(\widetilde{\Omega})$, where χ_D denotes the characteristic function of a domain D.

Our aim is to pass to the limit with $\varepsilon \to 0$ in (4.9). For the passage to the limit in its left-hand side, we adapt to our situation some ideas from [24]. To this end, we set

$$\begin{aligned} B_1^\varepsilon &= \{x = (x_1, x_2) \in \Omega^\varepsilon : L < x_2 < L + H^\varepsilon(x_1)\}, \\ B_2^\varepsilon &= \{x = (x_1, x_2) \in \Omega^\varepsilon : -L < x_2 < -L + H^\varepsilon(x_1)\}, \\ B_-^\varepsilon &= \{x = (x_1, x_2) \in \Omega^\varepsilon : 0 < x_2 < H^\varepsilon(x_1)\}, \\ B_+^\varepsilon &= \left\{x = (x_1, x_2) \in \Omega^\varepsilon : H^\varepsilon(x_1) < x_2 < \varepsilon^{k+1}\overline{h}\right\}. \end{aligned}$$

We first notice that $\Omega_1^\varepsilon = (\Omega_1 \setminus B_-^\varepsilon) \cup B_1^\varepsilon$ and $\Omega_2^\varepsilon = (\Omega_2 \cup B_-^\varepsilon) \setminus B_2^\varepsilon$. The oscillating interface Σ^ε, as well as the top and the bottom oscillating boundaries Σ_1^ε and Σ_2^ε of the domain Ω^ε, are contained in the sets $S^\varepsilon = (0, l) \times [0, \varepsilon^{k+1}\overline{h}]$ and, respectively, in $S_1^\varepsilon = (0, l) \times [L, L + \varepsilon^{k+1}\overline{h}]$ and $S_2^\varepsilon = (0, l) \times \left[-L, -L + \varepsilon^{k+1}\overline{h}\right]$. An important feature of the sets S^ε, S_1^ε, and S_2^ε is that, when ε tends to zero, their measures tend to zero. This is a crucial argument in the convergence process.

Let us notice that, for the passage to the limit in the left-hand side of (4.9), we also use [24, Remark 2.2], which ensures that the convergence results of the paper remain valid for the case of distinct diffusion matrices in the upper and lower part of the domain.

Passing to the limit in the first integral in the left-hand side of (4.9), we obtain

$$\int_{\widetilde{\Omega}} \chi_{\Omega_1^\varepsilon}(x) A_1^\varepsilon(x) \nabla \widetilde{u}_1^\varepsilon(x) \cdot \nabla v(x)\,\mathrm{d}x \to \int_{\Omega_1} A_1^{\mathrm{hom}} \nabla \widetilde{u}_1 \cdot \nabla v\,\mathrm{d}x. \tag{4.10}$$

Indeed, one has

$$\begin{aligned} &\int_{\widetilde{\Omega}} \chi_{\Omega_1^\varepsilon}(x) A_1^\varepsilon(x) \nabla \widetilde{u}_1^\varepsilon(x) \cdot \nabla v(x)\,\mathrm{d}x \\ &= \int_{\widetilde{\Omega}} \chi_{\Omega_1 \setminus B_-^\varepsilon}(x) A_1^\varepsilon(x) \nabla \widetilde{u}_1^\varepsilon(x) \cdot \nabla v(x)\,\mathrm{d}x + \int_{\widetilde{\Omega}} \chi_{B_1^\varepsilon}(x) A_1^\varepsilon(x) \nabla \widetilde{u}_1^\varepsilon(x) \cdot \nabla v(x)\,\mathrm{d}x. \end{aligned}$$

By using the hypothesis on the matrix A_1^ε and the estimates (4.1), we obtain

$$\Big| \int_{B_1^\varepsilon} A_1^\varepsilon(x) \nabla \widetilde{u}_1^\varepsilon(x) \cdot \nabla v(x)\,\mathrm{d}x \Big| \leqslant C \|\nabla v\|_{L^2(B_1^\varepsilon)} \to 0,$$

since B_1^ε is included in S_1^ε, whose measure tends to zero. We then use [24, Proposition 3.1], to obtain (4.10).

