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CURVED-PIPE FLOW WITH BOUNDARY CONDITIONS INVOLVING BERNOULLI PRESSURE

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Abstract. In this article, we study the steady-state flow of the incompressible viscous fluid through a thin distorted pipe with an arbitrary central curve. We prescribe the inflow and outflow boundary conditions involving the Bernoulli pressure with a given pressure drop. Using the multiscale expansion technique with respect to the pipe's thickness, we construct the higher-order asymptotic approximation of the flow given by the explicit formulae for the velocity and pressure. We also perform a detailed error analysis justifying the usage of the proposed solution and indicating its order of accuracy.

1. Introduction

Curved-pipe flows have been studied analytically for many years due to its obvious practical importance. The pioneering work is due to Dean [5] back in 1927 who first used the perturbation techniques to investigate the fluid flow through a curved pipe with circular cross-section. Having in mind that thin (or long) pipes naturally appear in numerous applications, it is no surprise that in the last two decades several results have been reported that propose new asymptotic models describing the effective flows through thin pipe-like domains. The models for steady-state flow of a Newtonian fluid through 3D curved pipes have been proposed and justified in [15, 17], whereas a system of thin pipes has been studied in [13, 16]. The non-Newtonian, micropolar fluid has been investigated in [6] and [7] in 2D domains such as a periodically constricted tubes and curvilinear channels. The 3D cases including a curved pipe and a multiple pipe system have been studied in [1, 20]. The corresponding rigorous results for non-steady flows have been reported in [18, 19] for a Newtonian fluid flowing through a system of thin pipes, while a curved-pipe flow has been addressed in [3]. The analysis of the non-steady micropolar fluid flow can be found in [2, 21].

The flow of the incompressible viscous Newtonian fluid is described by the nonlinear Navier-Stokes system and the concept of weak solutions is naturally introduced. When the inflows and outflows are described by the given velocity, the existence of the weak solution can be proved using the standard Galerkin method (see e.g.

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Temam [22]). From the perspective of applications, prescribing boundary conditions involving pressure is evidently more plausible. However, in such a setting, it is much more difficult to establish the solvability of the corresponding variational fomulation. One usually needs to restrict to the case of small boundary data in order to control the inertial term in Navier-Stokes equations (see e.g. [10, 16]).

The another approach would be to prescribe the dynamic boundary condition, namely the Bernoulli pressure $\Psi = p + \frac{1}{2}|\mathbf{u}|^2$, as in [4]. Quite recently, in [12], the authors considered the steady-state flow of a viscous fluid through a finite undeformed pipe with inflow and outflow boundary conditions involving the Bernoulli pressure. Using the method of Leray [14], they succeeded to prove the existence of the weak solution for arbitrary data and its uniqueness for small data. Relying on this result, the asymptotic approximation of the flow through a network of thin straight pipes has been rigorously derived in [11]. Inspired by these works, in this article we investigate the flow through a thin curved pipe subjected to inflow and outflow boundary conditions involving the Bernoulli pressure with a given pressure drop. Since the Bernoulli pressure represents an important quantity for moderately high Reynolds number, we seek for the inertial effects on the effective flow along with the effects of pipe's distortion.

This article is organized as follows. In Section 2, we formally describe the geometry of the curved pipe with an arbitrary central curve and a circular cross-section. To do so, we choose the so-called Germano's frame of reference (introduced in [8, 9]) in which the domain's cross-section posseses no rotation with respect to the tangent vector. The flow is assumed to be governed by the Navier-Stokes equations endowed with the no-slip boundary condition for the velocity prescribed on the pipe's lateral boundary. In Section 3, using the approach from [17, 20], we write the differential operators in curvilinear coordinates transforming the governing problem to an undeformed pipe. Motivated by the applications, we assume that the ratio between pipe's thickness and its length is small (and denoted by ε) meaning that we are considering the fluid flow in a pipe which is either thin or long Thus, in Section 4, we employ the two-scale asymptotic expansion in powers of ε and construct the asymptotic approximation for the velocity and pressure up to a second–order. By doing that, we are able to capture not only the effects of the pipe's geometry, but also the effects of the additional inertial term appearing due to the Bernoulli pressure inflow-outflow boundary conditions. It should be emphasized that the asymptotic approximation is provided in the explicit form and, therefore, it can be used as an useful check for numerical simulations. Lastly, to justify the usage of the proposed asymptotic solution and provide its order of accuracy, in Section 5, we prove the error estimates in suitable norms by using functional analysis techniques.

2. Formulation of the problem

2.1. **The domain.** Let $\Omega_\varepsilon^\alpha$ denote our thin domain representing a curved pipe characterized by a smooth central curve γ and a circular cross-section. The curve γ is taken to be a generic curve in $\mathbb{R}^3$, parameterized by its arc length $x_1 \in [0, l]$. We denote the corresponding natural parametrization as $\boldsymbol{\pi} \in C^3([0, l]; \mathbb{R}^3)$, satisfying $\boldsymbol{\pi}'(x_1) \neq 0$ for every $x_1 \in [0, l]$.

To formally describe $\Omega_\varepsilon^\alpha$, we start by defining a straight pipe with a circular cross-section. Introducing the small positive parameter $\varepsilon \ll 1$ and a unit circle

$B = B(0,1) \subset \mathbb{R}^2$, a thin straight pipe is given by

$$\Omega_\varepsilon = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \colon x_1 \in \langle 0, l\rangle,\ \mathbf{x}_* := (x_2, x_3) \in \varepsilon B\}\,.$$

We now introduce the appropriate frame of reference attached to the curve γ and pass from Ω_ε to $\Omega_\varepsilon^\alpha$ via the appropriate parametrization. Firstly, at each point $\boldsymbol{\pi}(x_1)$ of the curve γ, we introduce the standard Frenet's basis

$$\mathbf{t} = \boldsymbol{\pi}', \quad \mathbf{n} = \frac{1}{\kappa}\mathbf{t}', \quad \mathbf{b} = \mathbf{t} \times \mathbf{n}\,,$$

where $\mathbf{t}$ is the tangent, $\mathbf{n}$ the normal and $\mathbf{b}$ the binormal. The flexion of the curve γ is given by $\kappa(x_1) = |\boldsymbol{\pi}''(x_1)|$, whereas the torsion is denoted by $\tau(x_1) = -|\mathbf{b}'(x_1)|$. The Frenet's basis $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ satisfies the system

$$\mathbf{t}' = \kappa\mathbf{n}, \quad \mathbf{n}' = -\kappa\mathbf{t} + \tau\mathbf{b}, \quad \mathbf{b}' = -\tau\mathbf{n}\,. \tag{2.1}$$

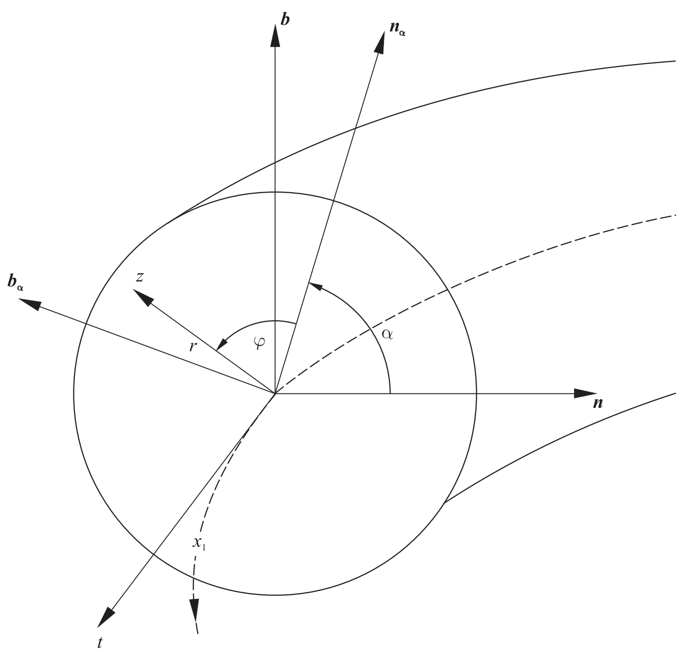

Figure 1. Pipe's cross-section with a frame of reference attached.

Let us now introduce the rotated unit vectors (see Figure 1)

$$\mathbf{n}_\alpha(x_1) = \cos\alpha(x_1)\mathbf{n}(x_1) + \sin\alpha(x_1)\mathbf{b}(x_1)\,,$$
$$\mathbf{b}_\alpha(x_1) = -\sin\alpha(x_1)\mathbf{n}(x_1) + \cos\alpha(x_1)\mathbf{b}(x_1)\,,$$

with rotation given by

$$\alpha(x_1) = -\int_{x_0}^{x_1} \tau(\xi)d\xi + \alpha_0,$$

where x_0 and α_0 are arbitrary constants. Defining the mapping $\mathbf{\Phi}_\varepsilon^\alpha \colon \Omega_\varepsilon \to \mathbb{R}^3$ by

$$\mathbf{\Phi}_\varepsilon^\alpha(\mathbf{x}) = \boldsymbol{\pi}(x_1) + x_2\,\mathbf{n}_\alpha(x_1) + x_3\,\mathbf{b}_\alpha(x_1)\,,$$

and putting

$$\Omega_\varepsilon^\alpha = \mathbf{\Phi}_\varepsilon^\alpha(\Omega_\varepsilon)\,,$$

we obtain our curved pipe with the central curve γ and circular cross section εB. Note that the variable x_1 follows the central curve of the pipe, while $x_* = (x_2, x_3)$ describes its cross-section. The local injectivity of $\mathbf{\Phi}_\varepsilon^\alpha$ can be easily established assuming ε is sufficiently small (see [20] for details).

Finally, by

$$\Gamma_\varepsilon^\alpha = \mathbf{\Phi}_\varepsilon^\alpha(\langle 0, l\rangle \times \varepsilon\,\partial B)\,, \quad \Sigma_\varepsilon^i = \mathbf{\Phi}_\varepsilon^\alpha(\{i\} \times \varepsilon\, B),\ i = 0, l\,,$$

we denote the pipe's lateral boundary and its ends, respectively.

