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# ASYMPTOTIC BEHAVIOR OF EIGENVALUES OF FOURTH-ORDER DIFFERENTIAL OPERATORS WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITIONS

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Abstract. We consider a spectral problem for a fourth-order differential equation with spectral parameter dependent boundary conditions. We determine the high energy eigenvalue behavior for this operator. Moreover, if the coefficient of differential equation is sufficiently smooth, we can obtain sharp eigenvalue asymptotic behavior. This behavior exhibits a non-standard high-frequency effect generated by the spectral parameter in the boundary conditions.

## 1. Introduction and main results

We consider the eigenvalue problem

$$\begin{gathered} y^{(4)}(x) - (p(x)y'(x))' = \lambda y(x), \\ y(0) = y'(0) = y''(1) = 0, \\ (a\lambda + b)y(1) - (c\lambda + d)Ty(1) = 0, \end{gathered} \tag{1.1}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $p$ is a real absolutely continuous function on $[0,1]$, $Ty = y''' - py'$, $a$, $b$, $c$, $d \in \mathbb{R}$, and $\sigma = bc - ad > 0$.

This problem arises in the boundary value problem describing the free bending vibrations of a homogeneous beam when the left end of this beam is fixed, and at the right end there is concentrated a load (see [24]). If $b = c = 0$ and $d = 1$, then this situation corresponds to the right end of beam having a load hanging, the mass of which is equal to $-a$ (see [13]).

Boundary value problems for a fourth-order differential equation with a spectral parameter in the boundary conditions have been studied quite actively in previous years. A general theory of boundary value problems for ordinary differential operators with a spectral parameter in the boundary conditions was constructed in [25]. There the main results are devoted to completeness, minimality, and basis property of system of eigenfunctions and associated functions for these operators. The spectral properties of boundary eigenvalue problems for differential equations of the form $Ny = \lambda Py$ investigated in [26]. There $N$ and $P$ are regular differential operators of order $n$ and $p$, with $n > p \geq 0$, and the boundary conditions depend on

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a spectral parameter polynomially. The main results there describe the completeness, minimality, and the Riesz basis properties of the corresponding eigenfunctions and associated eigenfunctions.

Basis properties for fourth-order differential operators (1.1) with a spectral parameter in the boundary conditions of different statements are investigated in [1, 2, 6, 14, 15]. General characteristic of the location of the eigenvalues on the real axis and oscillation properties of eigenfunctions for fourth-order differential operators with spectral parameter in three boundary conditions are obtained in [3]. Convergence of eigenfunction expansions for this operators are studied in [4, 5].

The asymptotic behavior of spectrum and a trace formula for fourth-order differential operator with unbounded operator coefficient and spectral parameter in boundary condition are given in [7, 8]. Sharp eigenvalue asymptotic behavior and a trace formula for this type operator without operator coefficient are obtained in [20, 21].

Fourth-order differential operators with squared spectral parameter in boundary conditions were considered in [12]. In this article we provide the simplicity and interlacing properties of the eigenvalues and the oscillation properties of the corresponding eigenfunctions.

The location of the spectrum and asymptotic behavior of the eigenvalues for fourth-order differential operators with spectral parameter dependent boundary conditions of different type were investigated in [16, 17, 18, 27].

The main goal of this article is to determine the asymptotic behavior of the eigenvalues of the spectral problem (1.1). This property of problem (1.1) was studied in [14]. But in that work only the main term of the asymptotics was established. In our manuscript we obtain sharp asymptotics of the eigenvalues for an absolutely continuous coefficient and for a smoother one. Moreover, we compare this asymptotic formula with the spectral asymptotics of the same operator with similar boundary conditions, but without a spectral parameter in the boundary condition and describe the non-standard effects that appear here.

Problem (1.1) can be reduced to the spectral problem for some linear operator $\mathcal{H}$. In the Hilbert space $H = L^2(0,1) \oplus \mathbb{C}$ with inner product

$$(\mathbf{y}, \mathbf{v}) = \int_0^1 y(x)\overline{v(x)}\,dx + \sigma^{-1} a\bar{b}, \quad \mathbf{y} = \begin{pmatrix} y(x) \\ a \end{pmatrix} \in H, \quad \mathbf{v} = \begin{pmatrix} v(x) \\ b \end{pmatrix} \in H,$$

we define the operator

$$\mathcal{H}\mathbf{y} = \mathcal{H}\begin{pmatrix} y(x) \\ cTy(1) - ay(1) \end{pmatrix} = \begin{pmatrix} (Ty(x))' \\ by(1) - dTy(1) \end{pmatrix}$$

on the domain

$$\begin{aligned} \operatorname{Dom}(\mathcal{H}) = \Big\{\mathbf{y} = \begin{pmatrix} y(x) \\ cTy(1) - ay(1) \end{pmatrix} \in H,\ y \in W^{4,1}(0,1),\ (Ty)' \in L^2(0,1), \\ y(0) = y'(0) = y''(1) = 0\Big\}, \end{aligned}$$

where $W^{4,1}(0,1)$ is the standard Sobolev space. This domain is dense in $H$ [25, Lemma 1.5]. The operator is well-defined in $H$. Therefore, we conclude that problem (1.1) is equivalent to the spectral problem

$$\mathcal{H}\mathbf{y} = \lambda\mathbf{y}, \quad \mathbf{y} \in \operatorname{Dom}(\mathcal{H}),$$

i.e. the eigenvalues $\lambda_n$ of the problem (1.1) and those of the operator $\mathcal{H}$ coincide (see also [25, Lemma 1.4]). Moreover, the problem (1.1) is regular in the sense of [25] and, in particular, it has a discrete spectrum. Aliev and Kerimov [14, Lemma 4.2] proved that the eigenvalues $\lambda_n$ of (1.1) form a countable set without finite limit points. Furthermore, all eigenvalues are simple.

We consider the differential equation

$$y^{(4)} - (py')' = \lambda y, \quad \lambda \in \mathbb{C}. \tag{1.2}$$

Now we define the fundamental solutions $\varphi_j$, $j = 1, 2, 3, 4$, of this equation. These solutions satisfy the conditions $\varphi_j^{(k-1)}(0, \lambda) = \delta_{jk}$, $k = 1, 2, 3$, $(\varphi_j''' - p\varphi_j')(0, \lambda) = \delta_{j4}$, where $\delta_{jk}$ is the Kronecker symbol. Note that $\varphi_j(x, \cdot)$, $x \in [0, 1]$, is an entire function. Moreover, the spectrum is

$$\sigma(\mathcal{H}) = \{\lambda \in \mathbb{C} : D(\lambda) = 0\},$$

where $D$ is an entire function defined by

$$D(\lambda) = \det \begin{pmatrix} \varphi_3''(1, \lambda) & \varphi_4''(1, \lambda) \\ (G\varphi_3)(1, \lambda) & (G\varphi_4)(1, \lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}. \tag{1.3}$$

Here and below we denote by $G$ the function

$$(G\varphi_j)(1, \lambda) = (a\lambda + b)\varphi_j(1, z) - (c\lambda + d)T\varphi_j(1, z), \quad j = 1, 2, 3, 4. \tag{1.4}$$

Now we consider the case $p = 0$. The function $D = D_0$ has the form

$$D_0(\lambda) = -\frac{cz^4 + d}{2}(1 + \cos z \cos iz) - \frac{az^4 + b}{2z^3}\big(\sin z \cos iz + i \cos z \sin iz\big), \tag{1.5}$$

where

$$z = \lambda^{1/4}, \quad \arg z \in \big(-\frac{\pi}{4}, \frac{\pi}{4}\big], \quad \text{as } \arg \lambda \in (-\pi, \pi].$$

The eigenvalues of (1.1) are the zeros of $D_0$ and have the following asymptotic behavior (see [14, Theorem 6.1]):

$$\lambda_n^0 = \begin{cases} \big(-3\pi/4 + \pi n\big)^4 + \mathcal{O}(n^2), & \text{if } c = 0, \\ \big(-3\pi/2 + \pi n\big)^4 + \mathcal{O}(n^2), & \text{if } c \neq 0, \end{cases} \quad n \in \mathbb{N}. \tag{1.6}$$

Recall that the eigenvalues $\lambda_n^0$ are simple [14, Theorem 2.1].

We denote the eigenvalues of the operator $\mathcal{H}$ by $\lambda_n$, $n \in \mathbb{N}$. We introduce the coefficients

$$f_0 = \int_0^1 f(x)\, dx, \quad \widehat{f}_{cn}(\varepsilon) = \int_0^1 f(x) \cos \pi(2n - \varepsilon)x\, dx,$$
$$\widehat{f}_{sn}(\varepsilon) = \int_0^1 f(x) \sin \pi(2n - \varepsilon)x\, dx,$$

for $n \in \mathbb{Z}$, where $\varepsilon$ is a positive constant. Our main result is devoted to the high energy asymptotic behavior of $\lambda_n$.

**Theorem 1.1.** *(i) Suppose that $p \in W^{1,1}(0, 1)$. The eigenvalues $\lambda_n$ are real, simple, and have the following asymptotic behaviour for $c \neq 0$:*

$$\lambda_n = \big(-\frac{3\pi}{2} + \pi n\big)^4 + \big(-\frac{3\pi}{2} + \pi n\big)^2 p_0 + \mathcal{O}(n), \tag{1.7}$$

*as $n \to +\infty$. If $c = 0$, then the asymptotic behavior of the eigenvalue has the form*

$$\lambda_n = \big(-\frac{3\pi}{4} + \pi n\big)^4 + \big(-\frac{3\pi}{4} + \pi n\big)^2 \big(p_0 - \frac{2d}{a}\big) + \mathcal{O}(n), \tag{1.8}$$

*as* $n \to +\infty$.

*(ii) Suppose that* $p \in W^{3,1}(0,1)$ *and* $p(0) = p(1)$. *If* $c \neq 0$, *then*

$$\begin{aligned} \lambda_n = &\big(-\frac{3\pi}{2} + \pi n\big)^4 + \big(-\frac{3\pi}{2} + \pi n\big)^2 p_0 + 3p(1)\pi(2n-3) \\ &+ \frac{p_0^2 - \|p\|^2}{8} - 2p'(1) + \frac{4a}{c} - \frac{\rho_{1,n} + \widehat{p}'''_{cn}(3)}{4\pi(2n-3)} + \mathcal{O}(n^{-2}), \end{aligned} \tag{1.9}$$

*as* $n \to +\infty$, *where*

$$\begin{aligned} \rho_{1,n} = &\int_0^1 e^{-\pi(2n-3)s}\big(p'''(1-s) - p'''(s)\big)\,ds \\ &- 8\int_0^1 e^{-\pi(n-3/2)s}\big(p'''(s) + p'''(1-s)\big)\sin\big(-\frac{3\pi}{2} + \pi n\big)s\,ds. \end{aligned} \tag{1.10}$$

*If* $c = 0$, *then the eigenvalues* $\lambda_n$ *have the asymptotic behavior*

$$\begin{aligned} \lambda_n = &\big(-\frac{3\pi}{4} + \pi n\big)^4 + \big(-\frac{3\pi}{4} + \pi n\big)^2\Big(p_0 - \frac{2d}{a}\Big) \\ &- \big(-\frac{3\pi}{4} + \pi n\big)\Big(p(1) + \frac{d^2}{a^2}\Big) + \frac{p_0^2 - \|p\|^2}{8} - \frac{p'(1)}{2} - \frac{p_0 d}{2a} + \frac{d^2}{2a^2} \\ &- \frac{d^3}{3a^3} - \frac{7dp(1)}{2a} + \frac{\rho_{2,n} - \widehat{p}'''_{cn}(3/2)}{2\pi(4n-3)} + \mathcal{O}(n^{-2}), \end{aligned} \tag{1.11}$$

*as* $n \to +\infty$, *with*

$$\rho_{2,n} = \int_0^1 e^{-\pi(2n-3/2)s}p'''(s)\,ds + 8\int_0^1 e^{-\pi(n-3/4)s}p'''(s)\sin\pi(n-3/4)s\,ds. \tag{1.12}$$

**Remark 1.2.** In this manuscript, we do not discuss the numbering of eigenvalues in a circle of large radius. The asymptotic behavior of the eigenvalues from Theorem 1.1 will be true only starting from some sufficiently large $n$. To obtain the information about the number of zeros of operator $\mathcal{H}$, we need to obtain the number of zeros of the function $D_0$ in a circle of large radius and the estimates for the difference $D - D_0$. However, it is not trivial for higher-order operators.

Now we discuss the main features of the considered operator and the main advantages of our primary results. Note that the formulas (1.7) and (1.8) improve well-known asymptotics (1.6) from [14, Theorem 6.1]. Namely, we establish the second and the third terms of the eigenvalue asymptotics. If the coefficient $p$ is smooth, then we obtain a more detailed asymptotic behavior of the eigenvalues (1.9) and (1.11). We discuss a structure of these asymptotics. It contains the term $\rho_{1,n}$ or $\rho_{2,n}$ of the form (1.10) and (1.12), respectively. These terms correlate with the corresponding terms from well-known spectral asymptotics for an operator of the form (1.1) with boundary conditions $y(0) = y(1) = y'(0) = y'(1) = 0$ (see details in [22, Theorem 1]), i.e. with identical fixation at the left end. At the same time, in comparison with [22, Theorem 1], additional constant terms appear in (1.9) and (1.11). These terms are nontrivial, but they are typical specifically for problems with a spectral parameter in the boundary condition. This effect also previously appeared when studying the properties of fourth-order operators with a free term [20, 21]. It follows from (1.8) that in some problems this nontrivial effect is already observed for the case of non-smooth coefficient $p$. However, it does not appear in problems for a fourth-order operator without a spectral parameter in the boundary conditions (see [9, 22, 23], and the references therein).

Consider the shifted operator $\mathcal{H}_t(p_t)$ defined by (1.1), where $p_t = p(\cdot + t)$, $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$. We denote by $\mu_n(t)$ the eigenvalues of the operator $\mathcal{H}_t$. The asymptotics (1.9) and (1.11) show that the series

$$\sum_{n=1}^{\infty}\Big(\lambda_n(t) - \lambda_n(0) - 3\pi(2n-3)\big(p(t)-p(0)\big) + 2\big(p'(t)-p'(0)\big)\Big),$$

for $c \neq 0$, and

$$\sum_{n=1}^{\infty}\Big(\lambda_n(t) - \lambda_n(0) + \big(-\frac{3\pi}{4} + \pi n\big)\big(p(t)-p(0)\big) + \frac{p'(t)-p'(0)}{2} + \frac{7d\big(p(t)-p(0)\big)}{2a}\Big),$$

for $c = 0$, converges. Thus, the regularized trace formula is correctly defined for this class of operators. However, at the moment it is an open question: to obtain exact trace formulas for both cases. We only determine the asymptotics of the characteristic function $D$ of the form (1.3) (see Lemma 4.1). To establish the main results we use the Birkhoff method [19, Ch. 2]. In this work, we apply a combination of the matrix version of this method for higher-order operators (see [10, 23]) and the semiclassical method from [11]. This combination allows us to determine the second and the third term in the asymptotic behavior of the eigenvalues, as well as to obtain a precisely controlled remainder term.

The plan of this article is as follows. In Section 2 we study the properties of the characteristic function $D$ and the fundamental matrix of equation (1.2). Moreover, in this section we provide the representation of this fundamental matrix. In Section 3 we obtain the asymptotical formulas for the eigenvalues $\lambda_n$ in the case $p \in W^{1,1}(0,1)$. The case $p \in W^{3,1}(0,1)$ is considered in Section 4.

## 2. Properties of the fundamental solutions

In this section we introduce other fundamental solutions $\phi_j$, $j = 1,2,3,4$. These solutions are different from the solutions $\varphi_j$, $j = 1,2,3,4$, but the asymptotic behavior of $\phi_j$, $j = 1,2,3,4$, can be well controlled. We transform the equation (1.2). Using this transformation, we can determine the representation of the fundamental matrix and hold asymptotic analysis of the fundamental matrix.