For the second integral in the left-hand side of (4.9), we obtain

$$\int_{\widetilde{\Omega}} \chi_{\Omega_2^\varepsilon}(x) A_2^\varepsilon(x) \nabla \widetilde{u}_2^\varepsilon(x) \cdot \nabla v(x)\,\mathrm{d}x \to \int_{\Omega_2} A_2^{\mathrm{hom}} \nabla \widetilde{u}_2 \cdot \nabla v\,\mathrm{d}x. \tag{4.11}$$

Indeed, one has

$$\int_{\widetilde{\Omega}} \chi_{\Omega_2^\varepsilon}(x) A_2^\varepsilon(x) \nabla \widetilde{u}_2^\varepsilon(x) \cdot \nabla v(x)\,\mathrm{d}x$$

$$\begin{aligned}&= \int_{\widetilde{\Omega}} \chi_{\Omega_2} A_2^\varepsilon(x)\nabla\widetilde{u}_2^\varepsilon(x)\cdot\nabla v(x)\,\mathrm{d}x + \int_{\widetilde{\Omega}} \chi_{B_-^\varepsilon}(x) A_2^\varepsilon(x)\nabla\widetilde{u}_2^\varepsilon(x)\cdot\nabla v(x)\,\mathrm{d}x\\ &\quad - \int_{\widetilde{\Omega}} \chi_{B_2^\varepsilon}(x) A_2^\varepsilon(x)\nabla\widetilde{u}_2^\varepsilon(x)\cdot\nabla v(x)\,\mathrm{d}x\\ &= J_1 + J_2 - J_3.\end{aligned}$$

Classical convergence results in the theory of homogenization give us

$$J_1 = \int_{\widetilde{\Omega}} \chi_{\Omega_2}(x) A_2^\varepsilon(x)\nabla\widetilde{u}_2^\varepsilon(x)\cdot\nabla v(x)\,\mathrm{d}x \to \int_{\Omega_2} A_2^{\rm hom}\nabla\widetilde{u}_2\cdot\nabla v\,\mathrm{d}x.$$

For J_2, by using the hypothesis on the matrix A_2^ε and estimates (4.1), one has

$$J_2 = |\int_{B_-^\varepsilon} A_2^\varepsilon(x)\nabla\widetilde{u}_2^\varepsilon(x)\cdot\nabla v(x)\,\mathrm{d}x| \leqslant C\|\nabla v\|_{L^2(B_-^\varepsilon)} \to 0,$$

since B_-^ε is included in S_2^ε, whose measure goes to zero. The value of J_3 is zero by the construction of the extension $\widetilde{u}^\varepsilon$. Hence, we arrive at (4.11).

The first term in the right-hand side of (4.9) obviously gives

$$\int_{\widetilde{\Omega}} \chi_{\Omega^\varepsilon} f(x)v(x)\,\mathrm{d}x \to \int_\Omega fv\,\mathrm{d}x. \tag{4.12}$$

For dealing with the second term in the right-hand side of (4.9), namely the surface term, we first express it as a one-dimensional integral in the coordinate x_1. Then, due to the $\mathcal{C}^1$ regularity of the function h on the interval $[0,1]$, we can use the one-dimensional version of periodic unfolding operator on fixed domains in [19, Chapter 1], whose definition and main properties are briefly recalled in the Appendix. More precisely, we use the results in the Appendix for the particular values $m=1$, $\omega=(0,l)$, and $Y=(0,1)$. Applying Propositions 5.3 and 5.4, we obtain, since g is 1-periodic,

$$\begin{aligned}\int_{\Sigma^\varepsilon} g^\varepsilon(x)v(x)\,\mathrm{d}\sigma_x &= \int_0^l g\big(\frac{x_1}{\varepsilon}\big)v\left(x_1,\varepsilon^{k+1}h\big(\frac{x_1}{\varepsilon}\big)\right)\sqrt{1+|\varepsilon^k h'\left(\frac{x_1}{\varepsilon}\right)|^2}\,\mathrm{d}x_1\\ &= \int_0^l\int_0^1 \mathcal{T}^\varepsilon(g)(x_1,y_1)\mathcal{T}^\varepsilon(V^\varepsilon)(x_1,y_1)\mathcal{T}^\varepsilon(W^\varepsilon)(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1\\ &= \int_0^l\int_0^1 g(y_1)\mathcal{T}^\varepsilon(V^\varepsilon)(x_1,y_1)\mathcal{T}^\varepsilon(W^\varepsilon)(x_1,y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1.\end{aligned}$$