2.2. **Governing system.** As explained in the introduction, we study the steady-state flow of the incompressible viscous fluid in a thin curved pipe $\Omega_\varepsilon^\alpha$. The usual no-slip boundary condition for the velocity is imposed on the pipe's lateral boundary $\Gamma_\varepsilon^\alpha$, while we prescribe the dynamic boundary condition involving the Bernoulli pressure on the pipe's ends Σ_ε^i. To close the problem, we take the tangential velocity to be zero on Σ_ε^i, which is not a serious restriction since the only part that counts is the normal part, due to the Saint-Venant principle for thin domains (see e.g. [15]). Therefore, the governing system reads

$$\begin{gathered}
-\nu\Delta\mathbf{u}_\varepsilon + (\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon + \nabla p_\varepsilon = \frac{1}{\varepsilon^2}\,\mathbf{f} \quad \text{in } \Omega_\varepsilon^\alpha\,,\\
\operatorname{div}\mathbf{u}_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon^\alpha\,,\\
\mathbf{u}_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon^\alpha\,,\\
\mathbf{u}_\varepsilon \times \mathbf{t} = 0 \quad \text{on } \Sigma_\varepsilon^i,\ i = 0, l\,,\\
-\nu\partial_t\mathbf{u}_\varepsilon\cdot\mathbf{t} + (p_\varepsilon + \frac{1}{2}|\mathbf{u}_\varepsilon|^2) = \frac{c^i}{\varepsilon^2} \quad \text{on } \Sigma_\varepsilon^i,\ i = 0, l\,.
\end{gathered} \tag{2.2}$$

where ν is a positive constant (the viscosity of the fluid), $\mathbf{u}_\varepsilon$ is the velocity, p_ε is the pressure, $\partial_t g = \nabla g\cdot\mathbf{t}$ denotes the tangential derivative of g (with respect to the curve γ), c^i are some constants and $\mathbf{f} \in L^2(\Omega_\varepsilon^\alpha)$ is a given function (external force).

It should be noted that, using the boundary condition $(2.2)_4$, and the incompressibility equation, it follows that $-\nu\partial_t\mathbf{u}_\varepsilon\cdot\mathbf{t}|_{\Sigma_\varepsilon^i} = 0$. Now, employing the identity

$$\frac{1}{2}(\nabla\mathbf{u}_\varepsilon^2) = \mathbf{u}_\varepsilon\cdot(\nabla\mathbf{u}_\varepsilon)^t = \sum_{k=1}^n u_k\nabla u_k\,,$$

the system (2.2) can be rewritten as

$$\begin{gathered}
-\nu\Delta\mathbf{u}_\varepsilon + (\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon\cdot(\nabla\mathbf{u}_\varepsilon)^t + \nabla\Psi_\varepsilon = \frac{1}{\varepsilon^2}\,\mathbf{f} \quad \text{in } \Omega_\varepsilon^\alpha\,,\\
\operatorname{div}\mathbf{u}_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon^\alpha\,,\\
\mathbf{u}_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon^\alpha\,,\\
\mathbf{u}_\varepsilon \times \mathbf{t} = 0 \quad \text{on } \Sigma_\varepsilon^i,\ i = 0, l,\\
\Psi_\varepsilon = p_i \quad \text{on } \Sigma_\varepsilon^i,\ i = 0, l\,.
\end{gathered} \tag{2.3}$$

In (2.3), a new quantity has been naturally introduced, namely the Bernoulli pressure given by

$$\Psi_\varepsilon = p_\varepsilon + \frac{1}{2}|\mathbf{u}_\varepsilon|^2\,,$$

while p_i denotes the constants $\frac{c^i}{\varepsilon^2}$. Using the idea from [12], it is straightforward to prove that problem (2.3) admits at least one weak solution which is unique under the small data assumption (see [11] for details). The goal of this article is to investigate the asymptotic behavior of the fluid flow described by (2.3) via rigorous asymptotic analysis with respect to the small parameter ε.

3. Curvilinear coordinates

To ensure comprehensive understanding, we provide a succinct summary of formulating the problem (2.3) in curvilinear coordinates (x_i). We follow the procedure in [17, 20] and begin by introducing the necessary geometric tools. We define the covariant basis as the gradient of the function mapping $\mathbf{\Phi}_\varepsilon^\alpha$, namely,

$$\mathbf{a}_i(x) := \frac{\partial \mathbf{\Phi}_\varepsilon^\alpha}{\partial x_i}(\mathbf{x})\,.$$

Taking into account the Frenet system (2.1), we deduce

$$\mathbf{a}_1 = (1 - \kappa\,(\mathbf{e}_\alpha \cdot \mathbf{x}_*))\,\mathbf{t}\,,$$

where

$$\mathbf{x} = (x_1, \mathbf{x}_*)\ ,\ \mathbf{e}_\alpha = (\cos\alpha, -\sin\alpha)\ ,\quad \mathbf{e}_\alpha^\perp = (\sin\alpha, \cos\alpha)\ .$$

Note that in computing the vector $\mathbf{a}_1$, we used the fact that $\alpha' = \tau$.

The contravariant basis is defined as the dual to the covariant basis, i.e. $\mathbf{a}^i \cdot \mathbf{a}_j = \delta_{ij}$. We thus have

$$\mathbf{a}^1 = \frac{1}{1 - \kappa\,(\mathbf{e}_\alpha \cdot \mathbf{x}_*)}\,\mathbf{t}\,,\quad \mathbf{a}^2 = \cos\alpha\mathbf{n} + \sin\alpha\mathbf{b}\,,\quad \mathbf{a}^3 = -\sin\alpha\mathbf{n} + \cos\alpha\mathbf{b}\,.$$

It should be observed that

$$\nabla\mathbf{\Phi}_\varepsilon^\alpha = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\ ,\quad (\nabla\mathbf{\Phi}_\varepsilon^\alpha)^{-1} = \begin{bmatrix}(\mathbf{a}^1)^t\\ (\mathbf{a}^2)^t\\ (\mathbf{a}^3)^t\end{bmatrix}\ .$$

Christoffel's symbols are defined as

$$\Gamma_{jk}^i = \mathbf{a}^i \cdot \frac{\partial \mathbf{a}_k}{\partial x_j}\,,$$

being symmetric in lower indices. For the non-zero ones, we obtain

$$\begin{gathered}
\Gamma_{11}^1 = -\frac{\left(\kappa'(e_\alpha \cdot x_*) + \kappa\tau(e_\alpha^\perp \cdot x_*)\right)}{1 - \kappa(e_\alpha \cdot x_*)} = \mathcal{O}(\varepsilon)\\
\Gamma_{12}^1 = \Gamma_{21}^1 = -\frac{\kappa\cos\alpha}{1 - \kappa(e_\alpha \cdot x_*)} = -\kappa\cos\alpha + \mathcal{O}(\varepsilon)\,,\\
\Gamma_{13}^1 = \Gamma_{31}^1 = \frac{\kappa\sin\alpha}{1 - \kappa(e_\alpha \cdot x_*)} = \kappa\sin\alpha + \mathcal{O}(\varepsilon)\\
\Gamma_{11}^2 = \kappa\,(1 - \kappa(e_\alpha \cdot x_*))\cos\alpha = \kappa\cos\alpha + \mathcal{O}(\varepsilon)\,,\\
\Gamma_{11}^3 = -\kappa\,(1 - \kappa(e_\alpha \cdot x_*))\sin\alpha = -\kappa\sin\alpha + \mathcal{O}(\varepsilon)\ .
\end{gathered}$$

Moreover, we use the following formulae (see e.g. [20, Appendix] for its proof).

$$(\nabla s)^T \circ \mathbf{\Phi}_\varepsilon^\alpha = (\nabla\mathbf{\Phi}_\varepsilon^\alpha)^{-T}(\nabla S)^T\,, \tag{3.1}$$

$$(\nabla\mathbf{v}) \circ \mathbf{\Phi}_\varepsilon^\alpha = (\nabla\mathbf{\Phi}_\varepsilon^\alpha)^{-T}\Big(\big[\frac{\partial V^k}{\partial x_l}\big]_{k,l} - V^j\mathbf{\Gamma}^j\Big)(\nabla\mathbf{\Phi}_\varepsilon^\alpha)^{-1}\,,\quad \mathbf{\Gamma}^i = [\Gamma_{jk}^i]_{j,k}\,, \tag{3.2}$$

$$
\begin{aligned}
(\Delta \mathbf{v}) \circ \mathbf{\Phi}_\varepsilon^\alpha = (\nabla \mathbf{\Phi}_\varepsilon^\alpha)^{-T} \Big(\frac{\partial}{\partial x_i} \big(\big[\frac{\partial V^k}{\partial x_l} \big]_{k,\,l} - \ V^j \, \mathbf{\Gamma}^j \big) - \big(\big[\frac{\partial V^k}{\partial x_l} \big]_{k,\,l} - \ V^j \, \mathbf{\Gamma}^j \big) \hat{\mathbf{\Gamma}}_i \\
- \hat{\mathbf{\Gamma}}_i^T \big(\big[\frac{\partial V^k}{\partial x_l} \big]_{k,\,l} - \ V^j \, \mathbf{\Gamma}^j \big) \Big) (\nabla \mathbf{\Phi}_\varepsilon^\alpha)^{-1} \mathbf{a}^i \,, \quad \hat{\mathbf{\Gamma}}_i = [\Gamma_{ik}^j]_{j,k} \,,
\end{aligned} \tag{3.3}
$$

$$
\big((\operatorname{rot} \mathbf{v}) \circ \mathbf{\Phi}_\varepsilon^\alpha \big) \times \mathbf{c} = (\nabla \mathbf{\Phi}_\varepsilon^\alpha)^{-T} \big(\operatorname{rot} \mathbf{V} \times (\nabla \mathbf{\Phi}_\varepsilon^\alpha)^{-1} \mathbf{c} \big) \,, \quad \mathbf{c} \in \mathbf{R}^3 \,, \tag{3.4}
$$

for the scalar field $S = s \circ \mathbf{\Phi}_\varepsilon^\alpha$ and the vector field $\mathbf{V} = \mathbf{v} \circ \mathbf{\Phi}_\varepsilon^\alpha$. In the above identities, the summation is taken over the repeated indices, while $V^i = \mathbf{V} \cdot \mathbf{a}_i$ denote the contravariant components.