Recall that $z = \lambda^{1/4}$, $z \in \overline{\mathcal{Z}}$, $\lambda \in \mathbb{C}$, where

$$\overline{\mathcal{Z}} = \big\{z \in \mathbb{C} : \arg z \in \big(-\frac{\pi}{4}, \frac{\pi}{4}\big]\big\}, \quad \mathcal{Z} = \big\{z \in \mathbb{C} : \arg z \in \big(-\frac{\pi}{4}, \frac{\pi}{4}\big)\big\}.$$

If $\lambda \in \mathbb{C}_+$, then $z \in \mathcal{Z}_+$, where

$$\mathcal{Z}_+ = \big\{z \in \mathbb{C} : \arg z \in \big(0, \frac{\pi}{4}\big)\big\}.$$

Define the numbers $\omega_1 = -\omega_4 = i$, $\omega_2 = -\omega_3 = 1$. Therefore,

$$\mathrm{Re}(i\omega_1 z) \leqslant \mathrm{Re}(i\omega_2 z) \leqslant \mathrm{Re}(i\omega_3 z) \leqslant \mathrm{Re}(i\omega_4 z), \quad z \in \mathcal{Z}_+.$$

It is easy to see that equation (1.2) with $p = 0$ has the fundamental solutions

$$\phi_j^0(x,z) = e^{izx\omega_j}, \quad j = 1,2,3,4. \tag{2.1}$$

Consider the perturbed equation (1.2). Let $r > 0$ be large enough and let $z \in \mathcal{Z}_+(r)$, where $\mathcal{Z}_+(r) = \{z \in \mathcal{Z}_+ : |z| > r\}$, $r > 0$. Then equation (1.2) has the fundamental solutions $\phi_j(x,z)$, $j = 1,2,3,4$, $x \in [0,1]$, $z \in \mathcal{Z}_+(r)$, satisfying the asymptotics

$$\begin{gathered}\phi_j(x,z) = \phi_j^0(x,z)(1+\mathcal{O}(z^{-1})), \quad \phi_j'(x,z) = (\phi_j^0)'(x,z)(1+\mathcal{O}(z^{-1})),\\ \phi_j''(x,z) = (\phi_j^0)''(x,z)(1+\mathcal{O}(z^{-1})), \quad \phi_j'''(x,z) = (\phi_j^0)'''(x,z)(1+\mathcal{O}(z^{-1})),\end{gathered} \tag{2.2}$$

as $|z| \to \infty$, uniformly in $x \in [0,1]$ (see [19]).

Now we define the fundamental matrix $A(x,z)$, $x \in [0,1]$, $z \in \mathcal{Z}_+(r)$, of equation (1.2) by

$$A = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \phi_1' & \phi_2' & \phi_3' & \phi_4' \\ \phi_1'' & \phi_2'' & \phi_3'' & \phi_4'' \\ \phi_1''' - p\phi_1' & \phi_2''' - p\phi_2' & \phi_3''' - p\phi_1' & \phi_4''' - p\phi_4' \end{pmatrix}. \tag{2.3}$$

This matrix-valued function satisfies the equation

$$A' = \mathcal{P}A, \quad \text{where} \quad \mathcal{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & p & 0 & 1 \\ \lambda & 0 & 0 & 0 \end{pmatrix}. \tag{2.4}$$

We rewrite the function $D$, given by (1.3), in terms of the fundamental solutions $\phi_j$, where the matrix-valued function $\phi$ has the form

$$\phi(z) = \begin{pmatrix} \phi_1(0,z) & \phi_2(0,z) & \phi_3(0,z) & \phi_4(0,z) \\ \phi_1'(0,z) & \phi_2'(0,z) & \phi_3'(0,z) & \phi_4'(0,z) \\ \phi_1''(1,z) & \phi_2''(1,z) & \phi_3''(1,z) & \phi_4''(1,z) \\ (G\phi_1)(1,z) & (G\phi_2)(1,z) & (G\phi_3)(1,z) & (G\phi_4)(1,z) \end{pmatrix}, \tag{2.5}$$

where $G$ has the form (1.4).

Repeating the arguments from [10, Lemma 3.2], we derive that the determinant $D$ can be represented in the form

$$D(\lambda) = \frac{\det \phi(z)}{\det A(0,z)}, \quad \lambda = z^4. \tag{2.6}$$

The function $\det \phi$ is analytic in $\mathcal{Z}_+(r)$. Hence the function $D$ is entire and the identity (2.6) can be extended analytically from $\mathcal{Z}_+(r)$ onto the whole complex plane. Note that the function $\det \phi$ is not an entire function of the variable $\lambda$, but its asymptotics at high energy is well controlled.

Consider the unperturbed case $p = 0$. Let $z \in \mathcal{Z}_+$. Therefore, the fundamental matrix $A = A_0$ satisfies

$$A_0 = \begin{pmatrix} \phi_1^0 & \phi_2^0 & \phi_3^0 & \phi_4^0 \\ (\phi_1^0)' & (\phi_2^0)' & (\phi_3^0)' & (\phi_4^0)' \\ (\phi_1^0)'' & (\phi_2^0)'' & (\phi_3^0)'' & (\phi_4^0)'' \\ (\phi_1^0)''' & (\phi_2^0)''' & (\phi_3^0)''' & (\phi_4^0)''' \end{pmatrix} = \Omega Y_0, \quad Y_0 = \operatorname{diag}(\phi_1^0, \phi_2^0, \phi_3^0, \phi_4^0),$$

where $\phi_j^0$, $j = 1,2,3,4$, are defined by (2.1) and

$$\Omega = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -z & iz & -iz & z \\ z^2 & -z^2 & -z^2 & z^2 \\ -z^3 & -iz^3 & iz^3 & z^3 \end{pmatrix}. \tag{2.7}$$

Using the identity $\phi_1^0\phi_2^0\phi_3^0\phi_4^0 = 1$, we obtain

$$\det A_0(0,z) = \det \Omega = -16iz^6. \tag{2.8}$$

The matrix-valued function $\phi = \phi_0$ has the form

$$\begin{aligned}\phi_0(z) &= \begin{pmatrix} \phi_1^0(0,z) & \phi_2^0(0,z) & \phi_3^0(0,z) & \phi_4^0(0,z) \\ (\phi_1^0)'(0,z) & (\phi_2^0)'(0,z) & (\phi_3^0)'(0,z) & (\phi_4^0)'(0,z) \\ (\phi_1^0)''(1,z) & (\phi_2^0)''(1,z) & (\phi_3^0)''(1,z) & (\phi_4^0)''(1,z) \\ (G\phi_1^0)(1,z) & (G\phi_2^0)(1,z) & (G\phi_3^0)(1,z) & (G\phi_4^0)(1,z) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ -z & iz & -iz & z \\ z^2e^{-z} & -z^2e^{iz} & -z^2e^{-iz} & z^2e^{z} \\ (G\phi_1^0)(1,z) & (G\phi_2^0)(1,z) & (G\phi_3^0)(1,z) & (G\phi_4^0)(1,z) \end{pmatrix}.\end{aligned}$$

Then

$$\det \phi_0(z) = 8iz^6(cz^4+d)(1+\cos z\cos iz) + 8iz^3(az^4+b)\big(\sin z\cos iz + i\cos z\sin iz\big).$$

The identities (2.6) and (2.8) show that the entire function $D_0$ in this case satisfies (1.5).

It follows from formula (2.6) that to obtain the asymptotics of $D$ we need to analyse the asymptotics of the function $\det\phi$. Now we derive the formula for the fourth-order determinant $\det\phi$. More precisely we express $\det\phi$ in terms of linear combinations for product of second-order determinants.

**Lemma 2.1.** *Suppose that $p \in W^{1,1}(0,1)$, and $|z| \to \infty$. Then*

$$\det\phi(z) = \begin{cases} e^{2\operatorname{Re} z}\mathcal{O}(z^{10}), & \text{for } c \neq 0, \\ e^{2\operatorname{Re} z}\mathcal{O}(z^{7}), & \text{for } c = 0, \end{cases} \quad z \in \mathcal{Z}_+. \tag{2.9}$$

*Moreover,*

$$\det\phi(z) = \begin{cases} \gamma_1(z)\gamma_2(z) + \gamma_3(z)\gamma_4(z) + \mathcal{O}(z^{10}), & \text{for } c \neq 0, \\ \gamma_1(z)\gamma_2(z) + \gamma_3(z)\gamma_4(z) + \mathcal{O}(z^{7}), & \text{for } c = 0, \end{cases} \quad z \in \mathcal{Z}_+, \tag{2.10}$$

*where*

$$\gamma_1(z) = \det\begin{pmatrix} \phi_1(0,z) & \phi_2(0,z) \\ \phi_1'(0,z) & \phi_2'(0,z) \end{pmatrix}, \tag{2.11}$$

$$\gamma_2(z) = \det\begin{pmatrix} \phi_3''(1,z) & \phi_4''(1,z) \\ (G\phi_3)(1,z) & (G\phi_4)(1,z) \end{pmatrix}, \tag{2.12}$$

$$\gamma_3(z) = \det\begin{pmatrix} \phi_3(0,z) & \phi_1(0,z) \\ \phi_3'(0,z) & \phi_1'(0,z) \end{pmatrix}, \tag{2.13}$$

$$\gamma_4(z) = \det\begin{pmatrix} \phi_2''(1,z) & \phi_4''(1,z) \\ (G\phi_2)(1,z) & (G\phi_4)(1,z) \end{pmatrix}, \tag{2.14}$$

*and $G\phi_j$, $j = 1,2,3,4$, have the form* (1.4).

*Proof.* Let $z \in \mathcal{Z}_+$ and $|z| \to \infty$. Then $0 \leqslant \operatorname{Im} z \leqslant \operatorname{Re} z$. Consider the definition (2.5). Direct calculations imply

$$\begin{aligned}
\det \phi(z) &= \det \begin{pmatrix} \phi_1(0,z) & \phi_2(0,z) \\ \phi_1'(0,z) & \phi_2'(0,z) \end{pmatrix} \det \begin{pmatrix} \phi_3''(1,z) & \phi_4''(1,z) \\ (G\phi_3)(1,z) & (G\phi_4)(1,z) \end{pmatrix} \\
&\quad + \det \begin{pmatrix} \phi_3(0,z) & \phi_1(0,z) \\ \phi_3'(0,z) & \phi_1'(0,z) \end{pmatrix} \det \begin{pmatrix} \phi_2''(1,z) & \phi_4''(1,z) \\ (G\phi_2)(1,z) & (G\phi_4)(1,z) \end{pmatrix} \\
&\quad + \det \begin{pmatrix} \phi_1(0,z) & \phi_4(0,z) \\ \phi_1'(0,z) & \phi_4'(0,z) \end{pmatrix} \det \begin{pmatrix} \phi_2''(1,z) & \phi_3''(1,z) \\ (G\phi_2)(1,z) & (G\phi_3)(1,z) \end{pmatrix} \\
&\quad + \det \begin{pmatrix} \phi_2(0,z) & \phi_3(0,z) \\ \phi_2'(0,z) & \phi_3'(0,z) \end{pmatrix} \det \begin{pmatrix} \phi_1''(1,z) & \phi_4''(1,z) \\ (G\phi_1)(1,z) & (G\phi_4)(1,z) \end{pmatrix} \\
&\quad + \det \begin{pmatrix} \phi_4(0,z) & \phi_2(0,z) \\ \phi_4'(0,z) & \phi_2'(0,z) \end{pmatrix} \det \begin{pmatrix} \phi_1''(1,z) & \phi_3''(1,z) \\ (G\phi_1)(1,z) & (G\phi_3)(1,z) \end{pmatrix} \\
&\quad + \det \begin{pmatrix} \phi_3(0,z) & \phi_4(0,z) \\ \phi_3'(0,z) & \phi_4'(0,z) \end{pmatrix} \det \begin{pmatrix} \phi_1''(1,z) & \phi_2''(1,z) \\ (G\phi_1)(1,z) & (G\phi_2)(1,z) \end{pmatrix}.
\end{aligned} \tag{2.15}$$

Let $c \neq 0$. Here and below we denote $1 + \mathcal{O}(z^{-1})$ by $[1]_{-1}$. Consider the term $(G\phi_1)(1,z)$. Using (1.4) and (2.2), we have

$$\begin{aligned}
&(G\phi_1)(1,z) \\
&= (az^4+b)\phi_1(1,z) - (cz^4+d)\big(\phi_1'''(1,z) - p(1)\phi_1'(1,z)\big) \\
&= (az^4+b)e^{-z}[1]_{-1} - (cz^4+d)\big(-z^3e^{-z}[1]_{-1} + p(1)ze^{-z}[1]_{-1}\big) \\
&= e^{-z}\Big((az^4+b)[1]_{-1} + (cz^4+d)z^3[1]_{-1}\Big) = cz^7e^{-z}[1]_{-1}.
\end{aligned} \tag{2.16}$$

Similar arguments give

$$\begin{gathered}
(G\phi_2)(1,z) = ciz^7e^{iz}[1]_{-1}, \quad (G\phi_3)(1,z) = -ciz^7e^{-iz}[1]_{-1}, \\
(G\phi_4)(1,z) = -cz^7e^{z}[1]_{-1}.
\end{gathered} \tag{2.17}$$

Then the estimates $0 \leqslant \operatorname{Im} z \leqslant \operatorname{Re} z$ and the asymptotics (2.2), (2.16), and (2.17) imply

$$\begin{aligned}
&\det \begin{pmatrix} \phi_3(0,z) & \phi_4(0,z) \\ \phi_3'(0,z) & \phi_4'(0,z) \end{pmatrix} \det \begin{pmatrix} \phi_1''(1,z) & \phi_2''(1,z) \\ (G\phi_1)(1,z) & (G\phi_2)(1,z) \end{pmatrix} \\
&= \det \begin{pmatrix} [1]_{-1} & [1]_{-1} \\ -iz[1]_{-1} & z[1]_{-1} \end{pmatrix} \det \begin{pmatrix} z^2e^{-z}[1]_{-1} & -z^2e^{iz}[1]_{-1} \\ cz^7e^{-z}[1]_{-1} & ciz^7e^{iz}[1]_{-1} \end{pmatrix} \\
&= e^{-z+iz}\mathcal{O}(z^{10}) = e^{-\operatorname{Re} z - \operatorname{Im} z}\mathcal{O}(z^{10}) = e^{-2\operatorname{Re} z}\mathcal{O}(z^{10})
\end{aligned}$$

and

$$\begin{aligned}
&\det \begin{pmatrix} \phi_4(0,z) & \phi_2(0,z) \\ \phi_4'(0,z) & \phi_2'(0,z) \end{pmatrix} \det \begin{pmatrix} \phi_1''(1,z) & \phi_3''(1,z) \\ (G\phi_1)(1,z) & (G\phi_3)(1,z) \end{pmatrix} \\
&= \det \begin{pmatrix} [1]_{-1} & [1]_{-1} \\ z[1]_{-1} & iz[1]_{-1} \end{pmatrix} \det \begin{pmatrix} z^2e^{-z}[1]_{-1} & -z^2e^{-iz}[1]_{-1} \\ cz^7e^{-z}[1]_{-1} & -ciz^7e^{-iz}[1]_{-1} \end{pmatrix} \\
&= e^{-z-iz}\mathcal{O}(z^{10}) = e^{-\operatorname{Re} z + \operatorname{Im} z}\mathcal{O}(z^{10}) = \mathcal{O}(z^{10}).
\end{aligned}$$