Here, we denoted

$$\begin{aligned}V^\varepsilon(x_1) &= v\left(x_1,\varepsilon^{k+1}h\big(\frac{x_1}{\varepsilon}\big)\right),\\ W^\varepsilon(x_1) &= \sqrt{1+|\varepsilon^k h'\left(\frac{x_1}{\varepsilon}\right)|^2}.\end{aligned}$$

According to (3.13), one has

$$v\left(\cdot,\varepsilon^{k+1}h\big(\frac{\cdot}{\varepsilon}\big)\right) \to v(\cdot,0) \quad \text{strongly in } \ L^2(0,l).$$

This, together with Proposition 5.3(6) and (8), leads to

$$\mathcal{T}^\varepsilon(V^\varepsilon)(x_1,y_1) \to v(x_1,0) \quad \text{strongly in } L^2((0,l)\times(0,1)).$$

By Proposition 5.4 applied to h', we have

$$\mathcal{T}^\varepsilon(h')(x_1,y_1) \to h'(y_1) \quad \text{strongly in } L^2((0,l)\times(0,1)).$$

For $k > 0$, one has

$$\mathcal{T}^\varepsilon(\varepsilon^k h')(x_1, y_1) = \varepsilon^k \mathcal{T}^\varepsilon(h')(x_1, y_1) \to 0 \quad \text{strongly in } L^2((0,l) \times (0,1)).$$

Using Proposition 5.3(6) and the properties of Nemytskii's operator, we obtain

$$\mathcal{T}^\varepsilon(W^\varepsilon)(x_1, y_1) \rightharpoonup 1 \quad \text{weakly in } L^2((0,l) \times (0,1)).$$

Therefore, for $k > 0$, for the term involving the flux jump, we obtain

$$\begin{aligned}\int_{\Sigma^\varepsilon} g^\varepsilon(x_1) v(x_1, x_2)\,\mathrm{d}\sigma_x &\to \int_0^l \int_0^1 g(y_1) v(x_1, 0)\,\mathrm{d}x_1\,\mathrm{d}y_1 \\ &= \int_0^1 g(y_1)\,\mathrm{d}y_1 \int_0^l v(x_1, 0)\,\mathrm{d}x_1 \\ &= \int_0^1 g(y_1)\,\mathrm{d}y_1 \int_{\Sigma_0} v(x_1, x_2)\,\mathrm{d}\sigma.\end{aligned} \tag{4.13}$$

For $k = 0$, the term involving the flux jump reads

$$\int_{\Sigma^\varepsilon} g^\varepsilon(x_1) v(x_1, x_2)\,\mathrm{d}\sigma_x = \int_0^l g\big(\frac{x_1}{\varepsilon}\big) v\big(x_1, \varepsilon h\big(\frac{x_1}{\varepsilon}\big)\big) \sqrt{1 + |h'\left(\frac{x_1}{\varepsilon}\right)|^2}\,\mathrm{d}x_1.$$

By applying again the one-dimensional unfolding operator and using similar arguments as above, we obtain

$$\int_{\Sigma^\varepsilon} g^\varepsilon(x_1) v(x_1, x_2)\,\mathrm{d}\sigma_x \to \int_0^1 g(y_1) \sqrt{1 + |h'(y_1)|^2}\,\mathrm{d}y_1 \int_{\Sigma_0} v(x_1, x_2)\,\mathrm{d}\sigma. \tag{4.14}$$

It remains now to obtain the boundary conditions on $\partial\Omega$ for the limit function $u = \widetilde{u}|_\Omega$. On the lateral boundaries, the homogeneous Dirichlet condition is obviously kept at the limit. For the bottom and the upper boundaries of Ω, we shall prove that the prescribed homogeneous Dirichlet boundary condition will be also preserved at the limit. More precisely, we shall prove that $u = 0$ on $(0,l) \times \{-L\}$ and on $(0,l) \times \{L\}$.