Introducing

$$
\begin{gathered}
\mathbf{V}_\varepsilon = \mathbf{u}_\varepsilon \circ \mathbf{\Phi}_\varepsilon^\alpha = V_\varepsilon^1 \mathbf{a}^1 + V_\varepsilon^2 \mathbf{a}^2 + V_\varepsilon^3 \mathbf{a}^3 \,, \\
P_\varepsilon = \Psi_\varepsilon \circ \mathbf{\Phi}_\varepsilon^\alpha \,,
\end{gathered} \tag{3.5}
$$

we now proceed with expressing each differential operator occurring in equations (2.3) in curvilinear coordinates.

Let $\lambda := 1 - \kappa(\mathbf{e}_\alpha \cdot \mathbf{x}_*)$. The second-order term $-\nu \Delta \mathbf{u}_\varepsilon$ takes the form

$$
\begin{aligned}
&(-\nu \Delta \mathbf{u}_\varepsilon) \circ \mathbf{\Phi}_\varepsilon^\alpha \\
&= -\nu \Big[\Delta V_\varepsilon^1 + \kappa \cos\alpha \Big(\frac{\partial V_\varepsilon^1}{\partial x_2} - \frac{\partial V_\varepsilon^2}{\partial x_1} \Big) + \kappa \sin\alpha \Big(\frac{\partial V_\varepsilon^3}{\partial x_1} - \frac{\partial V_\varepsilon^1}{\partial x_3} \Big) \\
&\quad + \frac{\partial}{\partial x_1} \big(V_\varepsilon^3 \kappa \sin\alpha - V_\varepsilon^2 \kappa \cos\alpha \big) - \kappa^2 V_\varepsilon^1 \Big] \mathbf{a}^1 \\
&- \nu \Big(\frac{1}{\lambda^2} - 1 \Big) \Big[\frac{\partial^2 V_\varepsilon^1}{\partial x_1^2} + \frac{\partial}{\partial x_1} \big(V_\varepsilon^3 \kappa \sin\alpha - V_\varepsilon^2 \kappa \cos\alpha \big) - \kappa \cos\alpha \Big(\frac{\partial V_\varepsilon^1}{\partial x_2} + \frac{V_\varepsilon^2}{\partial x_1} \Big) \\
&\quad + \kappa \sin\alpha \Big(\frac{\partial V_\varepsilon^1}{\partial x_3} - \frac{\partial V_\varepsilon^3}{\partial x_1} \Big) - 2\kappa^2 V_\varepsilon^1 \Big] \mathbf{a}^1 - \frac{\nu}{\lambda^2} \Big[\Delta V_\varepsilon^2 + \frac{\partial}{\partial x_1} \big(V_\varepsilon^1 \kappa \cos\alpha \big) \\
&\quad - \kappa \cos\alpha \Big(\frac{\partial V_\varepsilon^2}{\partial x_2} + V_\varepsilon^2 \kappa \cos\alpha - V_\varepsilon^3 \kappa \sin\alpha - \frac{\partial V_\varepsilon^1}{\partial x_1} \Big) + \kappa \sin\alpha \frac{\partial V_\varepsilon^2}{\partial x_3} \Big] \mathbf{a}^2 \\
&\quad + \nu \Big(\frac{1}{\lambda^2} - 1 \Big) \Big[\frac{\partial^2 V_\varepsilon^2}{\partial x_2^2} + \frac{\partial^2 V_\varepsilon^2}{\partial x_3^2} \Big] \mathbf{a}^2 - \frac{\nu}{\lambda^2} \Big[\Delta V_\varepsilon^3 - \frac{\partial}{\partial x_1} \big(V_\varepsilon^1 \kappa \sin\alpha \big) \\
&\quad - \kappa \cos\alpha \frac{\partial V_\varepsilon^3}{\partial x_2} + \kappa \sin\alpha \Big(\frac{\partial V_\varepsilon^3}{\partial x_3} + V_\varepsilon^2 \kappa \cos\alpha - V_\varepsilon^3 \kappa \sin\alpha - \frac{\partial V_\varepsilon^1}{\partial x_1} \Big) \Big] \mathbf{a}^3 \\
&\quad + \nu \Big(\frac{1}{\lambda^2} - 1 \Big) \Big[\frac{\partial^2 V_\varepsilon^3}{\partial x_2^2} + \frac{\partial^2 V_\varepsilon^3}{\partial x_3^2} \Big] \mathbf{a}^3
\end{aligned}
$$

The terms including the pressure and the external force are transformed as follows

$$
\begin{gathered}
\nabla \psi_\varepsilon \circ \mathbf{\Phi}_\varepsilon^\alpha = \frac{\partial P_\varepsilon}{\partial x_1} \mathbf{a}^1 + \frac{\partial P_\varepsilon}{\partial x_2} \mathbf{a}^2 + \frac{\partial P_\varepsilon}{\partial x_3} \mathbf{a}^3 \,, \\
\mathbf{f} \circ \mathbf{\Phi}_\varepsilon^\alpha = f_1 \lambda \mathbf{a}^1 + (f_2 \cos\alpha + f_3 \sin\alpha) \, \mathbf{a}^2 + (f_3 \cos\alpha - f_2 \sin\alpha) \, \mathbf{a}^3 \,.
\end{gathered}
$$

To rewrite the inertial terms $(\mathbf{u}_\varepsilon \cdot \nabla) \mathbf{u}_\varepsilon$ and $\mathbf{u}_\varepsilon \cdot (\nabla \mathbf{u}_\varepsilon)^t$, we start by writing the velocity gradient as

$$
\nabla \mathbf{u}_\varepsilon \circ \mathbf{\Phi}_\varepsilon^\alpha
$$

$$= \begin{bmatrix} \frac{1}{\lambda^2}\big(\frac{\partial V_\varepsilon^1}{\partial x_1} - V_\varepsilon^2\kappa\cos\alpha + V_\varepsilon^3\kappa\sin\alpha\big) & \frac{\partial V_\varepsilon^1}{\partial x_2} + V_\varepsilon^1\kappa\cos\alpha & \frac{\partial V_\varepsilon^1}{\partial x_3} - V_\varepsilon^1\kappa\sin\alpha \\ \frac{1}{\lambda^2}\big(\frac{\partial V_\varepsilon^2}{\partial x_1} + V_\varepsilon^1\kappa\cos\alpha\big) & \frac{\partial V_\varepsilon^2}{\partial x_2} & \frac{\partial V_\varepsilon^2}{\partial x_3} \\ \frac{1}{\lambda^2}\big(\frac{\partial V_\varepsilon^3}{\partial x_1} - V_\varepsilon^1\kappa\sin\alpha\big) & \frac{\partial V_\varepsilon^3}{\partial x_2} & \frac{\partial V_\varepsilon^3}{\partial x_3} \end{bmatrix}$$

Therefore, the term $(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon$ becomes

$$\begin{aligned}
&\Big[\frac{V_\varepsilon^1}{\lambda^3}\Big(\frac{\partial V_\varepsilon^1}{\partial x_1} - V_\varepsilon^2\kappa\cos\alpha + V_\varepsilon^3\kappa\sin\alpha\Big) + \big(V_\varepsilon^2\cos\alpha - V_\varepsilon^3\sin\alpha\big)\Big(\frac{\partial V_\varepsilon^1}{\partial x_2} + V_\varepsilon^1\kappa\cos\alpha\Big) \\
&+ \big(V_\varepsilon^2\sin\alpha + V_\varepsilon^3\cos\alpha\big)\Big(\frac{\partial V_\varepsilon^1}{\partial x_3} - V_\varepsilon^1\kappa\sin\alpha\Big)\Big]\mathbf{a}^1 \\
&+ \Big[\frac{V_\varepsilon^1}{\lambda^3}\Big(\frac{\partial V_\varepsilon^2}{\partial x_1} + V_\varepsilon^1\kappa\cos\alpha\Big) + \big(V_\varepsilon^2\cos\alpha - V_\varepsilon^3\sin\alpha\big)\frac{\partial V_\varepsilon^2}{\partial x_2} \\
&+ \big(V_\varepsilon^2\sin\alpha + V_\varepsilon^3\cos\alpha\big)\frac{\partial V_\varepsilon^2}{\partial x_3}\Big]\mathbf{a}^2 + \Big[\frac{V_\varepsilon^1}{\lambda^3}\Big(\frac{\partial V_\varepsilon^3}{\partial x_1} - V_\varepsilon^1\kappa\sin\alpha\Big) \\
&+ \big(V_\varepsilon^2\cos\alpha - V_\varepsilon^3\sin\alpha\big)\frac{\partial V_\varepsilon^3}{\partial x_2} + \big(V_\varepsilon^2\sin\alpha + V_\varepsilon^3\cos\alpha\big)\frac{\partial V_\varepsilon^3}{\partial x_3}\Big]\mathbf{a}^3\,.
\end{aligned}$$