Moreover,

$$\det \begin{pmatrix} \phi_2(0,z) & \phi_3(0,z) \\ \phi_2'(0,z) & \phi_3'(0,z) \end{pmatrix} \det \begin{pmatrix} \phi_1''(1,z) & \phi_4''(1,z) \\ (G\phi_1)(1,z) & (G\phi_4)(1,z) \end{pmatrix}$$

$$= \det\begin{pmatrix} [1]_{-1} & [1]_{-1} \\ iz[1]_{-1} & -iz[1]_{-1} \end{pmatrix} \det\begin{pmatrix} z^2 e^{-z}[1]_{-1} & z^2 e^{z}[1]_{-1} \\ cz^7 e^{-z}[1]_{-1} & -cz^7 e^{z}[1]_{-1} \end{pmatrix} = \mathcal{O}(z^{10})$$

and

$$\begin{aligned} &\det\begin{pmatrix} \phi_1(0,z) & \phi_4(0,z) \\ \phi_1'(0,z) & \phi_4'(0,z) \end{pmatrix} \det\begin{pmatrix} \phi_2''(1,z) & \phi_3''(1,z) \\ (G\phi_2)(1,z) & (G\phi_3)(1,z) \end{pmatrix} \\ &= \det\begin{pmatrix} [1]_{-1} & [1]_{-1} \\ -z[1]_{-1} & z[1]_{-1} \end{pmatrix} \det\begin{pmatrix} -z^2 e^{iz}[1]_{-1} & -z^2 e^{-iz}[1]_{-1} \\ ciz^7 e^{iz}[1]_{-1} & -ciz^7 e^{-iz}[1]_{-1} \end{pmatrix} = \mathcal{O}(z^{10}). \end{aligned}$$

Substituting these asymptotics into (2.15), we get the first formula in (2.10). Using (2.2), (2.16), (2.17), and the estimates $0 \leqslant \operatorname{Im} z \leqslant \operatorname{Re} z$ again, we obtain

$$\begin{aligned} &\gamma_1(z)\gamma_2(z) + \gamma_3(z)\gamma_4(z) \\ &= \det\begin{pmatrix} [1]_{-1} & [1]_{-1} \\ -z[1]_{-1} & iz[1]_{-1} \end{pmatrix} \det\begin{pmatrix} -z^2 e^{-iz}[1]_{-1} & z^2 e^{z}[1]_{-1} \\ -ciz^7 e^{-iz}[1]_{-1} & -cz^7 e^{z}[1]_{-1} \end{pmatrix} \\ &\quad + \det\begin{pmatrix} [1]_{-1} & [1]_{-1} \\ -iz[1]_{-1} & -z[1]_{-1} \end{pmatrix} \det\begin{pmatrix} -z^2 e^{iz}[1]_{-1} & z^2 e^{z}[1]_{-1} \\ ciz^7 e^{iz}[1]_{-1} & -cz^7 e^{z}[1]_{-1} \end{pmatrix} \\ &= e^{z-iz}\mathcal{O}(z^{10}) + e^{z+iz}\mathcal{O}(z^{10}) = e^{z-iz}\mathcal{O}(z^{10}) \\ &= e^{\operatorname{Re} z + \operatorname{Im} z}\mathcal{O}(z^{10}) = e^{2\operatorname{Re} z}\mathcal{O}(z^{10}). \end{aligned}$$

This yields the first asymptotics in (2.9).

Let $c = 0$. We first compute the elements $(G\phi_j)(1,z)$, $j = 1,2,3,4$. The formulas (1.4) and (2.2) give

$$(G\phi_1)(1,z) = (az^4 + b)e^{-z}[1]_{-1} - d\big(-z^3 e^{-z}[1]_{-1} + p(1)ze^{-z}[1]_{-1}\big) = az^4 e^{-z}[1]_{-1}.$$

Using similar arguments, we get

$$(G\phi_2)(1,z) = az^4 e^{iz}[1]_{-1}, \quad (G\phi_3)(1,z) = az^4 e^{-iz}[1]_{-1}, \quad (G\phi_4)(1,z) = az^4 e^{z}[1]_{-1}.$$

Repeating the above procedure as in the case $c \neq 0$, we obtain the second asymptotics in (2.9) and (2.10). □

Now we investigate the properties of the fundamental matrix $A$ of equation (2.4). Since the contribution of bounded and decreasing elements of the matrix $A$ completely disappears against the background of the contribution of increasing ones, the asymptotic analysis of the matrix $A$ is a rather difficult problem. It is clearly seen that the matrix $\mathcal{P}$ in equation (2.4) contains growing elements. We transform this equation into a first-order differential equation in such a way that the increasing terms are separated into a separate term, and the right side of the equation is decreasing as $z^{-1}$. Further, we apply the Birkhoff method [19, Chapter 2] to this first-order equation. In this manuscript we use the matrix version of this scheme from [10, § 2] (see also [23]). Applying this modification, we reduce this equation to the equivalent Fredholm integral equation with a "small kernel". Using the method of simple iterations, we find a solution of the Fredholm integral equation. This solution allows us to obtain the representation (2.35) of the fundamental matrix $A$. The main feature of (2.35) is that all growing elements are separated into a separate diagonal matrix. This makes it possible to control their contribution to the asymptotic behavior of the eigenvalues. If the coefficient $p$ has the necessary additional smoothness, then we can transform (2.4) into a first-order differential equation of the same type as before, but with a better rate of decrease of the right-hand side. In order to do this we use semiclassical method [11, Chapter V.1.3].

Then we again apply the Birkhoff method and obtain a representation (2.37) of the fundamental matrix $A$.

Now we transform equation (2.4). Introduce the matrix-valued function $Y_1(x,z)$ by

$$A(x,z)=\Omega(z)Y_1(x,z),\quad (x,z)\in[0,1]\times\mathcal{Z}_+(r), \tag{2.18}$$

and the matrix

$$\mathcal{T}=\operatorname{diag}(\omega_1,\omega_2,\omega_3,\omega_4)=\operatorname{diag}(i,1,-1,-i).$$

Here $\Omega$ is defined by (2.7).

**Lemma 2.2.** *Suppose that $p\in W^{1,1}(0,1)$ and $z\in\mathcal{Z}_+(r)$, where $r>0$ is large enough. Then the matrix-valued function $Y_1$, defined by* (2.18)*, satisfies the equation*

$$Y_1'-izT_1Y_1=\frac{1}{z}\Phi_1Y_1, \tag{2.19}$$

*where $T_1$ and $\Phi_1$ are the matrix-valued functions of the form*

$$T_1=\mathcal{T}-\frac{p}{4z^2}\mathcal{T}^3,\quad \Phi_1=F_1=\frac{p}{4}(P+i\mathcal{T}^3), \tag{2.20}$$

*with*

$$P=\begin{pmatrix}-1 & i & -i & 1\\ 1 & -i & i & -1\\ 1 & -i & i & -1\\ -1 & i & -i & 1\end{pmatrix},\quad Q=\begin{pmatrix}-1 & -1 & -1 & -1\\ i & i & i & i\\ -i & -i & -i & -i\\ 1 & 1 & 1 & 1\end{pmatrix}.$$

The proof of the above lemma can be found in [23, Lemma 1]. Now we consider the case of the smooth coefficient. Let $p\in W^{3,1}(0,1)$. Using the method [11, Ch. V.1.3], we transform equation (2.19) so that the matrix coefficient on the right side decreases as $z^{-4}$. We introduce a new unknown matrix-valued function $Y_4(x,z)$ by

$$Y_1(x,z)=\Big(\mathbb{I}_4+\frac{\mathcal{W}(x,z)}{z^2}\Big)Y_4(x,z),\quad (x,z)\in[0,1]\times\mathcal{Z}_+(r), \tag{2.21}$$

where $Y_1$ is the solution of equation (2.19) and $\mathcal{W}$ has the form

$$\mathcal{W}=pW_1+\frac{p'}{z}W_2+\frac{p''}{32z^2}Q_1-\frac{p^2}{64z^2}Q_2. \tag{2.22}$$

We choose the matrices $W_1$, $W_2$, $Q_1$, and $Q_2$ in such a way that the coefficient on the right side of equation (2.25) decreases as $z^{-4}$. Thus,

$$W_1=\frac{1}{8}\begin{pmatrix}0 & 1+i & 1-i & 1\\ -1+i & 0 & -1 & -1-i\\ -1-i & -1 & 0 & -1+i\\ 1 & 1-i & 1+i & 0\end{pmatrix},\quad W_2=\frac{1}{16}\begin{pmatrix}0 & -2 & -2 & -1\\ 2i & 0 & i & 2i\\ -2i & -i & 0 & -2i\\ 1 & 2 & 2 & 0\end{pmatrix}, \tag{2.23}$$

and

$$Q_1=\begin{pmatrix}0 & 2-2i & 2+2i & 1\\ 2+2i & 0 & 1 & 2-2i\\ 2-2i & 1 & 0 & 2+2i\\ 1 & 2+2i & 2-2i & 0\end{pmatrix},\quad Q_2=\begin{pmatrix}-2 & 1-i & 1+i & 0\\ 1+i & 2i & 0 & 1-i\\ 1-i & 0 & -2i & 1+i\\ 0 & 1+i & 1-i & 2\end{pmatrix}. \tag{2.24}$$

We have the following result.

**Lemma 2.3.** *Suppose that $p \in W^{3,1}(0,1)$ and $z \in \mathcal{Z}_+(r)$, where $r > 0$ is large enough. Then the matrix-valued function $Y_4$, given by (2.21), satisfies the equation*

$$Y_4' - izT_4Y_4 = \frac{1}{z^4}\Phi_4 Y_4, \tag{2.25}$$

*where*

$$T_4 = \mathcal{T} - \frac{p}{4z^2}\mathcal{T}^3 + \frac{p^2}{32z^4}\mathcal{T} + \frac{ipp'}{64z^5}(-3\mathbb{I}_4 + 4i\mathcal{T}), \tag{2.26}$$

$$\Phi_4 = F_4 + \mathcal{O}(z^{-1}), \quad F_4 = -\frac{p'''}{32}Q_1 + \frac{pp'}{64}(Q_3 + 3\mathbb{I}_4 - 4i\mathcal{T}), \tag{2.27}$$

*as $|z| \to \infty$, uniformly in $x \in [0,1]$. The matrix $Q_1$ has the form (2.24) and $Q_3$ is a matrix.*

The proof of this lemma can be found in [23, Lemma 4].

**Remark 2.4.** The matrix $Q_3$ has a specific form. It is not important for further calculations, since this term is $\mathcal{O}(z^{-1})$.

To transform the differential equations (2.19) and (2.25) we use the Birkhoff method. Moreover, we obtain the representation of the fundamental matrix $A$. The formula (2.3) shows that the matrix $A$ contains exponentially increasing, bounded, and decreasing entries at high energy. Using the Birkhoff method, we extract exponentially increasing elements into a separate diagonal matrix. It is known (see [10, Theorem 4.5] and [23, Lemma 5]) that the matrix-valued function

$$Y_\sigma(x,z) = \mathcal{X}(x,z)e^{iz\int_0^x T_\sigma(s,z)\,ds}, \quad \sigma = 1,4,$$

satisfies the differential equations (2.19) and (2.25) if and only if $\mathcal{X}$ is a solution of the integral equations

$$\mathcal{X} = \mathbb{I}_4 + \frac{1}{z^\sigma}K\mathcal{X}, \quad \sigma = 1,4, \tag{2.28}$$

where $T_1$ and $T_4$ have the form (2.20) and (2.26), respectively, and $K$ is an integral operator in the space $C[0,1]$ of $4\times 4$ matrix-valued functions defined by

$$(K\mathcal{X})_{lj}(x,z) = \int_0^1 K_{lj}(x,s,z)(\Phi_\sigma\mathcal{X})_{lj}(s,z)\,ds, \quad l,j = 1,2,3,4,\ \sigma = 1,4,$$

for all $\mathcal{X} \in C[0,1]$, $(x,z) \in [0,1]\times\mathcal{Z}_+(r)$, and

$$\begin{gathered} K_{lj}(x,s,z) = \begin{cases} e^{iz\int_s^x(\Theta_l(u,z)-\Theta_j(u,z))\,du}\chi(x-s), & l<j,\\ -e^{iz\int_s^x(\Theta_l(u,z)-\Theta_j(u,z))\,du}\chi(s-x), & l\geqslant j, \end{cases}\\ \chi(s) = \begin{cases} 1, & s\geqslant 0,\\ 0, & s<0. \end{cases} \end{gathered} \tag{2.29}$$

Here $\Phi_1$ and $\Phi_4$ satisfy (2.20) and (2.27), respectively. Note that the integral operator $K$ is a contraction for $z \in \mathcal{Z}_+(r)$, where $r > 0$ is large enough. The equations (2.28) have a unique solution $\mathcal{X}(\cdot,z) \in C[0,1]$. Moreover, each matrix-valued function $\mathcal{X}(x,\cdot)$, $x \in [0,1]$, is analytic on $\mathcal{Z}_+(r)$ and satisfies the asymptotics

$$\mathcal{X}(x,z) = \mathbb{I}_4 + \frac{1}{z^\sigma}(K\mathbb{I}_4)(x,z) + \mathcal{O}(z^{-2\sigma}), \quad \sigma = 1,4, \tag{2.30}$$

as $|z| \to \infty$, $z \in \mathcal{Z}_+$, uniformly in $x \in [0,1]$, where

$$(K\mathbb{I}_4)_{lj}(x,z) = \int_0^1 K_{lj}(x,s,z)\Phi_{\sigma,lj}(s,z)\,ds, \quad l,j=1,2,3,4,\ \sigma=1,4.$$

Relations (2.20) and (2.27) show that in equations (2.19) and (2.25) the function $K\mathbb{I}_4$ has the asymptotics $K\mathbb{I}_4 = \mathcal{B}_\sigma + \mathcal{O}(z^{-1})$, $\sigma = 1,4$, where the matrix-valued functions $\mathcal{B}_\sigma(x,z)$, $(x,z) \in [0,1]\times\mathcal{Z}_+$, $\sigma = 1,4$, are defined by

$$\mathcal{B}_{\sigma,lj}(x,z) = \int_0^1 K_{lj}(x,s,z)F_{\sigma,lj}(s)\,ds.$$

Moreover, the formula (2.29) yields

$$\mathcal{B}_{\sigma,jj}(x,z) = 0, \quad j=1,2,3,4, \tag{2.31}$$

$$\mathcal{B}_{\sigma,lj}(x,z) = -\int_x^1 e^{-iz(s-x)(\omega_l-\omega_j)}F_{\sigma,lj}(s)\,ds, \quad 1 \leqslant j < l \leqslant 4, \tag{2.32}$$

$$\mathcal{B}_{\sigma,lj}(x,z) = \int_0^x e^{iz(x-s)(\omega_l-\omega_j)}F_{\sigma,lj}(s)\,ds, \quad 1 \leqslant l < j \leqslant 4. \tag{2.33}$$

Recall that the functions $F_\sigma$, $\sigma = 1,4$, are defined by (2.20) and (2.27). Therefore, the asymptotics (2.30) gives

$$\mathcal{X}(x,z) = \mathbb{I}_4 + \frac{\mathcal{B}_\sigma(x,z)}{z^\sigma} + \mathcal{O}(z^{-\sigma-1}), \quad \sigma = 1,4.$$

**Remark 2.5.** Note that the matrix-valued functions $\mathcal{B}_\sigma(x,\cdot)$, $x \in [0,1]$, $\sigma = 1,4$, are analytic and bounded in $\mathcal{Z}_+$.

Using these results, we represent the matrix $A$ of equation (1.2) as a product of the bounded matrix $\mathcal{X}$, the simple matrix $\Omega$, and the diagonal matrix $\exp\{iz\int_0^x T_\sigma(s,z)\,ds\}$. Note that all exponentially increasing terms contain into this diagonal matrix. To obtain sharp eigenvalue asymptotics we provide two factorizations of the fundamental matrix. It is more convenient for obtaining sharp eigenvalue asymptotics.