We first prove that $u = 0$ on $\Sigma_2 = (0,l) \times \{-L\}$. Since, by construction, $\widetilde{u}_2^\varepsilon = 0$ on Σ_2, one has

$$\|\widetilde{u}_2\|_{L^2(\Sigma_2)} = \|\widetilde{u}_2^\varepsilon - \widetilde{u}_2\|_{L^2(\Sigma_2)} \leqslant C\|\nabla\widetilde{u}_2^\varepsilon - \nabla\widetilde{u}_2\|_{L^2(\Omega_2)},$$

where we use the compactness of the trace operator. By using the convergence (4.2), we pass to the limit and we obtain $\|\widetilde{u}_2\|_{L^2(\Sigma_2)} = 0$, which implies that $u_2 = 0$ on Σ_2.

To obtain the boundary condition for u on $\Sigma_1 = (0,l) \times \{L\}$, we notice that, since by construction, $\widetilde{u}_1^\varepsilon(\cdot, L + H^\varepsilon(x_1)) = 0$, we have

$$\begin{aligned}\|\widetilde{u}_1(\cdot, L)\|_{L^2(0,l)} &= \|\widetilde{u}_1^\varepsilon(\cdot, L + H^\varepsilon(x_1)) - \widetilde{u}_1(\cdot, L)\|_{L^2(0,l)} \\ &\leqslant \|\widetilde{u}_1^\varepsilon(\cdot, L + H^\varepsilon(x_1)) - \widetilde{u}_1^\varepsilon(\cdot, L)\|_{L^2(0,l)} + \|\widetilde{u}_1^\varepsilon - \widetilde{u}_1\|_{L^2(\Sigma_1)} \\ &\leqslant C\sqrt{\varepsilon^{k+1}} + C\|\nabla\widetilde{u}_1^\varepsilon - \nabla\widetilde{u}_1\|_{L^2(\Omega_1)}\end{aligned}$$

where we use an adaptation of (3.13), estimate (4.1) and the compactness of the trace operator. By using the convergence (4.2), we pass to the limit and we obtain $\|\widetilde{u}_1(\cdot, L)\|_{L^2(0,l)} = 0$, which implies $\widetilde{u}_1 = 0$ on Σ_1. Hence, $u_1 = 0$ on Σ_1.

Collecting convergences (4.10), (4.11), (4.12), (4.13), and (4.14), we are led to

$$\int_{\Omega_1} A_1^{\text{hom}} \nabla u_1 \cdot \nabla v\,\mathrm{d}x + \int_{\Omega_2} A_2^{\text{hom}} \nabla u_2 \cdot \nabla v\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x + G \int_{\Sigma_0} v\,\mathrm{d}\sigma_x, \tag{4.15}$$

which is equivalent to the limit transmission problem (4.6), where G is given by (4.7).

Since the limit problem (4.6) admits a unique solution u, the convergences (4.2) hold for the whole sequence. □

Remark 4.3. If $k > 0$, the microscopic and the homogenized problems are well-posed for the same range of values of the generalized contrasts, namely $\kappa > 1$. On the contrary, for $k = 0$, the well-posedness of the homogenized problem is ensured as soon as $\kappa > 1$, while the one of the microscopic problem holds only for $\kappa > 1 + 2\overline{h'} + 4\overline{h'}^2$.

Remark 4.4. The upper bounds obtained in (3.8) for the norms of the lifting operators are still valid in the case $k < 0$. Hence, they provide a well -posedness result for the microscopic problem for fixed ε for $k < 0$. However, as these bounds blow-up as $\varepsilon \to 0$, the uniform **T**-coercivity fails and the macroscopic problem (2.7) might be ill-posed as $\varepsilon \to 0$.