Similarly, the term $\mathbf{u}_\varepsilon\cdot(\nabla\mathbf{u}_\varepsilon)^t$ becomes

$$\begin{aligned}
&\Big[\frac{V_\varepsilon^1}{\lambda^3}\Big(\frac{\partial V_\varepsilon^1}{\partial x_1} - V_\varepsilon^2\kappa\cos\alpha + V_\varepsilon^3\kappa\sin\alpha\Big) + \frac{1}{\lambda^2}\big(V_\varepsilon^2\cos\alpha - V_\varepsilon^3\sin\alpha\big)\Big(\frac{\partial V_\varepsilon^2}{\partial x_1} + V_\varepsilon^1\kappa\cos\alpha\Big) \\
&+ \frac{1}{\lambda^2}\big(V_\varepsilon^2\sin\alpha + V_\varepsilon^3\cos\alpha\big)\Big(\frac{\partial V_\varepsilon^3}{\partial x_1} - V_\varepsilon^1\kappa\sin\alpha\Big)\Big]\mathbf{a}^1 + \Big[\frac{V_\varepsilon^1}{\lambda}\Big(\frac{\partial V_\varepsilon^1}{\partial x_2} + V_\varepsilon^1\kappa\cos\alpha\Big) \\
&+ \big(V_\varepsilon^2\cos\alpha - V_\varepsilon^3\sin\alpha\big)\frac{\partial V_\varepsilon^2}{\partial x_2} + \big(V_\varepsilon^2\sin\alpha + V_\varepsilon^3\cos\alpha\big)\frac{\partial V_\varepsilon^3}{\partial x_2}\Big]\mathbf{a}^2 \\
&+ \Big[\frac{V_\varepsilon^1}{\lambda}\Big(\frac{\partial V_\varepsilon^1}{\partial x_3} - V_\varepsilon^1\kappa\sin\alpha\Big) + \big(V_\varepsilon^2\cos\alpha - V_\varepsilon^3\sin\alpha\big)\frac{\partial V_v^2 e}{\partial x_3} \\
&+ \big(V_\varepsilon^2\sin\alpha + V_\varepsilon^3\cos\alpha\big)\frac{\partial V_\varepsilon^3}{\partial x_3}\Big]\mathbf{a}^3\,.
\end{aligned}$$

Finally, taking the trace in the expression for the velocity gradient gives

$$\operatorname{div}\mathbf{v}\circ\mathbf{\Phi}_\varepsilon^\alpha = \frac{1}{\lambda^2}\Big(\frac{\partial V_\varepsilon^1}{\partial x_1} - V_\varepsilon^2\kappa\cos\alpha + V_\varepsilon^3\kappa\sin\alpha\Big) + \frac{\partial V_\varepsilon^2}{\partial x_2} + \frac{\partial V_\varepsilon^3}{\partial x_3}\,. \tag{3.6}$$

4. Asymptotic expansion

To construct the approximation of the solution, we now expand the unknown velocity components and the pressure in the two-scale asymptotic expansions in powers of ε, namely,

$$\begin{aligned}
V_\varepsilon^i(\mathbf{x}) &= V_0^i\Big(x_1, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big) + \varepsilon V_1^i\Big(x_1, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big) + \varepsilon^2 V_2^i\Big(x_1, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big) + \dots \\
P_\varepsilon(\mathbf{x}) &= \frac{1}{\varepsilon^2}P_0(x_1) + \frac{1}{\varepsilon}P_1\Big(x_1, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big) + P_2\Big(x_1, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\Big) + \dots
\end{aligned} \tag{4.1}$$

The idea is to use the derived expressions of differential operators in curvilinear coordinates (see Section 3), substitute the expansions (4.1) and collect the terms with equal powers of ε. By doing that, we shall obtain the recursive sequence of linear problems that, combining with the boundary conditions, we aim to solve explicitly.

The $\mathcal{O}(1/\varepsilon^2)$-order term. Let us introduce the dilated variable $y_* = x_*/\varepsilon$ capturing the fast changes of the solution on the pipe's cross-section. In the sequel, by $\Omega = \langle 0,l\rangle \times B$ and $\Gamma = \langle 0,l\rangle \times \partial B$, we denote the corresponding rescaled regions and employ the following notation for the formal partial differential operators

$$\nabla_{\boldsymbol{y}_*} = \Big(\frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}\Big), \quad \Delta_{\boldsymbol{y}_*}\mathbf{V} = \frac{\partial^2 \mathbf{V}}{\partial y_2^2} + \frac{\partial^2 \mathbf{V}}{\partial y_3^2},$$
$$\operatorname{div}_{\boldsymbol{y}_*}\mathbf{V} = \frac{\partial V^2}{\partial y_2} + \frac{\partial V^3}{\partial y_3}, \quad \mathbf{V} = (V^1, V^2, V^3).$$

By collecting terms of order $\mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$ in $(2.3)_1$, we arrive at

$$\begin{aligned}&-\nu(\Delta_{y_*}V_0^1\mathbf{a}^1 + \Delta_{y_*}V_0^2\mathbf{a}^2 + \Delta_{y_*}V_0^3\mathbf{a}^3) + \frac{\partial P_0}{\partial x_1}\mathbf{a}^1 + \frac{\partial P_1}{\partial y_2}\mathbf{a}^2 + \frac{\partial P_1}{\partial x_3}\mathbf{a}^3\\ &= f_1\lambda\mathbf{a}_1 + (\cos\alpha f_2 + \sin\alpha f_3)\,\mathbf{a}^2 + (-\sin\alpha f_2 + \cos\alpha f_3)\,\mathbf{a}^3 \quad \text{in } \Omega\,,\end{aligned}$$

Taking into account the incompressibility condition ($\operatorname{div}_{\mathbf{y}_*}\mathbf{V}_0 = 0$ in Ω), the zero boundary condition for the velocity on Γ and the pressure boundary condition $P_0(i) = p_i$, $(i = 0, l)$, we deduce that

$$V_0^1(\mathbf{y}_*) = \frac{2}{\pi}\left(1 - |\mathbf{y}_*|^2\right)F_0^*, \quad V_0^2 = 0, \quad V_0^3 = 0.$$
$$P_0(x_1) = -\frac{8\nu}{\pi}F_0^* x_1 + \int_0^{x_1} f_1(\xi)\,d\xi + p_0.$$
$$P_1(x_1,\mathbf{y}_*) = f_2(x_1)\left(\mathbf{e}_\alpha(x_1)\mathbf{y}_*\right) + f_3(x_1)\left(\mathbf{e}_\alpha^\perp(x_1)\mathbf{y}_*\right)\,,$$

where $F_0^* = \frac{\pi}{8\mu\ell}\big(p_0 - p_l + \int_0^l f_1(\xi)d\xi\big)$ is a constant. As expected, we obtained the zero-order approximation for the velocity given as the Poiseuille-type solution with no effects of the pipe's distortion and inertial terms. Thus, we aim to construct the higher-order correctors.

The $\mathcal{O}(1/\varepsilon)$-order term. Grouping the terms of order $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$ from the momentum equation, we obtain the following equation satisfied by the first velocity component

$$\begin{aligned}&-\nu\Big(\Delta_{y_*}V_1^1 + \kappa\cos\alpha\frac{\partial V_0^1}{\partial y_2} - \kappa\sin\alpha\frac{\partial V_0^1}{\partial y_3} + \kappa\left(\mathbf{e}_\alpha\cdot\mathbf{y}_*\right)\Delta_{y_*}V_0^1\Big)\\ &+ \kappa\left(\mathbf{e}_\alpha\cdot\mathbf{y}_*\right)\frac{\partial P_0}{\partial x_1} + \frac{\partial P_1}{\partial x_1} = 0\,,\end{aligned}$$

leading to the problem

$$-\nu\Delta_{y_*}V_1^1 = y_2 H_1(x_1) + y_3\,H_2(x_2) \quad \text{in } \Omega\,,$$
$$V_1^1 = 0 \quad \text{on } \Gamma\,,$$

where

$$H_1(x_1) = -\cos\alpha\Big(\frac{4\nu\kappa}{\pi}F_0^* + \kappa f_1 + \frac{\partial f_2}{\partial x_1} - \tau f_3\Big) - \sin\alpha\Big(\tau f_2 + \frac{\partial f_3}{\partial x_1}\Big),$$
$$H_2(x_1) = \sin\alpha\Big(\frac{4\nu\kappa}{\pi}F_0^* + \kappa f_1 + \frac{\partial f_2}{\partial x_1} - \tau f_3\Big) - \cos\alpha\Big(\tau f_2 + \frac{\partial f_3}{\partial x_1}\Big)\,.$$

Following [20], we obtain

$$V_1^1 = \frac{1}{8\nu}\left(1 - |\mathbf{y}_*|^2\right)\left(y_2 H_1(x_1) + y_3 H_2(x_1)\right)\,.$$

It should be observed that the corrector V_1^1 feels the effects of the pipe's distortion through the explicit appearance of the flexion κ and torsion τ.

For the remaining two velocity components, we obtain

$$
\begin{aligned}
&-\nu\Big(\Delta_{y_*}V_1^2-\kappa\cos\alpha\frac{\partial V_0^2}{\partial y_2}+\kappa\sin\alpha\frac{\partial V_0^2}{\partial y_3}\Big)+\frac{\partial P_2}{\partial y_2}\\
&\quad -V_0^1\frac{\partial V_0^1}{\partial y_2}+\big(V_0^2\sin\alpha+V_0^3\cos\alpha\big)\Big(\frac{\partial V_0^2}{\partial y_3}-\frac{\partial V_0^3}{\partial y_2}\Big)=0\quad\text{in }\Omega\,,\\
&-\nu\Big(\Delta_{y_*}V_1^3-\kappa\cos\alpha\frac{\partial V_0^3}{\partial y_2}+\kappa\sin\alpha\frac{\partial V_0^3}{\partial y_3}\Big)+\frac{\partial P_2}{\partial y_3}\\
&\quad -V_0^1\frac{\partial V_0^1}{\partial y_3}+\big(V_0^2\cos\alpha-V_0^3\sin\alpha\big)\Big(\frac{\partial V_0^3}{\partial y_2}-\frac{\partial V_0^2}{\partial y_3}\Big)=0,\quad\text{in }\Omega\,.
\end{aligned}
$$

In view of the zero-order approximation obtained above, the problem satisfied by (V_1^2,V_1^3) becomes

$$
\begin{gathered}
-\nu\Delta_{y_*}V_1^2+\frac{\partial P_2}{\partial y_2}-V_0^1\frac{\partial V_0^1}{\partial y_2}=0\quad\text{in }\Omega\,,\\
-\nu\Delta_{y_*}V_1^3+\frac{\partial P_2}{\partial y_3}-V_0^1\frac{\partial V_0^1}{\partial y_3}=0\quad\text{in }\Omega\,,\\
\frac{\partial V_1^2}{\partial y_2}+\frac{\partial V_1^3}{\partial y_3}=0\quad\text{in }\Omega,\\
V_1^2=V_1^3=0\quad\text{on }\Gamma\,.
\end{gathered}
$$

implying the solution is given by

$$
V_1^2=V_1^3=0,\quad P_2(\mathbf{y}_*)=\frac{2}{\pi^2}(F_0^*)^2\left(1-|\mathbf{y}_*|^2\right)^2\,.
$$

The computed first-order corrector still contains no contribution from the inertial term originating from the imposed Bernoulli pressure at the pipe's ends. Therefore, we need to continue the computation.