For $\sigma = 1,4$, $l$, $j = 1,2,3,4$, we introduce the functions

$$\zeta_{\sigma,lj}(x,z) = \frac{\mathcal{B}_{\sigma,lj}(x,z)}{z^\sigma} + \frac{\mathcal{W}_{lj}(x,z)}{z^2}. \tag{2.34}$$

Note that these functions are analytic and bounded in $\mathcal{Z}_+$. If $p \in W^{1,1}(0,1)$, then $\sigma = 1$ and $\mathcal{W} = 0$ and if $p \in W^{3,1}(0,1)$, then $\sigma = 4$ and $\mathcal{W}$ is defined by (2.22).

Now we formulate the following lemma about factorization of the fundamental matrix $A$. Recall that the matrix $\Omega$ is given by (2.7), the diagonal $4\times 4$ matrix-valued functions $T_1$ and $T_4$ have the form (2.20) and (2.26), respectively, and the matrix-valued functions $\mathcal{B}_\sigma$, $\sigma = 1,4$, are defined by (2.31)–(2.33).

**Lemma 2.6.** *Suppose that $p \in W^{1,1}(0,1)$, $x \in [0,1]$, and $z \in \mathcal{Z}_+(r)$ for some $r > 0$ large enough. Then*

*(i) The fundamental matrix $A$ of equation (1.2) satisfies the asymptotics*

$$A(x,z) = \Omega(z)\Big(\mathbb{I}_4 + \frac{\mathcal{B}_1(x,z)}{z} + \mathcal{O}(z^{-2})\Big)e^{iz\int_0^x T_1(s,z)\,ds}, \tag{2.35}$$

*uniformly in $x \in [0,1]$. Moreover, the function $\det A(0,z)$ has the asymptotics*

$$\det A(0,z) = -16iz^6(1+\mathcal{O}(z^{-1})), \tag{2.36}$$

*as* $|z| \to \infty$.

*(ii) Suppose that* $p \in W^{3,1}(0,1)$. *Then*

$$A(x,z) = \Omega(z)\Big(\mathbb{I}_4 + \frac{\mathcal{W}(x,z)}{z^2}\Big)\Big(\mathbb{I}_4 + \frac{\mathcal{B}_4(x,z)}{z^4} + \mathcal{O}(z^{-5})\Big)e^{iz\int_0^x T_4(s,z)\,ds}. \tag{2.37}$$

*(iii) The fundamental solutions* $\phi_j$, $j = 1,2,3,4$, *given by* (2.2), *have the asymptotics*

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \phi_1' & \phi_2' & \phi_3' & \phi_4' \\ \phi_1'' & \phi_2'' & \phi_3'' & \phi_4'' \\ \phi_1''' - p\phi_1' & \phi_2''' - p\phi_2' & \phi_3''' - p\phi_1' & \phi_4''' - p\phi_4' \end{pmatrix} \tag{2.38}$$

$$= \Omega \begin{pmatrix} (1 + [\zeta_{\sigma,11}]_{-\sigma-1})c_{\sigma,1} & [\zeta_{\sigma,12}]_{-\sigma-1}c_{\sigma,2} & [\zeta_{\sigma,13}]_{-\sigma-1}c_{\sigma,3} & [\zeta_{\sigma,14}]_{-\sigma-1}c_{\sigma,4} \\ [\zeta_{\sigma,21}]_{-\sigma-1}c_{\sigma,1} & (1 + [\zeta_{\sigma,22}]_{-\sigma-1})c_{\sigma,2} & [\zeta_{\sigma,23}]_{-\sigma-1}c_{\sigma,3} & [\zeta_{\sigma,24}]_{-\sigma-1}c_{\sigma,4} \\ [\zeta_{\sigma,31}]_{-\sigma-1}c_{\sigma,1} & [\zeta_{\sigma,32}]_{-\sigma-1}c_{\sigma,2} & (1 + [\zeta_{\sigma,33}]_{-\sigma-1})c_{\sigma,3} & [\zeta_{\sigma,34}]_{-\sigma-1}c_{\sigma,4} \\ [\zeta_{\sigma,41}]_{-\sigma-1}c_{\sigma,1} & [\zeta_{\sigma,42}]_{-\sigma-1}c_{\sigma,2} & [\zeta_{\sigma,43}]_{-\sigma-1}c_{\sigma,3} & (1 + [\zeta_{\sigma,44}]_{-\sigma-1})c_{\sigma,4} \end{pmatrix},$$

*as* $|z| \to \infty$, $z \in \mathcal{Z}_+$, *uniformly in* $x \in [0,1]$, *where*

$$c_{\sigma,j}(x,z) = e^{iz\int_0^x T_{\sigma,j}(s,z)\,ds}, \quad [\zeta_{\sigma,lj}]_{-\sigma-1} = \zeta_{\sigma,lj} + \mathcal{O}(z^{-\sigma-1}),$$

*for* $l, j = 1,2,3,4$, $\sigma = 1,4$, *and* $\zeta_{\sigma,lj}$, $\sigma = 1,4$, $l$, $j = 1,2,3,4$, *are given by* (2.34).

The proof of this lemma is similar to [23, Lemma 6], we omit it.

## 3. Eigenvalue asymptotics in the case $p \in W^{1,1}(0,1)$

3.1. **Asymptotics of the functions** $\gamma_j$**,** $j = 1,2,3,4$**.** The main goal of this section is to obtain eigenvalue asymptotics of the operator $\mathcal{H}$ at high energy. But first we deduce more convenient form for the functions $\gamma_j$, $j = 1,2,3,4$, defined by (2.12)–(2.14). The formulas (2.20) and (2.26) imply

$$\begin{gathered} \alpha_\sigma(z) = \int_0^1 T_{\sigma,2}(s,z)\,ds = -\int_0^1 T_{\sigma,3}(s,z)\,ds, \\ \beta_\sigma(z) = \int_0^1 T_{\sigma,1}(s,z)\,ds = -\int_0^1 T_{\sigma,4}(s,z)\,ds, \quad z \in \mathcal{Z}_+, \end{gathered} \tag{3.1}$$

for $\sigma = 1,4$, where $T_{\sigma,j}$ are entries of the matrices $T_\sigma = (T_{\sigma,j})_{j=1}^4$. Therefore, the functions $\alpha_\sigma$ and $\beta_\sigma$ have the form

$$\begin{gathered} \alpha_1(z) = 1 - \frac{p_0}{4z^2}, \quad \beta_1(z) = i + \frac{ip_0}{4z^2}, \\ \alpha_4(z) = 1 + \frac{\|p\|^2}{32z^4} - \frac{p_0}{4z^2}, \quad \beta_4(z) = i + \frac{i\|p\|^2}{32z^4} + \frac{ip_0}{4z^2}. \end{gathered} \tag{3.2}$$

In the following lemma, one assumes that the functions $\zeta_{\sigma,lj}$, $\sigma = 1,4$, $l$, $j = 1,2,3,4$, satisfy (2.34). Now we introduce the functions

$$\begin{aligned} \varkappa_{\sigma,1}(z) = {}& \Big(\zeta_{\sigma,22} - i\zeta_{\sigma,32} + (1-i)\zeta_{\sigma,42} + \zeta_{\sigma,11} + (1+i)\zeta_{\sigma,31} + i\zeta_{\sigma,41}\Big)(0,z) \\ & + \frac{(1-i)}{2}\big(M_{\sigma,1} + \widetilde{M}_{\sigma,1}\big)(0,z), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \varkappa_{\sigma,2}(z) = {}& \Big(\zeta_{\sigma,11} + (1-i)\zeta_{\sigma,21} - i\zeta_{\sigma,41} + i\zeta_{\sigma,23} + \zeta_{\sigma,33} + (1+i)\zeta_{\sigma,43}\Big)(0,z) \\ & + \frac{(1+i)}{2}\big(M_{\sigma,2} + \widetilde{M}_{\sigma,2}\big)(0,z), \end{aligned} \tag{3.4}$$

$$\begin{aligned}\varkappa_{\sigma,3}(z) = \Big(&\zeta_{\sigma,14}+\zeta_{\sigma,44}+\zeta_{\sigma,23}+\zeta_{\sigma,33}+(\zeta_{\sigma,23}+\zeta_{\sigma,33})(\zeta_{\sigma,14}+\zeta_{\sigma,44})\\ &-(\zeta_{\sigma,13}+\zeta_{\sigma,43})(\zeta_{\sigma,24}+\zeta_{\sigma,34})\Big)(1,z),\end{aligned} \tag{3.5}$$

$$\begin{aligned}\varkappa_{\sigma,6}(z) = \Big(&\zeta_{\sigma,22}+\zeta_{\sigma,32}+\zeta_{\sigma,14}+\zeta_{\sigma,44}+(\xi_{\sigma,22}+\xi_{\sigma,32})(\xi_{\sigma,14}+\xi_{\sigma,44})\\ &-(\xi_{\sigma,12}+\xi_{\sigma,42})(\xi_{\sigma,24}+\xi_{\sigma,34})\Big)(1,z),\end{aligned} \tag{3.6}$$

$$\varkappa_{\sigma,4}(z)=\psi_1(z)+\frac{1-i}{2}\big(M_{\sigma,3}+\widetilde{M}_{\sigma,3}\big)(1,z),\quad \varkappa_{\sigma,5}(z)=\overline{\psi_1(z)}+\mathcal{O}(z^{-4}), \tag{3.7}$$

$$\varkappa_{\sigma,7}(z)=\psi_2(z)+\frac{1+i}{2}\big(M_{\sigma,5}+\widetilde{M}_{\sigma,5}\big)(1,z),\quad \varkappa_{\sigma,8}(z)=\overline{\psi_2(z)}+\mathcal{O}(z^{-4}), \tag{3.8}$$

where

$$\psi_1(z)=\Big(i\zeta_{\sigma,14}+(-1-i)\zeta_{\sigma,24}+\zeta_{\sigma,44}+(-1+i)\zeta_{\sigma,13}-i\zeta_{\sigma,23}+\zeta_{\sigma,33}\Big)(1,z),$$

$$\psi_2(z)=\Big((-1-i)\zeta_{\sigma,12}+\zeta_{\sigma,22}+i\zeta_{\sigma,32}-i\zeta_{\sigma,14}+(-1+i)\xi_{\sigma,34}+\xi_{\sigma,44}\Big)(1,z),$$

and

$$\begin{aligned}M_{\sigma,1}(0,z)&=(-\zeta_{\sigma,12}+i\zeta_{\sigma,22}-i\zeta_{\sigma,32}+\zeta_{\sigma,42})(\zeta_{\sigma,11}+\zeta_{\sigma,21}+\zeta_{\sigma,31}+\zeta_{\sigma,41})(0,z),\\ \widetilde{M}_{\sigma,1}(0,z)&=(\zeta_{\sigma,12}+\zeta_{\sigma,22}+\zeta_{\sigma,32}+\zeta_{\sigma,42})(\zeta_{\sigma,11}-i\zeta_{\sigma,21}+i\zeta_{\sigma,31}-\zeta_{\sigma,41})(0,z),\\ M_{\sigma,2}(0,z)&=(\zeta_{\sigma,11}-i\zeta_{\sigma,21}+i\zeta_{\sigma,31}-\zeta_{\sigma,41})(\zeta_{\sigma,13}+\zeta_{\sigma,23}+\zeta_{\sigma,33}+\zeta_{\sigma,43})(0,z),\\ \widetilde{M}_{\sigma,2}(0,z)&=(\zeta_{\sigma,11}+\zeta_{\sigma,21}+\zeta_{\sigma,31}+\zeta_{\sigma,41})(-\zeta_{\sigma,13}+i\zeta_{\sigma,23}-i\zeta_{\sigma,33}+\zeta_{\sigma,43})(0,z),\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}M_{\sigma,3}(1,z)&=(-\zeta_{\sigma,13}-i\zeta_{\sigma,23}+i\zeta_{\sigma,33}+\zeta_{\sigma,43})(\zeta_{\sigma,14}-\zeta_{\sigma,24}-\zeta_{\sigma,34}+\zeta_{\sigma,44})(1,z),\\ \widetilde{M}_{\sigma,3}(1,z)&=(-\zeta_{\sigma,13}+\zeta_{\sigma,23}+\zeta_{\sigma,33}-\zeta_{\sigma,43})(-\zeta_{\sigma,14}-i\zeta_{\sigma,24}+i\zeta_{\sigma,34}+\zeta_{\sigma,44})(1,z),\\ M_{\sigma,4}(1,z)&=(-i\zeta_{\sigma,14}+i\zeta_{\sigma,24}+i\zeta_{\sigma,34}-i\zeta_{\sigma,44})(-i\zeta_{\sigma,12}+\zeta_{\sigma,22}-\zeta_{\sigma,32}+i\zeta_{\sigma,42})(1,z),\\ \widetilde{M}_{\sigma,4}(1,z)&=(-\zeta_{\sigma,12}+\zeta_{\sigma,22}+\zeta_{\sigma,32}-\zeta_{\sigma,42})(-\zeta_{\sigma,14}-i\zeta_{\sigma,24}+i\zeta_{\sigma,34}+\zeta_{\sigma,44})(1,z).\end{aligned} \tag{3.10}$$

Now we formulate the main result of this subsection.

**Lemma 3.1.** *Suppose that $p\in W^{1,1}(0,1)$, $|z|\to\infty$. Then the functions $\gamma_1$ and $\gamma_3$ have the form*

$$\begin{aligned}\gamma_1(z)&=(1+i)z\Big(1+\varkappa_{\sigma,1}(z)+\mathcal{O}(z^{-\sigma-1})\Big),\\ \gamma_3(z)&=(-1+i)z\Big(1+\varkappa_{\sigma,2}(z)+\mathcal{O}(z^{-\sigma-1})\Big),\end{aligned} \tag{3.11}$$

*for $z\in\mathcal{Z}_+$, where $\varkappa_{\sigma,1}$, $\varkappa_{\sigma,2}$, $\sigma=1,4$, satisfy* (3.3) *and* (3.4).

*If $c\neq 0$, then the functions $\gamma_2$ and $\gamma_4$ are defined by*

$$\begin{aligned}\gamma_2(z)=z^9e^{-i\alpha_\sigma z-i\beta_\sigma z}\big(&c(1+i)\big(1+\varkappa_{\sigma,4}(z)\big)+\frac{c(-1+i)p(1)}{z^2}\big(1+\varkappa_{\sigma,5}(z)\big)\\ &-\frac{2a}{z^3}+\frac{d(1+i)}{z^4}+\mathcal{O}(z^{-\sigma-1})\big),\end{aligned} \tag{3.12}$$

$$\begin{aligned}\gamma_4(z)=z^9e^{i\alpha_\sigma z-i\beta_\sigma z}\big(&c(1-i)\big(1+\varkappa_{\sigma,7}(z)\big)+\frac{c(-1-i)p(1)}{z^2}\big(1+\varkappa_{\sigma,8}(z)\big)\\ &-\frac{2a}{z^3}+\frac{d(1-i)}{z^4}+\mathcal{O}(z^{-\sigma-1})\big),\end{aligned} \tag{3.13}$$

*where* $\varkappa_{\sigma,j}$, $j = 3, 4, 5, 6, 7, 8$, *are given by* (3.5)–(3.8).