5. Appendix: background on the unfolding method

In this Appendix, we collect some useful results from [19, Chapter 1] on the periodic unfolding method. Consider, for simplicity, the bounded domain $\omega = (0, l)^m \subset \mathbb{R}^m$. Let ε be a sequence of strictly positive numbers such that $l/\varepsilon \in \mathbb{N}^*$ and let $Y = (0, 1)^m$. Thus, the domain $\omega \subset \mathbb{R}^m$ is obtained as the union of an entire number of $\varepsilon-$shrinked and translated cells Y.

Let us notice that for $x \in \mathbb{R}^m$, by denoting $[x]$ the entire part of x in $\mathbb{Z}^m$, then $x - [x] \in Y$. Set $\{x\} = x - [x]$, for $x \in \mathbb{R}^m$. In particular, for any $x \in \mathbb{R}^m$ and any $\varepsilon > 0$, one has

$$x = \varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon \left\{\frac{x}{\varepsilon}\right\}. \tag{5.1}$$

Definition 5.1 ([19, Definition 1.2])**.** For each function φ Lebesgue–measurable on ω, the periodic unfolding operator $\mathcal{T}^\varepsilon$ is defined by

$$\mathcal{T}^\varepsilon(\varphi)(x, y) = \varphi\Big(\varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon y\Big) \quad \text{for a.e. } (x, y) \in \Omega \times Y.$$

The function $\mathcal{T}^\varepsilon(\varphi)$ is Lebesgue–measurable on $\omega \times Y$. One has the following property.

Proposition 5.2 ([19, Prop. 1.12])**.** *Let $p \in (1, +\infty)$ and let $\{v^\varepsilon\}$ be a bounded sequence in $L^p(\omega)$. Then, the sequence $\{\mathcal{T}^\varepsilon(v^\varepsilon)\}$ is bounded in $L^p(\omega \times Y)$ and, if*

$$\mathcal{T}^\varepsilon(v^\varepsilon) \rightharpoonup v \textit{ weakly in } L^p(\omega \times Y),$$

then

$$v^\varepsilon \rightharpoonup \int_Y v(x, y)\, \mathrm{d}y \textit{ weakly in } L^p(\omega).$$

The unfolding operator $\mathcal{T}^\varepsilon$ has the following properties.

Proposition 5.3 ([19, Chapter 1])**.** *Properties of operator $\mathcal{T}^\varepsilon : L^p(\omega) \to L^p(\omega \times Y)$, $p \in (1, \infty)$:*

1. *The operator $\mathcal{T}^\varepsilon$ is linear and continuous.*
2. *One has $\mathcal{T}^\varepsilon(vw) = \mathcal{T}^\varepsilon(v)\mathcal{T}^\varepsilon(w)$, for v, w Lebesgue–measurable functions.*

(3) *(Integration formula) If* $v \in L^1(\omega)$, *then*

$$\int_\omega v(x)\ \mathrm{d}x = \int_{\omega\times Y} \mathcal{T}^\varepsilon(v)(x,y)\ \mathrm{d}x\ \mathrm{d}y.$$

(4) *If* $\{v^\varepsilon\}$ *is a sequence of functions in* $L^1(\omega)$, *then*

$$\lim_{\varepsilon\to 0}\int_\omega v^\varepsilon(x)\, dx = \lim_{\varepsilon\to 0}\int_{\omega\times Y} \mathcal{T}^\varepsilon(v^\varepsilon)(x,y)\ \mathrm{d}x\ \mathrm{d}y.$$

(5) *We have the estimate*

$$\|\mathcal{T}^\varepsilon(v)\|_{L^p(\omega\times Y)} \leqslant \|v\|_{L^p(\omega)}.$$

(6) *If* N *is a real-valued continuous function and* v *is a Lebesgue–measurable function, then*

$$\mathcal{T}^\varepsilon(N(v)) = N(\mathcal{T}^\varepsilon(v)).$$

Convergence results.