The $\mathcal{O}(1)$-order term. Grouping the $\mathcal{O}(1)$-terms from the momentum equation yields the following equation for the first component

$$
\begin{aligned}
&-\nu\Big(\frac{\partial^2 V_0^1}{\partial x_1^2}+\Delta_{y_*}V_2^1+\kappa\cos\alpha\frac{\partial V_1^1}{\partial y_2}-\kappa\sin\alpha\frac{\partial V_1^1}{\partial y_3}-\kappa^2V_0^1\Big)\\
&-\nu\kappa\left(\mathbf{e}_\alpha\cdot\mathbf{y}_*\right)\Big(\Delta_{y_*}V_1^1-\kappa\cos\alpha\frac{\partial V_0^1}{\partial y_2}+\kappa\sin\alpha\frac{\partial V_0^1}{\partial y_3}\Big)\\
&-\nu\kappa^2(\mathbf{e}_\alpha\cdot\mathbf{y}_*)^2\Delta_{\mathbf{y}_*}V_0^1+\kappa^2(\mathbf{e}_\alpha\cdot\mathbf{y}_*)^2\frac{\partial P_0}{\partial x_1}+\kappa\left(\mathbf{e}_\alpha\cdot\mathbf{y}_*\right)\frac{\partial P_1}{\partial x_1}+\frac{\partial P_2}{\partial x_1}=0\quad\text{in }\Omega\,.
\end{aligned}
$$

If we apply the derived expressions for the zero and the first-order approximation and include the no-slip condition, we arrive at

$$
\begin{gathered}
\Delta_{y_*}V_2^1=A_1y_2^2+A_2y_2y_3+A_3y_3^2+A_4\quad\text{in }\Omega\,,\\
V_2^1=0\quad\text{on }\Gamma\,,
\end{gathered}
\tag{4.2}
$$

where

$$
A_1=\frac{11H_1}{8\nu}\kappa\cos\alpha-\frac{H_2}{8\nu}\kappa\sin\alpha-\frac{2\kappa^2F_0^*}{\pi}-\kappa^2\cos^2\alpha\frac{4F_0^*}{\pi}+\frac{\kappa^2}{\nu}\cos^2\alpha f_1
$$

$$
\begin{aligned}
&\quad + \frac{\kappa}{\nu}\left[f_2'\cos^2\alpha + f_3'\cos\alpha\sin\alpha + \tau f_2\cos\alpha\sin\alpha - \tau f_3\cos^2\alpha\right] ,\\
A_2 &= \frac{5H_2}{4\nu}\,\kappa\cos\alpha - \frac{5H_1}{4\nu}\,\kappa\sin\alpha + \frac{8\kappa F_0^*}{\pi}\kappa\sin\alpha\cos\alpha - \frac{2\kappa^2}{\nu}\cos\alpha\sin\alpha f_1\\
&\quad + \frac{\kappa}{\nu}\big[-2\cos\alpha\sin\alpha f_2' + (\cos^2\alpha-\sin^2\alpha)f_3' + (\cos^2\alpha-\sin^2\alpha)\tau f_2\\
&\quad + 2\cos\alpha\sin\alpha\tau f_3\big] ,\\
A_3 &= \frac{H_1}{8\nu}\,\kappa\cos\alpha - \frac{11H_2}{8\nu}\kappa\sin\alpha - \frac{2\kappa^2F_0^*}{\pi} - \kappa^2\sin^2\alpha\frac{4F_0^*}{\pi} + \frac{\kappa^2}{\nu}\sin^2\alpha f_1\\
&\quad + \frac{\kappa}{\nu}\left[f_2'\sin^2\alpha - f_3'\cos\alpha\sin\alpha - \tau f_2\cos\alpha\sin\alpha - \tau f_3\sin^2\alpha\right] ,\\
A_4 &= -\kappa\cos\alpha\frac{H_1}{8\nu} + \kappa\sin\alpha\frac{H_2}{8\nu} + 2\frac{\kappa^2F_0^*}{\pi} .
\end{aligned}
$$

As in [21], the system (4.2) can be explicitly solved as

$$
V_2^1(x_1,\mathbf{y}_*) = \left(|\mathbf{y}_*|^2-1\right)\left[B_1y_2^2 + B_2y_2y_3 + B_3y_3^2 + B_4\right] ,
$$

where

$$
B_1 = \frac{7A_1-A_3}{96} ,\quad B_2 = \frac{A_2}{12} ,\quad B_3 = \frac{7A_3-A_1}{96} ,\quad B_4 = \frac{A_4}{4} + \frac{A_1+A_3}{32} .
$$

Finally, in the remaining two components of the momentum equation, we deduce the contribution of the inertial terms. Indeed, collecting the $\mathcal{O}(1)$ terms in the momentum equation, we obtain

$$
\begin{aligned}
&-\nu\Big(\Delta_{y_*}V_2^2 - \kappa\cos\alpha\frac{\partial V_1^2}{\partial y_2} + \kappa\sin\alpha\frac{\partial V_1^2}{\partial y_3} + \Big(2\kappa\frac{\partial V_0^1}{\partial x_1} + \kappa' V_0^1\Big)\cos\alpha + \kappa\tau V_0^1\sin\alpha\Big)\\
&+ \frac{\partial P_3}{\partial y_2} + \left(V_0^1\right)^2\kappa\cos\alpha - \left(V_0^1\right)^2\kappa\cos\alpha - \kappa(\mathbf{e}_\alpha\cdot\mathbf{y}_*)V_0^1\frac{\partial V_0^1}{\partial y_2}\\
&= V_1^1\frac{\partial V_0^1}{\partial y_2} - V_0^1\frac{\partial V_1^1}{\partial y_2} = 0 \quad \text{in } \Omega ,\\
&-\nu\Big(\Delta_{y_*}V_2^3 + \kappa\sin\alpha\frac{\partial V_1^3}{\partial y_3} - \kappa\cos\alpha\frac{\partial V_1^3}{\partial y_2} - \Big(2\kappa\frac{\partial V_0^1}{\partial x_1} + \kappa' V_0^1\Big)\sin\alpha + \kappa\tau V_0^1\cos\alpha\Big)\\
&+ \frac{\partial P_3}{\partial y_3} - \left(V_0^1\right)^2\kappa\sin\alpha + \left(V_0^1\right)^2\kappa\sin\alpha - \kappa(\mathbf{e}_\alpha\cdot\mathbf{y}_*)V_0^1\frac{\partial V_0^1}{\partial y_3}\\
&- V_1^1\frac{\partial V_0^1}{\partial y_3} - V_0^1\frac{\partial V_1^1}{\partial y_3} = 0 \quad \text{in } \Omega ,
\end{aligned}
$$

which, since $V_1^2 = V_1^3 = 0$ and $V_0^1 = V_0^1(\mathbf{y}_*)$, reduces to the problem

$$
\begin{gathered}
-\nu\left(\Delta_{y_*}V_2^2 + \kappa' V_0^1\cos\alpha + \kappa\tau V_0^1\sin\alpha\right) + \frac{\partial P_3}{\partial y_2}\\
-\kappa(\mathbf{e}_\alpha\cdot\mathbf{y}_*)V_0^1\frac{\partial V_0^1}{\partial y_2} - V_1^1\frac{\partial V_0^1}{\partial y_2} - V_0^1\frac{\partial V_1^1}{\partial y_2} = 0 \quad \text{in } \Omega ,\\
-\nu\left(\Delta_{y_*}V_2^3 - \kappa' V_0^1\sin\alpha + \kappa\tau V_0^1\cos\alpha\right) + \frac{\partial P_3}{\partial y_3}\\
-\kappa(\mathbf{e}_\alpha\cdot\mathbf{y}_*)V_0^1\frac{\partial V_0^1}{\partial y_3} - V_1^1\frac{\partial V_0^1}{\partial y_3} - V_0^1\frac{\partial V_1^1}{\partial y_3} = 0 \quad \text{in } \Omega ,\\
\frac{\partial V_1^1}{\partial x_1} + \left(\kappa'\left(\mathbf{e}_\alpha\cdot\mathbf{y}_*\right) + \kappa\tau\left(\mathbf{e}_\alpha^\perp\cdot\mathbf{y}_*\right)\right)V_0^1 + \frac{\partial V_2^2}{\partial y_2} + \frac{\partial V_3^3}{\partial y_3} = 0 \quad \text{in } \Omega ,
\end{gathered}
$$

$$V_2^2 = V_2^3 = 0 \quad \text{on } \Gamma\,.$$

To simplify the proof layout, we split the desired solution in two particular solutions