*If* $c = 0$, *then the functions* $\gamma_2$ *and* $\gamma_4$ *satisfy*

$$\begin{aligned}\gamma_2(z) &= z^6 e^{-i\alpha_\sigma z - i\beta_\sigma z}\Big( - 2a\big(1 + \varkappa_{\sigma,3}(z)\big) + \frac{d(1+i)}{z}\big(1 + \varkappa_{\sigma,4}(z)\big) \\ &\quad + \frac{d(-1+i)p(1)}{z^3} - \frac{2b}{z^4} + \mathcal{O}(z^{-\sigma-1})\Big),\end{aligned} \tag{3.14}$$

$$\begin{aligned}\gamma_4(z) &= z^6 e^{i\alpha_\sigma z - i\beta_\sigma z}\Big( - 2a\big(1 + \varkappa_{\sigma,6}(z)\big) + \frac{d(1-i)}{z}\big(1 + \varkappa_{\sigma,7}(z)\big) \\ &\quad + \frac{d(-1-i)p(1)}{z^3} - \frac{2b}{z^4} + \mathcal{O}(z^{-\sigma-1})\Big).\end{aligned} \tag{3.15}$$

*Proof.* Let $z \in \mathcal{Z}_+$, $|z| \to \infty$. Substitute (2.38) into the first formula from (2.12). Then

$$\begin{aligned}\gamma_1(z) = iz&\Big(1 + (\zeta_{\sigma,11} + \zeta_{\sigma,21} + \zeta_{\sigma,31} + \zeta_{\sigma,41})(0, z) + \mathcal{O}(z^{-\sigma-1})\Big) \\ &\times \Big(1 + (i\zeta_{\sigma,12} + \zeta_{\sigma,22} - \zeta_{\sigma,32} - i\zeta_{\sigma,42})(0, z) + \mathcal{O}(z^{-\sigma-1})\Big) \\ &+ z\Big(1 + (\zeta_{\sigma,12} + \zeta_{\sigma,22} + \zeta_{\sigma,32} + \zeta_{\sigma,42})(0, z) + \mathcal{O}(z^{-\sigma-1})\Big) \\ &\times \Big(1 + (\zeta_{\sigma,11} - i\zeta_{\sigma,21} + i\zeta_{\sigma,31} - \zeta_{\sigma,41})(0, z) + \mathcal{O}(z^{-\sigma-1})\Big).\end{aligned}$$

Now the first equation in (3.11) follows immediately.

Consider the first definition from (2.14). Arguments similar to this provide the second equation in (3.11).

Now we prove the first formula from (3.12). We define the functions

$$\begin{aligned}\xi_1(z) &= \phi_3''(1, z)\phi_4(1, z) - \phi_3(1, z)\phi_4''(1, z), \\ \xi_2(z) &= \phi_3'''(1, z)\phi_4''(1, z) - \phi_3''(1, z)\phi_4'''(1, z), \\ \xi_3(z) &= \phi_3''(1, z)\phi_4'(1, z) - \phi_3'(1, z)\phi_4''(1, z).\end{aligned}$$

Equalities (1.4) and (2.12) imply

$$\begin{aligned}&\gamma_2(z) \\ &= \phi_3''(1, z)\Big((az^4 + b)\phi_4(1, z) - (cz^4 + d)\big(\phi_4'''(1, z) - p(1)\phi_4'(1, z)\big)\Big) \\ &\quad - \phi_4''(1, z)\Big((az^4 + b)\phi_3(1, z) - (cz^4 + d)\big(\phi_3'''(1, z) - p(1)\phi_3'(1, z)\big)\Big) \\ &= z^4\Big(a\xi_1(z) + c\xi_2(z) + cp(1)\xi_3(z) + \frac{b}{z^4}\xi_1(z) + \frac{d}{z^4}\xi_2(z) + \frac{dp(1)}{z^4}\xi_3(z)\Big).\end{aligned} \tag{3.16}$$

We consider all the terms in the last equality separately. Using again (2.38), we obtain

$$\begin{aligned} &\xi_1(z)\\ &= -z^2e^{-i\alpha_\sigma z-i\beta_\sigma z}\Big(1+(-\zeta_{\sigma,13}+\zeta_{\sigma,23}+\zeta_{\sigma,33}-\zeta_{\sigma,43})(1,z)+\mathcal{O}(z^{-\sigma-1})\Big)\\ &\quad\times\Big(1+(\zeta_{\sigma,14}+\zeta_{\sigma,24}+\zeta_{\sigma,34}+\zeta_{\sigma,44})(1,z)+\mathcal{O}(z^{-\sigma-1})\Big)\\ &\quad-z^2e^{-i\alpha_\sigma z-i\beta_\sigma z}\Big(1+(\zeta_{\sigma,13}+\zeta_{\sigma,23}+\zeta_{\sigma,33}+\zeta_{\sigma,43})(1,z)+\mathcal{O}(z^{-\sigma-1})\Big)\\ &\quad\times\Big(1+(\zeta_{\sigma,14}-\zeta_{\sigma,24}-\zeta_{\sigma,34}+\zeta_{\sigma,44})(1,z)+\mathcal{O}(z^{-\sigma-1})\Big)\\ &= -2z^2e^{-i\alpha_\sigma z-i\beta_\sigma z}\Big(1+\varkappa_{\sigma,3}(z)+\mathcal{O}(z^{-\sigma-1})\Big), \end{aligned}\tag{3.17}$$

where $\varkappa_{\sigma,3}$ has the form (3.5). Similarly,

$$\begin{aligned} &\xi_2(z)\\ &= iz^5e^{-i\alpha_\sigma z-i\beta_\sigma z}\Big(1+(i\zeta_{\sigma,13}-\zeta_{\sigma,23}+\zeta_{\sigma,33}-i\zeta_{\sigma,43})(1,z)+\mathcal{O}(z^{-\sigma-1})\Big)\\ &\quad\times\Big(1+(\zeta_{\sigma,14}-\zeta_{\sigma,24}-\zeta_{\sigma,34}+\zeta_{\sigma,44})(1,z)+\mathcal{O}(z^{-\sigma-1})\Big)\\ &\quad+z^5e^{-i\alpha_\sigma z-i\beta_\sigma z}\Big(1+(-\zeta_{\sigma,13}+\zeta_{\sigma,23}+\zeta_{\sigma,33}-\zeta_{\sigma,43})(1,z)+\mathcal{O}(z^{-\sigma-1})\Big)\\ &\quad\times\Big(1+(-\zeta_{\sigma,14}-i\zeta_{\sigma,24}+i\zeta_{\sigma,34}+\zeta_{\sigma,44})(1,z)+\mathcal{O}(z^{-\sigma-1})\Big)\\ &= (1+i)z^5e^{-i\alpha_\sigma z-i\beta_\sigma z}\Big(1+\varkappa_{\sigma,4}(z)+\mathcal{O}(z^{-\sigma-1})\Big), \end{aligned}\tag{3.18}$$

and

$$\begin{aligned} &\xi_3(z)\\ &= -z^3e^{-i\alpha_\sigma z-i\beta_\sigma z}\Big(1+(-\zeta_{\sigma,13}+\zeta_{\sigma,23}+\zeta_{\sigma,33}-\zeta_{\sigma,43})(1,z)+\mathcal{O}(z^{-\sigma-1})\Big)\\ &\quad\times\Big(1+(-\zeta_{\sigma,14}+i\zeta_{\sigma,24}-i\zeta_{\sigma,34}+\zeta_{\sigma,44})(1,z)+\mathcal{O}(z^{-\sigma-1})\Big)\\ &\quad+iz^3e^{-i\alpha_\sigma z-i\beta_\sigma z}\Big(1+(-i\zeta_{\sigma,13}-\zeta_{\sigma,23}+\zeta_{\sigma,33}+i\zeta_{\sigma,43})(1,z)+\mathcal{O}(z^{-\sigma-1})\Big)\\ &\quad\times\Big(1+(\zeta_{\sigma,14}-\zeta_{\sigma,24}-\zeta_{\sigma,34}+\zeta_{\sigma,44})(1,z)+\mathcal{O}(z^{-\sigma-1})\Big)\\ &= (-1+i)z^3e^{-i\alpha_\sigma z-i\beta_\sigma z}\Big(1+\varkappa_{\sigma,5}(z)+\mathcal{O}(z^{-\sigma-1})\Big), \end{aligned}\tag{3.19}$$

where $\varkappa_{\sigma,4}$ and $\varkappa_{\sigma,5}$ satisfy (3.7). We substitute (3.17)–(3.19) into (3.16) and take out the factor $z^5e^{-i\alpha_\sigma z-i\beta_\sigma z}$. Then

$$\begin{aligned} \gamma_2(z) = z^9e^{-i\alpha_\sigma z-i\beta_\sigma z}\big(&c(1+i)\big(1+\varkappa_{\sigma,4}(z)\big)+\frac{c(-1+i)p(1)}{z^2}\big(1+\varkappa_{\sigma,5}(z)\big)\\ &-\frac{2a}{z^3}\big(1+\varkappa_{\sigma,3}(z)\big)+\frac{d(1+i)}{z^4}+\mathcal{O}(z^{-\sigma-1})\big). \end{aligned}$$

The identity (2.34) implies that $\zeta_{1,lj}(\cdot,z)=\mathcal{O}(z^{-1})$ and $\zeta_{4,lj}(\cdot,z)=\mathcal{O}(z^{-2})$. This yields the formula (3.12) immediately. Arguments similar to this provide the asymptotics (3.13).

It remains only to prove formulas (3.14) and (3.15). If $c = 0$, then we rewrite (3.16) in the form

$$\gamma_2(z) = z^4\Big(a\xi_1(z) + \frac{b}{z^4}\xi_1(z) + \frac{d}{z^4}\xi_2(z) + \frac{dp(1)}{z^4}\xi_3(z)\Big).$$

Substituting (3.17)–(3.19) into this definition and using the relations $\zeta_{1,lj}(\cdot,z) = \mathcal{O}(z^{-1})$ and $\zeta_{4,lj}(\cdot,z) = \mathcal{O}(z^{-2})$, we obtain (3.14). Arguments similar to this provide the equation (3.15). □

3.2. **Sharp eigenvalue asymptotics.** Now we determine the eigenvalue asymptotics for the case $p \in W^{1,1}(0,1)$. This corresponds to the case $\sigma = 1$. Recall that the eigenvalues $\lambda_n$ of the operator $\mathcal{H}$ are zeros of the entire function $D$ given by (1.3). The definition (2.6) and the asymptotics (2.36) show that the large eigenvalues are zeros of the function $\det\phi$. Therefore, using the formulas (2.10), we obtain the eigenvalue asymptotics of the operator $\mathcal{H}$.

**Lemma 3.2.** *Let $p \in W^{1,1}(0,1)$. Then the eigenvalues $\lambda_n$ satisfy the asymptotics* (1.7) *in the case $c \neq 0$ and the asymptotics* (1.8) *in the case $c = 0$.*

*Proof.* Let $c \neq 0$ and $\lambda = z^4 = \lambda_n$, $n \to +\infty$. It follows from [14, Theorem 6.1] that $z = -3\pi/2 + \pi n + \delta_n$, $\delta_n = \mathcal{O}(n^{-1})$. The relations (3.11)–(3.13) give

$$\begin{gathered}
\gamma_1(z) = (1+i)z\Big(1 + \varkappa_{1,1}(z) + \mathcal{O}(n^{-2})\Big),\\
\gamma_3(z) = (-1+i)z\Big(1 + \varkappa_{1,2}(z) + \mathcal{O}(n^{-2})\Big),\\
\gamma_2(z) = c(1+i)z^9 e^{-i\alpha_1 z - i\beta_1 z}\Big(1 + \varkappa_{1,4}(z) + \mathcal{O}(n^{-2})\Big),\\
\gamma_4(z) = c(1-i)z^9 e^{i\alpha_1 z - i\beta_1 z}\Big(1 + \varkappa_{1,7}(z) + \mathcal{O}(n^{-2})\Big),
\end{gathered}$$

where $\varkappa_{1,j}$, $j = 1,2,4,7$, have the form (3.3), (3.4), (3.7), and (3.8). Substituting these asymptotics into (2.10), we obtain

$$\begin{aligned}
\det\phi(z) = 2icz^{10}e^{-i\beta_1 z}\big(&e^{-i\alpha_1 z}\Big(1 + \varkappa_{1,1}(z) + \varkappa_{1,4}(z) + \varkappa_{1,1}(z)\varkappa_{1,4}(z)\Big)\\
&+ e^{i\alpha_1 z}\Big(1 + \varkappa_{1,2}(z) + \varkappa_{1,7}(z) + \varkappa_{1,2}(z)\varkappa_{1,7}(z)\Big) + \mathcal{O}(n^{-2})\big),
\end{aligned} \tag{3.20}$$

where $\alpha_1$ and $\beta_1$ satisfy (3.2). The identity $z = -3\pi/2 + \pi n + \delta_n$ and the formulas (3.2) give

$$\begin{aligned}
e^{\pm i\alpha_1 z} &= e^{\pm iz \mp ip_0/(4z)} = \pm i(-1)^n e^{\pm i\delta_n \mp ip_0/(4z)}\\
&= \pm i(-1)^n\Big(1 \pm i\delta_n \mp \frac{ip_0}{2\pi(2n-3)} + \mathcal{O}(n^{-2})\Big).
\end{aligned}$$

Moreover, the identity (2.34) yields

$$\zeta_{1,kj}(x,z) = \mathcal{O}(n^{-1}), \quad \zeta_{1,kj}(x,z)\zeta_{1,ls}(y,z) = \mathcal{O}(n^{-2}), \tag{3.21}$$

for $k,j,l,\ s = 1,2,3,4$, $x,y \in [0,1]$, and $z \in \mathcal{Z}_+$. Therefore,

$$\varkappa_{1,j}(x,z)\varkappa_{1,k}(y,z) = \mathcal{O}(n^{-2}) \tag{3.22}$$

for $k,\ j = 1,2,3,4,5,6,7,8$. Substituting these asymptotics into (3.20), we obtain

$$\begin{aligned}
\det\phi(z) = 2(-1)^{n+1}cz^{10}e^{-3\pi/2+\pi n}\big(&2i\delta_n - \frac{ip_0}{\pi(2n-3)} + \varkappa_{1,2}(z)\\
&+ \varkappa_{1,7}(z) - \varkappa_{1,1}(z) - \varkappa_{1,4}(z) + \mathcal{O}(n^{-2})\big).
\end{aligned}$$

Then the equation $\det\phi(z)=0$ implies

$$\delta_n=\frac{p_0}{2\pi(2n-3)}-\frac{1}{2i}\big(\varkappa_{1,2}(z)+\varkappa_{1,7}(z)-\varkappa_{1,1}(z)-\varkappa_{1,4}(z)\big)+\mathcal{O}(n^{-2}).$$

Thus, the identity $z=-3\pi/2+\pi n+\delta_n$ yields

$$\begin{aligned}z=-\frac{3\pi}{2}+\pi n+\frac{p_0}{2\pi(2n-3)}-\frac{1}{2i}\big(&\varkappa_{1,2}(z)+\varkappa_{1,7}(z)-\varkappa_{1,1}(z)\\&-\varkappa_{1,4}(z)\big)+\mathcal{O}(n^{-2}).\end{aligned}\tag{3.23}$$

Integrating by parts and using the identity $z=-3\pi/2+\pi n+\delta_n$, we obtain $\mathcal{B}_{1,21}(0,z)=\mathcal{O}(n^{-1})$. Similar arguments imply for $\mathcal{B}_{1,41}(0,z)=\mathcal{O}(n^{-1})$, $\mathcal{B}_{1,23}(0,z)=\mathcal{O}(n^{-1})$, $\mathcal{B}_{1,43}(0,z)=\mathcal{O}(n^{-1})$. Therefore, $\varkappa_{1,2}(z)=\mathcal{O}(n^{-2})$. Repeating this procedure for $\varkappa_{1,1}$, $\varkappa_{1,4}$, and $\varkappa_{1,7}$, we obtain

$$\varkappa_{1,2}(z)+\varkappa_{1,7}(z)-\varkappa_{1,1}(z)-\varkappa_{1,4}(z)=\mathcal{O}(n^{-2}).$$

It remains to compute $\varkappa_{1,2}(z)+\varkappa_{1,7}(z)-\varkappa_{1,1}(z)-\varkappa_{1,4}(z)$. The formulas (3.4), (3.8), the identity (2.34) with $\mathcal{W}=0$, asymptotics (3.21), and (2.31) give

$$\begin{aligned}\varkappa_{1,2}(z)=\frac{1}{z}\Big(&(1-i)\mathcal{B}_{1,21}(0,z)-i\mathcal{B}_{1,41}(0,z)\\&+i\mathcal{B}_{1,23}(0,z)+(1+i)\mathcal{B}_{1,43}(0,z)\Big)+\mathcal{O}(z^{-2}).\end{aligned}\tag{3.24}$$

The definitions (2.32) and (2.33) with (2.20) imply $\mathcal{B}_{1,23}(0,z)=\mathcal{B}_{1,32}(1,z)=0$ and

$$\mathcal{B}_{1,21}(0,z)=-\int_0^1 e^{(-1-i)zs}F_{1,21}(s)\,ds=-\frac{1}{4}\int_0^1 e^{(-1-i)zs}p(s)\,ds.$$

It remains to substitute this expression into (3.23). Then

$$z=-\frac{3\pi}{2}+\pi n+\frac{p_0}{2\pi(2n-3)}+\mathcal{O}(n^{-2}).$$

This gives (1.7).