(7) *If* $v \in L^p(\omega)$, *then* $\mathcal{T}^\varepsilon(v) \to v$ *strongly in* $L^p(\omega\times Y)$.
(8) *If* $v^\varepsilon \to v$ *strongly in* $L^p(\omega)$, *then* $\mathcal{T}^\varepsilon(v^\varepsilon) \to v$ *strongly in* $L^p(\omega\times Y)$.

Proposition 5.4 ([19, Proposition 1.5])**.** *For a Lebesgue–measurable function* h *on* Y, *extended by* Y*-periodicity to the whole of* $\mathbb{R}^m$, *we define the sequence* $\{h^\varepsilon\}$ *by*

$$h^\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right) \quad \text{for a.e. } x \in \mathbb{R}^m.$$

Then $\mathcal{T}^\varepsilon(h^\varepsilon)(x,y) = h(y)$, *for a.e.* $x \in \mathbb{R}^m$. *If* h *belongs to* $L^p(Y)$, $p \in [1,+\infty)$, *and if* ω *is bounded, then*

$$\mathcal{T}^\varepsilon(h^\varepsilon) \to h \quad \text{strongly in } L^p(\omega\times Y).$$

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References

[1] G. Allaire; *Shape optimization by the homogenization method*, vol. 146 of Applied Mathematical Sciences, Springer-Verlag, New York, 2002.
[2] Y. Amirat, O. Bodart, U. De Maio, A. Gaudiello; *Asymptotic approximation of the solution of the Laplace equation in a domain with highly oscillating boundary*, SIAM J. Math. Anal., 35 (2004), pp. 1598–1616.
[3] J. M. Arrieta, M. C. Pereira; *Elliptic problems in thin domains with highly oscillating boundaries*, Bol. Soc. Esp. Mat. Apl., SEMA, 51 (2010), pp. 17–24.
[4] J. M. Arrieta, M. C. Pereira; *Homogenization in a thin domain with an oscillatory boundary*, J. Math. Pures Appl. (9), 96 (2011), pp. 29–57.
[5] J. Avila, S. Monsurrò, F. Raimondi; *Homogenization of an eigenvalue problem through rough surfaces*, Asymptotic Anal., 137 (2024), pp. 97–121.
[6] A. G. Belyaev, A. Pyatnitskii, G. A. Chechkin; *Averaging in a perforated domain with an oscillating third boundary condition*, Sbornik: Mathematics, 192 (2001), pp. 933–949.
[7] A. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr.; *T-coercivity for scalar interface problems between dielectrics and metamaterials*, ESAIM Math. Model. Numer. Anal., 46 (2012), pp. 1363–1387.
[8] A.-S. Bonnet-Ben Dhia, P. J. Ciarlet, C. M. Zwölf; *Time harmonic wave diffraction problems in materials with sign-shifting coefficients*, J. Comput. Appl. Math., 234 (2010), pp. 1912–1919.
[9] E. Bonnetier, C. Dapogny, F. Triki; *Homogenization of the eigenvalues of the Neumann-Poincaré operator*, Arch. Rational Mech. Anal., 234 (2019), pp. 777–855.

[10] G. Bouchitté, H. Lidouh, J.-C. Michel, P. Suquet; *Might boundary homogenization help to understand friction?*, in Proc. Contact Mechanics Int. Symp., A. Curnier, ed., 1992, pp. 175–192.

[11] R. Bunoiu, L. Chesnel, K. Ramdani, M. Rihani; *Homogenization of Maxwell's equations and related scalar problems with sign-changing coefficients*, Annales de la Faculté des sciences de Toulouse: Mathématiques, Ser. 6, 30 (2021), pp. 1075–1119.

[12] R. Bunoiu, K. Ramdani; *Homogenization of materials with sign changing coefficients*, Commun. Math. Sci., 14 (2016), pp. 1137–1154.

[13] R. Bunoiu, K. Ramdani, C. Timofte; *T-coercivity for the asymptotic analysis of scalar problems with sign-changing coefficients in thin periodic domains*, Electronic Journal of Differential Equations, 2021 (2021), No. 59, pp. 1–22.