$$V_2^2 := \overline{V}_2^2 + \widetilde{V}_2^2, \quad V_2^3 := \overline{V}_2^3 + \widetilde{V}_2^3, \quad P_3 := \overline{P}_3 + \widetilde{P}_3\,,$$

which satisfy

$$\begin{gathered}
-\nu\left(\Delta_{y_*}\overline{V}_2^2 + \kappa' V_0^1\cos\alpha + \kappa\tau V_0^1\sin\alpha\right) + \frac{\partial\overline{P}_3}{\partial y_2} = 0 \quad \text{in } \Omega\,,\\
-\nu\left(\Delta_{y_*}\overline{V}_2^3 - \kappa' V_0^1\sin\alpha + \kappa\tau V_0^1\cos\alpha\right) + \frac{\partial\overline{P}_3}{\partial y_3} = 0 \quad \text{in } \Omega\,,\\
\frac{\partial V_1^1}{\partial x_1} + \left(\kappa'\left(\mathbf{e}_\alpha\cdot\mathbf{y}_*\right) + \kappa\tau\left(\mathbf{e}_\alpha^\perp\cdot\mathbf{y}_*\right)\right)V_0^1 + \frac{\partial\overline{V}_2^2}{\partial y_2} + \frac{\partial\overline{V}_3^3}{\partial y_3} = 0 \quad \text{in } \Omega\,,\\
\overline{V}_2^2 = \overline{V}_2^3 = 0 \quad \text{on } \Gamma\,,
\end{gathered} \tag{4.3}$$

and

$$\begin{gathered}
-\nu\Delta_{y_*}\widetilde{V}_2^2 + \frac{\partial P_3}{\partial y_2} - \kappa(\mathbf{e}_\alpha\cdot\mathbf{y}_*)V_0^1\frac{\partial V_0^1}{\partial y_2} - V_1^1\frac{\partial V_0^1}{\partial y_2} - V_0^1\frac{\partial V_1^1}{\partial y_2} = 0 \quad \text{in } \Omega\,,\\
-\nu\Delta_{y_*}\widetilde{V}_2^3 + \frac{\partial P_3}{\partial y_3} - \kappa(\mathbf{e}_\alpha\cdot\mathbf{y}_*)V_0^1\frac{\partial V_0^1}{\partial y_3} - V_1^1\frac{\partial V_0^1}{\partial y_3} - V_0^1\frac{\partial V_1^1}{\partial y_3} = 0 \quad \text{in } \Omega\,,\\
\frac{\partial\widetilde{V}_2^2}{\partial y_2} + \frac{\partial\widetilde{V}_3^3}{\partial y_3} = 0 \quad \text{in } \Omega\,,\\
\widetilde{V}_2^2 = \widetilde{V}_2^3 = 0 \quad \text{on } \Gamma\,.
\end{gathered} \tag{4.4}$$

When comparing with the second order corrector from [21], the difference is exactly in the particular solutions $\widetilde{V}_2^2, \widetilde{V}_2^3$ and $\widetilde{P}_3$ which satisfy (4.4). Moreover, by expanding (4.4), we obtain

$$\begin{aligned}
\Delta_{y_*}\widetilde{V}_2^2 - \frac{1}{\nu}\frac{\partial\widetilde{P}_3}{\partial y_2} &= \frac{8(F_0^*)^2\kappa}{\nu^2\pi^2}(1-|\mathbf{y}_*|^2)(y_2\cos\alpha - y_3\sin\alpha)y_2\\
&\quad - \frac{\partial}{\partial y_2}\left(\frac{F_0^*}{4\nu^2\pi}\left(1-|\mathbf{y}_*|^2\right)^2(y_2H_1(x_1) + y_3\,H_2(x_1))\right)\,,\\
\Delta_{y_*}\widetilde{V}_2^3 - \frac{1}{\nu}\frac{\partial\widetilde{P}_3}{\partial y_3} &= \frac{8(F_0^*)^2\kappa}{\nu^2\pi^2}(1-|\mathbf{y}_*|^2)(y_2\cos\alpha - y_3\sin\alpha)y_3\\
&\quad - \frac{\partial}{\partial y_3}\left(\frac{F_0^*}{4\nu^2\pi}\left(1-|\mathbf{y}_*|^2\right)^2(y_2H_1(x_1) + y_3H_2(x_1))\right)\,,\\
&\frac{\partial\widetilde{V}_2^2}{\partial y_2} + \frac{\partial\widetilde{V}_3^3}{\partial y_3} = 0 \quad \text{in } \Omega\,,\\
&\widetilde{V}_2^2 = \widetilde{V}_2^3 = 0 \quad \text{on } \Gamma\,.
\end{aligned} \tag{4.5}$$

In view of the above, we seek for the solution of the form

$$\begin{gathered}
\overline{V}_2^2(x_1,\mathbf{y}_*) = (1-|\mathbf{y}_*|^2)\left(B_5y_2^2 + B_6y_2y_3 + B_7y_3^2 + B_8\right)\,,\\
\overline{V}_2^3(x_1,\mathbf{y}_*) = (1-|\mathbf{y}_*|^2)\left(B_9y_2^2 + B_{10}y_2y_3 + B_{11}y_3^2 + B_{12}\right),\\
\overline{P}_3(x_1,\mathbf{y}_*) = M_1y_2^3 + M_2y_3^2 + M_3y_2y_3^2 + M_4y_2^2y_3 + M_5y_2 + M_6y_3\,,
\end{gathered}$$

$$
\begin{aligned}
\widetilde{V}_2^2(x_1,\mathbf{y}_*) &= (1-|\mathbf{y}_*|^2)\Big(C_1y_2^4+C_2y_2^3y_3+C_3y_2^2y_3^2+C_4y_2y_3^3+C_5y_3^4+C_6y_2^2\\
&\quad + C_7y_2y_3+C_8y_3^2+C_9\Big)\,,\\
\widetilde{V}_2^3(x_1,\mathbf{y}_*) &= (1-|\mathbf{y}_*|^2)\Big(D_1y_3^4+D_2y_3^3y_2+D_3y_3^2y_2^2+D_4y_3y_2^3+D_5y_2^4\\
&\quad + D_6y_3^2+D_7y_3y_2+D_8y_2^2+D_9\Big)\,,\\
\widetilde{P}_3(x_1,\mathbf{y}_*) &= \frac{F_0^*}{4\nu^2\pi}\Big(1-|\mathbf{y}_*|^2\Big)^2(y_2H_1(x_1)+y_3H_2(x_1))+L_1y_2^5+L_2y_2^4y_3\\
&\quad + L_3y_2^3y_3^2+L_4y_2^2y_3^3+L_5y_2y_3^4+L_6y_3^5+L_7y_2^3+L_8y_2^2y_3\\
&\quad + L_9y_2y_3^2+L_{10}y_3^3
\end{aligned}
$$

such that $\overline{V}_2^2, \overline{V}_2^3$ and $\overline{P}_3$ satisfy (4.3), and $\widetilde{V}_2^2, \widetilde{V}_2^3$, $\widetilde{P}_3$ satisfy (4.5). Denoting

$$
a:=\frac{2\nu}{\pi}\left(\kappa'\cos\alpha+\kappa\tau\sin\alpha\right)F_0^*\,,\quad b:=\frac{2\nu}{\pi}\left(\kappa\tau\cos\alpha-\kappa'\sin\alpha\right)F_0^*\,,
$$
$$
c:=\frac{8(F_0^*)^2\kappa}{\nu^2\pi^2}\cos\alpha\,,\quad d:=-\frac{8(F_0^*)^2\kappa}{\nu^2\pi^2}\sin\alpha\,,
$$

after tedious but straightforward calculation, we obtain the sought coefficients:

$$
B_5=\frac{-a}{96}\,,\quad B_6=\frac{b}{24}\,,\quad B_7=\frac{-5a}{96}\,,\quad B_8=\frac{a}{96}\,,\quad B_9=\frac{-5b}{96}\,,\quad B_{10}=\frac{a}{24}\,,
$$
$$
B_{11}=\frac{-b}{96}\,,\quad B_{12}=\frac{b}{96}\,,\quad M_1=\nu\frac{-a}{4}\,,\quad M_2=\nu\frac{-b}{4}\,,\quad M_3=\nu\frac{-a}{4}\,,
$$
$$
M_4=\nu\,\frac{-b}{4}\,,\quad M_5=\nu\,\frac{5a}{6}\,,\quad M_6=\nu\,\frac{5b}{6}\,,\quad C_1=\frac{-c}{1152}\,,\quad C_2=\frac{d}{192}\,,
$$
$$
C_3=\frac{-c}{144}\,,\quad C_4=\frac{d}{192}\,,\quad C_5=\frac{-7c}{1152}\,,\quad C_6=\frac{5c}{1152}\,,\quad C_7=\frac{-d}{64}\,,
$$
$$
C_8=\frac{23c}{1152}\,,\quad C_9=\frac{7c}{1152}\,,\quad D_1=\frac{-d}{1152}\,,\quad D_2=\frac{c}{192}\,,\quad D_3=\frac{-d}{144}\,,
$$
$$
D_4=\frac{c}{192}\,,\quad D_5=\frac{-7d}{1152}\,,\quad D_6=\frac{5d}{1152}\,,\quad D_7=\frac{-c}{64}\,,\quad D_8=\frac{23d}{1152}\,,
$$
$$
D_9=\frac{7d}{1152}\,,\quad L_1=\nu\frac{5c}{24}\,,\quad L_2=\nu\frac{5d}{24}\,,\quad L_3\nu=\nu\frac{5c}{12}\,,\quad L_4=\nu\frac{5d}{12}\,,
$$
$$
L_5=\nu\frac{5c}{24}\,,\quad L_6=\nu\frac{5d}{24}\,,\quad L_7=\nu\frac{-3c}{8}\,,\quad L_8=\nu\frac{-3d}{8}\,,
$$
$$
L_9=\nu\frac{-3c}{8}\,,\quad L_{10}=\nu\frac{-3d}{8}
$$