Let $c=0$ and $\lambda=z^4=\lambda_n$, $n\to+\infty$. It follows from [14, Theorem 6.1] that $z=-3\pi/4+\pi n+\delta_n$, $\delta_n=\mathcal{O}(n^{-1})$. The relations (3.11), (3.14), and (3.15) give

$$\begin{gathered}\gamma_1(z)=(1+i)z\Big(1+\varkappa_{1,1}(z)+\mathcal{O}(n^{-2})\Big),\\ \gamma_2(z)=-2az^6e^{-i\alpha z-i\beta z}\Big(1+\varkappa_{1,3}(z)-\frac{d(1+i)}{2az}+\mathcal{O}(n^{-2})\Big),\\ \gamma_3(z)=(-1+i)z\Big(1+\varkappa_{1,2}(z)+\mathcal{O}(n^{-2})\Big),\\ \gamma_4(z)=-2az^6e^{i\alpha z-i\beta z}\Big(1+\varkappa_{1,6}(z)-\frac{d(1-i)}{2az}+\mathcal{O}(n^{-2})\Big),\end{gathered}$$

where $\varkappa_{1,j}$, $j=1,2,3,6$, have the form (3.3)–(3.6). Substituting these expressions into (2.10), we obtain

$$\begin{aligned}&\det\phi(z)\\&=-2az^7e^{-i\beta_1 z}\big((1+i)e^{-i\alpha_1 z}\Big(1+\varkappa_{1,1}(z)+\varkappa_{1,3}(z)+\varkappa_{1,1}(z)\varkappa_{1,3}(z)-\frac{d(1+i)}{2az}\Big)\\&\quad+(-1+i)e^{i\alpha_1 z}\Big(1+\varkappa_{1,2}(z)+\varkappa_{1,6}(z)+\varkappa_{1,2}(z)\varkappa_{1,6}(z)-\frac{d(1-i)}{2az}\Big)+\mathcal{O}(n^{-2})\big),\end{aligned}$$

where $\alpha_1$ and $\beta_1$ satisfy (3.2). The identity $z = -3\pi/4 + \pi n + \delta_n$ and the formulas (3.2) give

$$\begin{aligned} e^{\pm i\alpha_1 z} = e^{\pm iz \mp ip_0/(4z)} &= \frac{(-1)^n(-1\mp i)\sqrt{2}}{2} e^{\pm i\delta_n \mp ip_0/(4z)} \\ &= \frac{(-1)^n(-1\mp i)\sqrt{2}}{2}\Big(1 \pm i\delta_n \mp \frac{ip_0}{\pi(4n-3)} + \mathcal{O}(n^{-2})\Big). \end{aligned}$$

Therefore, these equalities and (3.22) imply

$$\begin{aligned} \det\phi(z) = 2\sqrt{2}(-1)^{n+1} a z^7 e^{-3\pi/4+\pi n}\big(2i\delta_n - \frac{2ip_0 a - 4id}{a\pi(4n-3)} + \varkappa_{1,2}(z) + \varkappa_{1,6}(z) \\ - \varkappa_{1,1}(z) - \varkappa_{1,3}(z) + \mathcal{O}(n^{-2})\big). \end{aligned}$$

The equality $\det\phi(z) = 0$ implies

$$\delta_n = \frac{p_0 a - 2d}{a\pi(4n-3)} - \frac{1}{2i}\big(\varkappa_{1,2}(z) + \varkappa_{1,6}(z) - \varkappa_{1,1}(z) - \varkappa_{1,3}(z)\big) + \mathcal{O}(n^{-2}).$$

Thus, the identity $z = -3\pi/4 + \pi n + \delta_n$ yields

$$\begin{aligned} z = -\frac{3\pi}{4} + \pi n + \frac{p_0}{\pi(4n-3)} - \frac{2d}{a\pi(4n-3)} - \frac{1}{2i}\big(\varkappa_{1,2}(z) + \varkappa_{1,6}(z) \\ - \varkappa_{1,1}(z) - \varkappa_{1,3}(z)\big) + \mathcal{O}(n^{-2}). \end{aligned} \tag{3.25}$$

Repeating the process as in the case $c \neq 0$ and using the relation $z = -3\pi/4 + \pi n + \delta_n$, we obtain

$$\varkappa_{1,2}(z) + \varkappa_{1,6}(z) - \varkappa_{1,1}(z) - \varkappa_{1,3}(z) = \mathcal{O}(n^{-2}),$$

Substituting this expression into (3.25), we have

$$z = -\frac{3\pi}{4} + \pi n - \frac{2d}{a\pi(4n-3)} + \frac{p_0 - \rho_{2,n} + \widehat{p}_{sn}(3/2)}{\pi(4n-3)} + \mathcal{O}(n^{-2}).$$

This gives (1.8). □

## 4. Eigenvalue asymptotics in the case $p \in W^{3,1}(0,1)$

Now we determine sharp eigenvalue asymptotics of the operator $\mathcal{H}$ in the case $p \in W^{3,1}(0,1)$. This corresponds to the case $\sigma = 4$. Again we obtain the asymptotics for the function $\det\phi$ of the form (2.10). Using this asymptotics, we get the eigenvalue asymptotics of the operator $\mathcal{H}$.

**Lemma 4.1.** *Suppose that $p \in W^{3,1}(0,1)$ and $p(0) = p(1)$. Then the eigenvalues $\lambda_n$ satisfy (1.9) for $c \neq 0$ and (1.11) for $c = 0$.*

*Proof.* Let $c \neq 0$ and $\lambda = z^4 = \lambda_n$, $n \to +\infty$. It follows from Lemma 3.2 that

$$z = -\frac{3\pi}{2} + \pi n + \frac{p_0}{2\pi(2n-3)} + \delta_n, \quad \delta_n = \mathcal{O}(n^{-2}). \tag{4.1}$$

Substituting (3.11)–(3.13) with $\sigma = 4$ into (2.10), we obtain

$$\begin{aligned} \det\phi(z) = 2icz^{10}e^{-i\beta_4 z}\big(e^{-i\alpha_4 z}\Big(1 + \varkappa_{4,1}(z) + \varkappa_{4,4}(z) + \varkappa_{4,1}(z)\varkappa_{4,4}(z) \\ - \frac{2a}{c(1+i)z^3} + \frac{d}{cz^4} + \frac{ip(1)}{z^2}\big(1 + \varkappa_{4,1}(z) + \varkappa_{4,5}(z)\big)\Big) \\ + e^{i\alpha_4 z}\Big(1 + \varkappa_{4,2}(z) + \varkappa_{4,7}(z) + \varkappa_{4,2}(z)\varkappa_{4,7}(z) \\ - \frac{ip(1)}{z^2}\big(1 + \varkappa_{4,2}(z) + \varkappa_{4,8}(z)\big) - \frac{2a}{c(1-i)z^3} + \frac{d}{cz^4}\Big) + \mathcal{O}(z^{-5})\big), \end{aligned} \tag{4.2}$$

where $\alpha_4$, $\beta_4$ satisfy (3.2) and $\varkappa_{4,j}$, $j = 1, 2, 4, 5, 7, 8$, have the form (3.3)–(3.8).

Now we compute $\varkappa_{4,1}(z) + \varkappa_{4,4}(z) + \varkappa_{4,1}(z)\varkappa_{4,4}(z)$. Formulas (3.3), (3.7), identity (2.34) with $\sigma = 4$, and (2.31) give

$$\begin{aligned} \varkappa_{4,1}(z) = \frac{1}{z^2}\Big(\mathcal{W}_{22} - i\mathcal{W}_{32} + (1-i)\mathcal{W}_{42} + \mathcal{W}_{11} + (1+i)\mathcal{W}_{31} + i\mathcal{W}_{41}\Big)(0,z) \\ + \frac{1-i}{2}\big(M_1 + \widetilde{M}_1\big)(0,z) + \frac{1}{z^4}\Big( - i\mathcal{B}_{4,32}(0,z) \\ + (1-i)\mathcal{B}_{4,42}(0,z) + (1+i)\mathcal{B}_{4,31}(0,z) + i\mathcal{B}_{4,41}(0,z)\Big) + \mathcal{O}(z^{-5}), \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\varkappa_{4,4}(z) \\ &= \frac{1}{z^2}\Big(i\mathcal{W}_{14} + (-1-i)\mathcal{W}_{24} + \mathcal{W}_{44} + (-1+i)\mathcal{W}_{13} - i\mathcal{W}_{23} + \mathcal{W}_{33}\Big)(1,z) \\ &+ \frac{1-i}{2}\big(M_3 + \widetilde{M}_3\big)(1,z) + \frac{1}{z^4}\Big(i\mathcal{B}_{4,14}(1,z) \\ &+ (-1-i)\mathcal{B}_{4,24}(1,z) + (-1+i)\mathcal{B}_{4,13}(1,z) - i\mathcal{B}_{4,23}(1,z)\Big) + \mathcal{O}(z^{-5}), \end{aligned} \tag{4.4}$$

where $\mathcal{W}$, $M_j$, $\widetilde{M}_j$, $j = 1, 3$, have the form (2.22), (3.9), (3.10), respectively. Direct calculations imply

$$\begin{aligned} &\frac{1}{z^2}\Big(\mathcal{W}_{22} - i\mathcal{W}_{32} + (1-i)\mathcal{W}_{42} + \mathcal{W}_{11} + (1+i)\mathcal{W}_{31} + i\mathcal{W}_{41}\Big)(0,z) \\ &= \frac{p(1)}{z^2}\Upsilon(W_1) + \frac{p'(1)}{z^3}\Upsilon(W_2) + \frac{p''(1)}{32z^4}\Upsilon(Q_1) - \frac{p^2(1)}{64z^4}\Upsilon(Q_2), \end{aligned}$$

where

$$\Upsilon(\mathcal{A}) = \mathcal{A}_{1,22} - i\mathcal{A}_{1,32} + (1-i)\mathcal{A}_{1,42} + \mathcal{A}_{1,11} + (1+i)\mathcal{A}_{1,31} + i\mathcal{A}_{1,41}$$

and $W_1$, $W_2$, $Q_1$, $Q_2$ have the form (2.23) and (2.24). Therefore, these formulas yield

$$\begin{aligned} &\frac{1}{z^2}\Big(\mathcal{W}_{22} - i\mathcal{W}_{32} + (1-i)\mathcal{W}_{42} + \mathcal{W}_{11} + (1+i)\mathcal{W}_{31} + i\mathcal{W}_{41}\Big)(0,z) \\ &= -\frac{ip(1)}{4z^2} + \frac{(3-3i)p'(1)}{16z^3} + \frac{p''(1)}{4z^4} - \frac{(1+i)p^2(1)}{32z^4}. \end{aligned} \tag{4.5}$$

Similar arguments give

$$\begin{aligned} &\frac{1}{z^2}\Big(i\mathcal{W}_{14} + (-1-i)\mathcal{W}_{24} + \mathcal{W}_{44} + (-1+i)\mathcal{W}_{13} - i\mathcal{W}_{23} + \mathcal{W}_{33}\Big)(1,z) \\ &= \frac{3ip(1)}{4z^2} + \frac{(5-5i)p'(1)}{16z^3} - \frac{p''(1)}{4z^4} + \frac{(1+i)p^2(1)}{32z^4}. \end{aligned}$$

Moreover, using the identities (3.9), (3.10), and (2.23), we obtain

$$\begin{aligned}
&\frac{1-i}{2}\big(M_1+\widetilde{M_1}\big)(0,z)\\
&=\frac{(1-i)p^2(1)}{128z^4}\Big(\big(-W_{1,12}+iW_{1,22}-iW_{1,32}+W_{1,42}\big)\\
&\quad\times\big(W_{1,11}+W_{1,21}+W_{1,31}+W_{1,41}\big)\\
&\quad+\big(W_{1,12}+W_{1,22}+W_{1,32}+W_{1,42}\big)\big(W_{1,11}-iW_{1,21}+iW_{1,31}-W_{1,41}\big)\Big)\\
&=\frac{p^2(1)}{64z^4}
\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}
&\frac{1-i}{2}\big(M_3+\widetilde{M_3}\big)(1,z)\\
&=\frac{(1-i)p^2(1)}{128z^4}\Big(\big(-W_{1,13}-iW_{1,23}+iW_{1,33}+W_{1,43}\big)\\
&\quad\times\big(W_{1,14}-W_{1,24}-W_{1,34}+W_{1,44}\big)\\
&\quad+\big(-W_{1,13}+W_{1,23}+W_{1,33}-W_{1,43}\big)\big(-W_{1,14}-iW_{1,24}+iW_{1,34}+W_{1,44}\big)\Big)\\
&=\frac{9p^2(1)}{64z^4}.
\end{aligned}$$

The definitions (2.32) and (2.33) with (2.27) imply $\mathcal{B}_{1,23}(0,z)=\mathcal{B}_{1,32}(1,z)=0$ and

$$\mathcal{B}_{4,32}(0,z)=-\int_0^1 e^{i2zs}F_{4,32}(s)\,ds=\frac{1}{32}\int_0^1 e^{i2zs}p'''(s)\,ds+\mathcal{O}(z^{-1}).\tag{4.7}$$

Similar arguments yield

$$\mathcal{B}_{4,42}(0,z)=\frac{1+i}{16}\int_0^1 e^{(-1+i)zs}p'''(s)\,ds+\mathcal{O}(z^{-1}),\tag{4.8}$$

$$\mathcal{B}_{4,31}(0,z)=\frac{1-i}{16}\int_0^1 e^{(-1+i)zs}p'''(s)\,ds+\mathcal{O}(z^{-1}),\tag{4.9}$$

$$\mathcal{B}_{4,41}(0,z)=\frac{1}{32}\int_0^1 e^{-2zs}p'''(s)\,ds+\mathcal{O}(z^{-1}),\tag{4.10}$$

$$\mathcal{B}_{4,14}(1,z)=-\frac{1}{32}\int_0^1 e^{-2z(1-s)}p'''(s)\,ds+\mathcal{O}(z^{-1}),\tag{4.11}$$

$$\mathcal{B}_{4,24}(1,z)=\frac{-1+i}{16}\int_0^1 e^{(-1+i)z(1-s)}p'''(s)\,ds+\mathcal{O}(z^{-1}),\tag{4.12}$$

$$\mathcal{B}_{4,23}(1,z)=-\frac{1}{32}\int_0^1 e^{i2z(1-s)}p'''(s)\,ds+\mathcal{O}(z^{-1}),\tag{4.13}$$

$$\mathcal{B}_{4,13}(1,z)=\frac{-1-i}{16}\int_0^1 e^{(-1+i)z(1-s)}p'''(s)\,ds+\mathcal{O}(z^{-1}).\tag{4.14}$$