[14] R. Bunoiu, K. Ramdani, C. Timofte; *T-coercivity for the homogenization of sign-changing coefficients scalar problems with extreme contrasts*, Mathematical Reports, 24 (2022), pp. 113–123.

[15] R. Bunoiu, K. Ramdani, C. Timofte; *Homogenization of a transmission problem with sign-changing coefficients and interfacial flux jump*, Commun. Math. Sci., 21 (2023), pp. 2029–2049.

[16] R. Bunoiu, C. Timofte; *Homogenization of a thermal problem with flux jump*, Netw. Heterog. Media, 11 (2016), pp. 545–562.

[17] R. Bunoiu, C. Timofte; *Upscaling of a diffusion problem with interfacial flux jump leading to a modified Barenblatt model*, ZAMM, Z. Angew. Math. Mech., 99 (2019), p. e201800018.

[18] G. A. Chechkin, A. Friedman, A. L. Piatnitski; *The boundary-value problem in domains with very rapidly oscillating boundary*, J. Math. Anal. Appl., 231 (1999), pp. 213–234.

[19] D. Cioranescu, A. Damlamian, G. Griso; *The periodic unfolding method*, vol. 3 of Series in Contemporary Mathematics, Springer, Singapore, 2018.

[20] A. Damlamian, K. Pettersson; *Homogenization of oscillating boundaries*, Discrete Contin. Dyn. Syst., 23 (2009), pp. 197–219.

[21] J. I. Díaz, D. Gómez-Castro, T. A. Shaposhnikova; *Nonlinear reaction-diffusion processes for nanocomposites. Anomalous improved homogenization*, vol. 39 of De Gruyter Ser. Nonlinear Anal. Appl., Berlin: De Gruyter, 2021.

[22] P. Donato, D. Giachetti; *Existence and homogenization for a singular problem through rough surfaces*, SIAM J. Math. Anal., 48 (2016), pp. 4047–4086.

[23] P. Donato, E. C. Jose, D. Onofrei; *Asymptotic analysis of a multiscale parabolic problem with a rough fast oscillating interface*, Archive of Applied Mechanics, 89 (2019), pp. 437–465.

[24] P. Donato, A. L. Piatnitski; *On the effective interfacial resistance through rough surfaces*, Communications on Pure and Applied Analysis, 9 (2010), pp. 1295–1310.

[25] T. Fatima, E. Ijioma, T. Ogawa, A. Muntean; *Homogenization and dimension reduction of filtration combustion in heterogeneous thin layers*, Netw. Heterog. Media, 9 (2014), pp. 709–737.

[26] M. Gahn, M. Neuss-Radu, P. Knabner; *Homogenization of reaction–diffusion processes in a two-component porous medium with nonlinear flux conditions at the interface*, SIAM Journal on Applied Mathematics, 76 (2016), pp. 1819–1843.

[27] D. Gómez, S. A. Nazarov, E. Pérez; *Spectral stiff problems in domains with a strongly oscillating boundary*, in Integral methods in science and engineering. Computational and analytic aspects. In Selected papers of the 11th international conference on integral methods in science and engineering (IMSE 2010), Brighton, UK, July 12–14, 2010, Basel: Birkhäuser, 2011, pp. 159–172.

[28] E. Ijioma, A. Muntean, T. Ogawa; *Pattern formation in reverse smouldering combustion: a homogenisation approach*, Combustion Theory and Modelling, 17 (2013), pp. 185–223.

[29] S. Monsurrò, C. Perugia, F. Raimondi; *Homogenization of a nonlinear elliptic problem with imperfect rough interface*, 2023.

[30] E. Shamonina, L. Solymar; *Metamaterials: How the subject started*, Metamaterials, 1 (2007), pp. 12–18.

[31] D. R. Smith, J. B. Pendry, M. C. K. Wiltshire; *Metamaterials and negative refractive index*, Science, 305 (2004), pp. 788–792.

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