To conclude, the second-order velocity corrector takes the form

$$
\mathbf{V}_2=\left(V_2^1,V_2^2,V_2^3\right)=\left(V_2^1,\overline{V}_2^2+\widetilde{V}_2^2,\overline{V}_2^3+\widetilde{V}_2^3\right)\,.
$$

5. Error analysis

5.1. **Asymptotic approximation.** Summarizing the calculation performed in Section 4, we write the asymptotic approximation of the solution to the problem (2.3) as

$$
\begin{gathered}
\boldsymbol{\nu}_\varepsilon(\mathbf{z}):=\mathbf{V}_\varepsilon^{\text{approx}}(\mathbf{x}),\\
\mathbf{V}_\varepsilon^{\text{approx}}(\mathbf{x})=V_0^1\left(\frac{\mathbf{x}_*}{\varepsilon}\right)\mathbf{a}_1+\varepsilon V_1^1\left(x_1,\frac{\mathbf{x}_*}{\varepsilon}\right)\mathbf{a}_1+\varepsilon^2\mathbf{V}_2\left(x_1,\frac{x_*}{\varepsilon}\right)\,,
\end{gathered}
\tag{5.1}
$$

$$
\begin{aligned}
&q_\varepsilon(\mathbf{z}) := P_\varepsilon^{\text{approx}}(\mathbf{x})\,, \\
&P_\varepsilon^{\text{approx}}(\mathbf{x}) = \frac{1}{\varepsilon^2}P_0(x_1) + \frac{1}{\varepsilon}P_1\left(x_1, \frac{\mathbf{x}_*}{\varepsilon}\right) + P_2\left(x_1, \frac{\mathbf{x}_*}{\varepsilon}\right)\,,
\end{aligned} \tag{5.2}
$$

for $\mathbf{z} := \mathbf{\Phi}_\varepsilon^\alpha(\mathbf{x})$. In the following, we aim to evaluate the difference between the original solution (which cannot be found) and the approximate one (given by (5.1)–(5.2)) in a suitable functional norm.

5.2. **A priori estimates.** Before proving the a priori estimate for the velocity, we first recall some technical results which can be proved in a standard manner by taking into account the dependence of the domain on the small parameter ε (see [15], [16] for details). Throughout Section 5, $C > 0$ denotes a generic constant independent of ε.

Lemma 5.1. *Poincare's inequality holds,*

$$\|\varphi\|_{L^2(\Omega_\varepsilon^\alpha)} \le C\varepsilon\|\nabla\varphi\|_{L^2(\Omega_\varepsilon^\alpha)}\,,$$

for all $\varphi \in H^1(\Omega_\varepsilon^\alpha)$ *such that* $\varphi = 0$ *on* $\Gamma_\varepsilon^\alpha$.

Lemma 5.2. *Let* $K \in L_0^2(\Omega_\varepsilon^\alpha)$. *Then the problem*

$$
\begin{gathered}
\operatorname{div}\varphi_\varepsilon = K \quad \text{in } \Omega_\varepsilon^\alpha\,, \\
\varphi_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon^\alpha
\end{gathered}
$$

admits a solution satisfying

$$\|\nabla\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} \le \frac{C}{\varepsilon}\|K\|_{L^2(\Omega_\varepsilon^\alpha)}\,.$$

We now prove the a priori estimate for the velocity which we need in the error analysis.

Proposition 5.3. *Let* $(\mathbf{u}_\varepsilon, \Psi_\varepsilon)$ *be the solution of* (2.3). *Then there exists a constant* C *such that*

$$\|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} \le C. \tag{5.3}$$

Proof. Employing $\mathbf{u}_\varepsilon$ as a test function in the momentum equation gives

$$
\begin{aligned}
\nu\int_{\Omega_\varepsilon^\alpha}|\nabla\mathbf{u}_\varepsilon|^2 &= p_0\int_{\Sigma_\varepsilon^0}\mathbf{u}_\varepsilon\cdot\mathbf{t}(0) - p_l\int_{\Sigma_\varepsilon^l}\mathbf{u}_\varepsilon\cdot\mathbf{t}(l) - \int_{\Omega_\varepsilon^\alpha}(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\cdot\mathbf{u}_\varepsilon \\
&\quad + \int_{\Omega_\varepsilon^\alpha}\mathbf{u}_\varepsilon\cdot(\nabla\mathbf{u}_\varepsilon)^t\cdot\mathbf{u}_\varepsilon + \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon^\alpha}\mathbf{f}\mathbf{u}_\varepsilon
\end{aligned}
$$

Using Lemma 5.1 and the fact that $|\Omega_\varepsilon^\alpha| = \mathcal{O}(\varepsilon^2)$, we have

$$
\begin{aligned}
&\Big|p_0\int_{\Sigma_\varepsilon^0}\mathbf{u}_\varepsilon\cdot\mathbf{t}(0) - p_l\int_{\Sigma_\varepsilon^l}\mathbf{u}_\varepsilon\cdot\mathbf{t}(l)\Big| \\
&= \Big|\int_{\Omega_\varepsilon^\alpha}\operatorname{div}\Big(\Big(p_0 + \frac{p_l - p_0}{l}x_1\Big)\mathbf{u}_\varepsilon\Big)\Big| \\
&\le C\varepsilon^2\|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\,.
\end{aligned}
$$

Similarly,

$$\frac{1}{\varepsilon^2}\Big|\int_{\Omega_\varepsilon^\alpha}\mathbf{f}\mathbf{u}_\varepsilon\Big| \le \frac{1}{\varepsilon^2}\|\mathbf{f}\|_{L^2(\Omega_\varepsilon^\alpha)}\|\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} \le C\|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\,.$$

For the remaining terms, by straight forward calculation, we deduce

$$
\begin{aligned}
&(\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon \cdot \mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon \cdot (\nabla \mathbf{u}_\varepsilon)^t \cdot \mathbf{u}_\varepsilon \\
&= \big((\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon \cdot (\nabla \mathbf{u}_\varepsilon)^t\big) \cdot \mathbf{u}_\varepsilon \\
&= \Big(\big[\sum_{i=1}^3 u_i \partial_i u_1 \sum_{i=1}^3 u_i \partial_i u_2 \ \sum_{i=1}^3 u_i \partial_i u_3\big] \\
&\quad - \big[\sum_{i=1}^3 u_i \partial_1 u_i \ \sum_{i=1}^3 u_i \partial_2 u_i \sum_{i=1}^3 u_i \, \partial_3 u_i\big]\Big) \cdot \mathbf{u}_\varepsilon = 0\,.
\end{aligned}
$$

implying

$$
\int_{\Omega_\varepsilon^\alpha} (\mathbf{u}_\varepsilon \cdot \nabla)\mathbf{u}_\varepsilon \cdot \mathbf{u}_\varepsilon - \int_{\Omega_\varepsilon^\alpha} \mathbf{u}_\varepsilon \cdot (\nabla \mathbf{u}_\varepsilon)^t \cdot \mathbf{u}_\varepsilon = 0\,.
$$

Collecting the above, we deduce (5.3). ☐

5.3. **Error estimates.** We are now in the position to formulate and prove the main result of this section providing the order of accuracy of the proposed asymptotic solution. Since we work in a thin-domain setting, we express the error estimates in the rescaled norm $|\Omega_\varepsilon^\alpha|^{-1/2}\|\cdot\|_{L^2(\Omega_\varepsilon^\alpha)}$. The goal is to derive the satisfactory error estimates acknowledging the contributions of the pipe's distortion and the Bernoulli pressure on the effective flow.

Theorem 5.4. *The following estimates hold:*

$$
|\Omega_\varepsilon^\alpha|^{-1/2}\|\mathbf{u}_\varepsilon - \boldsymbol{\nu}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} \le C\varepsilon^3\,, \tag{5.4}
$$

$$
|\Omega_\varepsilon^\alpha|^{-1/2}\|\psi_\varepsilon - q_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)/\mathbf{R}} \le C\varepsilon\,. \tag{5.5}
$$

where $\boldsymbol{\nu}_\varepsilon$ and q_ε are given by (5.1)–(5.2)..

Proof. The pair $(\boldsymbol{\nu}_\varepsilon, q_\varepsilon)$ satisfies

$$
\begin{gathered}
-\nu\Delta\boldsymbol{\nu}_\varepsilon + (\boldsymbol{\nu}_\varepsilon \cdot \nabla)\boldsymbol{\nu}_\varepsilon - \boldsymbol{\nu}_\varepsilon \cdot (\nabla\boldsymbol{\nu}_\varepsilon)^t + \nabla q_\varepsilon = \frac{1}{\varepsilon^2}\mathbf{f} + \mathbf{E}_\varepsilon \quad \text{in } \Omega_\varepsilon^\alpha\,, \\
\operatorname{div}\boldsymbol{\nu}_\varepsilon = \pi_\varepsilon \ \text{ in } \Omega_\varepsilon^\alpha\,, \\
\boldsymbol{\nu}_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon^\alpha\,,
\end{gathered}
$$

where the remainder $\|\mathbf{E}_\varepsilon\|_{L^\infty(\Omega_\varepsilon^\alpha)} \le C\varepsilon$ and thus, since $|\Omega_\varepsilon^\alpha| = \mathcal{O}(\varepsilon^2)$, we obtain $\|\mathbf{E}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} = \mathcal{O}(\varepsilon^2)$. Analogously, we have $\|\pi_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} = \mathcal{O}(\varepsilon^3)$. Denoting the differences

$$
\mathbf{R}_\varepsilon = \mathbf{u}_\varepsilon - \boldsymbol{\nu}_\varepsilon, \quad r_\varepsilon = p_\varepsilon - q_\varepsilon\,,
$$

we obtain

$$
\begin{gathered}
\begin{aligned}
&-\nu\Delta\mathbf{R}_\varepsilon + \nabla r_\varepsilon + (\mathbf{R}_\varepsilon \cdot \nabla)\,\mathbf{u}_\varepsilon + (\boldsymbol{\nu}_\varepsilon \cdot \nabla)\,\mathbf{R}_\varepsilon - \Big(\mathbf{R}_\varepsilon(\nabla\mathbf{u}_\varepsilon)^t + \boldsymbol{\nu}_\varepsilon\,(\nabla\mathbf{R}_\varepsilon)^t\Big) \\
&= -\mathbf{E}_\varepsilon \quad \text{in } \Omega_\varepsilon^\alpha\,,
\end{aligned} \\
\operatorname{div}\mathbf{R}_\varepsilon = -\pi_\varepsilon \quad \text{in } \Omega_\varepsilon^\alpha\,, \\
\mathbf{R}_\varepsilon = 0 \quad \text{on } \Gamma_\varepsilon^\alpha\,.
\end{gathered} \tag{5.6}
$$