Substituting these expressions into (4.3) and (4.4), we obtain

$$\begin{aligned}
&\varkappa_{4,1}(z)+\varkappa_{4,4}(z)+\varkappa_{4,1}(z)\varkappa_{4,4}(z)\\
&=\frac{ip(1)}{2z^2}+\frac{(1-i)p'(1)}{2z^3}+\frac{11p^2(1)}{32z^4}+\frac{\eta_1(z)}{z^4}+\mathcal{O}(z^{-5}),
\end{aligned}\tag{4.15}$$

where

$$\begin{aligned}\eta_1(z) &= \frac{i}{32}\int_0^1 e^{-2zs}\big(p'''(s)-p'''(1-s)\big)\,ds+\frac{1}{4}\int_0^1 e^{(-1+i)zs}\big(p'''(s)+p'''(1-s)\big)\,ds\\ &\quad+\frac{i}{32}\int_0^1 p'''(s)\big(e^{i2z(1-s)}-e^{i2zs}\big)\,ds.\end{aligned}\tag{4.16}$$

Arguments similar to this provide

$$\varkappa_{4,1}(z)+\varkappa_{4,5}(z)=-\frac{ip(1)}{2z^2}+\mathcal{O}(z^{-3}),\quad \varkappa_{4,2}(z)+\varkappa_{4,8}(z)=\frac{ip(1)}{2z^2}+\mathcal{O}(z^{-3}),\tag{4.17}$$

and

$$\begin{aligned}&\varkappa_{4,2}(z)+\varkappa_{4,7}(z)+\varkappa_{4,2}(z)\varkappa_{4,7}(z)\\ &=-\frac{ip(1)}{2z^2}+\frac{(1+i)p'(1)}{2z^3}+\frac{11p^2(1)}{32z^4}+\frac{\eta_2(z)}{z^4}+\mathcal{O}(z^{-5}),\end{aligned}\tag{4.18}$$

where

$$\begin{aligned}\eta_2(z)&=\frac{i}{32}\int_0^1 e^{-2zs}\big(p'''(1-s)-p'''(s)\big)\,ds\\ &\quad+\frac{1}{4}\int_0^1 e^{(-1-i)zs}\big(p'''(s)+p'''(1-s)\big)\,ds.\end{aligned}\tag{4.19}$$

Now we substitute (4.15), (4.17), and (4.18) into (4.2). Then

$$\begin{aligned}&\det\phi(z)\\ &=2icz^{10}e^{-i\beta_4 z}\Big(e^{-i\alpha_4 z}\Big(1+\frac{3ip(1)}{2z^2}+\frac{(1-i)p'(1)}{2z^3}+\frac{27p^2(1)}{32z^4}+\frac{\eta_1(z)}{z^4}\\ &\quad-\frac{2a}{c(1+i)z^3}+\frac{d}{cz^4}\Big)+e^{i\alpha_4 z}\Big(1-\frac{3ip(1)}{2z^2}+\frac{(1+i)p'(1)}{2z^3}\\ &\quad+\frac{27p^2(1)}{32z^4}+\frac{\eta_2(z)}{z^4}-\frac{2a}{c(1-i)z^3}+\frac{d}{cz^4}\Big)+\mathcal{O}(z^{-5})\Big),\end{aligned}\tag{4.20}$$

where $\alpha_4$ and $\beta_4$ have the form (3.2).

Now we compute the asymptotic behavior of the eigenvalues $\lambda_n$, as $n\to+\infty$. Recall that the zeros of the function (4.20) are the eigenvalues $\lambda_n$. Then (4.1) and (3.2) imply

$$\begin{aligned}e^{\pm i\alpha_4 z}=e^{\pm iz\pm i\|p\|^2/(32z^3)\mp ip_0/(4z)}&=\pm i(-1)^n e^{\pm i\delta_n+\mathcal{O}(z^{-3})}\\ &=\pm i(-1)^n\Big(1\pm i\delta_n+\mathcal{O}(n^{-3})\Big).\end{aligned}$$

Substituting these into (4.20), we obtain

$$\begin{aligned}\det\phi(z)&=2icz^{10}e^{-3\pi/2+\pi n}\Big(e^{-i\alpha_4 z}\Big(1+\frac{3ip(1)}{2z^2}\Big)+e^{i\alpha_4 z}\Big(1-\frac{3ip(1)}{2z^2}\Big)+\mathcal{O}(n^{-3})\Big)\\ &=4icz^{10}(-1)^{n+1}e^{-3\pi/2+\pi n}\Big(\delta_n-\frac{6p(1)}{\pi^2(2n-3)^2}+\mathcal{O}(n^{-3})\Big).\end{aligned}$$

The equation $\det\phi(z)=0$ yields

$$\delta_n=\frac{6p(1)}{\pi^2(2n-3)^2}+\mathcal{O}(n^{-3}).$$

Then (4.1) yields

$$z=-\frac{3\pi}{2}+\pi n+\frac{p_0}{2\pi(2n-3)}+\frac{6p(1)}{\pi^2(2n-3)^2}+\delta_n,\quad \delta_n=\mathcal{O}(n^{-3}). \tag{4.21}$$

Now we improve the asymptotics (4.21). The formulas (4.21) and (3.2) give

$$\begin{aligned}\alpha_4 z&=z-\frac{p_0}{4z}+\frac{\|p\|^2}{32z^3}=-\frac{3\pi}{2}+\pi n+\delta_n+\frac{p_0}{2\pi(2n-3)}+\frac{6p(1)}{\pi^2(2n-3)^2}\\&\quad+\frac{\|p\|^2}{32z^3}-\frac{p_0}{4z}+\mathcal{O}(n^{-4})\\&=-\frac{3\pi}{2}+\pi n+\delta_n+\frac{6p(1)}{\pi^2(2n-3)^2}+\frac{2p_0^2+\|p\|^2}{4\pi^3(2n-3)^3}+\mathcal{O}(n^{-4}).\end{aligned}$$

Then

$$e^{\pm i\alpha_4 z}=\pm i(-1)^n\Big(1\pm i\delta_n\pm\frac{6ip(1)}{\pi^2(2n-3)^2}\pm\frac{i\big(\|p\|^2+2p_0^2\big)}{\pi^3(2n-3)^3}+\mathcal{O}(n^{-4})\Big).$$

Substituting these into (4.20), we obtain

$$\begin{aligned}\det\phi(z)&=2cz^{10}(-1)^{n+1}e^{-3\pi/2+\pi n}\big(2i\delta_n+\frac{8ip'(1)}{\pi^3(2n-3)^3}\\&\quad+\frac{i(\|p\|^2+2p_0^2)}{2\pi^3(2n-3)^3}-\frac{16ia}{c\pi^3(2n-3)^3}+\mathcal{O}(n^{-4})\big).\end{aligned}$$

The equation $\det\phi(z)=0$ implies

$$\delta_n=-\frac{4p'(1)}{\pi^3(2n-3)^3}-\frac{\|p\|^2+2p_0^2}{4\pi^3(2n-3)^3}+\frac{8a}{c\pi^3(2n-3)^3}+\mathcal{O}(n^{-4}).$$

Then identity (4.21) yields

$$\begin{aligned}z&=-\frac{3\pi}{2}+\pi n+\frac{p_0}{2\pi(2n-3)}+\frac{6p(1)}{\pi^2(2n-3)^2}-\frac{4p'(1)}{\pi^3(2n-3)^3}\\&\quad-\frac{\|p\|^2+2p_0^2}{4\pi^3(2n-3)^3}+\frac{8a}{c\pi^3(2n-3)^3}+\delta_n,\quad \delta_n=\mathcal{O}(n^{-4}).\end{aligned} \tag{4.22}$$

Now we improve this asymptotic behavior. Relations (4.22) and (3.2) give

$$\begin{aligned}\alpha_4 z&=z-\frac{p_0}{4z}+\frac{\|p\|^2}{32z^3}\\&=-\frac{3\pi}{2}+\pi n+\delta_n+\frac{p_0}{2\pi(2n-3)}+\frac{6p(1)}{\pi^2(2n-3)^2}-\frac{4p'(1)}{\pi^3(2n-3)^3}\\&\quad-\frac{\|p\|^2+2p_0^2}{4\pi^3(2n-3)^3}+\frac{8a}{c\pi^3(2n-3)^3}+\frac{\|p\|^2}{32z^3}-\frac{p_0}{4z}\\&=-\frac{3\pi}{2}+\pi n+\delta_n+\frac{6p(1)}{\pi^2(2n-3)^2}-\frac{4p'(1)}{\pi^3(2n-3)^3}\\&\quad+\frac{8a}{c\pi^3(2n-3)^3}+\frac{6p_0p(1)}{\pi^4(2n-3)^4}+\mathcal{O}(n^{-5}).\end{aligned}$$

Then

$$e^{\pm i\alpha_4 z}=\pm i(-1)^n\Big(1\pm i\delta_n\pm\frac{6ip(1)}{\pi^2(2n-3)^2}\mp\frac{4ip'(1)}{\pi^3(2n-3)^3}\pm\frac{8ia}{c\pi^3(2n-3)^3}$$

$$\pm \frac{6ip_0p(1)}{\pi^4(2n-3)^4} - \frac{18p^2(1)}{\pi^4(2n-3)^4} + \mathcal{O}(n^{-5})\Big).$$

Substituting these expression into (4.20), we obtain

$$\begin{aligned}\det\phi(z) = 2cz^{10}(-1)^{n+1}e^{-3\pi/2+\pi n}\Big(2i\delta_n + \frac{36ip_0p(1)}{\pi^4(2n-3)^4} \\ + \frac{16\big(\eta_2(z)-\eta_1(z)\big)}{\pi^4(2n-3)^4} + \mathcal{O}(n^{-5})\Big).\end{aligned}$$

The relations (4.16), (4.19), and (4.22) imply

$$\begin{aligned}\eta_2(z)-\eta_1(z) &= \frac{i}{16}\int_0^1 e^{-2zs}\big(p'''(1-s)-p'''(s)\big)\,ds \\ &\quad + \frac{1}{4}\int_0^1 \big(p'''(s)+p'''(1-s)\big)\Big(e^{(-1-i)zs}-e^{(-1+i)zs}\Big)\,ds \\ &\quad - \frac{i}{32}\int_0^1 p'''(s)\Big(e^{i2z(1-s)}-e^{i2zs}\Big)\,ds \\ &= \frac{i\rho_{1,n}}{16} + \frac{i}{16}\int_0^1 p'''(s)\cos 2zs\,ds + \mathcal{O}(n^{-1}) \\ &= \frac{i\big(\rho_{1,n}+\widehat{p}'''_{cn}(3)\big)}{16} + \mathcal{O}(n^{-1}),\end{aligned}$$

where $\rho_{1,n}$ has the form (1.10). Then

$$\det\phi(z) = 2cz^{10}(-1)^{n+1}e^{-3\pi/2+\pi n}\Big(2i\delta_n + \frac{36ip_0p(1)}{\pi^4(2n-3)^4} + \frac{i\big(\rho_{1,n}+\widehat{p}'''_{cn}(3)\big)}{\pi^4(2n-3)^4} + \mathcal{O}(n^{-5})\Big).$$

The equation $\det\phi(z)=0$ implies

$$\delta_n = -\frac{18p_0p(1)}{\pi^4(2n-3)^4} - \frac{\rho_{1,n}+\widehat{p}'''_{cn}(3)}{2\pi^4(2n-3)^4} + \mathcal{O}(n^{-5}).$$

Then identity (4.22) yields

$$\begin{aligned}z = -\frac{3\pi}{2} + \pi n + \frac{p_0}{2\pi(2n-3)} + \frac{6p(1)}{\pi^2(2n-3)^2} - \frac{4p'(1)}{\pi^3(2n-3)^3} \\ - \frac{\|p\|^2+2p_0^2}{4\pi^3(2n-3)^3} + \frac{8a}{c\pi^3(2n-3)^3} - \frac{18p_0p(1)}{\pi^4(2n-3)^4} - \frac{\rho_{1,n}+\widehat{p}'''_{cn}(3)}{2\pi^4(2n-3)^4} + \mathcal{O}(n^{-5}).\end{aligned}$$

This gives (1.9).

Let $c=0$ and $\lambda = z^4 = \lambda_n$, $n\to+\infty$. It follows from Lemma 3.2 that

$$z = -\frac{3\pi}{4} + \pi n + \frac{p_0}{\pi(4n-3)} - \frac{2d}{a\pi(4n-3)} + \delta_n, \quad \delta_n = \mathcal{O}(n^{-2}). \tag{4.23}$$

Substituting (3.11), (3.14), (3.15) with $\sigma = 4$ into (2.10), we obtain

$$\begin{aligned}
&\det\phi(z)\\
&= -2az^7e^{-i\beta_4 z}\Big((1+i)e^{-i\alpha_4 z}\Big(1+\varkappa_{4,1}(z)+\varkappa_{4,3}(z)+\varkappa_{4,1}(z)\varkappa_{4,3}(z)\\
&\quad -\frac{d(1+i)}{2az}\big(1+\varkappa_{4,1}(z)+\varkappa_{4,4}(z)\big)-\frac{d(-1+i)p(1)}{2az^3}+\frac{b}{az^4}\Big)\\
&\quad +(-1+i)e^{i\alpha_4 z}\Big(1+\varkappa_{4,2}(z)+\varkappa_{4,6}(z)+\varkappa_{4,2}(z)\varkappa_{4,6}(z)\\
&\quad -\frac{d(1-i)}{2az}\big(1+\varkappa_{4,2}(z)+\varkappa_{4,7}(z)\big)-\frac{d(-1-i)p(1)}{2az^3}+\frac{b}{az^4}\Big)+\mathcal{O}(z^{-5})\Big),
\end{aligned}\tag{4.24}$$

where $\alpha_4$, $\beta_4$ satisfy (3.2) and $\varkappa_{4,j}$, $j = 1,2,3,4,6,7$, have the form (3.3)–(3.8).