Let us now introduce the test function $\mathbf{d}_\varepsilon$ as the solution of the auxiliary problem

$$
\begin{gathered}
\operatorname{div}\mathbf{d}_\varepsilon = r_\varepsilon \quad \text{in } \Omega_\varepsilon^\alpha\,, \\
\mathbf{d}_\varepsilon = 0 \quad \text{in } \partial\Omega_\varepsilon^\alpha\,.
\end{gathered}
$$

Since the pressure is determined up to an additive constant, we can suppose $\int_{\Omega_\varepsilon^\alpha} r_\varepsilon = 0$, so the problem admits the solution satisfying

$$\|\nabla \mathbf{d}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} \le \frac{C}{\varepsilon}\|r_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\,, \tag{5.7}$$

because of Lemma 5.2. Using $\mathbf{d}_\varepsilon$ as the test-function in (5.6) yields

$$\begin{aligned}\|r_\varepsilon\|^2_{L^2(\Omega_\varepsilon^\alpha)} &= \nu\int_{\Omega_\varepsilon^\alpha}\nabla R_\varepsilon\nabla\mathbf{d}_\varepsilon + \int_{\Omega_\varepsilon^\alpha}\mathbf{E}_\varepsilon\,\mathbf{d}_\varepsilon\\ &\quad - \int_{\Omega_\varepsilon^\alpha}\Big[(\mathbf{R}_\varepsilon\cdot\nabla)\,\mathbf{u}_\varepsilon + (\nu_\varepsilon\cdot\nabla)\,\mathbf{R}_\varepsilon - \Big(\mathbf{R}_\varepsilon(\nabla\mathbf{u}_\varepsilon)^t + \boldsymbol{\nu}_\varepsilon\,(\nabla\mathbf{R}_\varepsilon)^t\Big)\Big]\,\mathbf{d}_\varepsilon\,.\end{aligned}$$

The first two integrals are estimated using (5.7) and the fact that $\|\mathbf{E}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} = \mathcal{O}(\varepsilon^2)$ and $\|\pi_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} = \mathcal{O}(\varepsilon^3)$:

$$\Big|\int_{\Omega_\varepsilon^\alpha}\nabla\mathbf{R}_\varepsilon\nabla\mathbf{d}_\varepsilon\Big| \le \|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\|\nabla\mathbf{d}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} \le \frac{C}{\varepsilon}\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\|r_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\,,$$

$$\Big|\int_{\Omega_\varepsilon^\alpha}\mathbf{E}_\varepsilon\,\mathbf{d}_\varepsilon\Big| \le \|\mathbf{E}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\|\mathbf{d}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} \le C\varepsilon^2\|r_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\,.$$

To estimate the remaining terms, we will need the special case of the Gagliardo-Nirenberg interpolation inequality

$$\|f\|_{L^4} \le C\|f\|^{1/4}_{L^2}\|\nabla f\|^{3/4}_{L^2}\,. \tag{5.8}$$

From this, (5.3), and the Poincare's inequality, we obtain

$$\begin{aligned}\Big|\int_{\Omega_\varepsilon^\alpha}(\mathbf{R}_\varepsilon\cdot\nabla)\,\mathbf{u}_\varepsilon\mathbf{d}_\varepsilon\Big| &\le C\|\nabla\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\|\mathbf{R}_\varepsilon\|_{L^4(\Omega_\varepsilon^\alpha)}\|\mathbf{d}_\varepsilon\|_{L^4(\Omega_\varepsilon^\alpha)}\\ &\le C\varepsilon^{1/2}\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\|\nabla\mathbf{d}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\\ &\le \frac{C}{\varepsilon^{1/2}}\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\|r_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\,,\\ \Big|\int_{\Omega_\varepsilon^\alpha}(\nu_\varepsilon\cdot\nabla)\mathbf{R}_\varepsilon\,\mathbf{d}_\varepsilon\Big| &\le \|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\|\nu_\varepsilon\|_{L^4}\|\mathbf{d}_\varepsilon\|_{L^4(\Omega_\varepsilon^\alpha)}\\ &\le C\varepsilon^{1/2}\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\|\nabla\mathbf{d}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\\ &\le \frac{C}{\varepsilon^{1/2}}\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\|r_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\,.\end{aligned}$$

In the same way, for the last two terms, we have

$$\Big|\int_{\Omega_\varepsilon^\alpha}\Big(\mathbf{R}_\varepsilon(\nabla\mathbf{u}_\varepsilon)^t + \nu_\varepsilon\,(\nabla\mathbf{R}_\varepsilon)^t\Big)\,\mathbf{d}_\varepsilon\Big| \le \frac{C}{\varepsilon^{1/2}}\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\|r_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\,.$$

Collecting these estimates, we obtain

$$\|r_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} \le \frac{C}{\varepsilon}\|\nabla\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\,. \tag{5.9}$$

Now we go back to the momentum equation (5.6) and use $\mathbf{R}_\varepsilon$ as a test function. As a result, we obtain

$$\begin{aligned}\nu\|\nabla\mathbf{R}_\varepsilon\|^2_{L^2(\Omega_\varepsilon^\alpha)} &= \int_{\Omega_\varepsilon^\alpha} r_\varepsilon\,\mathrm{div}\mathbf{R}_\varepsilon - \int_{\Omega_\varepsilon^\alpha}\mathbf{E}_\varepsilon\mathbf{R}_\varepsilon\\ &\quad + \int_{\Omega_\varepsilon^\alpha}\big[(\mathbf{R}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon + (\nu_\varepsilon\cdot\nabla)\mathbf{R}_\varepsilon - \big(\mathbf{R}_\varepsilon(\nabla\mathbf{u}_\varepsilon)^t + \boldsymbol{\nu}_\varepsilon(\nabla\mathbf{R}_\varepsilon)^t\big)\big]\mathbf{R}_\varepsilon\,.\end{aligned}$$

Using Lemma 5.1, the a priori estimate (5.3), (5.8) and (5.9), we obtain

$$\Big|\int_{\Omega_\varepsilon^\alpha} r_\varepsilon \operatorname{div} \mathbf{R}_\varepsilon\Big| = \Big|\int_{\Omega_\varepsilon^\alpha} r_\varepsilon \pi_\varepsilon\Big| \le \|r_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\|\pi_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} \le C\varepsilon^2\|\nabla \mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}$$

$$\Big|\int_{\Omega_\varepsilon^\alpha} \mathbf{E}_\varepsilon \mathbf{R}_\varepsilon\Big| \le \|\mathbf{E}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\|\mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} \le C\varepsilon^3\|\nabla \mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\,,$$

$$\Big|\int_{\Omega_\varepsilon^\alpha} (\mathbf{R}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon\,\mathbf{R}_\varepsilon\Big| \le C\,\|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\|\mathbf{R}_\varepsilon\|^2_{L^4(\Omega_\varepsilon^\alpha)} \le C\varepsilon^{1/2}\|\nabla \mathbf{R}_\varepsilon\|^2_{L^2(\Omega_\varepsilon^\alpha)}\,,$$

$$\Big|\int_{\Omega_\varepsilon^\alpha} (\nu_\varepsilon\cdot\nabla)\mathbf{R}_\varepsilon\mathbf{R}_\varepsilon\Big| \le \|\nabla \mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\|\boldsymbol{\nu}_\varepsilon\|_{L^4}\|\mathbf{R}_\varepsilon\|_{L^4(\Omega_\varepsilon^\alpha)} \le C\varepsilon^{1/2}\|\nabla \mathbf{R}_\varepsilon\|^2_{L^2(\Omega_\varepsilon^\alpha)}\,,$$

$$\Big|\int_{\Omega_\varepsilon^\alpha} \Big(\mathbf{R}_\varepsilon(\nabla \mathbf{u}_\varepsilon)^t + \boldsymbol{\nu}_\varepsilon\,(\nabla \mathbf{R}_\varepsilon)^t\Big)\,\mathbf{R}_\varepsilon\Big| \le C\varepsilon^{1/2}\|\nabla \mathbf{R}_\varepsilon\|^2_{L^2(\Omega_\varepsilon^\alpha)}\,.$$

Putting the above estimates together yields

$$\nu\|\nabla \mathbf{R}_\varepsilon\|^2_{L^2(\Omega_\varepsilon^\alpha)} \le C\varepsilon^{1/2}\|\nabla \mathbf{R}_\varepsilon\|^2_{L^2(\Omega_\varepsilon^\alpha)} + C\varepsilon^2\|\nabla \mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)}\,.$$

Using Young's inequality we can estimate the second term on the right-hand side as

$$\varepsilon^2\|\nabla \mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} \le \frac{\varepsilon^3}{2} + \frac{\varepsilon}{2}\|\nabla \mathbf{R}_\varepsilon\|^2_{L^2(\Omega_\varepsilon^\alpha)}$$

leading to

$$\|\nabla \mathbf{R}_\varepsilon\|_{L^2(\Omega_\varepsilon^\alpha)} \le C\,\varepsilon^3\,.$$

This proves (5.4), whereas the estimate for the pressure (5.5) then follows directly from (5.9). □

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