Now we compute $\varkappa_{4,1}(z) + \varkappa_{4,3}(z) + \varkappa_{4,1}(z)\varkappa_{4,3}(z)$. The formula (3.5), identity (2.34) with $\sigma = 4$, and (2.31) give

$$\begin{aligned}
&\varkappa_{4,3}(z)\\
&= \frac{1}{z^2}\Big(\mathcal{W}_{14}+\mathcal{W}_{44}+\mathcal{W}_{23}+\mathcal{W}_{33}\Big)(1,z)+\frac{1}{z^4}\Big(\mathcal{B}_{4,14}(1,z)+\mathcal{B}_{4,23}(1,z)\Big)\\
&\quad +\frac{p^2(1)}{64z^4}\Big((W_{1,23}+W_{1,33})(W_{1,14}+W_{1,44})\\
&\quad -(W_{1,13}+W_{1,43})(W_{1,24}+W_{1,34})\Big)+\mathcal{O}(z^{-5}),
\end{aligned}\tag{4.25}$$

where $\mathcal{W}$ and $W_1$ have the form (2.22), (2.23), respectively. Direct calculations imply

$$\begin{aligned}
&\frac{1}{z^2}\Big(\mathcal{W}_{14}+\mathcal{W}_{44}+\mathcal{W}_{23}+\mathcal{W}_{33}\Big)(1,z)\\
&+\frac{p^2(1)}{64z^4}\Big((W_{1,23}+W_{1,33})(W_{1,14}+W_{1,44})-(W_{1,13}+W_{1,43})(W_{1,24}+W_{1,34})\Big)\\
&=\frac{(-1+i)p'(1)}{16z^3}+\frac{p''(1)}{16z^4}+\frac{(1+2i)p^2(1)}{64z^4}.
\end{aligned}$$

This statement, the identities (4.8), (4.14), and (4.25) imply

$$\begin{aligned}
\varkappa_{4,3}(z) &= \frac{(-1+i)p'(1)}{16z^3}+\frac{p''(1)}{16z^4}+\frac{(1+2i)p^2(1)}{64z^4}\\
&\quad -\frac{1}{32z^4}\int_0^1 e^{-2z(1-s)}p'''(s)\,ds-\frac{1}{32z^4}\int_0^1 e^{i2z(1-s)}p'''(s)\,ds+\mathcal{O}(z^{-5}).
\end{aligned}$$

Therefore, using this asymptotics and the formulas (4.5)–(4.14), we obtain

$$\begin{aligned}
&\varkappa_{4,1}(z)+\varkappa_{4,3}(z)+\varkappa_{4,1}(z)\varkappa_{4,3}(z)\\
&= -\frac{ip(1)}{4z^2}+\frac{(1-i)p'(1)}{8z^3}+\frac{5p''(1)}{16z^4}+\frac{\eta_3(z)}{z^4}+\mathcal{O}(z^{-5}),
\end{aligned}\tag{4.26}$$

where

$$\begin{aligned}
\eta_3(z) &= \frac{1}{4}\int_0^1 e^{(-1+i)zs}p'''(s)\,ds-\frac{i}{32}\int_0^1 e^{i2zs}p'''(s)\,ds-\frac{1}{32}\int_0^1 e^{i2z(1-s)}p'''(s)\,ds\\
&\quad +\frac{i}{32}\int_0^1 e^{-2zs}p'''(s)\,ds-\frac{1}{32}\int_0^1 e^{-2z(1-s)}p'''(s)\,ds.
\end{aligned}\tag{4.27}$$

Arguments similar to this provide the asymtotics

$$\begin{aligned}\varkappa_{4,1}(z)+\varkappa_{4,4}(z)&=\frac{ip(1)}{2z^2}+\frac{(1-i)p'(1)}{2z^3}+\mathcal{O}(z^{-4}),\\ \varkappa_{4,2}(z)+\varkappa_{4,7}(z)&=-\frac{ip(1)}{2z^2}+\frac{(1+i)p'(1)}{2z^3}+\mathcal{O}(z^{-4}),\end{aligned}\tag{4.28}$$

and

$$\begin{aligned}&\varkappa_{4,2}(z)+\varkappa_{4,6}(z)+\varkappa_{4,2}(z)\varkappa_{4,6}(z)\\ &=\frac{ip(1)}{4z^2}+\frac{(1+i)p'(1)}{8z^3}+\frac{5p''(1)}{16z^4}+\frac{\eta_4(z)}{z^4}+\mathcal{O}(z^{-5}),\end{aligned}\tag{4.29}$$

where

$$\begin{aligned}\eta_4(z)=&-\frac{i}{32}\int_0^1 e^{-2zs}p'''(s)\,ds-\frac{1}{32}\int_0^1 e^{-2z(1-s)}p'''(s)\,ds\\ &+\frac14\int_0^1 e^{(-1-i)zs}p'''(s)\,ds.\end{aligned}\tag{4.30}$$

Now we substitute (4.26), (4.28), and (4.29) into (4.24). Then

$$\begin{aligned}\det\phi(z)=-2az^7e^{-i\beta_4 z}\Big(&(1+i)e^{-i\alpha_4 z}\Big(1-\frac{d(1+i)}{2az}-\frac{ip(1)}{4z^2}+\frac{(1-i)p'(1)}{8z^3}\\ &+\frac{3dp(1)(1-i)}{4az^3}+\frac{5p''(1)}{16z^4}-\frac{dp'(1)}{2az^4}+\frac{\eta_3(z)}{z^4}+\frac{b}{az^4}\Big)\\ &+(-1+i)e^{i\alpha_4 z}\Big(1-\frac{d(1-i)}{2az}+\frac{ip(1)}{4z^2}+\frac{(1+i)p'(1)}{8z^3}\\ &+\frac{3dp(1)(1+i)}{4az^3}+\frac{5p''(1)}{16z^4}-\frac{dp'(1)}{2az^4}+\frac{\eta_4(z)}{z^4}+\frac{b}{az^4}\Big)+\mathcal{O}(z^{-5})\Big),\end{aligned}\tag{4.31}$$

where $\alpha_4$ and $\beta_4$ have the form (3.2).

Now we compute the asymptotic behavior of the eigenvalues $\lambda_n$, as $n\to+\infty$. Recall that the zeros of the function (4.31) are the eigenvalues $\lambda_n$. Then formulas (4.23) and (3.2) imply

$$\begin{aligned}e^{\pm i\alpha_4 z}&=e^{\pm iz\pm i\|p\|^2/(32z^3)\mp ip_0/(4z)}\\ &=\frac{(-1)^n\sqrt2(-1\mp i)}{2}\Big(1\pm i\delta_n\mp\frac{2d}{a\pi(4n-3)}+\mathcal{O}(n^{-3})\Big).\end{aligned}$$

Substituting these into (4.31), we obtain

$$\begin{aligned}&\det\phi(z)\\ &=-2az^7e^{-3\pi/4+\pi n}\big((1+i)e^{-i\alpha_4 z}\Big(1-\frac{d(1+i)}{2az}-\frac{ip(1)}{4z^2}\Big)\\ &\quad+(-1+i)e^{i\alpha_4 z}\Big(1-\frac{d(1-i)}{2az}+\frac{ip(1)}{4z^2}\Big)+\mathcal{O}(z^{-3})\big)\\ &=2\sqrt2(-1)^{n+1}az^7e^{-3\pi/4+\pi n}\big(2i\delta_n+\frac{8ip(1)}{\pi^2(4n-3)^2}+\frac{8id^2}{a^2\pi^2(4n-3)^2}+\mathcal{O}(n^{-3})\big).\end{aligned}$$

The equation $\det\phi(z)=0$ yields

$$\delta_n=-\frac{4p(1)}{\pi^2(4n-3)^2}-\frac{4d^2}{a^2\pi^2(4n-3)^2}+\mathcal{O}(n^{-3}).$$

Then identity (4.23) yields

$$\begin{gathered} z=-\frac{3\pi}{4}+\pi n+\frac{p_0}{\pi(4n-3)}-\frac{2d}{a\pi(4n-3)}-\frac{4p(1)}{\pi^2(4n-3)^2} \\ -\frac{4d^2}{a^2\pi^2(4n-3)^2}+\delta_n, \\ \delta_n=\mathcal{O}(n^{-3}). \end{gathered} \tag{4.32}$$

Now we improve the asymptotic (4.32). The formulas (4.32) and (3.2) give

$$\begin{aligned} \alpha_4 z &= z-\frac{p_0}{4z}+\frac{\|p\|^2}{32z^3} \\ &= -\frac{3\pi}{4}+\pi n+\delta_n+\frac{p_0+\mathfrak{A}}{\pi(4n-3)}+\frac{\mathfrak{B}}{\pi^2(4n-3)^2}+\frac{\|p\|^2}{32z^3}-\frac{p_0}{4z}+\mathcal{O}(n^{-4}) \\ &= -\frac{3\pi}{4}+\pi n+\delta_n+\frac{\mathfrak{A}}{\pi(4n-3)}+\frac{\mathfrak{B}}{\pi^2(4n-3)^2}+\frac{\mathfrak{C}}{\pi^3(4n-3)^3} \\ &\quad +\mathcal{O}(n^{-4}), \end{aligned}$$

where

$$\mathfrak{A}:=-\frac{2d}{a},\quad \mathfrak{B}:=-4p(1)-\frac{4d^2}{a^2},\quad \mathfrak{C}:=2\|p\|^2+4p_0^2-\frac{8p_0 d}{a}. \tag{4.33}$$

Then

$$\begin{aligned} e^{\pm i\alpha_4 z}=\frac{(-1)^n\sqrt{2}(-1\mp i)}{2}\Big(&1\pm i\delta_n\pm\frac{i\mathfrak{A}}{\pi(4n-3)}\pm\frac{i\mathfrak{B}}{\pi^2(4n-3)^2}\pm\frac{i\mathfrak{C}}{\pi^3(4n-3)^3} \\ &+\frac{1}{2}\Big(\frac{i\mathfrak{A}}{\pi(4n-3)}+\frac{i\mathfrak{B}}{\pi^2(4n-3)^2}\Big)^2\mp\frac{i\mathfrak{A}^3}{6\pi^3(4n-3)^3}+\mathcal{O}(n^{-4})\Big). \end{aligned}$$

Substituting these expressions into (4.31), we obtain

$$\det\phi(z)=2\sqrt{2}(-1)^{n+1}az^7e^{-3\pi/4+\pi n}\Big(2i\delta_n-\frac{2i\mathfrak{D}}{\pi^3(4n-3)^3}+\mathcal{O}(n^{-4})\Big),$$

where

$$\mathfrak{D}:=-2\|p\|^2-4p_0^2-8p'(1)-\frac{56p(1)d}{a}+\frac{16p_0 d}{a}-\frac{16d^2}{a^2}-\frac{16d^3}{3a^3}. \tag{4.34}$$

The equation $\det\phi(z)=0$ implies

$$\delta_n=\frac{\mathfrak{D}}{\pi^3(4n-3)^3}+\mathcal{O}(n^{-4}).$$

Then identity (4.32) yields

$$\begin{gathered} z=-\frac{3\pi}{4}+\pi n+\frac{p_0+\mathfrak{A}}{\pi(4n-3)}+\frac{\mathfrak{B}}{\pi^2(4n-3)^2}+\frac{\mathfrak{D}}{\pi^3(4n-3)^3}+\delta_n, \\ \delta_n=\mathcal{O}(n^{-4}). \end{gathered} \tag{4.35}$$

Now we improve this asymptotics. The relations (4.35) and (3.2) give

$$\begin{aligned} \alpha_4 z &= z-\frac{p_0}{4z}+\frac{\|p\|^2}{32z^3} \\ &= -\frac{3\pi}{4}+\pi n+\delta_n+\frac{p_0}{\pi(4n-3)}+\frac{\mathfrak{A}}{\pi(4n-3)}+\frac{\mathfrak{B}}{\pi^2(4n-3)^2} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\mathfrak{D}}{\pi^3(4n-3)^3}+\frac{\|p\|^2}{32z^3}-\frac{p_0}{4z}\\
&=-\frac{3\pi}{4}+\pi n+\delta_n+\frac{\mathfrak{A}}{\pi(4n-3)}+\frac{\mathfrak{B}}{\pi^2(4n-3)^2}+\frac{\mathfrak{F}}{\pi^3(4n-3)^3}\\
&+\frac{4\mathfrak{B}p_0}{\pi^4(4n-3)^4}+\mathcal{O}(n^{-5}),
\end{aligned}
$$

where

$$
\mathfrak{F}:=-\frac{56p(1)d}{a}+\frac{8p_0d}{a}-\frac{16d^2}{a^2}-\frac{16d^3}{3a^3}-8p'(1).
$$

Then

$$
\begin{aligned}
e^{\pm i\alpha_4 z}=\frac{(-1)^n\sqrt{2}(-1\mp i)}{2}\Big(&1\pm i\delta_n\pm\frac{i\mathfrak{A}}{\pi(4n-3)}\pm\frac{i\mathfrak{B}}{\pi^2(4n-3)^2}\pm\frac{i\mathfrak{F}}{\pi^3(4n-3)^3}\\
&\pm\frac{4\mathfrak{B}p_0}{\pi^4(4n-3)^4}+\frac{1}{2}\Big(\frac{i\mathfrak{A}}{\pi(4n-3)}+\frac{i\mathfrak{B}}{\pi^2(4n-3)^2}+\frac{i\mathfrak{F}}{\pi^3(4n-3)^3}\Big)^2\\
&\pm\frac{1}{6}\Big(\frac{i\mathfrak{A}}{\pi(4n-3)}+\frac{i\mathfrak{B}}{\pi^2(4n-3)^2}\Big)^3+\frac{\mathfrak{A}^4}{24\pi^4(4n-3)^4}+\mathcal{O}(n^{-5})\Big).
\end{aligned}
$$

Substituting these expressions into (4.31), we obtain

$$
\begin{aligned}
\det\phi(z)=2\sqrt{2}(-1)^{n+1}az^7e^{-3\pi/4+\pi n}\Big(&2i\delta_n-\frac{96id^2p_0}{a^2\pi^4(4n-3)^4}+\frac{192id^3}{a^3\pi^4(4n-3)^4}\\
&-\frac{96ip_0p(1)}{\pi^4(4n-3)^4}+\frac{192idp(1)}{a\pi^4(4n-3)^4}+\frac{256\big(\eta_4(z)-\eta_3(z)\big)}{\pi^4(4n-3)^4}+\mathcal{O}(n^{-5})\Big).
\end{aligned}
$$

Now we obtain $\eta_4(z)-\eta_3(z)$. Then equations (4.27), (4.30), and (4.35) give

$$
\begin{aligned}
\eta_4(z)-\eta_3(z)&=-\frac{i}{16}\int_0^1 e^{-2zs}p'''(s)\,ds+\frac{1}{4}\int_0^1 e^{-zs}p'''(s)\big(e^{-izs}-e^{izs}\big)\,ds\\
&\quad+\frac{1}{32}\int_0^1 p'''(s)\big(ie^{i2zs}+e^{i2z(1-s)}\big)\,ds\\
&=-\frac{i\rho_{2,n}}{16}+\frac{i}{16}\int_0^1 p'''(s)\cos\pi\Big(2n-\frac{3}{2}\Big)s\,ds+\mathcal{O}(n^{-1})\\
&=-\frac{i(\rho_{2,n}-\widehat{p}'''_{cn}(3/2))}{16}+\mathcal{O}(n^{-1}),
\end{aligned}
$$

where $\rho_{2,n}$ has the form (1.12). The equation $\det\phi(z)=0$ implies

$$
\begin{aligned}
\delta_n&=\frac{48d^2p_0}{a^2\pi^4(4n-3)^4}-\frac{96d^3}{a^3\pi^4(4n-3)^4}+\frac{48p_0p(1)}{\pi^4(4n-3)^4}-\frac{96dp(1)}{a\pi^4(4n-3)^4}\\
&\quad+\frac{8\big(\rho_{2,n}-\widehat{p}'''_{cn}(3/2)\big)}{\pi^4(4n-3)^4}+\mathcal{O}(n^{-5}).
\end{aligned}
$$

Then identity (4.35) yields

$$
\begin{aligned}
z&=-\frac{3\pi}{4}+\pi n+\frac{p_0}{\pi(4n-3)}+\frac{\mathfrak{A}}{\pi(4n-3)}+\frac{\mathfrak{B}}{\pi^2(4n-3)^2}+\frac{\mathfrak{D}}{\pi^3(4n-3)^3}\\
&\quad+\frac{48d^2p_0}{a^2\pi^4(4n-3)^4}-\frac{96d^3}{a^3\pi^4(4n-3)^4}+\frac{48p_0p(1)}{\pi^4(4n-3)^4}-\frac{96dp(1)}{a\pi^4(4n-3)^4}\\
&\quad+\frac{8\big(\rho_{2,n}-\widehat{p}'''_{cn}(3/2)\big)}{\pi^4(4n-3)^4}+\mathcal{O}(n^{-5}),
\end{aligned}
$$

where $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{D}$ have the form (4.33) and (4.34). This implies (1.11). □

*Proof of Theorem 1.1.* It follows from [14, Lemmas 4.1, 4.2] that the eigenvalues $\lambda_n$ are real and simple. The asymptotics (1.7), (1.8), (1.9), and (1.11) are proved in Lemmas 3.2 and 4.1. The proof is complete. □

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